Parameter Ranges for CW Passive Mode Locking

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Abstract—**CW passive mode locking of a homogeneously broadened laser is considered. In the coordinate plane, whose abscissa is proportional to the small-signal saturable absorber loading, and whose ordinate is proportional to the small-signal gain, the following regimes are laid out:**

1) **steady-state single-pulse mode-locking solutions;**
2) **stability against relaxation oscillations;**
3) **self-starting of mode locking.**

The assumption is made that CW mode locking can be obtained only for a choice of parameters for which all three regimes overlap. We require further that the overlap regime be reached by a monotonic increase of small-signal gain (pumping), without passing outside regime 2). Under these conditions one may state requirements on the system parameters for the obtainment of single-pulse mode locking by a saturable absorber. The analysis explains why it has been impossible to mode lock passively the CW Nd:YAG laser, but passive mode locking of the CW dye-laser system is possible.

I. Introduction

In two previous papers [1], [2] we presented the theory of CW mode locking of a homogeneously broadened laser by a "fast" and a "slow" saturable absorber. The relaxation time of the fast absorber was assumed to be short compared with the duration of the mode-locked pulse τ_p, and that of the slow absorber was assumed to be long compared with τ_p.

The existence of steady-state mode-locked pulse solutions does not guarantee that these will be realized in practice. In fact, the scarcity of successful experiments of mode locking of a homogeneously broadened CW laser by a saturable absorber suggests that, more likely than not, the mode locking may be prevented by some other instability such as the well-known relaxation oscillations [3],[1] or may not be self-starting. The only successful mode locking of CW, or quasi-CW, homogeneously broadened lasers by a saturable absorber was that of Ippen *et al.* [4] on the CW dye laser, and Jung *et al.* [5] on the CO_2 laser. We emphasize that we are discussing homogeneously broadened systems. Inhomogeneously broadened systems are much easier to mode lock because they tend to oscillate in the steady state in a number of axial modes. The sole function of mode locking is to provide an injection signal to phase the different oscillation frequencies with respect to each other. This is a relatively easy task.

In this paper we explore explicitly the requirements for the self-starting of the mode-locked state and for the suppression of relaxation oscillations. Since the relaxation time of the absorber will always be fast compared to the cavity round-trip time, even in the case of the slow saturable absorber, the analyses for both the fast and the slow absorber cases are identical. We shall overlay the parameter regimes of stability against relaxation oscillation, of self-starting of mode locking, and of the mode-locking solution. Only when these three regimes overlap can one expect self-starting and steady-state excitation of mode-locked pulses. We shall see that the parameters of the Nd:YAG laser are incompatible with CW mode-locked operation. This explains why no CW mode locking with a saturable absorber has been reported for this system. We shall also see that the CW dye-laser system fits rather snugly into the region of overlap of these three regimes.

We shall illustrate the regimes of stable mode locking in the q_0-g_0 plane, where[2] $q_0 = Q/Q_A^0$ is the small-signal value of the inverse Q of the saturable absorber Q normalized to the inverse cavity Q, and g_0 is the small-signal value of the gain per pass normalized to the cavity loss per pass. One traverses the q_0-g_0 plane in a vertical direction when the pump power to the laser medium is raised. At fixed pump power the q_0-g_0 plane is traversed horizontally through addition of absorber dye to the solvent in the absorber cell. In the previous papers we have not presented the data in this form. In the fast absorber case we used the variables $g_0/1 + q_0$ and $q_0K/1 + q_0$ where K is the parameter [1]

$$K \equiv \frac{1}{4}\frac{P_L}{P_A}\omega_L T_R \tag{1}$$

with P_L as the laser medium saturation power, P_A as the saturation power of the saturable absorber, ω_L as the laser linewidth, and T_R as the cavity round-trip time. K is a parameter representing the ease with which the saturable absorber is bleached; the smaller P_A compared with P_L, the easier is the bleaching of the saturable absorber. K also contains a measure of the cavity round-trip time normalized to the inverse laser linewidth. In the analysis of the fast absorber [1], the relaxation time of the laser medium T_L was assumed long compared with cavity round-trip time T_R, so that no appreciable gain modulation occurred within the duration of the pulse. The overall gain pulldown was due to the inversion depletion of many successive pulses. If this condition is relaxed and T_L is made comparable to T_R, it is still possible to find mode-locked pulses. We shall assume, in the present analysis, that $T_L \gg \tau_p$ but not necessarily $T_L \gg T_R$.

It is an easy task to replot the boundary of mode-locking solutions of [1, fig. 4] in the g_0-q_0 plane, now with K as a parameter. Fig. 1 shows a sequence of such curves labeled for

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[1] The "giant pulsing" is a fully developed relaxation oscillation. Ours is essentially the study of incipient giant pulsing.

[2] We use a subscript 0 on q to emphasize that it is a small-signal value. References [1] and [2] did not use a subscript.

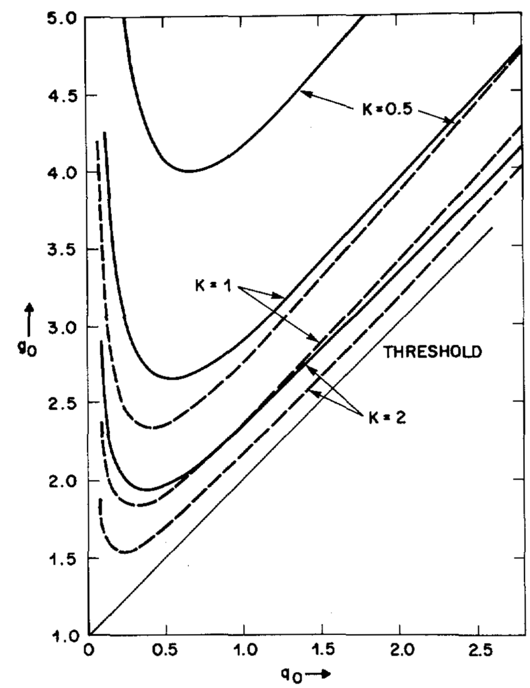

Fig. 1. Regimes of single-pulse mode locking by fast saturable absorber. Solid curves give stability boundary; dashed curves give boundary determined by $1/(\omega_L \tau_p) > 0.1$. The regimes between solid and dashed curve pair contain g_0-q_0 values that lead to single-pulse mode-locking solutions.

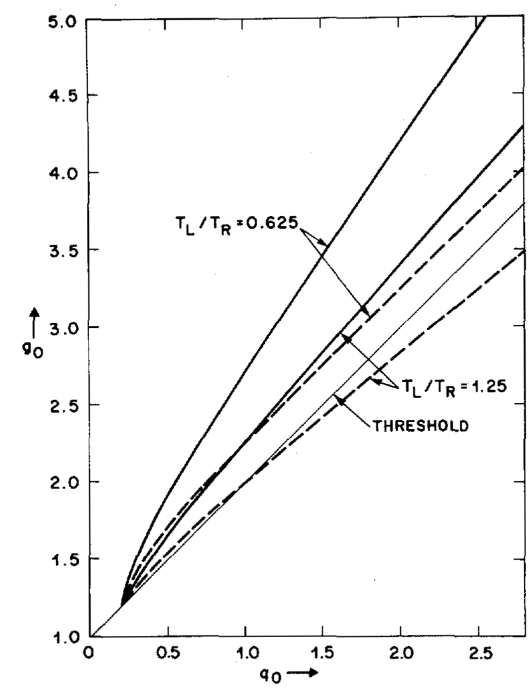

Fig. 2. Regimes of single-pulse mode locking by slow saturable absorber for $s = 6$. Solid curves give upper stability boundary; dashed curves give lower stability boundary. The regimes between solid and dashed curves contain g_0-q_0 values that lead to single-pulse mode-locking solutions.

different K values. Each curve pair bounds the region within which single-pulse mode-locking solutions can be obtained. The top boundary shown solid is the stability boundary of [1]. Above this boundary multiple-pulse solutions occur. The lower boundary shown dashed is drawn for an assumed pulse duration τ_p of magnitude $1/\omega_L \tau_p = 0.1$. Below the boundary the pulses are excessively long. Note the dependence of the mode-locking regime upon K. The regime gets very narrow for $K \geqslant 2$. Hence values of K of the order of unity should be striven for in practical systems so as not to make mode locking too sensitive to (unavoidable) variations in q_0 and g_0.

In the analysis of the slow absorber it was found to be necessary to include the relaxation process of the laser medium in order to assure stability of the mode-locked pulse with respect to (noise) perturbations following the pulse. This requirement was previously pointed out by New [6].

Fig. 2 shows regimes in the g_0-q_0 plane for which mode-locking solutions are found. For any given value of q_0 there is a range of gain (g_0) values for which single-pulse mode-locking solutions exist. The top boundary is shown solid and the bottom boundary dashed for two different values of the ratio T_L/T_R. We have picked

$$s = E_L/E_A = 6$$

where E_A is the saturation energy of the absorber and E_L that of the laser medium. The parameter s plays a role somewhat similar to that of K in the fast saturable absorber case. The lower boundary for $T_L/T_R = 1.25$ (T_L is the laser medium relaxation time) dips below the threshold line $g_0 = 1 + q_0$, indicative of the presence of a hysteresis effect for $q_0 > 1$: as the small-signal gain is raised by increased pumping and threshold is reached, the laser oscillates and mode locks. As the gain is lowered while the laser oscillates, the laser will keep oscillating and mode locking even when the threshold line is passed from above.

In Section II we determine the basic equations for the stability analysis. Section III is devoted to determination of the self-starting regime and Section IV determines the region in the g_0-q_0 plane for which no relaxation oscillations occur.

II. The Basic Equation

The fast absorber analysis [1] contains a relation between the nth and $(n + 1)$th pulse passing through the laser. This relation may be adapted both to the analysis of self-starting and the study of relaxation oscillations. The relation is [1]

$$v_{n+1}(t) = v_n(t - T_R) - \frac{\omega_0 T_R}{2Q}\left[1 + \frac{Q}{Q_A(t)} - g\left(1 + \frac{1}{\omega_L^2}\frac{d^2}{dt^2}\right) + \frac{g}{\omega_L}\frac{d}{dt}\right] v_n(t - T_R). \tag{2}$$

Here $v_n(t)$ is the pulse envelope at one of the end mirrors after nth passage, T_R is the cavity round-trip time, ω_0 is the optical carrier frequency, Q is the empty cavity Q, $1/Q_A(t)$ is the time-dependent contribution of the absorber to the inverse Q, and g is the normalized gain of the medium.

Whereas the detailed derivation of (2) appears in [1], the meaning of (2) is rather self-evident. We see from (2) that v_{n+1} is a delayed and modified version of v_n. The modification is treated as small—and hence additive. There is the loss

$$\Delta v_n|_{\text{loss}} = -\frac{\omega_0 T_R}{2}\left[\frac{1}{Q} + \frac{1}{Q_A(t)}\right] v_n(t - T_R)$$

due to the cavity Q and time-dependent saturable absorber Q.

The laser medium modifies v_n by

$$\Delta v_n|_{\text{gain}} = \frac{\omega_0 T_R}{2Q} g \left[1 + \frac{1}{\omega_L^2}\frac{d^2}{dt^2} - \frac{1}{\omega_L}\frac{d}{dt}\right] v_n(t - T_R).$$

This modification is of operator character which entails growth (first term), spreading via "diffusion in time" (second term), and a delay due to the change in dielectric susceptibility as caused by the laser medium (third term).

We note first of all that a startup of single mode-locked pulses is characterized by the period T_R; relaxation oscillations usually have a much longer period. The laser linewidth affects the process only for time durations of the order of the mode-locked pulsewidth, which is much shorter than T_R. Therefore the terms $(g/\omega_L^2)\, d^2/dt^2$ and $(g/\omega_L)\, d/dt$ can be disregarded in a study of self-starting and of relaxation oscillations. The time dependence of the gain cannot be disregarded, but is easily included by making the remaining g in (2) time dependent.

In the study of the stability of mode-locked solutions against small perturbations [1] we have used two time parameters, a fast time parameter in terms of which the short pulse was described, and a slow time parameter expressing the number of passages through the laser cavity. Once one disregards the operators d^2/dt^2 and d/dt, such a distinction is not necessary; the time t used henceforth describes both the oscillating behavior of a perturbation (real part of the frequency) as well as the growth or decay of the perturbation (imaginary part of the frequency).

If one considers the limit where the change per pass is small enough, so that

$$\frac{v_{n+1}(t) - v_n(t - T_R)}{T_R} \simeq \frac{d}{dt} v$$

then one has the basic equation for v

$$\frac{dv}{dt} = -\frac{\omega_0}{2Q}[1 + q - g]\, v \tag{3}$$

where we have defined the normalized inverse Q of the saturable absorber

$$Q/Q_A(t) \equiv q(t). \tag{4}$$

In addition to the equation for the field envelope $v(t)$ in the cavity, one needs equations for the time-dependent gain $g(t)$ and the time-dependent inverse Q of the absorber, i.e., $q(t)$. We assume that the population difference n_A of the saturable absorber obeys the rate equation

$$\frac{d}{dt} n_A = -\frac{n_A - n_A^0}{T_A} - n_A \frac{P}{E_A} \tag{5}$$

and a similar equation for the population difference of the laser medium. Here n_A^0 is the equilibrium difference, T_A is the relaxation time, and E_A is the saturation energy. Now

$$\frac{n_A}{n_A^0} = \frac{Q_A^0}{Q_A} \tag{6}$$

where Q_A^0 is the small-signal value of Q_A. Using (6) in (5) and the definition (4), we obtain the equation of motion for q

$$\frac{d}{dt} q = -\frac{q - q_0}{T_A} - q\frac{P}{E_A}. \tag{7}$$

The equation of motion for the gain is correspondingly

$$\frac{d}{dt} g = -\frac{1}{T_L}(g - g_0) - g\frac{P}{E_L}. \tag{8}$$

These equations assume that $1/Q_A$ and g follow the time variation of P in the same way as they adjust to its time average (i.e., the same saturation energy is used for both). This is only correct if the laser medium and saturable absorber medium are near the end mirrors. Modifications to take into account other positions will be considered later.

Equation (3) has a different interpretation when the period of the process under study is a submultiple of the cavity round-trip time T_R (self-starting of mode locking) from the one when the period of the process is much longer than T_R (relaxation oscillations).

The first case is, to a certain extent, analogous to the perturbation of a free-running oscillator, of frequency $2\pi/T_R$. The frequency of the perturbation is close to this "unperturbed" frequency. In the second case of a relaxation oscillation with a period much greater than T_R one has the analog of a system which selects its own frequency of oscillation; the frequency is not prescribed by the round-trip condition itself. A relaxation oscillation is one in which the energy inside the cavity grows at a rate usually much slower than the round-trip time. The laser gain gets progressively depleted, gain gives way to loss, the energy decays. The pumping restores the gain while the field in the cavity is low, and the process repeats itself. For a certain parameter range the regimes of relaxation oscillation and self-starting merge and cannot be distinguished. We shall show later that in practical systems this is never the case.

III. Study of Self-Starting

Let us set up the equations for self-starting of mode locking from a CW time-independent state of operation. Suppose that the amplitude of the field in the cavity consists of a constant time-independent part v_s and a perturbation δv which changes upon each round trip. The time-independent part must obey the relation, using (3),

$$[1 + q_s - g_s]\, v_s = 0 \tag{9}$$

where we use subscripts s to denote the steady state (in distinction of the zero-power small-signal values indicated by 0). Because $v_s \neq 0$, the coefficient in brackets must vanish. This is a condition from which we may evaluate the power of the steady state $P_s = |v_s|^2$. Using the time-independent forms of (7) and (8), one has from (9)

$$1 + \frac{q_0}{1 + \dfrac{P_s T_A}{E_A}} - \frac{g_0}{1 + \dfrac{P_s T_L}{E_L}} = 0. \tag{10}$$

Equation (10) yields two roots for P_s, one of which is negative if $1 + q_0 - g_0 < 0$, i.e., the system is above threshold. Hence only one root is acceptable. A perturbation of (3) gives

$$\frac{d}{dt}\,\delta v = -\frac{\omega_0}{2Q}\,[\delta q - \delta g]\,v_s. \tag{11}$$

We note first of all that self-starting of a mode-locked pulse is characterized by growth of a periodic perturbation on the time-independent laser output. The perturbation must have a period which is a submultiple of T_R. We set

$$\delta v = a \exp j\phi \exp j\,\frac{2\pi m}{T_R}\,t \tag{12}$$

where a and ϕ are time dependent. The time dependence of ϕ accounts for deviations of the repetition period from T_R.

The power δP caused by the perturbation is, with a convenient nomalization for v,

$$P = |v_s|^2 + \left[a \exp j\phi \exp\left(j\,\frac{2\pi m}{T_R}\,t\right) + a \exp j\phi \exp\left(-j\,\frac{2\pi m}{T_R}\,t\right)\right] v_s. \tag{13}$$

The perturbation of the gain may be written, using (8),

$$\delta g = -Y_L\,\frac{a}{v_s}\exp j\phi \exp j\,\frac{2\pi}{T_R}\,t + \text{c.c.} \tag{14}$$

where

$$Y_L \equiv \frac{g_s\,\dfrac{P_s}{P_L}}{1 + \dfrac{P_s}{P_L} + j\,\dfrac{2\pi m}{T_R}\,T_L} \tag{15}$$

and $P_L = E_L/T_L$. Similarly, one finds for δq from (7)

$$\delta q = -Y_A\,\frac{a}{v_s}\exp j\phi \exp j\,\frac{2\pi}{T_R}\,t + \text{c.c.} \tag{16}$$

where

$$Y_A = \frac{q_s\,\dfrac{P_s}{P_A}}{1 + \dfrac{P_s}{P_A} + j\,\dfrac{2\pi m}{T_R}\,T_A} \tag{17}$$

and $P_A = E_A/T_A$. The equation for the perturbation δv becomes, after separation of the positive and negative frequency components,

$$\dot{a} + j\dot{\phi}a = \frac{\omega_0}{2Q}\,(Y_A - Y_L)\,a. \tag{18}$$

The perturbation will grow, ($\dot{a} > 0$) mode locking will be self-starting, if

$$\text{Re}\,[Y_A - Y_L] > 0 \tag{19}$$

or

$$\frac{\dfrac{q_0}{P_A}}{\left(1 + \dfrac{P_s}{P_A}\right)^2 + \left(\dfrac{2\pi m T_A}{T_R}\right)^2} > \frac{\dfrac{g_0}{P_L}}{\left(1 + \dfrac{P_s}{P_L}\right)^2 + \left(\dfrac{2\pi m T_L}{T_R}\right)^2}. \tag{20}$$

One may eliminate P_s from (20) using (10), and thus one obtains a relation between g_0 and q_0 with $(2\pi m T_L/T_R)$, T_L/T_A, and P_L/P_A as parameters. A boundary for self-starting in the q_0-g_0 plane is shown in Fig. 3 which is plotted for the following assumed parameter values: $(2\pi m T_L/T_R) = 2$; $(T_L/T_A) = 4$; and $(P_L/P_A) = 0.5, 0.75$, and 1.0.

Self-starting of mode locking occurs within the q_0-g_0 plane below the curve(s) shown in Fig. 3. Only those portions of the curves above the threshold line are shown because a precondition for the analysis of self-starting of mode locking was the existence of a time-independent solution. In a monotonic increase of the pumping this is the case only when the threshold line has been passed. The curves change for different choices of parameters, but certain general characteristics may be noted. The threshold line is

$$g_0 = 1 + q_0. \tag{21}$$

At threshold the power approaches zero, and hence one may find the intercept of the self-starting boundary with the threshold line by eliminating g_0 from the two equations (20) and (21) with $P_s = 0$ and the inequality replaced by an equality. One finds

$$q_0 = \frac{1}{\dfrac{P_L}{P_A}\,\dfrac{1 + (2\pi m T_L/T_R)^2}{1 + (2\pi m T_A/T_R)^2} - 1}.$$

In order to obtain an intercept at a finite q_0, one requires that

$$\frac{P_L}{P_A}\,\frac{1 + (2\pi m T_L/T_R)^2}{1 + (2\pi m T_A/T_R)^2} > 1. \tag{22}$$

If there is no intercept, there are no regions of self-starting above the threshold line. Hence raising the pumping power

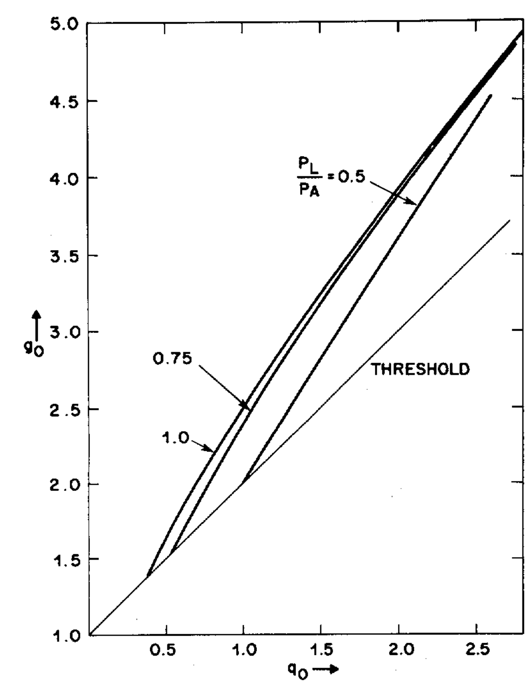

Fig. 3. Self-starting regimes in q_0-g_0 plane; the regimes to right of respective curves correspond to self-starting. Only regimes above threshold line are shown.

(g_0) until the threshold line is crossed will result in operation with a time-independent power output—no mode locking. Equation (22) shows that a value of P_L/P_A that is too small (the saturable absorber is too hard to saturate) can result in no self-starting of mode locking.

IV. Relaxation Oscillation

We consider next the question whether relaxation oscillations may exist in the system. If one finds such oscillations, one may expect, at the very least, that the mode-locked pulse train is modulated by the relaxation oscillation. It is more likely that the existence of relaxation oscillations suppresses mode locking if the period of the relaxation oscillation is too short to allow for the buildup of mode-locked pulses within one cycle of the oscillation.

The equation for the radiation inside the cavity is now a simple rate equation. It follows from (3) by multiplication by v^* and addition of the complex conjugate. One may normalize v so that $|v|^2 = P$, the power traveling in one direction in the cavity. The differential equation for P is

$$\frac{d}{dt} P = -\frac{1}{T_C} [1 + q - g] P \tag{23}$$

where

$$\frac{1}{T_C} = \frac{\omega_0}{Q}.$$

The perturbation of (23) gives

$$\frac{d}{dt} \delta P + \frac{1}{T_C} [\delta q - \delta g] P_s = 0 \tag{24}$$

where we have taken into account the fact that for the CW steady state the quantity $1 + q_s - g_s = 0$. If T_A is taken as short compared to the period of interest, a condition always met when relaxation oscillations tend to occur, then δq is an instantaneous function of $\delta P/P_A$

$$\delta q = -\frac{q_0}{\left(1 + \frac{P_s}{P_A}\right)^2} \frac{\delta P}{P_A}. \tag{25}$$

Using (25) in (24) and the differential equation (8) for δg, one has two coupled first-order differential equations for δP and δg. Assuming the time dependence $\exp st$, one obtains the determinantal equation

$$s^2 - s\left\{\frac{q_0 \frac{P_s}{P_A}}{\left(1 + \frac{P_s}{P_A}\right)^2 T_C} - \left(1 + \frac{P_s}{P_L}\right)\frac{1}{T_L}\right\} - \frac{1}{T_C T_L}\left[\frac{\left(1 + \frac{P_s}{P_L}\right)}{\left(1 + \frac{P_s}{P_A}\right)^2} q_0 \frac{P_s}{P_A} - \frac{g_0}{1 + \frac{P_s}{P_L}} \frac{P_s}{P_L}\right] = 0. \tag{26}$$

Instabilities are found (Re $s > 0$) when either the s-independent term is negative, i.e.,

$$\frac{q_0 \frac{1}{P_A}}{\left(1 + \frac{P_s}{P_A}\right)^2} > \frac{g_0 \frac{1}{P_L}}{\left(1 + \frac{P_s}{P_L}\right)^2} \tag{27}$$

or the coefficient of s is negative

$$\frac{q_0 \frac{P_s}{P_A}}{\left(1 + \frac{P_s}{P_A}\right)^2} > \left(1 + \frac{P_s}{P_L}\right)\frac{T_C}{T_L}. \tag{28}$$

Condition (27) corresponds to exponential growth of a perturbation. It has the simple interpretation

$$\left|\frac{d}{dP} q\right| > \left|\frac{d}{dP} g\right| \quad \text{at } P = P_S. \tag{29}$$

This condition is met when the loss characteristic $1 + q(P)$ and gain characteristic $g(P)$ intersect so that the slope of the loss characteristic is steeper than the slope of the gain characteristic. If the small-signal value of the gain g_0 is greater than $1 + q_0$, i.e., the system is above threshold, then the slope of g is always steeper than the slope of q at the intersection point $P = P_s$, and hence (27) is not satisfied for a system above threshold. Therefore condition (27) is not of interest in our present investigation which treats only self-starting systems ($g_0 > 1 + q_0$).

Condition (28) is of interest and gives regimes of instability in the q_0-g_0 plane. The regime of instability is circumscribed by the curve obtained from (28) by replacing the inequality by an equality. The curve has for one asymptote the threshold line $g_0 = 1 + q_0$, and for the other asymptote the straight line

$$g_0 = \frac{P_A}{P_L} q_0. \tag{30}$$

The minimum value of q_0 is found to occur at a power $P_{\min}$

$$P_{\min}/P_A = -\frac{1}{4}\frac{P_L}{P_A} + \sqrt{\left(\frac{1}{4}\frac{P_L}{P_A}\right)^2 + \frac{1}{2}\frac{P_L}{P_A}}$$

and has the value

$$q_{\min} = \frac{T_C}{T_L} \frac{(1 + P_{\min}/P_L)(1 + P_{\min}/P_A)^2}{P_{\min}/P_A}.$$

Thus the instability regime lies above the threshold line for $P_L/P_A < 1$ and covers more and more of the q_0-g_0 plane for smaller and smaller values of T_C/T_L. Fig. 4 shows some typical stability boundaries.

From (28) we see that an excessive value of q_0 leads to the instability, as is to be expected because the bleaching of the absorber is the cause of the instability. The smaller T_C/T_L, or the larger the laser medium relaxation time compared to the cavity relaxation time, the greater the tendency toward an instability. This may be understood by recalling the stabilizing influence of the gain changes. If T_L is too long, the gain change shifts out of phase with the change of absorption of the saturable absorber and hence the gain change cannot stabilize against the destabilization of the absorption change.

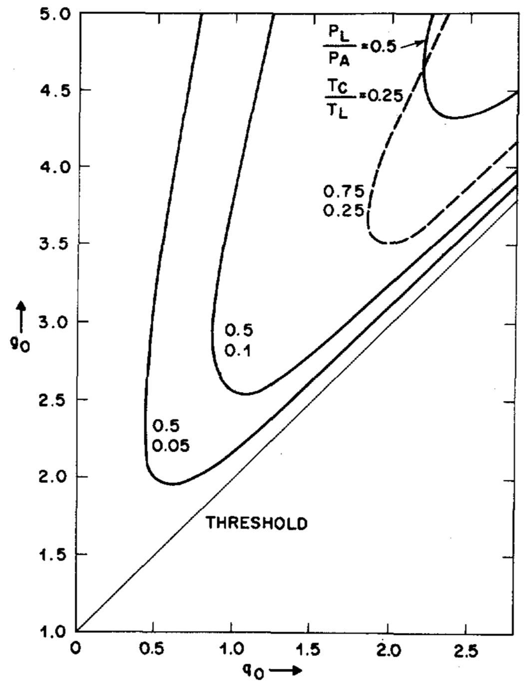

Fig. 4. Instability contours in q_0-g_0 plane for relaxation oscillations; $P_L/P_A < 1$, regions within curves are unstable.

The stability boundary of Fig. 4 illustrates the difficulty of avoiding relaxation oscillations in the Nd:YAG system. Indeed, for the values displayed in Table I, which are typical of the system of Liao and Weber [7] in which they attempted mode locking, we find that $P_L/P_A < 1$ except for the longest relaxation times of the saturable absorber. When $P_L/P_A < 1$ and $T_C/T_L \ll 1$ (as is by necessity the case for an Nd:YAG system with a cavity of reasonable length and loss), the relaxation oscillation regime covers pretty much all of the q_0-g_0 plane above threshold. Indeed, the regime stable with regard to relaxation oscillation is practically invisible on the scale of Fig. 4. Hence no mode locking is possible for $P_L/P_A < 1$. A different situation prevails when T_A approaches the longest value attempted, 100 ps. Then the relaxation oscillation instability region moves below the threshold line. We find, however, that the K parameter becomes extremely large ($K \to 630$). On the scale of Fig. 1 in this limit of large K, the mode-locking region is of "infinitesimal" extent, and hence stable mode locking is unachievable, even though no relaxation oscillations need occur.

It is worth noting that relaxation oscillations have been observed in various situations. The brief theory presented here explains properly the growing oscillations of [8] observed in the aftermath of passive Q switching. In the experiment of [8], $P_L/P_A < 1$, and hence the instability boundary occurred above the threshold line. As the gain and loss gradually recovered, the operating point moved upward and to the upper right in the g_0-q_0 plane and entered the instability region. The predicted period of the order of $2\pi\sqrt{T_C T_L}$ checks with the observed period of 10 μs, using the parameter values of the experiment [9].

One may ask why transient saturable absorber mode locking of the Nd:YAG or Nd:glass laser is possible [10], [11], whereas steady-state mode locking is prevented by relaxation

TABLE I

	Nd YAG	Dye Laser
T_L	230 μs	4 ns
T_A	10 <—> 100 ps	1.2 ns
T_R	10 ns	10 ns
σ_A	10^{-16} cm^2	3×10^{-16} cm^2
σ_L		10^{-16} cm^2
A_A	$[10\mu]^2$	$[10\mu]^2$
A_L	10^{-2} cm^2	$[20\mu]^2$
P_L/A_L	2.3×10^3w cm^{-2}	7.7×10^5w cm^{-2}
P_A/A_A	1.8×10^8w cm^{-2} <—> 1.8×10^7	8.4×10^5w cm^{-2}
P_L/P_A	.13 <—> 1.3	3.7
ω_L	2.1×10^{11} s^{-1}	
$K = \frac{1}{4}\frac{P_L}{P_A}\omega_L T_R$	63 <—> 630	

oscillations. Clearly, if one obtains mode-locked pulses before the relaxation oscillations set in, then transient mode locking is assured. But why, then, did Liao and Weber not observe the mode-locked substructure of the relaxation oscillations? We believe that the reason for this is that the excess gain achieved in transient excitation tends to be much higher than the gain in steady-state excitations. Hence mode locking can build up faster in the transient case than under CW excitation. The transient excitation is usually over by the time relaxation oscillations have a chance to build up.

V. Discussion

We have identified three regions in the q_0-g_0 plane (the plane of saturable absorber loading and small-signal gain) whose disposition determines whether one can obtain self-starting single mode-locked pulses without running into relaxation oscillations. The dependence of the three regions upon the system parameters determines the required choice of system parameters. Here we shall briefly review the ranges of parameters found appropriate by the analysis.

A. Fast Absorber Case

The determining parameter for the width of the single-pulse mode-locking regime is K

$$K = \frac{1}{4}\frac{P_L}{P_A}\omega_L T_R.$$

The larger K the narrower the regime in the q_0-g_0 plane within which mode-locked single-pulse solutions are obtained. Now, it is clear that generally $\omega_L T_R >> 1$, because ω_L will be of the order of the inverse pulsewidth $1/\tau_p$ or larger ($\omega_L \tau_p > 1$ as indicated by the analysis of [1, fig. 2]), and $T_R/\tau_p >> 1$. In order to keep K on the order of unity, one needs

$$P_L/P_A < 1. \tag{31}$$

The saturable absorber should not saturate easily compared with the laser medium. A further requirement for the fast absorber case is that

$$T_L \geqslant T_R \tag{32}$$

so that the laser medium does not fully recover between pulses. Indeed, the leading edge of the mode-locked pulse must experience loss, $1 + q_0 - g > 0$, so that it is stable against perturbations preceding the pulse. If the laser medium recovered fully between pulses, $T_L << T_R$, then $g = g_0$ and the condition holds $1 + q_0 - g_0 > 0$. This means that the laser is below threshold and can never get started.

Condition (31) applied to (30) implies that a regime of relaxation oscillations occurs above the threshold line. In order to prevent a severe overlap of the instability regime, one needs

$$T_C/T_L \gtrsim 1. \tag{33}$$

The cavity relaxation time should be of the order of or larger than the laser relaxation time.

Finally, in order to have self-starting of mode-locked pulses one must satisfy (22). This condition can be met in spite of (31) as long as

$$\frac{P_L}{P_A} > \frac{1 + \left(\frac{2\pi T_A}{T_R}\right)^2}{1 + \left(\frac{2\pi T_L}{T_R}\right)^2}. \tag{34}$$

B. Slow Absorber Case

Mode locking with a slow absorber requires that the laser medium do some pulse shaping of its own. In order that the pulse is stable against perturbations preceding the pulse, one requires, as in the fast absorber case, that the laser medium not recover fully between pulses, or

$$T_L/T_R \gtrsim 1. \tag{35}$$

Further, one requires that the s parameter be greater than unity

$$s = \frac{E_L}{E_A} > 1. \tag{36}$$

It has been assumed in the analysis of mode locking by a slow absorber that the absorber fully recovers between pulses or

$$T_A/T_R < 1. \tag{37}$$

This is a necessary condition since the absorber must recover faster than the laser medium; recovery of the laser medium before that of the absorber would lead to net gain preceding the occurrence of the next pulse and lead to instability of the pulse train.

In order to have self-starting of the mode-locked solution, one needs

$$\frac{P_L}{P_A} > \frac{1 + \left(\frac{2\pi T_A}{T_R}\right)^2}{1 + \left(\frac{2\pi T_L}{T_R}\right)^2}. \tag{38}$$

No direct requirement on P_L/P_A is found. Indeed, P_L/P_A could be made greater than unity, making it easy to meet (38) and, at the same time, preventing relaxation oscillations above the threshold line (compare Fig. 4). For this reason it is less difficult to satisfy the three conditions for a slow saturable absorber than for the fast absorber.

So far we have assumed implicitly that the laser medium is at one of the end mirrors of the cavity. This assumption is inherent in using the same saturation energy E_L for all time dependences of g. If the laser medium is in the center of the cavity, then a perturbation of frequency $2\pi/T_R$ does not excite a corresponding perturbation of g. This is a consequence of the fact that a frequency of $2\pi/T_R$ is produced by the beating of two adjacent axial modes. But two adjacent axial modes do not couple via a medium in the cavity center. An immediate consequence of this fact is that the gain will not tend to stabilize against absorber changes—perturbations of period $2\pi/T_R$ are all unstable; the mode locking is always self-starting. Thus when the laser medium is in the cavity center one may disregard conditions (34) and (38).

In one of the realizations of the Shank-Ippen mode-locked dye-laser system the laser medium was mixed with the absorber dye and near one of the end mirrors. In another realization the laser medium was put into the center of the cavity. Fig. 5 is a plot of regimes 1), steady-state single-pulse mode-locking solutions, and 3), self-starting of mode locking, for the Shank-Ippen system in the first configuration, using

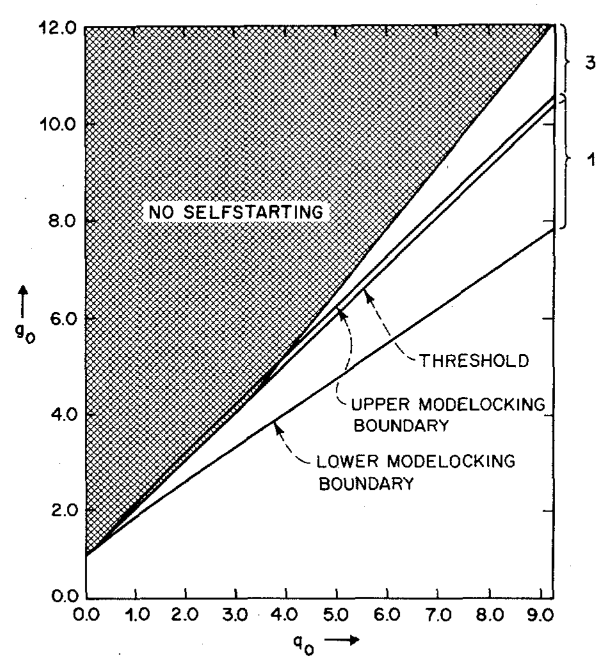

Fig. 5. Plot of regimes 1) and 3) for system of Shank and Ippen. Regime 2) occurs for $q_0 > 590$ and does not appear on this scale.

the values of Table I. Regime 2), the regime stable against relaxation oscillations, covers the plane above the threshold line in Fig. 5. In the second configuration one may simply disregard the self-starting boundary because the system is automatically self-starting. We have disregarded group velocity dispersion in plotting the mode-locking regime, thus overestimating slightly the extent of the regime [10].

We have mentioned earlier that relaxation oscillations and self-starting conditions might merge if the frequency of oscillation is comparable to $2\pi/T_R$. In this case the relaxation oscillations would not represent a threat to successful mode locking because they are indistinguishable from self-starting conditions. Now, (26) shows that the frequency of the relaxation oscillations is of the order of $1/\sqrt{T_C T_L}$. Hence when $\sqrt{T_C T_L} \simeq T_R/2\pi$, one could disregard the contour of stability against relaxation oscillations. But we have pointed out that $T_L > T_R$ so that the laser medium does not fully recover between pulses. Because $T_C >> T_R$ for low-loss systems, one finds that this condition is never met in practice.

VI. Conclusions

We have stated conditions for the self-starting of CW passively mode-locked pulse trains of single pulses. The available experimental facts on successful passive mode locking of CW laser systems seem to bear out our results. Also, we can explain why attempts at passive mode locking of the CW Nd:YAG laser have failed. In the process we have developed criteria for the choice of laser and saturable absorber parameters so that passive mode locking of the system will be successful.

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