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# Lasers as Toda oscillators: An experimental confirmation

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ABSTRACT

We study the transient oscillatory behavior of a Nd:YAG laser modulated in such a way that at the beginning of time evolution the population inversion is approximately equal to its threshold value and the photons present in the cavity are much less than at steady state. Under those conditions the laser dynamics is very regular. We measure the period and width of the pulses as a function of the pulse intensity finding a very good agreement with the predictions of the Toda oscillator model.


## 1. Introduction

Lasers known as class-B lasers, where the dynamics of inversion is much slower than that of photons, exhibit relaxation oscillations in the emitted power. In the framework of singlemode rate equations an analytic expression for the period of such oscillations can be readily obtained in the limit of small oscillations around the stationary output. Yet, since the first years of the laser era it became clear that such an expression is inadequate to describe the early stage of laser dynamics, where the oscillations are highly nonlinear [1]. Typically, the period between two spikes is at least 50% larger than the period of the small oscillations.

A very elegant analytical treatment of the problem was given by Oppo and Politi [2], who showed that the laser is equivalent to a dissipative anharmonic oscillator subjected to the Toda potential [3]. When dissipation is neglected simple semianalytical expressions for the period of the oscillations were found, which allows for an analytical approximation in the limit of large oscillation amplitude.

The Toda model was then applied to the study of the dynamics of modulated lasers [4–7] and of the statistical properties of class-B [8–10] or bad cavity lasers [11]. More recently, analytical approximated expressions for the period of the oscillations valid for every value of the oscillation amplitude were given in Ref. [12].

Although the Toda model of the laser is known since a long time, a convincing experimental demonstration of its validity based on the measurement of the period of the oscillations and the width of the pulses along the whole dynamical evolution of a laser is still missing. In this paper we show that very clear damped oscillations can be achieved by means of an amplitude modulator which allows to almost completely deplete the cavity without affecting the population inversion, thus realizing the ideal initial conditions for the Toda model. In that way we were able to obtain a series of experimental measurements, with different laser configurations, that fully confirm the predictions of the model and its universality.

## 2. The Toda model of laser dynamics

We first present the derivation of the Toda oscillator model from the laser equations as described in [2]. The laser equations are

$$\frac{dI}{dt}=k(AD-1)I, \tag{1}$$

$$\frac{dD}{dt}=-\gamma_{\parallel}[D(1+I)-1], \tag{2}$$

where $I$ and $D$ are, respectively, the dimensionless field intensity and population difference, $k$ and $\gamma_{\parallel}$ are their decay rates, and $A$ is the pump parameter ($A=1$ at threshold). The stationary intensity above threshold is $I_0=A-1$ and the frequency of the small relaxation oscillations around that state is $\omega_r=\sqrt{k\gamma_{\parallel}I_0}$. The rate equations can be reformulated as a nonlinear second order

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differential equation in $I$ with a further derivation with respect to time of the first equation and taking into account the second one. If we introduce the variable $x=\log(I/I_0)$, the dimensionless time $\tau=\omega_r t$, and the ratio $\gamma=\gamma_\parallel/k$ of the decay rates, the equation reads

$$\frac{d^2x}{d\tau^2}=1-e^x-\sqrt{\gamma}\left(\frac{1}{\sqrt{I_0}}+\sqrt{I_0}e^x\right)\frac{dx}{d\tau}. \quad (3)$$

We consider the limit $\gamma \ll 1$ where the last, dissipative terms become negligible. In that limit the laser is equivalent to a particle of unitary mass moving in the Toda potential $V(x)=e^x-x$ with total energy $E=p^2/2+V(x)$. For a given $E$, the position $x$ varies periodically between the two extrema $x_{min}=\log(I_{min}/I_0)<0$ and $x_{max}=\log(I_{max}/I_0)>0$. The conservative dynamics is independent of the laser parameters $A$ and $\gamma$. It depends only on the energy $E$ which, in turn, is related to $x_{max}$ and $x_{min}$ through the equations

$$E=V(x_{min})=V(x_{max}), \quad (4)$$

which simply express the fact that at the extrema of the oscillations the kinetic energy is null.

The period $T_1$ between two laser pulses coincides with the period of the Toda oscillator, i.e. the time taken by the particle to pass from $x_{min}$ to $x_{max}$ and back. The width $T_2$ of one pulse can be defined as the time interval during which $I>I_0$, which is the duration of that part of the orbit's particle where $x>0$. Since with our time scaling the period of the small oscillations is $T_0=2\pi$, we can write

$$T_{1,2}=2\int_{x_{min},0}^{x_{max}}\frac{dx}{p}=\frac{T_0}{\sqrt{2}\pi}\int_{x_{min},0}^{x_{max}}\frac{dx}{\sqrt{E-V(x)}}. \quad (5)$$

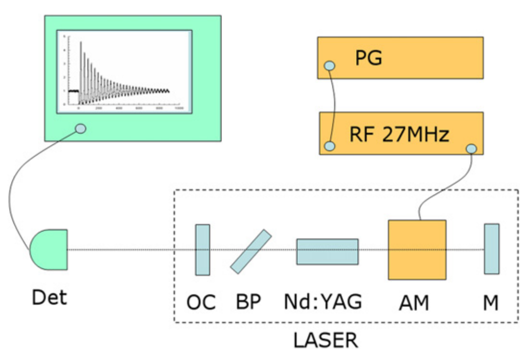


**Fig. 1.** Experimental setup of the Nd:YAG laser with an acousto-optic modulator.

Using Eq. (4) the ratios $T_{1,2}/T_0$ can be calculated as a function of one of the three quantities $E$, $x_{min}$, and $x_{max}$.

## 3. Experimental setup and observation of oscillatory behavior

A schematic diagram of the experimental setup is presented in Fig. 1. The radiation source is provided by a home-made Nd:YAG laser, a diode-side-pumped module, a brewster plate (BP) that selects the horizontal polarization, and the linear cavity mirrors OC (output coupler) and M. All the components are mounted on a slit and it is possible to change easily the distance between them. The Nd:YAG rod is 60 mm long, and its diameter is 2 mm. We call $I_d$ the current injected in the pump diode laser. The corresponding pump power for the Nd:YAG laser is given by $P=\eta(I_d-10.3\,A)$, with $\eta=5.1$ W/A.

An amplitude acousto-modulator (AM) is inserted in the laser cavity in order to observe the oscillatory behavior. The modulator is driven by a pulse of few μs containing a faster signal at frequency 27 MHz which increases the cavity losses due to diffraction. This signal is produced by a radio-frequency generator (RF 27 MHz), externally triggered by a home-made pulse generator (PG). The radiation is detected by a photodiode (Det) connected to a digital oscilloscope with a low pass filter with a cutoff frequency of 1 MHz at the input. The low pass filter removes the beatings at high frequency between the temporal modes of the laser.

The acousto-optic modulator acts as a fast loss modulator. In this case fast means that we increase the cavity losses for only $1-2$ μs in order to remove or reduce the power inside the cavity without changing the population inversion. This is possible because the decay time of the cavity is only few tens of ns whereas the the population inversion changes with a characteristic time equal to the spontaneous decay time (240 μs).

The presence of more than one longitudinal modes, which are not included in the model, does not constitute a problem, because it is known that it does not alter the oscillatory behavior of the laser [1,13].

In order to obtain a large set of data, the modulation depth was varied in the experiment from relatively low values, which induce regular oscillations, to relatively large ones, which allow to observe larger initial pulses, although the subsequent dynamics is more irregular, because the number of photons after modulation becomes small and the statistical fluctuations are more relevant. Two typical behaviors of the laser output are shown in

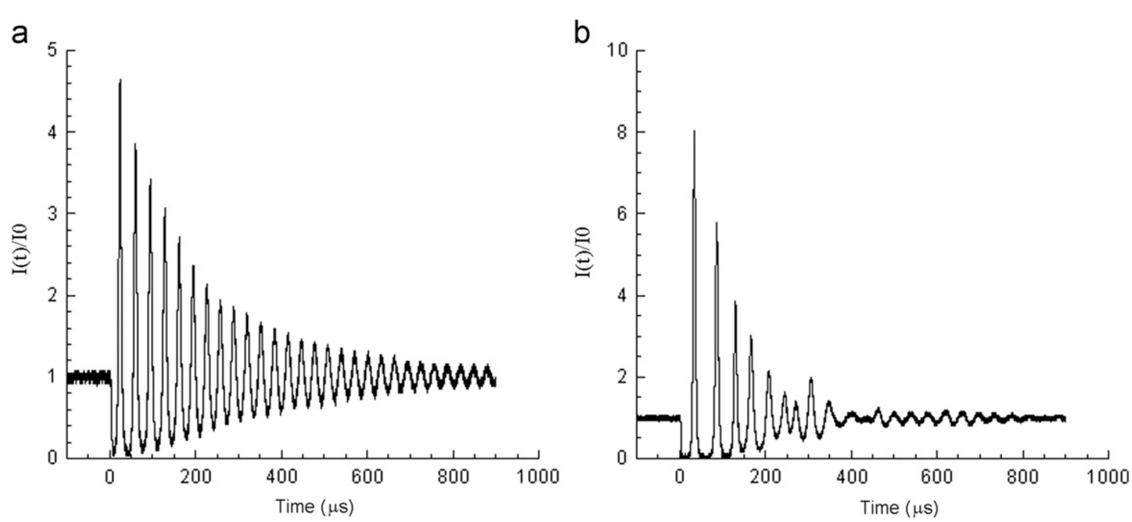


**Fig. 2.** Intensity dynamics with low (a) and high (b) loss modulation. $I_d$=11.5 A, f.s.r.=370 MHz, OC transmissivity=3%.

Fig. 2 for low (a) and high (b) loss modulation. When a low loss modulation is applied, at the end of the modulation signal the photons in the cavity are a few percent of those present in the stationary state. In this case the oscillations are regular and the first pulses are relatively less intense. When a high loss modulation is applied, the first pulses are more intense but the behavior becomes irregular and we can observe another frequency superposed to that of the relaxation oscillations, probably due to some beating between the spectrum of the modulation signal and the laser oscillation frequency. These measurements provide a smaller number of meaningful data but they allow to check the validity of the model for highly nonlinear oscillations.

## 4. Comparison with Toda model

Putting together measurements at low and high loss modulation we were able to reconstruct the behavior of $T_1/T_0$ and $T_2/T_0$ versus $x_{\max} = \log(I_{\max}/I_0)$. We notice that, while the pulse duration $T_2$ is associated unambiguously with one peak, the period $T_1$ is calculated between two peaks of different height because the oscillations are damped. Therefore, in order to compare the experimental values of $T_1/T_0$ with those of the model, where we have neglected damping, we calculated $I_{\max}$ as the average height of two subsequent pulses.

In Fig. 3 we can see the results for three different laser setups. We varied the pump, the transmissivity of the output coupler, and the free spectral range (f.s.r.). This means that in the three plots the stationary intensity $I_0$ of the laser and the period $T_0$ of the linear oscillations are different. Nevertheless, the theory predicts that the functional dependence of the ratios $T_1/T_0$ and $T_2/T_0$ on $x_{\max}$ is universal. Our measurements confirm this property of laser oscillations. In the three plots $T_1/T_0$ and $T_2/T_0$ represented, respectively, by the blue and red symbols, lie very close to the two curves given by Eq. (5). The agreement is very good up to values of $x_{\max}$ as large as about 2.7 ($I_{\max} \simeq 14I_0$) where $T_1/T_0$ and $T_2/T_0$ are very different from the values 1 and 0.5 of the small oscillations.

Analytical approximations for $T_1/T_0$ were provided in Ref. [8] as a function of $V(x_{\max})-1$ and in Ref. [12] as a function of $(x_{\max}-x_{\min})/2$. Here we did the same for both ratios, but using $x_{\max}$, which is a more directly measurable quantity, as independent variable. The approximated expressions, represented by the dashed lines in Fig. 3, are

$$\frac{T_1}{T_0} = 1 + \frac{x_{\max}^2}{24} + \frac{x_{\max}^3}{72} + \frac{x_{\max}^4}{256} + \cdots, \tag{6}$$

$$\frac{T_2}{T_0} = \frac{1}{2} - \frac{x_{\max}}{3\pi} + \left(\frac{1}{48} - \frac{1}{18\pi}\right)x_{\max}^2 + \cdots. \tag{7}$$

In the experimentally accessible range of $x_{\max}$ the approximated curves differ very little from the exact ones, and they can be safely used instead of them for a comparison between theory and experiment. According to Eq. (6), when $x_{\max} \simeq 2.3$ ($I_{\max} \simeq 10I_0$), the period $T_1$ of the nonlinear oscillations is 50% larger than the period $T_0$ of the linear oscillations and the pulse duration is about $T_0/4$.

Finally, Fig. 4 shows a comparison of an experimental time trace and a numerical one. In the experiment, the laser is initially in the steady state and then at $t=0$, under the action of the modulator, the intensity goes down to a few percent of the stationary value while the population inversion does not change appreciably. Accordingly, we set the initial conditions of the simulations to $I(0) = 0.043I_0$ and $D(0) = 1/A$. With these parameter the whole time evolution of the laser towards the stationary state is faithfully reproduced.

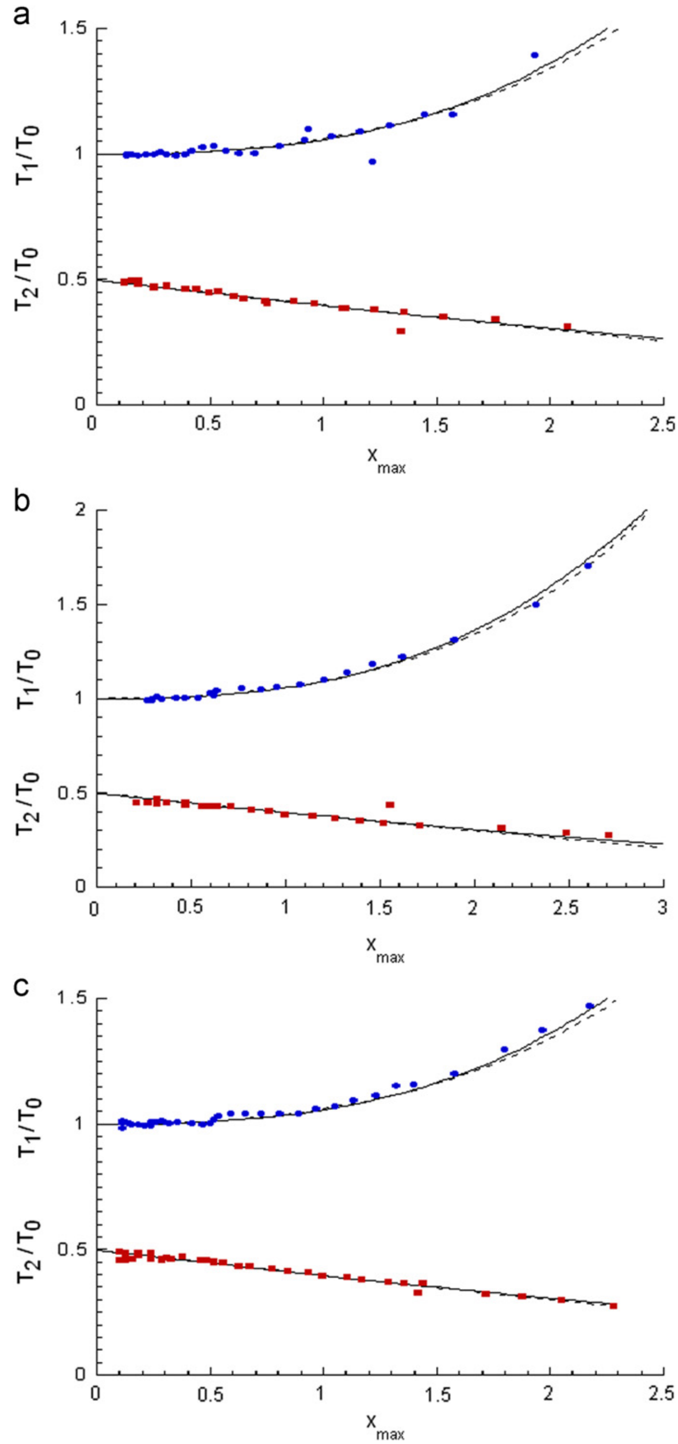


**Fig. 3.** The experimental values of $T_1/T_0$ (blue symbols) and $T_2/T_0$ (red symbols) as a function of $x_{\max}$ are compared to the exact (solid lines) and approximated (dashed lines) theoretical predictions. (a) $I_d$=11.5 A, f.s.r.=370 MHz, OC transmissivity=3%. (b) $I_d$=12.5 A, f.s.r.=370 MHz, OC transmissivity=8%. (c) $I_d$=13.5 A, f.s.r.=286 MHz, OC transmissivity=8%. The threshold current for the Nd:YAG laser in the three configurations is $I_{d,th} = 11$ A (a), $I_{d,th} = 12.1$ A (b), $I_{d,th} = 12.5$ A (c). (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this article.)

## 5. Conclusions

The Toda model for a class-B laser dates back to 1985 [2], yet in 2007 it was still necessary to admit that "an accurate experimental realization of the oscillator Toda remains a challenging task" [12]. We responded to the challenge, and realized a simple

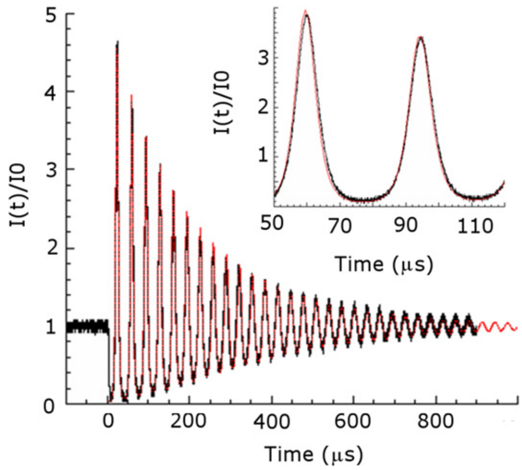


**Fig. 4.** An experimental time trace (black line) is compared to a numerical calculated one (red line). In the simulations $A = 1.665$, $\gamma = 3.05 \times 10^{-4}$ and the initial conditions are $I(0) = 0.043 I_0$ and $D(0) = 1/A$.

experimental setup which allows to observe Toda's oscillations in a properly modulated Nd:YAG laser. We collected results on the period and width of the pulses as a function of the peak intensity, obtained with different experimental configurations. All of them fit very well the universal laws of the model. We also provided simple analytical approximations of those laws, valid for peak intensities more than 10 times larger than the cw intensity, which make easier the comparison of experiment with theory.

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