Theory of mode locking with a fast saturable absorber*

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This paper presents a closed-form analysis of saturable absorber mode locking of a homogeneously broadened laser. A solution is obtained for the case of a short relaxation time of the saturable absorber. This pulse is a hyperbolic secant as a function of time. For each choice of parameters two pulse widths are found. A stability analysis shows that the solution of greater width is stable. The requirements for achieving mode locking with a fast saturable absorber are stated. The effect of a time-varying laser medium gain is investigated analytically.

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I. INTRODUCTION

A considerable amount of theoretical and experimental work has been done on mode locking by a saturable absorber (see Ref. 1 for a listing of the literature). Most of the experiments were done on transient systems. (For review articles see Refs. 2 and 3. Reference 3 is a summary of the Russian work.) The Russian literature in general and Letokhov's work[4–8] in particular concentrate on mode locking in systems like ruby and Nd-glass. The point of view in the theoretical developments is that the mode locking process is dominated by the (noise) initial conditions of the mode-locked pulse-train buildup. The same approach is taken in Refs. 9–13. However, both Letokhov[9] and Fleck[7] also presented some computer solutions for a sample in the pulse train after the laser and saturable absorber are well within the saturated regime, when the pulse shapes may be treated deterministically. This regime of operation ought to be amenable to a nonstatistical treatment, or quasi-steady-state, analysis.

Transient mode locking also has been achieved with TEA CO_2 lasers using SF_6, hot CO_2 cells, and Ge.[14–17]

The literature on steady-state mode locking by a saturable absorber is less extensive.[18–23] One example is the work of Smith with a He/Ne laser using a Ne cell, and Runge's work using a saturable dye. Dienes, Ippen, and Shank[14] demonstrated the production of picosecond pulses with a dye laser mode locked by a dye. Remarkable in all these experiments is the fact that the relaxation time of the absorbers used (dye, gas cell) was much longer than the pulse duration. The theory of this regime of mode locking will be discussed in another paper.

Forced mode locking has received a great deal of theoretical attention, notably the work by Siegmann and Kuizenga[24–26] which presents an analytic treatment of the process. Whereas some advances have been made in approaches to saturable absorber mode locking,[27–29] no closed-form analysis comparable to that of Refs.[24–26] exists to date. This paper attempts to remedy the situation. We present an analytic treatment of steady-state saturable absorber mode locking of a homogeneously broadened laser and present a closed-form solution for the pulse shape. The following assumptions are made:

(a) The relaxation time of the absorber is short compared with the pulse width.

(b) The absorption coefficient of the saturable absorber is evaluated from a rate equation. The dependence upon power of the absorption coefficient is expanded to first order in power.

(c) The relaxation time of the laser medium is long so that the laser gain is approximately time independent.

(d) The mode-locked pulse is only slightly modified (say 20%) upon *one* passage through any one of the components of the system.

(e) The spectrum of the pulse occupies only a narrow portion of the laser linewidth.

Assumption (b) may be the most serious one. We relax it at the end of this study, where we show that the qualitative conclusions still apply in the case of strong saturation of the absorber.

The analysis follows along the lines of Ref. 27. We extend this analysis through inclusion of the modulation function of the saturable absorber, self-consistent with the shape of the optical pulse in the cavity. Our analysis applies, strictly speaking, to cw mode locking of a homogeneously broadened laser. Yet, the CO_2 TEA laser mode locked by a Ge saturable absorber, which has a fast relaxation time, may be not too far from a quasi-steady state so that our analysis may apply to this problem[17] as well.

We contend further that the theory is relevant to an understanding of some aspects of the Nd-glass transient system. As a case in point we mention Auston's work[30] which uncovered rather well-defined pulse shapes using a fifth-order detection process. Characteristic of such an experiment is that it discriminates against all but largest (peak power) pulses. Thus, the statistics of the time of occurrence of these pulses are suppressed by the detection [i.e., the fluctuations of the time instants of occurrence of the pulses with highest peak power caused by the (noise) initial conditions or system irregularities do not affect the observation appreciably]. If one adopts the point of view that the pulses in a Nd-glass system are in a quasi-steady state beyond some initial time (fluctuating with respect to the initiation of the laser pumping), then the analysis presented here has relevance to this problem as well. In particular Letokhov's and Fleck's computer solutions should be representable by our closed-form solution (except that they allow for a nonzero relaxation time of the saturable absorber, a case not treated in this paper but left for another publication).

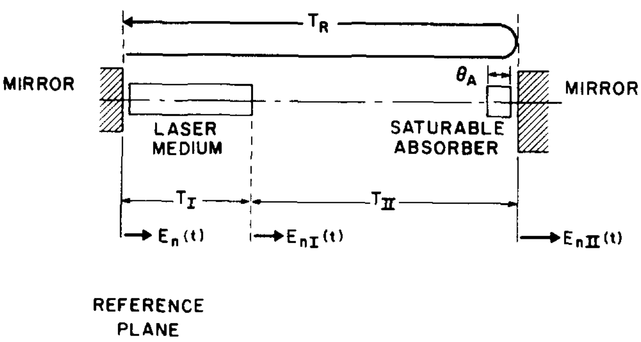

FIG. 1. Schematic of laser with saturable absorber.

In Sec. II we give a brief derivation of the fundamental equation, a second-order nonlinear differential equation. In Sec. III we derive the approximate constitutive laws for the saturable absorber and laser medium. Section IV combines the results to obtain the basic equation for mode locking with a "fast" saturable absorber. The pulse-shape solution is found to be a secant hyperbolic. Even though the solution is reminiscent of the self-induced transparency (SIT) solution, the physical model, and the resulting differential equations are distinct from those of SIT.

For any given choice of parameters one finds two values of pulse width and pulse energy. In Sec. V we show that the solution corresponding to the narrower width is unstable and hence does not correspond to a steady-state solution. In Sec. VI we show how one may choose the appropriate system parameters to achieve shortest possible pulses under the usual physical constraints upon the system. Section VII discusses the consequences of dropping assumption (b) above.

II. THE FUNDAMENTAL EQUATION

Much of the literature on passive mode locking treats mode locking in the time domain: the action on the mode-locked pulse within the laser cavity by each component of the system is analyzed, and the net modification of the pulse upon return to the starting reference plane is set equal to zero. We follow here the same approach, but make some additional approximations valid for near-threshold operation and for pulses with bandwidths narrow compared with the total laser linewidth. These approximations make the closed-form solution possible.

Consider the situation shown in Figure 1. The laser medium is at one end of the cavity, the saturable absorber is at the other end and confined to a small fraction of the total length of the cavity (i.e., $\theta_A/c\tau_P \ll 1$, where τ_P is the duration of the mode-locked pulse). At the start we assume a uniform intensity profile over the optical beam cross section. Later we show the generalization necessary in order to treat nonuniform (Gaussian) beam profiles. The positive-frequency part of the time-dependent electric field of the pulse after the nth transit, at the reference plane, is set equal to $E_n(t)$ and its Fourier transform is $E_n(\omega_k)$. The Fourier transform has a discrete spectrum with components spaced at $2\pi/T_P$, where T_P is the period of the pulse train.

After passage through the laser medium the Fourier spectrum of the pulse is

$$E_{n\mathrm{I}}(\omega_k) = \exp(-j\omega_k T_\mathrm{I}) \exp[G(\omega_k)] E_n(\omega_k), \tag{2.1}$$

where $G(\omega_k)$ is the frequency-dependent gain of the laser medium and T_I is the time delay in the laser medium. In the time domain

$$E_{n\mathrm{I}}(t) = \exp\left[G\left(\frac{d}{dt}\right)\right] E_n(t - T_\mathrm{I}), \tag{2.2}$$

where $\exp[G(d/dt)]$ is to be interpreted in terms of the expansion of the exponential $\exp[G(\omega_k)]$ in $j\omega_k$, and replacement of the nth power of $j\omega_k$ by d^n/dt^n. After passage of the pulse through the saturable absorber, the electric field is

$$E_{n\mathrm{II}}(t) = \exp[-L(t)] \exp\left[G\left(\frac{d}{dt}\right)\right] E_n(t - T_\mathrm{I} - T_\mathrm{II}), \tag{2.3}$$

where $L(t)$ is the power-dependent absorption coefficient of the saturable absorber, a function of time under mode-locked operation. When the pulse returns to the reference plane, the delay time is equal to the cavity round-trip time, $T_R = 2(T_\mathrm{I} + T_\mathrm{II})$, and the two exponentials have to be reapplied. In order to take the cavity loss into account one multiplies the result in addition by $\exp[-(\omega_0/2Q)T_R]$ which accounts for the exponential decay of the pulse as determined by the Q of the axial modes. Here ω_0 is the center frequency of the pulse spectrum which has been assumed narrow.

The return pulse at the reference plane, now by definition the $(n+1)st$ pulse of the train, is given by

$$E_{n+1}(t) = \exp\left(-\frac{\omega_0}{2Q} T_R\right) \exp\left[G\left(\frac{d}{dt}\right)\right] \times \exp[-2L(t)] \exp\left[G\left(\frac{d}{dt}\right)\right] E_n(t - 2T_\mathrm{I} - 2T_\mathrm{II}). \tag{2.4}$$

The first approximation we shall make is that the changes wrought on the pulse upon any one passage through the components of the system are small, so that the exponentials can be expanded to first order.

Further, we assume that the gain has a Lorentzian line shape and can be expanded to second order in the deviation from center frequency ω_0.

$$G(\omega_k) = G(\omega_0)\left(1 + j\frac{\omega_k - \omega_0}{\omega_L}\right)^{-1} \approx G(\omega_0)\left[1 - j\frac{\omega_k - \omega_0}{\omega_L} - \left(\frac{\omega_k - \omega_0}{\omega_L}\right)^2\right]. \tag{2.5}$$

Because the gain contains powers of $\omega - \omega_0$, it is convenient to write the electric field in terms of a slowly time-varying envelope $v(t)$ and the exponential $\exp(j\omega_0 t)$

$$E_n(t) = v_n(t) \exp(j\omega_0 t) \tag{2.6}$$

with the Fourier transform

$$E_n(\omega_k) = v_n(\omega_k - \omega_0). \tag{2.7}$$

Multiplication of $E_n(\omega_k)$ by $j(\omega_k - \omega_0)$ corresponds to $d/dt - j\omega_0$ in the time domain which may be interpreted by a time derivative of $v_n(t)$ alone. Using Eqs. (2.4)–(2.8), and expanding the exponentials, one obtains

$$v_{n+1}(t) = \left[1 - \frac{\omega_0}{2Q} T_R - 2L(t) + 2G(\omega_0)\left(1 + \frac{1}{\omega_L^2}\frac{d^2}{dt^2} - \frac{1}{\omega_L}\frac{d}{dt}\right)\right] v_n(t - T_R). \qquad (2.8)$$

It is convenient to simplify notation by introducing the following symbols:

$$\frac{2G(\omega_0)}{(\omega_0/2Q)T_R} \equiv g \qquad (2.9)$$

the gain normalized to the loss. Note that with no saturable absorber in the cavity, $g=1$ corresponds to threshold.

$$\frac{2L(t)}{(\omega_0/2Q)T_R} \equiv \frac{Q}{Q_A(t)}, \qquad (2.10)$$

where Q_A may be interpreted as the amplitude –, and hence time –, dependent Q as produced by the saturable absorber.

We then have for Eq. (2.8)

$$v_{n+1}(t) = v_n(t - T_R) - \frac{\omega_0}{2Q} T_R \times \left[1 + \frac{Q}{Q_A(t)} - g\left(1 + \frac{1}{\omega_L^2}\frac{d^2}{dt^2} - \frac{1}{\omega_L}\frac{d}{dt}\right)\right] v_n(t - T_R). \qquad (2.11)$$

The $(n+1)$st pulse is a delayed version of the nth pulse and has experienced the modifications expressed by the operator

$$-\frac{\omega_0}{2Q}\left[1 + \frac{Q}{Q_A(t)} - g\left(1 + \frac{1}{\omega_L^2}\frac{d^2}{dt^2} - \frac{1}{\omega_L}\frac{d}{dt}\right)\right].$$

The first term in parentheses is the effect of the linear cavity loss, the second term represents the "modulation" by the time-dependent inverse Q of the saturable absorber. The last term expresses the effect of the gain and dispersion of the laser medium. The dispersion causes "diffusion" along the time coordinate of the pulse as represented by the diffusion operator d^2/dt^2 and a time delay of $\omega_0 g/2Q\omega_L$ as represented by the operator $-(\omega_0 g/2Q\omega_L)(d/dt)$. Note that for small changes of the pulse envelope as assumed here, a time advance δT of a function $v(t)$ is represented by $v(t+\delta T) = v(t) + \delta T(dv/dt)$.

In the steady state, the "closure" condition requires that the pulse reproduce itself with a simple time delay after one period. The period T_P need not be equal to the cavity round-trip time T_R, because pulse envelope changes may lead to additional delays (or advances).

Here,

$$v_{n+1}(t) = v_n(t - T_R + \delta T), \qquad (2.12)$$

where δT is the time advance due to pulse envelope modulation. Introducing Eq. (2.12) into Eq. (2.11) and assuming, consistent with the previous approximations, that δT is small so that Eq. (2.12) can be expanded to first order in δT, we obtain

$$\left[1 + \frac{Q}{Q_A(t)} - g\left(1 + \frac{1}{\omega_L^2}\frac{d^2}{dt^2}\right) + \frac{g+\delta}{\omega_L}\frac{d}{dt}\right] v = 0, \qquad (2.13)$$

where

$$\delta \equiv \frac{\omega_L \delta T}{(\omega_0/2Q)T_R}. \qquad (2.14)$$

This is the desired equation for the steady-state pulse envelope $v(t)$.

The same result would have been obtained from the cavity-mode expansion approach making the assumption of high Q (slow growth and decay times of mode amplitudes due to media in cavity). One advantage of the cavity-mode expansion approach is that it permits the analysis of Gaussian modes with Gaussian transverse spatial dependence (in the high-Q approximation). If the mode pattern is known to be Gaussian, (a situation assured by the use of mode-suppressing irises) then the inverse Q's of the axial modes may be evaluated as weighted integrals of the imaginary parts of the susceptibilities of the media in the cavity.[31]

III. THE SATURABLE ABSORBER AND LASER GAIN

The rate equation for the population difference n between the lower and the upper levels of the saturable absorber is

$$\frac{\partial n}{\partial t} = -\frac{n - n_e}{T_A} - \sigma_A \frac{|v(t)|^2}{\hbar\omega_0 A_A} n. \qquad (3.1)$$

$v(t)$ is so normalized that $|v(t)|^2$ gives the sum of the powers in the two countertraveling waves in the cavity. Here T_A is the relaxation time of the absorbing medium, n_e is the equilibrium population difference, σ_A is the optical cross section of the absorbing particles, and A_A is the cross section of the optical beam within the absorber. We assumed a rapidly relaxing upper level. If the saturable absorber behaved like a two-level system with equal relaxation times of upper and lower levels, σ_A would have to be replaced by $2\sigma_A$.

If the relaxation time T_A of the absorber is fast compared with the rate of change of the intensity, the population difference is an instantaneous function of intensity. The time-dependent population difference n is then approximately

$$n = n_e\left(1 - \frac{|v(t)|^2}{P_A}\right), \qquad (3.2)$$

where we have defined the saturation power for the absorber by

$$P_A = \frac{\hbar\omega_0 A_A}{\sigma_A T_A} \qquad (3.3)$$

and where we have expanded n to first order in $|v(t)|^2$. The absorption of a pulse after single passage is

$$L(t) = \sigma_A \theta_A A_A n_e \left(1 - \frac{|v(t)|^2}{P_A}\right), \qquad (3.4)$$

where θ_A is the length of the absorber. The parameter Q/Q_A as defined by Eq. (2.10) is found to be

$$\frac{Q}{Q_A(t)} = q\left(1 - \frac{|v(t)|^2}{P_A}\right), \tag{3.5}$$

where

$$q = \frac{2\sigma_A \theta_A A_A n_e}{(\omega_0/2Q)T_R}$$

is the small-signal inverse Q of the saturable absorber normalized to the cavity Q. Thus, for the fast absorber, we obtain the differential equation

$$\left[1 + q + (g+\delta)\frac{1}{\omega_L}\frac{d}{dt} - g\left(1 + \frac{1}{\omega_L^2}\frac{d^2}{dt^2}\right)\right] v = q\frac{|v(t)|^2}{P_A} v. \tag{3.6}$$

If one assumes that the relaxation time of the homogeneously broadened laser medium is slow compared with the pulse repetition time, then g does not have any appreciable time dependence and is a monotonically decreasing function of the sum of the time average powers in the countertraveling waves, P. A convenient functional dependence of g upon P is

$$g = \frac{g_0}{1 + P/P_L}, \tag{3.7}$$

where g_0 is the small-signal value of g and P_L is the saturation power. This formula fits a number of saturation models. It applies to a laser medium that is short and near one of the mirrors. Then the mode patterns of all axial cavity modes are approximately equal and saturate the laser medium equally. Further the spectrum of the mode-locked pulse train must be narrow compared with the laser bandwidth so that the saturation power P_L is the same for all axial modes. In this case,

$$P_L = \frac{\hbar\omega_0 A_L}{\sigma_L T_L} \tag{3.8}$$

with σ_L equal to the optical cross section of the laser medium, T_L the relaxation time, and A_L equal to the cross section of the optical beam in the laser medium.

IV. SOLUTION

We shall now look for solutions of Eq. (3.6) corresponding to mode-locked pulses. The time-shift parameter δ is an adjustable parameter, to be determined from the character of the problem. It will become clear that no periodic solutions exist when $g + \delta \neq 0$ and thus we may state at the outset that the pulses must pick their repetition period so that $g + \delta = 0$. The remaining equation

$$\left[1 + q - g\left(1 + \frac{1}{\omega_L^2}\frac{d^2}{dt^2}\right)\right] v - q\frac{|v|^2}{P_A} v = 0 \tag{4.1}$$

can be recognized as the equation of motion of a particle of displacement $v(t)$ in a potential well

$$-\tfrac{1}{2}(1 + q - g)v^2 + \tfrac{1}{4}q\, v^4/P_A. \tag{4.2}$$

If the particle is launched at the well height 0, at the displacement

$$v_0 = [(2P_A/q)(1 + q - g)]^{1/2} \tag{4.3}$$

with zero velocity, it moves to the origin and stops there. This solution is pulselike and symmetric in time and has the time dependence

$$v(t) = \frac{v_0}{\cosh(t/\tau_P)} \tag{4.4}$$

where

$$q\frac{v_0^2}{P_A} = \frac{2g}{\omega_L^2 \tau_P^2} \tag{4.5}$$

and

$$1 + q - g = \frac{g}{\omega_L^2 \tau_P^2} \tag{4.6}$$

as can be ascertained by substitution of Eq. (4.4) in Eq. (4.1). This solution is an isolated pulse. A succession of periodic pulses, of any desired period T_P within the limits $\sqrt{2}\pi\{\omega_L[(1+q-g)/g]^{1/2}\}^{-1} < T_P < \infty$ is obtained by launching the particle at a lower height. If $\tau_P \ll T_P$, then the single pulse is an excellent approximation to one period of the periodic pulse train. Well-mode-locked pulses with good time separation are of this character.

Some salient features of this solution of passive mode locking are worth noting. The pulses have exponential tails which should be contrasted with the Gaussian time dependence of active (forced) mode locking.[25]

The pulse width τ_P, on the other hand, is set, as in the case of forced mode locking, by the curvature of the modulation function. In the Appendix we show that Kuizenga and Siegman's theory of forced mode locking predicts comparable pulse widths for a modulation function of the same curvature at the time of maximum pulse intensity.

From Eq. (4.6), we note that g is less than $1 + q$. This means that the laser is below threshold with respect to the linear loss (loss in absence of laser power). The laser can oscillate because the bleaching of the absorber reduces the loss below the linear loss. In fact, the condition $1 + q - g > 0$ is a necessary requirement for the stability of a train of isolated pulses. Indeed, if $1 + q - g$ were less than zero, with a long dead time between pulses, relatively slow noise perturbations which are unaffected by the gain frequency profile [the operator $g(d^2/dt^2)$] could grow between pulses. The solution obtained would not be stable.

In practice, the parameters specified by the system are the small-signal gain g_0, the saturation power of the laser P_L, the small-signal loading of the saturable absorber, $q = Q/Q_A^\circ$, and the saturation power of the saturable absorber, P_A. We shall now determine the pulse width and power as a function of these parameters. Since

$$\int_{-\infty}^{\infty} \frac{dt}{\cosh^2(t/\tau_P)} = 2\tau_P,$$

the energy in the pulse is given by $2\tau_P v_0^2$. The power P of the laser is then $P = 2\tau_P v_0^2/T_P$ where T_P is the pulse

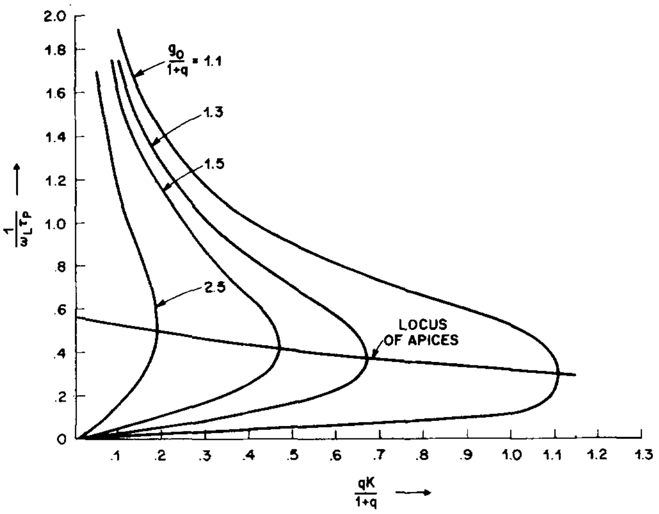

FIG. 2. The inverse pulse width versus $qK/(1+q)$.

repetition time. Introducing this expression into Eq. (4.5), one obtains an equation between P and τ_P

$$qK\frac{P}{P_L}=\frac{g}{\omega_L \tau_P}, \tag{4.7}$$

where we have defined the coefficient

$$K\equiv\tfrac{1}{4}(P_L/P_A)\omega_L T_P \tag{4.8}$$

a measure of laser saturation power normalized to the saturable absorber saturation power. Equations (4.6) and (4.7) supplemented by the dependence of the normalized gain g upon power P, Eq. (3.7), yield three equations for the three unknowns P/P_L, g, and the pulse width $\omega_L\tau_P$, in terms of q, K, and g_0. Equation (4.6) may be used to eliminate g, and from Eqs. (4.7) and (3.7) one may obtain an equation for $\omega_L\tau_P$ and one for P/P_L

$$\frac{g_0}{1+q}\left(\omega_L\tau_P+\frac{1}{\omega_L\tau_P}\right)^2=\omega_L\tau_P\left(\omega_L\tau_P+\frac{1}{\omega_L\tau_P}+\frac{1+q}{qK}\right), \tag{4.9}$$

$$\frac{qK}{1+q}\frac{P}{P_L}=\left(\omega_L\tau_P+\frac{1}{\omega_L\tau_P}\right)^{-1}. \tag{4.10}$$

Solutions of $1/\omega_L\tau_P$ vs $qK/(1+q)$ are shown in Fig. 2. It is clear from the figure that under certain conditions no mode-locking solutions are found. Indeed, that happens for a fixed excess gain parameter $g_0/(1+q)$ when $qK/(1+q)$ becomes too large, i.e., the Q of the absorber becomes too small or its saturation power P_A becomes too small.

When this happens, no single pulses are obtained. Double and multiple pulses are possible, however. Indeed K is proportional to the pulse repetition time T_P. Hence, a submultiple of T_P, T_P/m, provides a submultiple value for K, K/m, and solutions are found again from Fig. 2.

An asymptotic expression for $1/\omega_L\tau_P$ as a function of $qK/(1+q)$ is easily obtained and helps in the interpretation of Fig. 2. If one assumes that $qK/(1+q)\ll 1$, i.e., if one views the portion of the plots closest to the ordinate, and looks for solutions $1/\omega_L\tau_P\ll 1$, one obtains from Eq. (4.10)

$$\frac{1}{\omega_L\tau_P}\approx\left(\frac{g_0}{1+q}-1\right)\frac{qK}{1+q}. \tag{4.11}$$

Thus the lower branches of the curves in Fig. 2 are approximately straight lines.

The power (or pulse energy) of the mode-locked solutions is plotted in Fig. 3. The two branches of solutions having different pulse widths for the same $qK/(1+q)$ also have different powers, the lower power corresponds to the *larger* pulse width. The lower branches approach asymptotically the levels $g_0/(1+q)$ for $qK(1+q)\to 0$. These are the values of power obtained for weak action of the saturable absorber, i.e., pulses long compared to the inverse laser linewidth.

Since we have found in general two solutions for each set of parameters corresponding to different pulse widths and intensities one would expect that in general one of these solutions would be unstable. This question will be investigated in Sec. V.

V. STABILITY STUDY

In order to ascertain whether pulse solutions obtained thus far are stable against small perturbations, one must set up an equation for the time evolution of such a perturbation. Equation (2.11) is a convenient starting point for this purpose. The left-hand side of the equation exhibits the change of a pulse of time dependence $v_n(t)$ after one passage through the system.

We have for the change of the pulse in one transit

$$\Delta v_n=-\frac{\omega_0}{2Q}T_R\left[1+\frac{Q}{Q_A(t)}-g\left(1+\frac{1}{\omega_L^2}\frac{d^2}{dt^2}-\frac{1}{\omega_L}\frac{d}{dt}\right)\right]v_n(t). \tag{5.1}$$

Treating the transit number n as an almost continuous variable, one may replace the left-hand side of Eq. (5.1) by $\Delta v/\Delta n$. Alternately, one may introduce a slow time parameter $T=nT_R$ and consider v to be a function of both the short-term and long-term time variables t and T, respectively. We may write for the above

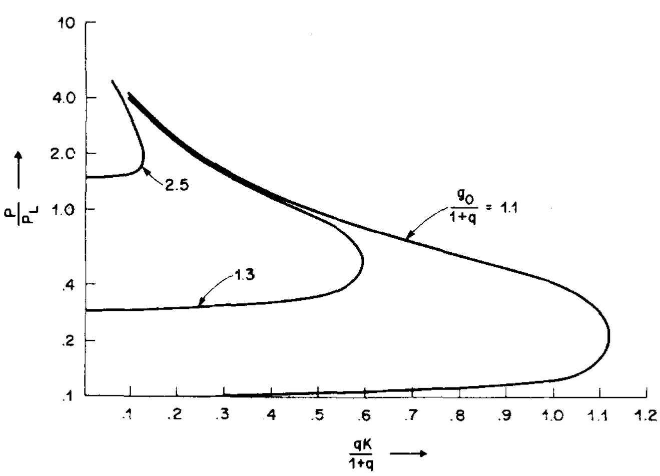

FIG. 3. Power (proportional to pulse energy) versus $qK/(1+q)$.

$$-\left[1+\frac{Q}{Q_A(t)}-g\left(1+\frac{1}{\omega_L^2}\frac{\partial^2}{\partial t^2}-\frac{1}{\omega_L}\frac{\partial}{\partial t}\right)\right]v=\frac{2Q}{\omega_0}\frac{\partial}{\partial T}v, \tag{5.2}$$

where we have introduced partial derivatives to indicate differentiation with respect to either one of the time variables. Equation (5.2) is the starting point for a perturbation analysis. We assume

$$v(t)=v_s(t)+\delta v(t,T), \tag{5.3}$$

where $v_s(t)$ is the steady-state solution and $\delta v(t,T)$ is a perturbation. Note that $\partial v_s/\partial T\neq 0$ because a time shift of v_s is permitted per pass and in fact

$$\frac{\partial v_s}{\partial T}=\delta_s\frac{1}{\omega_L}\frac{\partial v_s}{\partial t} \tag{5.4}$$

by definition of δ, Eq. (2.14).

Further, note that Q, Q_A, and g are functions of the power and get perturbed by the perturbation δv. Using the functional dependence Eq. (3.7) of g upon P, we have

$$\delta g=-\frac{g}{1+P/P_L}\frac{1}{P_L}\;2\int v_s\mathrm{Re}\delta v\,dt\quad T_R^{-1}. \tag{5.5}$$

Similarly, one has

$$\delta\left(\frac{Q}{Q_A}v\right)=q\,\delta v-\delta\left(q\frac{|v|^2v}{P_A}\right)$$
$$=q\,\delta v-q\frac{v_0^2}{P_A\cosh^2(t/\tau_P)}[2\,\mathrm{Re}\delta v+\delta v]. \tag{5.6}$$

Equation (5.5) assumes that the gain perturbation δg follows the perturbation "instantaneously". This implies that the equations to follow will be valid only if they predict rates of change of δv slow compared with the laser medium relaxation time T_L. Expanding Eq. (5.2) to first order in the perturbation, taking into account that the zeroth-order equation is automatically satisfied by v_s, and using Eqs. (4.5), (4.6), (5.5), and (5.6), we finally obtain

$$\frac{\partial}{\partial T}\delta v=-\frac{\omega_0}{2Q}\frac{g}{\omega_L^2\tau_P^2}\left[\left(1-\tau_P^2\frac{\partial^2}{\partial t^2}\right)\delta v-\frac{2}{\cosh^2(t/\tau_P)}(2\,\mathrm{Re}\delta v+\delta v)\right]$$
$$-\frac{\omega_0}{2Q}g\frac{P/P_L}{1+P/P_L}\left(\int\frac{\mathrm{Re}\delta v}{\cosh(t/\tau_P)}\frac{dt}{\tau_P}\right)$$
$$\times\left(1+\frac{1}{\omega_L}\frac{\partial^2}{\partial t^2}-\frac{1}{\omega_L}\frac{\partial}{\partial t}\right)\frac{1}{\cosh(t/\tau_P)}. \tag{5.7}$$

This equation expresses the time rate of change of the pulse perturbation with respect to the slow-time variable T. The first term in brackets is the rate of change of δv caused by cavity loss, linear absorber loss, and laser medium gain; in particular, the second derivative term expresses the laser medium dispersion. The second term in brackets gives the contribution of the absorber saturation. The last term, due to the gain change δg, adds to the perturbation a modified replica of the steady-state solution.

The time development of two particular perturbations deserve special attention. One of them is

$$\delta v=a_d(T)\frac{\sinh(t/\tau_P)}{\cosh^2(t/\tau_P)} \tag{5.8}$$

with $a_d(T)$ real, i.e., in phase with v_s. This perturbation is orthogonal to v_s and hence produces no change of g. It turns out that δv of Eq. (5.8) is an eigenfunction of the operator in brackets in Eq. (5.7) with eigenvalue zero. Therefore, one finds

$$\frac{da_d}{dT}=0.$$

Such a perturbation is neither stabilized nor destabilized. Note that $\sinh(t/\tau_P)[\cosh^2(t/\tau_P)]^{-1}$ is the time derivative of the steady-state solution. Addition of δv to v_s corresponds to a time shift of the pulse. We thus found that there is no stabilization of the pulse timing. This is an expected result because saturable absorber mode locking is not synchronized by an external timing signal and has no time reference.

The other perturbation of interest is one in quadrature with the steady state

$$\delta v=\frac{a_q(T)}{\cosh(t/\tau_P)} \tag{5.9}$$

with $a_q(T)$ pure imaginary. The terms with $\mathrm{Re}\delta v$ drop out of Eq. (5.7). The function (5.9) is an eigenfunction of the remaining operator in the brackets with eigenvalue zero. Thus, we find again

$$\frac{\partial a_q}{\partial T}=0.$$

Quadrature perturbations are neither stabilized nor destabilized. They correspond to a carrier phase perturbation. Phase perturbations experience "no restoring force", just like in the case of the simple van der Pol oscillator.[32]

A general stability test requires the proof that no perturbation of the steady state will grow indefinitely. For this purpose one may expand the perturbation in a complete set of orthogonal functions u_n.

$$\delta v=\textstyle\sum_n a_n(T)u_n(t), \tag{5.10}$$

introduce the expansion in Eq. (5.7), multiply the resulting equation by u_m, and integrate over one period (approximated by the infinite time interval). One obtains an equation of the form

$$\frac{da_m}{dT}=\sum_n M_{mn}a_n. \tag{5.11}$$

Stability is proven when, for an assumed time dependence $\exp(sT)$, one shows that all eigenvalues s have nonpositive real parts. This is a rather difficult task. Here we shall be satisfied with an investigation of the initial growth of the energy $\delta W\equiv\int|\delta v|^2dt$ of the perturbation δv. We shall assume that the steady-state solution is stable if no initial growth of δW occurs for arbitrary choice of the shape of the perturbation.

To derive the equation for the energy evolution, we multiply Eq. (5.7) by δv^*, integrate over one pulse, use integration by parts, and add the complex conjugate

$$\frac{\partial}{\partial T}\int|\delta v|^2dt$$
$$=-\frac{\omega_0 g}{2Q\omega_L^2\tau_P^2}\left[\int dt\left(|\delta v|^2+\tau_P^2\left|\frac{\partial\delta v}{\partial t}\right|^2\right)\right.$$

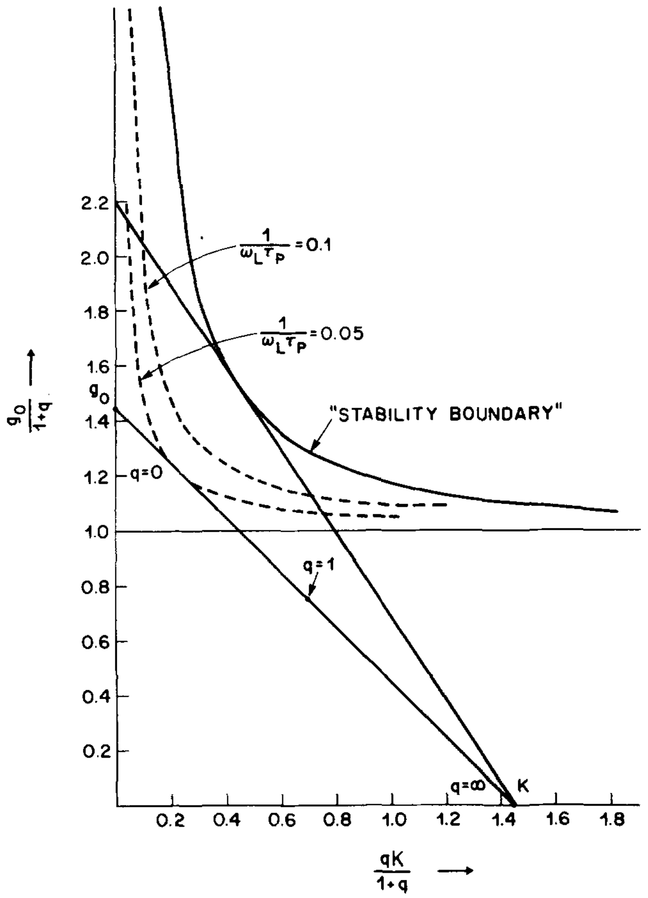

FIG. 4. Graph for determination of mode-locking regime.

$$-2\int\frac{[3(\mathrm{Re}\,\delta v)^2+(\mathrm{Im}\,\delta v)^2]}{\cosh^2(t/\tau_P)}\,dt\Bigg]-\frac{\omega_0}{2Q}\frac{P/P_L}{1+P/P_L}$$

$$\times\int\frac{\mathrm{Re}\,\delta v}{\cosh(t/\tau_P)}\frac{dt}{\tau_P}\int dt\,\delta v^*\left(1-\frac{1}{\omega_L}\frac{\partial}{\partial t}+\frac{1}{\omega_L^2}\frac{\partial^2}{\partial t^2}\right)$$

$$\times\frac{1}{\cosh(t/\tau_P)}+\text{complex conjugate.}\tag{5.12}$$

Let us interpret this energy equation. The first integral in the brackets is the contribution of the cavity loss, small-signal absorber loss, and laser medium gain (without gain change). The net effect is a stabilizing one (i.e., negative definite). The effect is the greater the "faster" the perturbation, due to the laser gain dispersion. The saturable absorber contribution is positive definite, destabilizing; the in-phase perturbations are more "dangerous" than the quadrature perturbations because the former are multiplied by 3. The last term is the contribution of the gain change caused by a perturbation. It is generally stabilizing, unless the contribution of the operation $-(1/\omega_L)(\partial/\partial t)+(1/\omega_L^2)(\partial^2/\partial t^2)$ becomes comparable to unity. This is again the effect of the gain dispersion which decreases the gain change δg with decreasing width of the perturbation pulse. Two general conclusions may be drawn from the discussion:

(a) In-phase perturbations (real values of δv) tend to be more unstable.

(b) The amount of destabilization as well as stabilization is a function of the perturbing pulse width.

Hence, we shall study in-phase perturbations of varying width given by

$$\delta v=\frac{a}{\cosh^\nu(t/\tau_P)},\tag{5.13}$$

where ν is a parameter ($\nu>0$) to be varied to ascertain the stability as a function of pulse width. Introducing Eq. (5.13) in Eq. (5.12), one obtains the requirement for positive definiteness of the right-hand side:

$$\frac{qK}{1+q}\leq f(\nu)\left[1-g(\nu)\frac{1}{\omega_L^2\tau_P^2}\right]\left[\frac{1}{\omega_L\tau_P}\left(1+\frac{1}{\omega_L^2\tau_P^2}\right)\right]^{-1}$$

$$-\left(\omega_L\tau_P+\frac{1}{\omega_L\tau_P}\right)^{-1},\tag{5.14}$$

where

$$f(\nu)=\frac{I^2(\nu+1)}{(6+\nu^2)I(2\nu+2)-(1+\nu^2)I(2\nu)}$$

and

$$g(\nu)=\frac{2I(\nu+3)}{I(\nu+1)}-1$$

with

$$I(\nu)=\int\frac{dx}{\cosh^\nu x}=\frac{2^{\nu-1}\Gamma^2(\frac{1}{2}\nu)}{\Gamma(\nu)}.$$

Here the functions obtained from Eq. (5.14) with the equality sign determine the range of values in the $qK/(1+q)$-$1/\omega_L\tau_P$ plane of Fig. 2 for which stable operation is to be expected. The family of functions, pertaining to different values of ν, possesses an envelope which bounds them from below. This envelope has been computed and coincides, for all practical purposes, with the locus of apices obeying the equation

$$\frac{qK}{1+q}=\left(1-\frac{3}{(\omega_L\tau_P)^2}\right)\left[\frac{2}{\omega_L\tau_P}\left(1+\frac{1}{\omega_L^2\tau_P^2}\right)\right]^{-1}.$$

Stable operation occurs below the locus. Hence, of the two solutions for pulse width, the one with a larger value for the pulse width is stable.

VI. MODE-LOCKING CRITERIA

Thus far we have investigated mode-locking solutions and their stability for a given set of parameters. It is of interest to reverse the investigation and try to choose system parameters so as to achieve mode locking. Once mode locking is achieved, one ususally wants to know how to optimize the parameters to achieve shorter pulses, maximum pulse energy, or both.

We have shown in Sec. V that the upper branch of solutions in Fig. 2 is unstable. Figure 4 shows the locus of apices of Fig. 2 as the "stability boundary" in the $g_0/(1+q)$-$qK/(1+q)$ plane. The range of parameter values lying outside this boundary corresponds to multiple-pulse operation. Also shown are loci of constant $1/\omega_L\tau_P$. Suppose one has a system of given small-signal gain and linear cavity loss. Suppose one varies the insertion loss of the saturable absorber (e.g., a dye cell using variable mole fractions of dye in solution.) Then one varies the parameter q, at fixed K and g_0. In Fig. 4 one describes the straight line intersecting the abscissa at K, and the ordinate at g_0. With increasing q one moves from "northwest" to "southeast". The point for which the small-signal insertion loss of the absorber is equal to the net cavity loss is the half-point on the line. In order to get single-pulse mode locking one must be within the range limited by the stability boundary on one side and the locus $\omega_L T_R > (2)(1.76)\omega_L \tau_P$ on the other side (the cavity round-trip time must be larger than two half-power widths of the pulse). If this is not the case one will obtain undesired pulse overlap. Also, the present analysis fails to apply in the case of overlap. Figure 4 shows as the lower limit $1/\omega_L \tau_P = .05$, i.e., the case when the cavity round-trip time is assumed to be equal to 70 inverse laser linewidths. It is clear from the figure that for an excess small-signal gain g_0 of 1.44, the saturable absorber parameter K must be greater than 1.44 to obtain well-separated pulses. In order to obtain single pulses for $K = 1.44$, g_0 cannot exceed 2.2.

VII. GENERALIZATION OF ABSORBER AND LASER MEDIUM

Thus far we have assumed that the amplitude of the mode-locked pulse was small enough so that the time-dependent inverse Q of the saturable absorber could be expanded to first order in $|v|^2/P_A$.

If this assumption is dropped, one does not obtain closed-form solutions. Yet the resulting pulses are symmetric and have exponential tails. This follows from the fact that the differential equation for $v(t)$ is of second order and, in the tails, the saturable absorber contribution can be disregarded, making the equation linear in this limit.

In the preceding analysis, we have assumed that the laser gain is time independent, that the passage of a single steady-state pulse produces a negligible temporal variation of the gain, and that the over-all pull-down of the laser gain was accomplished by the passage of many successive pulses. This assumes a T_L very much greater than the pulse width. Let us assume now that the opposite extreme of a fast relaxation time T_L pertains. Under these conditions the gain coefficient $G(\omega_k)$ has a more complicated frequency dependence. Yet in the limit of near-threshold operation, in which an expansion is made of the exponential, and g is close to $1+q$, one may separate out the major effect of a time-dependent gain by replacing the constant g in Eq. (4.1) by a time-dependent $g(t)$, when it appears in the basic equation in the combination $1+q-g$. Where g multiplies $(1/\omega_L^2)(d^2/dt^2)$ one uses the time-independent small-signal gain g_0 before—or after—the arrival of the pulse. This is justified by the fact that the dispersive effect $(1/\omega_L^2)(d^2/dt^2)$ itself is relatively small (the Fourier spectrum of the pulse is narrow compared with the laser linewidth and hence small changes in the coefficient multiplying the operator may be neglected). Equation (4.1) then becomes

$$\left[1 + q\left(1 - \frac{|v|^2}{P_A}\right) - g_0\left(1 - \frac{|v|^2}{P_{\mathrm{LF}}}\right) - \frac{g_0}{\omega_L^2}\frac{d^2}{dt^2}\right] v = 0 \tag{7.1}$$

where P_{LF} is the saturation power of the "fast" laser medium.

This is the equation of motion of a particle in the force field

$$\left[1 + q - g_0 + \left(g_0 \frac{|v|^2}{P_{\mathrm{LF}}} - q\,\frac{|v|^2}{P_A}\right)\right] v.$$

The force must be repulsive near the origin in order to stop the particle at the origin.

$$1 + q - g_0 > 0. \tag{7.2}$$

If Eq. (7.2) is to be satisfied, the laser is below threshold for small-signal levels and can never get started. The model of a *fast* saturable absorber and a *fast* laser medium does not give self-starting mode-locking solutions.

It is not hard to generalize the model to a system which does have self-starting mode locking yet exhibits an instantaneous response of the laser medium to a pulse excitation. The laser medium must have two relaxation times: one that is long, leading to an over-all pull-down of the gain, and a fast relaxation time which results in an instantaneous change of the laser gain as a function of the intensity in the cavity. There are many examples of systems exhibiting two such relaxation times. In a molecular laser such as CO_2 there is a fast relaxation time corresponding to the rotational relaxation, and there are slower times corresponding to vibrational and translational relaxations. If the parameters of the system are such that the mode-locked pulses are long compared with the rotational relaxation time, but short compared with the vibrational or translational relaxation times, then there are indeed two influences on the gain g. One may write

$$g = \frac{g_0}{(1 + P/P_{\mathrm{LS}})}\left(1 - \frac{|v|^2}{P_{\mathrm{LF}}}\right) = g\left(1 - \frac{|v|^2}{P_{\mathrm{LF}}}\right). \tag{7.3}$$

Here we have introduced the saturation power P_{LS} associated with the slow relaxation.

For this revised system we may now study the conditions necessary to achieve mode locking.

The new "potential" replacing Eq. (4.2) becomes

$$-\frac{1}{2}(1 + q - g)\frac{|v|^2}{P_A} + \frac{1}{4}\left(q - g\frac{P_A}{P_{\mathrm{LF}}}\right)\frac{|v|^4}{P_A^2}.$$

Mode locking is achieved if and only if the coefficient of the fourth-order term is positive, when

$$g\frac{P_A}{P_{\mathrm{LF}}} < \frac{Q}{Q_A^0}, \tag{5.10}$$

i.e., when the saturation power P_{LF} of the laser medium which determines the instantaneous response of the medium is not too small.

VIII. CONCLUSIONS

We have found a closed-form solution, a hyperbolic secant as a function of time, for the problem of mode locking of a homogeneously broadened laser by a saturable absorber. Previous analyses were usually done by computer, yielding no closed-form solutions. The fact that the present analysis permitted an analytic solution is attributable to the approximations: Our model implies small modification of the mode-locked pulse within one transit through the cavity.

The relation between inverse pulse width, saturable absorber parameters, and laser medium parameters could be approximated by [compare Eq. (4.11)]

$$\frac{1}{\omega_L \tau_P} \approx \left(\frac{g_0}{1+q} - 1\right) \frac{q}{1+q} \frac{P_L}{4P_A} \omega_L T_P,$$

where g_0 is the small-signal gain divided by cavity loss, q is the small-signal saturable absorber loss divided by cavity loss, P_L and P_A are saturation powers of laser and saturable absorber, respectively, ω_L is the laser linewidth, and T_P is the pulse repetition time. We have investigated the effect of strong bleaching of the absorber and found no qualitative change of the pulse solution. The wings are still exponentials; the pulse is symmetric.

A *fast* laser medium in conjunction with a fast saturable absorber cannot "self-start" as a mode-locked system. To have a self-starting system one needs at least one slow time constant for the laser medium relaxation. Further, the instantaneous response of the laser medium must not be excessive.

Gurevich and Pakhin found also a secant hyperbolic solution for mode locking by a saturable absorber.[33] However, they treated the limit of a slow saturable absorber in which the bandwidth of the laser medium does not determine the pulse width. In their case there is net gain after passage of the pulse. Hence, perturbations following the pulse grow and the solution is therefore unstable.

Experiments reported in the literature on steady-state mode locking with a fast saturable absorber do not present sufficiently detailed pictures of pulse shapes that could be meaningfully matched against the secant hyperbolic. However, in connection with pulse-shape measurements of a transient mode-locking experiment with a Nd-glass laser, Auston states that the best match to his experimentally observed pulse shapes was a secant hyperbolic.[30] This fact lends credence to our contention that our steady-state mode-locking analysis is relevant to the prediction of transient mode-locking as long as the assumption of a quasi-steady state is justified. This will be the case if the laser system is not excited excessively above threshold, precisely the situation when "clean" mode-locked pulse trains tend to be observed experimentally.[34]

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APPENDIX

To the extent that any form of mode locking—passive or active—compensates for the dispersive effect of amplification, one would expect similar results for the pulse width resulting from any of the theories of mode locking (provided, of course that the approximations made are similar). Kuizenga and Siegman[25] have obtained a formula for the pulse width at half-intensity

$$\tau_P = 2(\sqrt{2}\ln 2)^{1/2}\left(\frac{g_0}{\delta_l}\right)^{1/4}\left(\frac{1}{\omega_m \Delta\omega}\right)^{1/2}. \tag{A1}$$

If one makes an expansion of the exponentials in the theory of Kuizenga and Siegman, their change of the pulse per pass due to loss modulation and nondispersive gain is

$$\Delta v = [a(t)g]v = \{\delta_l[1-\cos(\omega_m t)] - g\}v. \tag{A2}$$

The saturable absorber, in our theory, causes a change per pass

$$\Delta v = \frac{\omega_0}{2Q} T_R \left[1 + q\left(1 - \frac{v_0^2}{P_A \cosh^2(t/\tau_P)}\right) - g\right] v. \tag{A3}$$

At the instant of minimum loss, using an expansion to second order in t, we obtain from the two theories:

Kuizenga and Siegman

$$\Delta v \approx (\tfrac{1}{2}\delta_l \omega_m^2 t^2 - g)v \tag{A4}$$

Ours

$$\Delta v \approx \frac{\omega_0}{2Q} T_R \left[1 + q - \frac{v_0^2}{P_A}\left(1 - \frac{t^2}{\tau_P^2}\right) - g\right]. \tag{A5}$$

Comparison of Eqs. (A4) and (A5) leads to the identification of the g of Ref. 21 (henceforth denoted by g_{KS}) with our $(\omega_0/2Q)T_R g$, and of $\frac{1}{2}\delta_l\omega_M^2$ with $\omega_0 T_R v_0^2/2QP_A\tau_P^2$ Further $\Delta\omega$ of Ref. 25 corresponds to our ω_L. Now, our pulse width obeys the relation

$$\tau_P = \frac{1}{\omega_L}\left(\frac{2g}{qv_0^2/P_A}\right)^{1/2}. \tag{A6}$$

Using the correspondence between v_0^2 and δ_l we find from Eq. (A6) for the half-power width $1.762\tau_P$

$$1.762\tau_P = 1.762\left(\frac{g_{KS}}{\delta_l}\right)^{1/4}\left(\frac{2}{\omega_m\Delta\omega}\right)^{1/2} \tag{A7}$$

which is comparable to Eq. (A1). We conclude that the mechanism for determining pulse widths is really the same as that in forced mode locking except for the fact the modulation is not prescribed *a priori*, but must be determined self-consistently.

Note that the tails of pulses produced by passive mode locking are quite different from those of active mode locking.

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