Physical origin of the Gouy phase shift

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We show explicitly that the well-known Gouy phase shift of any focused beam originates from transverse spatial confinement, which, through the uncertainty principle, introduces a spread in the transverse momenta and hence a shift in the expectation value of the axial propagation constant. A general expression is given for the Gouy phase shift in terms of expectation values of the squares of the transverse momenta. Our result also explains the phase shift in front of the Kirchhoff diffraction integral. © 2001 Optical Society of America

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The Gouy phase shift is the well-known $n\pi/2$ axial phase shift that a converging light wave experiences as it passes through its focus in propagating from $-\infty$ to $+\infty$. Here the dimension n equals 1 for a line focus (cylindrical wave) and equals 2 for a point focus (spherical wave). This phase anomaly was first observed by Gouy[1–3] and was shown to exist for any waves, including acoustic waves, that pass through a focus. The Gouy phase shift plays an important role in optics. It explains the phase advance for the secondary Huygens wavelets emanating from a primary wave front. It also determines the resonant frequencies of transverse modes in laser cavities.[3] In nonlinear optics the Gouy shift affects the efficiency of the generation of odd-order harmonics with focused beams. It also plays a role in the lateral trapping force at the focus of optical tweezers and leads to phase velocities that exceed the speed of light in vacuum. Recently we pointed out the effect of the Gouy phase shift on the temporal profile of a single-cycle electromagnetic pulse[4,5] and made a direct observation of the polarity reversal that results from a Gouy phase shift of π.[6] Another direct observation of a $\pi/2$ Gouy phase shift of terahertz beams in a cylindrical focusing geometry was reported recently.[7]

Although Gouy made his discovery more than 100 years ago, efforts are still being made to provide a simple and satisfying physical interpretation of this phase anomaly. An earlier paper[8] provided an intuitive explanation of this phase anomaly based on the geometrical properties of Gaussian beams. However, that argument cannot explain the $\pi/2$ phase shift for cylindrical focusing. In a recent paper an interpretation of the Gouy phase shift as a geometrical quantum effect was also proposed.[9] Whereas this interpretation is satisfying in its simplicity, the connection to quantum mechanics appears unnecessary because Gouy showed that the phase jump exists for all waves, including sound waves. It has also been suggested that the Gouy phase shift is a manifestation of a general Berry phase, which is an additional geometric (topological) phase acquired by a system after a cyclic adiabatic evolution in parameter space.[10] The parameter that is cycled in the case of the Gouy phase is the complex wave-front radius of curvature q associated with a Gaussian beam.[11,12] This sophisticated modern interpretation requires knowledge of such concepts as anholonomy and is far from being intuitive.

In this Letter we provide a simple intuitive explanation of the physical origin of the Gouy phase shift. We show explicitly that the Gouy phase shift of any focused beam originates from the transverse spatial confinement, which, through the uncertainty principle, introduces a spread in the transverse momenta and hence a shift in the expectation value of the axial propagation constant. A general expression is given for the Gouy phase shift in terms of the expectation values of the transverse momenta. It yields the correct values for both line and point focusing and also explains the phase shift in front of the Kirchhoff diffraction integral.

Consider a monochromatic wave of frequency ω and wave number $k = \omega/c$ propagating along the z direction. For an infinite plane wave, the momentum, which is proportional to k, is z directed and has no transverse components. The spread in transverse momentum is zero and hence, by the uncertainty principle, the spread in transverse position is infinite. A finite beam, however, will have a spread in transverse momentum because it is made up of an angular spectrum of plane waves obtainable by means of a Fourier transform. The wave number is related to these transverse components through

$$k^2 = {k_x}^2 + {k_y}^2 + {k_z}^2, \tag{1}$$

where k_x, k_y, and k_z are the wave-vector components along the coordinate axes. Inasmuch as k $(=\omega/c)$ is constant, the presence of the transverse components reduces the magnitude of the axial component from its value of $k_z = k$ for an infinite plane wave propagating along z. Because of the finite spread in wave-vector components, it is appropriate to deal with averages or expectation values defined by

$$\langle \xi \rangle \equiv \frac{\int_{-\infty}^{+\infty} \xi |f(\xi)|^2 \mathrm{d}\xi}{\int_{-\infty}^{+\infty} |f(\xi)|^2 \mathrm{d}\xi}, \tag{2}$$

where $f(\xi)$ is the distribution of the variable ξ. Then from Eq. (1) we can define an effective axial

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propagation constant for a finite beam through the second moment as

$$\bar{k}_z \equiv \frac{\langle k_z^2 \rangle}{k} = k - \frac{\langle k_x^2 \rangle}{k} - \frac{\langle k_y^2 \rangle}{k} \cdot \tag{3}$$

The effective propagation constant defined in Eq. (3) is associated with the overall propagation phase $\phi(z)$ on axis through $\bar{k}_z \equiv \partial\phi(z)/\partial z$.[13] The first term yields the phase kz of an infinite plane wave propagating along z. The last two terms give rise to the Gouy phase shift:

$$\phi_G = -\frac{1}{k}\int^z \{\langle k_x^2\rangle + \langle k_y^2\rangle\}\mathrm{d}z \,. \tag{4}$$

Hence the Gouy shift is the expectation value of the axial phase shift owing to the transverse momentum spread. In what follows, we shall apply our principle result, Eq. (4), to several situations.

Consider a monochromatic beam with a Gaussian transverse distribution given by

$$f(x,y) = \sqrt{\frac{2}{\pi}}\,\frac{1}{w(z)}\exp\left[-\frac{x^2+y^2}{w^2(z)}\right], \tag{5}$$

where

$$w^2(z) = w_0^2\left[1 + \left(\frac{z}{z_R}\right)^2\right] \tag{6}$$

is the beam radius and w_0 is the minimum spot size (beam waist) at $z = 0$. The Rayleigh range is defined by $z_R = \pi w_0^2/\lambda$, where λ is the wavelength. The angular spectrum of plane waves, or, equivalently, the distribution of transverse wave-vector components, is given by the Fourier transform

$$\tilde{F}(k_x,k_y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y) \times \exp(-ik_x x - ik_y y)\mathrm{d}x\mathrm{d}y \,. \tag{7}$$

On substituting the Gaussian distribution for $f(x,y)$ we find the spectrum

$$\tilde{F}(k_x,k_y) = \frac{w(z)}{\sqrt{2\pi}}\exp\left[-\frac{w^2(z)}{4}(k_x^2+k_y^2)\right], \tag{8}$$

which is also Gaussian and centered about $k_x = k_y = 0$. Both the functions $f(x,y)$ and $\tilde{F}(k_x,k_y)$ are normalized such that

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|f(x,y)|^2\mathrm{d}x\mathrm{d}y = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|\tilde{F}(k_x,k_y)|^2\mathrm{d}k_x\mathrm{d}k_y = 1\,. \tag{9}$$

Thus, using Eq. (2), we have

$$\langle k_x^2\rangle = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} k_x^2|\tilde{F}(k_x,k_y)|^2\mathrm{d}k_x\mathrm{d}k_y = \frac{1}{w^2(z)} = \langle k_y^2\rangle\,. \tag{10}$$

The Gouy phase shift for the Gaussian beam is then given by

$$\phi_G = -\frac{1}{k}\int^z\{\langle k_x^2\rangle + \langle k_y^2\rangle\}\mathrm{d}z = -\frac{2}{k}\int^z \frac{1}{w^2(z)}\mathrm{d}z\,. \tag{11}$$

The factor of 2 in Eq. (11) is related to the number of transverse dimensions, with each dimension contributing $1/w^2(z)$ to the mean-square transverse momentum. It is unity for one-dimensional focusing. This explains why the Gouy shift of a cylindrical wave is half that of a spherical wave. On carrying out the integration in Eq. (11) we obtain the standard result for the Gouy shift of a fundamental Gaussian beam[3]:

$$\phi_G = -\arctan(z/z_R)\,. \tag{12}$$

For a Gaussian beam that evolves from $-\infty$ to $+\infty$ through a point focus, Eq. (12) predicts a phase shift of π. These results can also be used to explain the $\pi/2$ phase shift in front of the Kirchhoff diffraction integral for two-dimensional diffracting screens and the $\pi/4$ phase shift for one-dimensional screens as the diffracted beams propagate from 0 to $+\infty$. Both phase shifts originate from the transverse spatial restriction imposed by the diffracting apertures.

The result of Eq. (11) can be generalized to complex values of the beam radius as defined by $w_c^2(z) = \lambda(z_R + iz)/\pi$.[3,13] Using this definition in Eq. (11), we obtain the complex Gouy shift:

$$\phi_G^c(z) = \phi_G(z) + i\ln\sqrt{z^2 + z_R^2}\,. \tag{13}$$

The real part of $\phi_G^c(z)$ is the ordinary Gouy phase; the imaginary part reduces the amplitude of the distribution $f(x,y)$ by a factor of $\sqrt{z^2+z_R^2}$ as a result of diffraction.

We now show that Eq. (4) predicts the phase anomaly not only for fundamental Gaussian beams but also for higher-order transverse modes and hence is valid for arbitrary field distributions. One complete set of transverse modes is described by Hermite–Gaussian beams[3,13]:

$$f_{mn}(x,y) = C_{mn}\frac{\sqrt{2}}{w(z)}\Theta_m\left[\frac{\sqrt{2}x}{w(z)}\right]\Theta_n\left[\frac{\sqrt{2}y}{w(z)}\right], \tag{14}$$

where the normalization coefficient is given by

$$C_{mn} = \left(\frac{1}{\pi 2^{m+n} m!n!}\right)^{1/2} \tag{15}$$

and $\Theta_m()$ is the Hermite–Gaussian of mth order:

$$\Theta_m(\xi) \equiv H_m(\xi)\exp(-\xi^2/2). \tag{16}$$

Here $H_m(\xi)$ is the mth-order Hermite polynomial. The Fourier transform of Eq. (14) is found to be

$$\tilde{F}_{mn}(k_x, k_y) = (-i)^{m+n} C_{mn} \frac{w(z)}{\sqrt{2}} \Theta_m\left[\frac{w(z)k_x}{\sqrt{2}}\right] \times \Theta_n\left[\frac{w(z)k_y}{\sqrt{2}}\right]. \tag{17}$$

The Hermite polynomials are orthonormal and satisfy the recursion relation

$$H_{n+1} - 2\xi H_n + 2nH_{n-1} = 0. \tag{18}$$

One can use this property, along with Eq. (2), to derive the expectation values

$$\langle {k_x}^2 \rangle_{mn} = \frac{2}{w^2(z)}\left(m + \frac{1}{2}\right),$$

$$\langle {k_y}^2 \rangle_{mn} = \frac{2}{w^2(z)}\left(n + \frac{1}{2}\right). \tag{19}$$

Substituting Eq. (19) into Eq. (4), we obtain

$$\phi_{G,mn}(z) = -(m + n + 1)\arctan(z/z_R), \tag{20}$$

which is identical to the expression given in Ref. 3 for the Gouy shift of a higher-order transverse mode (mn). Each transverse dimension offers a phase shift of

$$-(p + 1/2)\arctan(z/z_R), \qquad p = m \text{ for } x \text{ axis}, \quad p = n \text{ for } y \text{ axis} \tag{21}$$

owing to the transverse momentum in that direction. This is larger than that of the fundamental Gaussian because of the more rapid transverse variation.

As was mentioned above, the spread in transverse momentum and the transverse spatial extent of a finite beam are related by an uncertainty principle:

$$\Delta k_x \Delta x \geq \text{const.}, \tag{22}$$

where the operator Δ denotes the variance defined by

$$\Delta\xi \equiv [\langle(\xi - \langle\xi\rangle)^2\rangle]^{1/2}. \tag{23}$$

For a deterministic function $f(\xi)$, the quantity in Eq. (23) represents the root-mean-square width of the function. For a signal of zero mean, Eq. (23) yields $(\Delta\xi)^2 = \langle\xi^2\rangle$. Thus we find the uncertainty relation or the space–bandwidth product of the mth-order Hermite–Gaussian mode:

$$(\Delta k_x)_m(\Delta x)_m = m + 1/2. \tag{24}$$

The higher-order mode occupies a larger volume in the phase space (x, k_x).

In conclusion, we have provided a general expression and physical explanation of the Gouy phase shift by showing that the Gouy phase can be derived from the transverse momenta of the monochromatic wave for both spherical and cylindrical focusing. Consequently, we concluded that the Gouy shift of finite beams stems from transverse spatial confinement. This conclusion applies to the phase shift in front of the Kirchhoff diffraction integral as well. Our result is valid for a medium with constant refractive index but can be generalized to lenslike media for which the refractive index varies quadratically with radial position. In that case we find that for a stable propagating mode the Gouy shift is a linear function of distance z and can take on any arbitrary value.

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References

1. C. R. Gouy, C. R. Acad. Sci. Paris **110,** 1251 (1890).
2. C. R. Gouy, Ann. Chim. Phys. Ser. 6, **24,** 145 (1891).
3. A. E. Siegman, *Lasers* (University Science, Mill Valley, Calif., 1986). Note that Gouy's name is misspelled as "Guoy" in this classic reference.
4. S. Feng, H. G. Winful, and R. W. Hellwarth, Opt. Lett. **23,** 385 (1998).
5. S. Feng, H. G. Winful, and R. W. Hellwarth, Opt. Lett. **23,** 1141 (1998), erratum of Ref. 4.
6. A. B. Ruffin, J. V. Rudd, J. F. Whitaker, S. Feng, and H. G. Winful, Phys. Rev. Lett. **83,** 3410 (1999).
7. R. W. McGowan, R. A. Cheville, and D. Grischkowsky, Appl. Phys. Lett. **76,** 670 (2000).
8. R. W. Boyd, J. Opt. Soc. Am. **70,** 877 (1980).
9. P. Hariharan and P. A. Robinson, J. Mod. Opt. **43,** 219 (1996).
10. M. V. Berry, Proc. R. Soc. London Ser. A **392,** 45 (1984).
11. D. Subbarao, Opt. Lett. **20,** 2162 (1995).
12. R. Simon and N. Mukunda, Phys. Rev. Lett. **70,** 880 (1993).
13. H. A. Haus, *Waves and Fields in Optoelectronics* (Prentice-Hall, Englewood Cliffs, N. J., 1984).