FM and AM Mode Locking of the Homogeneous Laser—Part I: Theory

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Abstract—A new general analysis for mode-locked operation of a homogeneously broadened laser with either internal phase (FM) or amplitude (AM) modulation is presented in this paper. In this analysis, a complex Gaussian pulse is followed through one pass around the laser cavity. Approximations are made to the line shape and modulation characteristics to keep the pulse Gaussian. After one round trip, a self-consistent solution is required. This yields simple analytic expressions for the pulse length, frequency chirp, and bandwidth of the mode-locked pulses. The analysis is further extended to include effects of detuning of the modulator, in which case analytical expressions are obtained for the phase shift of the pulse within the modulation cycle, the shift of the pulse spectrum off line center, the change in pulse length, and the change in power output. Numerical results for a typical Nd:YAG laser are given. In the case of the FM mode-locked laser it is found that there is a frequency chirp on the pulse and that this causes pulse compression and stretching when the modulator is detuned. Etalon effects and dispersion effects are also considered.

I. Introduction

THE phenomenon of mode locking of a laser by an internal phase or amplitude perturbation to obtain short optical pulses is well known and has been investigated theoretically and experimentally by several authors.

Theoretical studies of the mode-locked laser with an inhomogeneously Doppler-broadened atomic line have been done by DiDomenico [1], Yariv [2], and Crowell [3], all of whom discuss a linearized solution to the problem. More detailed nonlinear calculations for the FM-type mode locking [4] and AM-type mode locking [5] have been presented by Harris and McDuff. In all the above analyses the coupled-mode-equation approach has been used, assuming that the axial modes saturate independently. This has led to a good understanding of mode-locked gas lasers, in particular the He–Ne laser [6] and argon laser [7].

Mode locking has also been observed in solid-state Nd:YAG lasers by DiDomenico *et al.* [8], using an amplitude modulator and by Osterink and Foster [9] using a phase modulator; and in solid-state ruby lasers using an internal amplitude modulator by Deutsch, Pantell, and Kohn [10], [11]. In analyzing these lasers and any other homogeneously broadened lasers, the use of the coupled-mode equations is complicated by the fact that the axial modes do not saturate independently due to the homogeneous broadening, and also by the fact that a very large number of coupled axial modes are usually generated. Haken and Pauthier [12] have suggested one new analytical approach that can be used for the homogeneous AM-type mode locking.

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In this paper we present a totally different approach to the homogeneously broadened system [13], working completely in the time domain, rather than the frequency or coupled-mode domain. We assume that there is a short pulse inside the laser cavity, and we follow this pulse once around the laser cavity, through the active medium and the modulator. We then require a self-consistent solution, i.e., no net change in the complete round trip. This approach is very similar to that used some time ago by Cutler [14] to analyze the microwave regenerative pulse generator. A similar approach has recently been used by Gunn [15] to analyze the homogeneously broadened laser with internal amplitude modulation.

In our analysis we assume that the pulse is Gaussian, and we make necessary approximations to the atomic line shape and intracavity modulation functions to keep the pulse Gaussian. One essential approximation is that the bandwidth of the pulse is small compared to the atomic linewidth. Experimental observations in the Nd:YAG laser show that this approximation is quite reasonable.

In this paper we will first consider some properties of Gaussian pulses, the active medium and the FM and AM modulators. The self-consistent solution then leads to a simple expression for the mode-locked pulsewidth, showing the dependence on all the important laser parameters such as linewidth, modulation frequency, depth of modulation, and saturated gain. Detuning of the modulator for the FM and AM cases is also considered as well as modifications to the theory for etalon effects and dispersion.

Where applicable, numerical results for a typical Nd:YAG laser will be presented.

II. Gaussian Pulses

We will consider the most general Gaussian optical pulse given by

$$E(t) = \tfrac{1}{2}E_0 \exp(-\alpha t^2) \exp[j(\omega_p t + \beta t^2)]. \tag{1}$$

The term α determines the Gaussian envelope of the pulse and the term $j\beta t$ is a linear frequency shift during the pulse (chirp). A complex constant γ can be defined as

$$\gamma = \alpha - j\beta, \tag{2}$$

so that

$$E(t) = \tfrac{1}{2}E_0 \exp(-\gamma t^2) \exp(j\omega_p t).$$

The Fourier transform of this pulse is given by [16]

$$E(\omega) = (E_0/2)\sqrt{\pi/\gamma} \exp[-(\omega - \omega_p)^2/4\gamma]. \tag{3}$$

The pulsewidth can be defined in various ways. In this paper we define the pulsewidth (τ_p) as the time between half-intensity points, and from (1) it follows that

$$\tau_p = \sqrt{(2 \ln 2)/\alpha}. \tag{4}$$

The bandwidth or spectral width of the (Δf_p) is defined as the frequency between half-power points of the pulse spectrum, and from (3) we get[1]

$$\Delta f_p = (1/\pi)\sqrt{2 \ln 2[(\alpha^2 + \beta^2)/\alpha]} \text{ Hz}. \tag{5}$$

Note how the frequency chirp contributes to the total bandwidth. The pulsewidth–bandwidth product is a parameter often used to characterize pulses, and for the Gaussian pulses used in this analysis, the pulsewidth–bandwidth product is given by

$$\tau_p \cdot \Delta f_p = (2 \ln 2/\pi)\sqrt{1 + (\beta/\alpha)^2}. \tag{6}$$

Two important special cases are $\beta = 0$ and $|\beta| = \alpha$ and for these two cases $\tau_p \cdot \Delta f_p = 0.440$ and $\tau_p \cdot \Delta f_p = 0.626$. We will see later that these cases apply to AM and FM modulation, respectively.

III. Active Medium

For a laser with a homogeneously broadened line such as the Nd:YAG laser [17], it can be shown (Appendix I) that the amplitude gain is given by

$$g_a(\omega) = \exp g/[1 + 2j(\omega - \omega_a)/\Delta\omega] \tag{7}$$

where g is the *saturated* amplitude gain through the active medium at line center (ω_a) for one round trip in the cavity. The midband gain coefficient is related to other material constants is shown in Appendix I. It should be noted here that L_c in (89) is twice the length of the active medium for an ordinary Fabry–Perot laser cavity, while L_c is equal to the active medium length for the ring-type laser cavity.

If we now consider the case where the bandwidth of the pulse is much less than the linewidth, we can expand (7) as follows.

$$g_a(\omega) \simeq G \exp\left[-2jg\left(\frac{\omega - \omega_a}{\Delta\omega}\right) - 4g\left(\frac{\omega - \omega_a}{\Delta\omega}\right)^2\right], \tag{8}$$

where $G = e^g$ and we have assumed that $(\omega - \omega_a)/\Delta\omega < 1$ so that we can neglect higher order terms. The line shape has now become Gaussian, and a Gaussian pulse going through an active medium with this line shape will remain Gaussian.

In (8), we have expanded the line shape about the center frequency ω_a. If we expand the line shape about any other frequency ω_p, we get

$$g_a(\omega) \simeq \exp\frac{g}{1 + 2j\eta}\left[1 - \left(\frac{2j}{1 + 2j\eta}\right)\left(\frac{\omega - \omega_p}{\Delta\omega}\right) - \frac{4}{(1 + 2j\eta)^2}\left(\frac{\omega - \omega_p}{\Delta\omega}\right)^2\right] \tag{9}$$

where $\omega_a = \omega_p + \nu$, and $\eta = \nu/\Delta\omega$, the shift of the spectrum center frequency ω_p normalized to the atomic linewidth $\Delta\omega$.

We will see later that when we consider detuning of the modulator, the pulse spectrum does indeed shift away from line center, and hence we must consider the line shape given by (9). Note that (8) and (9) have the same form, with g replaced by $g/(1 + 2j\eta)$ and $\Delta\omega$ replaced by $\Delta\omega(1 + 2j\eta)$ in (8).

IV. Modulator

The intracavity modulator may in practice be either a phase modulator or an amplitude modulator.

A. *Phase Modulator*

The internal phase modulator introduces a sinusoidally varying phase perturbation $\delta(t)$ such that the round-trip transmission through the modulator is given by

$$\exp[-j\delta(t)] = \exp(-j2\delta_c \cos\omega_m t), \tag{10}$$

where ω_m is the modulation frequency and δ_c is the effective single-pass phase retardation of the modulator. For an ordinary Fabry–Perot laser cavity with an intracavity modulator, it can be shown that [4]

$$\delta_c = \left(\frac{L}{a}\frac{2}{\pi}\sin\frac{a}{L}\frac{\pi}{2}\right)\left(\cos\frac{Z_0\pi}{L}\right)\delta_m, \tag{11}$$

where L is the length of the cavity, a the length of the modulator crystal, Z_0 the distance of the modulator to a mirror, and δ_m the peak phase retardation through the crystal. Usually $a/L \ll 1$ and $\delta_c \simeq \delta_m \cos(\pi Z_0/L)$. For the ring-type cavity, the pulse passes through the modulator only once per round trip, and we should drop the 2 in (10). Also, $\delta_c \simeq \delta_m$ for the ring cavity.

For the ideal mode-locking case, one can visualize short pulses passing through the modulator consecutively at one or other extremum of the phase variation. If we make the approximation that the pulse is short compared to the modulation period, then the transmission through the modulator is given by

$$\exp[-j\delta(t)] \simeq \exp(\mp j2\delta_c \pm j\delta_c\omega_m^2 t^2). \tag{12}$$

We note that there can exist two possible solutions for the FM case, one for each extremum of the phase variation. For further reference to these two possibilities we shall call the mode of operation corresponding to the top sign in (12) the positive mode and the other mode the negative mode. Note that the positive mode corresponds to the maximum phase variation and the negative mode to the minimum phase variation.

We can also consider the more general case, when the

[1] We adopt the convention that ω denotes circular frequencies in radians and f denotes frequencies in hertz. We will use ω in the analyses, and change to f in the final results.

pulse goes through the modulator at a phase angle θ from the ideal case. The transmission through the modulator can now be written as

$$\exp[-j\delta(t)] \simeq \exp[\mp j2\delta_c \cos\theta \pm j2\delta_c \sin\theta(\omega_m t) \pm j\delta_c \cos\theta(\omega_m t)^2]. \quad (13)$$

For θ positive, the pulse lags behind the modulation signal. We can interpret the terms in the exponent as follows. The first term is an additional phase shift that changes the optical length of the cavity, so that the optical length is now given by

$$L_0 = L \pm (\delta_c/\pi)\lambda_a \cos\theta. \quad (14)$$

The second term is a Doppler frequency shift and the third term gives a linear frequency chirp to the pulse. It is the last term that causes the mode locking.

B. Amplitude Modulator

For the amplitude modulator an idealized modulation characteristic will be assumed, where the amplitude transmission through the modulator is given by

$$a(t) = \exp(-2\delta_l \sin^2 \omega_m t). \quad (15)$$

The ideal mode-locking case is now when the pulse passes through the modulator at the instant of maximum transmission. This occurs twice in every cycle of the modulation signal, and hence the modulation frequency is half that for the phase-modulation case. With the assumption that the pulse is short compared to the modulation period, (15) becomes

$$a(t) \simeq \exp[-2\delta_l(\omega_m t)^2]. \quad (16)$$

For the more general case where the pulse passes through the modulator at a phase angle θ from the ideal case, (15) can be written as

$$a(t) \simeq \exp -[2\delta_l \sin^2\theta + 2\delta_l \sin 2\theta(\omega_m t) + 2\delta_l \cos 2\theta(\omega_m t)^2]. \quad (17)$$

The modulation characteristics of actual amplitude modulators may be considerably different, but it may be assumed that the amplitude transmission can always be written as follows

$$a(t) = \exp -[2\delta_0 + 2\delta_1(\omega_m t) + 2\delta_2(\omega_m t)^2 \cdots], \quad (18)$$

where δ_0, δ_1, and δ_2 depend on the phase angle θ and the depth of modulation. The constants will have to be evaluated for particular modulators such as the acoustic and electrooptic modulators. It can be shown, however, that for $\theta = 0$, $\delta_0 = 0$ and $\delta_1 = 0$.

V. Self-Consistent Solution With No Detuning

We can now consider the pulse going through the active medium and the modulator, and for the approximations we have made in the previous sections, a Gaussian pulse will remain Gaussian and a self-consistent solution becomes possible. The models for analyzing the mode-locked laser are shown in Fig. 1, and we see that the Fabry–Perot and ring cavities are entirely equivalent, except for small differences noted in the previous sections. We will only consider the Fabry–Perot type cavity, but in all cases the results for a ring cavity can be obtained by dividing the depth of modulation by 2.

If $E_1(t)$ is the pulse entering the active medium, then the Fourier transform of the pulse coming out is given by

$$E_2(\omega) = g_a(\omega)E_1(\omega) = \frac{E_0 G}{2}\sqrt{\frac{\pi}{\gamma}} \exp\left[-2jg\left(\frac{\omega-\omega_a}{\Delta\omega}\right) - 4g\left(\frac{\omega-\omega_a}{\Delta\omega}\right)^2 - (\omega-\omega_p)^2/4\gamma\right]. \quad (19)$$

Here we have considered the ideal case where the pulse is on line center ($\omega_p = \omega_a$). This will be the case if the pulse passes through the FM modulator with no Doppler shift or the AM modulator at minimum loss. In this case, (19) can be written as

$$E_2(\omega) = \frac{E_0 G}{2}\sqrt{\frac{\pi}{\gamma}} \cdot \exp[-A(\omega-\omega_a)^2] \exp[-jB(\omega-\omega_a)], \quad (20)$$

where

$$A = 1/(4\gamma) + 4g/\Delta\omega^2 \quad (20a)$$

and

$$B = 2g/\Delta\omega. \quad (20b)$$

Transforming into the time domain, the pulse becomes

$$E_2(t) = (E_0 G/4\sqrt{\gamma A}) \exp[-(t-B)^2/4A] \exp(j\omega_a t). \quad (21)$$

We can now see what the effect of the active medium on the pulses is. Since the pulse spectrum is on line center, the term $4g/\Delta\omega^2$ in (20a) is real and hence $|A| > (1/4\gamma)$, which means the spectral width has been reduced. However, it does not follow that the pulsewidth is increased, because $1/4\gamma$ is in general complex, and for $\beta = \alpha$, the pulsewidth actually remains unchanged, assuming $4g/\Delta\omega^2$ is small. We will consider these conditions in more detail later. The constant B indicates a delay of the pulse envelope.

Now consider the effects of the modulator. The transmission of the modulator can generally be given by $\exp -[\delta_g(\omega_m t)^2]$ where $\delta_g = \mp j\delta_c$ for FM modulator and $\delta_g = 2\delta_l$ for the ideal AM modulator from (12) and (16). The peak of the pulse goes through the modulator at time $t = B$ and hence the pulse coming out of the modulator is given by

$$E_3(t) = E_2(t) \exp[-\delta_g \omega_m^2 (t-B)^2]. \quad (22)$$

Finally, the round trip for the pulse is completed by including an additional time delay $2L_0/c$ and an effective reflectivity r of a mirror, to include all losses in the cavity. The pulse after one round trip is then given by

$$E_4(t) = rE_3[t - (2L_0/c)]. \quad (23)$$

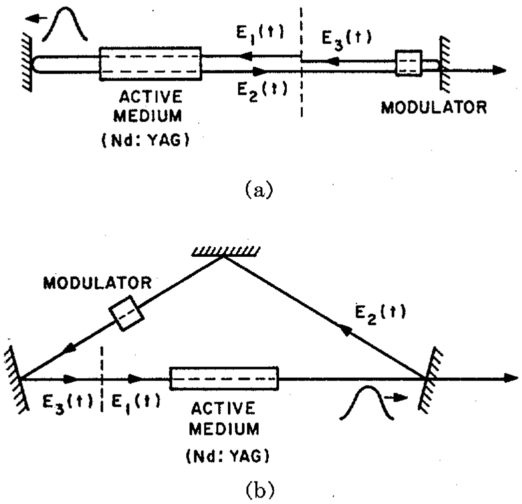

Fig. 1. Model for mode-locked laser. (a) Fabry–Perot-type cavity. (b) Ring-type cavity.

To obtain a self-consistent solution, the envelope of the pulse must go through the modulator at the same modulation phase every time. Hence, the total round trip time for the pulse is T_m, the modulator period, where $T_m = 2\pi/\omega_m$ for the phase modulator and $T_m = \pi/\omega_m$ for the amplitude modulator.

The self-consistency requirement now becomes

$$E_1(t - T_m)e^{-j\phi} = E_4(t). \tag{24}$$

The phase angle ϕ is included to allow for a possible phase shift (or phase precession [14]) of the optical signal with respect to the pulse envelope. From (1), (21), (22), and (23), the self-consistency condition equation can now be written as

$$E_0/2 \exp[-\gamma(t - T_m)^2] \exp[j\omega_a(t - T_m)] \exp(-j\phi)$$
$$= \frac{rE_0G}{4\sqrt{\gamma A}} \exp(-[t - B - (2L_0/c)]^2/4A)$$
$$\cdot \exp(-\delta_g\omega_m^2[t - B - (2L_0/c)]^2) \exp(j\omega_a[t - (2L_0/c)]). \tag{25}$$

From this equation it follows that

$$T_m = 2L_0/c + B \tag{26a}$$

$$\gamma = 1/(4A) + \delta_g\omega_m^2 \tag{26b}$$

$$e^{-j\phi} = (rG/2\sqrt{\gamma A}) \exp(j\omega_a B). \tag{26c}$$

These three equations combined with (20) now essentially solve the problem. From these equations we can obtain the desired analytical expressions for the modulation frequency, the pulsewidth, bandwidth, etc. First, consider (26b). We can substitute for A from (20a), and obtain a second-order equation in γ, which can be solved to give

$$\gamma = \frac{\delta_g\omega_m^2}{2} \pm \frac{1}{2}\sqrt{(\delta_g\omega_m^2)^2 + \frac{\delta_g\omega_m^2\,\Delta\omega^2}{4g}}. \tag{27}$$

The real part of γ must always be positive, and hence we only retain the positive sign in (27).

To see if further approximations are possible, consider the ratio of the second to the first term under the square root sign in (27), namely $\Delta\omega^2/4g\omega_m^2\delta_g$. For lasers with a wide linewidth such as Nd:YAG where $\Delta f \simeq 120$ GHz [17] the modulation frequency will be much less than the linewidth, i.e., $\omega_m \ll \Delta\omega$, and for practical values of g and δ_g we will usually have that

$$\Delta\omega^2/4g\omega_m^2\delta_g \gg 1. \tag{28}$$

With this approximation (27) becomes

$$\gamma \simeq (\omega_m\,\Delta\omega/4)\sqrt{\delta_g/g}. \tag{29}$$

We can now interpret this for the cases of FM and AM intracavity modulation.

A. FM Modulation

For the FM case $\delta_g = \mp j\delta_c$ and for γ_{FM} we now get

$$\gamma_{FM} = \alpha_{FM} - j\beta_{FM} = (1 \mp j)(\omega_m\,\Delta\omega/4)\sqrt{\delta_c/2g}. \tag{30}$$

This equation gives the values for α_{FM} and β_{FM} and from (4) and (5) we can now get the expressions for the pulsewidth and bandwidth[2]

$$\tau_{p0}(\text{FM}) = \frac{\sqrt{2\sqrt{2}\ln 2}}{\pi}\left(\frac{g_0}{\delta_c}\right)^{1/4}\left(\frac{1}{f_m\,\Delta f}\right)^{1/2} \tag{31a}$$

$$\Delta f_{p0}(\text{FM}) = \sqrt{2\sqrt{2}\ln 2}\left(\frac{\delta_c}{g_0}\right)^{1/4}(f_m\,\Delta f)^{1/2}. \tag{31b}$$

We can conclude the following from these two equations.

1) The pulsewidth–bandwidth product

$$\tau_{p0}(\text{FM})\cdot\Delta f_{p0}(\text{FM}) = 2\sqrt{2}\ln 2/\pi = 0.626.$$

2) The pulsewidth is inversely proportional to $(\delta_c)^{1/4}$, and since δ_c is proportional to the square root of P_m, the RF power into the modulator, $\tau_{p0}(\text{FM}) \propto (1/P_m)^{1/8}$ and hence the pulses shorten very slowly with increased modulator drive. This is not the best way to shorten the pulses.

3) The pulse length is proportional to $(f_m\Delta f)^{-1/2}$. It is sometimes assumed that the bandwidth of the pulse is approximately equal to the linewidth and it then follows that $\tau_p \propto 1/\Delta f$. However, the above expression for τ_{p0} shows clearly that this is not the case. Note that increasing the modulation frequency is the better way to shorten the pulses.

The exact modulation frequency can be obtained from (26a). Substituting for the optical length L_0 from (14) and B from (20b), we get for the FM case

$$f_{m0}(\text{FM}) = 1/[2L/c \pm 2\left(\frac{\delta_c}{\pi}\right)\frac{\lambda_a}{c} + 2g_0/\Delta\omega]. \tag{32}$$

The last two factors are perfectly understandable: the term $2(\delta_c/\pi)\lambda_a/c$ is the extremal "motion of the mirror,"

[2] We have here introduced the subscript 0 to indicate the values of τ_p, Δf_p, g, etc., at zero detuning of the modulator.

looking at phase modulation as a vibrating mirror, and the term $2g_0/\Delta\omega$ is the expected added dispersion or linear delay in the Lorentzian line. These two factors are small compared to $2L/c$, and hence the modulation frequency is approximately equal to $c/2L$.

From (26c) we can get the self-consistent value for g. Substituting for A from (26b), we get

$$rG\sqrt{1 \pm j\delta_c\omega_m/(\alpha - j\beta)} = \exp\left[-j(\phi + \omega_a B)\right].$$

Equating the magnitude parts of this equation, and letting $\beta = \mp\alpha$ and $\Delta f_p = 2\sqrt{\ln 2}\sqrt{\alpha/\pi}$ where Δf_p is the bandwidth of the pulse, we get

$$g_0 = \frac{1}{2}\ln\frac{1}{R} - \frac{1}{4}\ln\left[1 - \frac{16\ln 2\,\delta_c f_m^2}{\Delta f_{p0}^2} + \frac{1}{2}\left(\frac{16\ln 2\,\delta_c f_m^2}{\Delta f_{p0}^2}\right)^2\right] \quad (33)$$

where R is the effective (power) reflection of a mirror and includes all losses. Usually we will have that $\Delta f_p \gg \sqrt{\delta_c f_m}$ and hence we get an approximate value for g_0:

$$g_0 \simeq \tfrac{1}{2}\ln(1/R). \quad (34)$$

This value of g_0 can be used to calculate the pulsewidth and bandwidth, and then (33) can be used to get a better value for g_0. After a few iterations one can obtain the correct values for all the pulse parameters, but the approximate equation will in most cases be within a few percent of the correct value.

Equating the phase part of (26b), we can obtain ϕ, and we can show that $\phi \simeq -2g\omega_a/\Delta\omega$, and since $\omega_a \gg \Delta\omega$, ϕ is some large angle. However, the actual value for ϕ does not affect any of the pulse parameters, and we can neglect it.

B. AM Modulation

For the AM case $\delta_g = 2\delta_l$ and it follows that

$$\gamma_{\mathrm{AM}} = \alpha_{\mathrm{AM}} = (\omega_m\,\Delta\omega/4)\sqrt{2\delta_l/g}, \quad (35)$$

and $\beta_{\mathrm{AM}} = 0$. The pulsewidth and bandwidth are now given by

$$\tau_{p0}(\mathrm{AM}) = \frac{\sqrt{\sqrt{2}\ln 2}}{\pi}\left(\frac{g_0}{\delta_l}\right)^{1/4}\left(\frac{1}{f_m\,\Delta f}\right)^{1/2} \quad (36a)$$

$$\Delta f_{p0}(\mathrm{AM}) = \sqrt{2\sqrt{2}\ln 2}\left(\frac{\delta_l}{g_0}\right)^{1/4}(f_m\,\Delta f)^{1/2}. \quad (36b)$$

These equations are identical to those for the FM case, except that the pulsewidth is shorter by $\sqrt{2}$ for the AM case and hence the pulsewidth–bandwidth product

$$\tau_{p0}(\mathrm{AM})\cdot\Delta f_{p0}(\mathrm{AM}) = 2\ln 2/\pi = 0.440.$$

The exact modulation frequency is given by

$$f_{m0}(\mathrm{AM}) = \tfrac{1}{2}[1/(2L/c + 2g_0/\Delta\omega)]; \quad (37)$$

g_0 is given by

$$g_0 = \frac{1}{2}\ln\left(\frac{1}{R}\right) - \frac{1}{2}\ln\left[1 - 16\ln 2\,\delta_l\left(\frac{f_m}{\Delta f_{p0}}\right)^2\right].$$

As before, we usually have that $\Delta f_{p0} \ll f_m$, and hence we get

$$g_0 \simeq \tfrac{1}{2}\ln(1/R),$$

which is the same as for the FM case.

A simple interpretation can be given of the mode-locking process. We saw previously that the passage of the pulse through the active medium narrowed the spectral width of the pulse or alternatively changed the width of the pulse. One can now visualize this pulse going through an amplitude modulator where the pulse is shortened due to the time-varying transmission of the modulator. The equilibrium condition between the lengthening due to the active medium and shortening due to the modulator determines what the steady-state pulsewidth will be. A similar interpretation can be given for FM modulation, but it is now easier to visualize the process in the frequency domain. When the pulse passes through the FM modulator, a frequency chirp is put on the pulse. This frequency chirp increases the spectral width of the pulse and an equilibrium state is reached where the increase in spectral width due to the modulator is equal to the narrowing of the spectral width due to the active medium. It is interesting to notice that this equilibrium condition requires a steady-state frequency chirp on the pulse and further that the pulse envelope and frequency chirp contribute equally to the spectral width of the pulse (i.e., $\alpha = \beta$). The interpretation that is usually given for mode locking in an inhomogeneously broadened laser is that the modulator introduces some coupling between adjacent axial modes and that this coupling locks the phases of these modes in such a way as to give short pulses (and hence the terms mode locking or phase locking). This interpretation is not useful for the homogeneously broadened laser, since most of the axial modes are not present in the free-running laser. There are usually only a few axial modes, mostly due to spatial inhomogeneity and hence the term mode locking is somewhat of a misnomer, but we will retain it with a somewhat broadened meaning.

For a typical Nd:YAG laser with 10 percent round-trip loss (i.e., $R = 0.9$), a cavity length of 60 cm, and a linewidth of 120 GHz, the pulse length and bandwidth are given by $\tau_{p0}(\mathrm{FM}) = 39.0\ (1/\delta_c)^{1/4}$ ps and $\Delta f_p(\mathrm{FM}) = 16.1\ (\delta_c)^{1/4}$ GHz and for $\delta_c = 1$ radian, which is easily obtainable, pulses of 39 ps can be generated.

VI. Comparison With Free-Running Laser

There are basically two quantities of interest in comparing the free-running and mode-locked laser. First is the change in output power from the free-running to the mode-locked laser, and second how the axial-mode beats of the free-running laser compare with the modulation frequency for ideal mode locking as discussed in the previous section.

The basic condition for steady-state oscillation in a free-running laser is that the total round-trip gain must be exactly unity, i.e.,[3]

[3] The subscript f indicates parameters for the free-running laser.

$$r_f \exp(g_f/[1 + 2j(\omega - \omega_a)/\Delta\omega]) \exp[-j(\omega/c)2L] = \exp(-jq2\pi). \quad (38)$$

From the magnitude portion of the oscillation condition we get

$$g_f = \tfrac{1}{2}[1 + 4(\omega - \omega_a)^2/\Delta\omega^2] \ln(1/R_f). \quad (39)$$

Usually the free-running axial modes are close to line center and hence $g_f \simeq \frac{1}{2}\ln(1/R_f)$, where R_f is the effective power reflection of a mirror to account for all losses.

The change in output power can be obtained from the discussion in Appendix I and (116). If P_f is the free-running power and P_{ml} is the mode-locked power, then

$$\frac{P_{ml}}{P_f} = \frac{R_p(g_f/g_0) - 1}{(R_p - 1)} \frac{1}{S(0, \Delta f_{p0})}, \quad (40)$$

where R_p is the normalized pump power to the laser and $S(0, \Delta f_{p0})$ is the saturation function as defined by (113) and in this case we get that

$$S(0, \Delta f_{p0}) = \left[1 - \left(\frac{\Delta f_{p0}}{\Delta f}\right)^2 \frac{1}{2 \ln 2}\right]. \quad (41)$$

From (41), we see that small changes in g_0 will affect the power output, and hence we should use the complete equation for g_0 given by (33). The power output is thus given by

$$\frac{P_{ml}}{P_f} = \frac{R_p\left\{\dfrac{\ln(1/R_f)}{\ln(1/R) - \frac{1}{2}\ln\left\{1 - 16\ln 2\,\delta_c\left(\dfrac{f_{m0}}{\Delta f_{p0}}\right)^2 + \frac{1}{2}\left[16\ln 2\,\delta_c\left(\dfrac{f_{m0}}{\Delta f_{p0}}\right)^2\right]^2\right\}}\right\} - 1}{(R_p - 1)\left[1 - \dfrac{1}{2\ln 2}\left(\dfrac{\Delta f_{p0}}{\Delta f}\right)^2\right]} \quad (42)$$

where R_p is the normalized pump power, R_f the effective power reflection of the free-running laser, and R the effective power reflection for the mode-locked laser. R_f and R are defined to include all losses in the laser in both cases. Equation (42) will give the small changes in power output when the laser is mode locked. If we assume that the losses in the free-running and mode-locked laser are the same and $\Delta f > \Delta f_{p0}$ and $\Delta f_{p0} > f_{m0}$, then $P_{ml} = P_f$.

In actuality, however, the free-running laser will always maximize its output power, and particularly if there are small etalon effects in the cavity there will be some axial-mode selection in the free-running laser to minimize the losses. For this reason we may have that $R_f > R$ and there will be a decrease in output power for the mode-locked laser, particularly close to threshold and when δ_c is small.

From the phase portion of the oscillation condition (38), we get

$$\frac{2g_f[(\omega_q - \omega_a)/\Delta\omega]}{1 + \left(\dfrac{\omega_q - \omega_a}{\Delta\omega}\right)^2} + \left(\frac{\omega_q}{c}\right)2L = q2\pi \quad (43)$$

where ω_q is the oscillation frequency of the qth axial mode in the free-running laser. Substituting for g_f from (39), it can readily be shown that the axial-mode beats are given by

$$\Delta f_f = 1/[2L/c + \ln(1/R_f)/\Delta\omega]. \quad (44)$$

Comparing this with the modulation frequency for the FM laser, (32), we get that the axial-mode beat will be exactly between the modulation frequencies for the two modes of the FM laser if $g_0 = \frac{1}{2}\ln(1/R_f)$. This condition can be satisfied if etalon effects and pulling of the modulation frequency due to changes in g_0 [as given by (33)] are neglected. Due to etalon effects, however, $R_f > R$, and the modulation frequency will be lower than the axial-mode beat. Pulling of the modulation frequency due to additional changes in g_0 will further lower the modulation frequency. For the AM mode-locked laser, we will have the same effect as for the FM case.

In the actual free-running laser, the axial-mode beat will not be a single frequency, but due to mode pulling of the etalon effects, and dispersion of the host material and other crystals in the cavity, the axial-mode beats will be spread out.

VII. Self-Consistent Solution With Detuning

Detuning is defined as the frequency shift from the ideal mode-locking frequency (f_{m0}) and is given by

$$\Delta f_m = f_{m0} - f_m. \quad (45)$$

This is consistent with the definition of detuning given by Harris and McDuff [4]. Note that Δf_m is negative for $f_m > f_{m0}$ and vice versa.

The solutions for FM and AM modulation are now considerably different, and we will consider them separately.

A. FM Modulation

When the modulation frequency is detuned from the ideal frequency, the pulses pass through the modulator at some phase angle θ away from the extreme phase variation and the transmission through the modulator is given by (13). The pulses now experience some Doppler shift, and in consecutive passes through the modulator, the optical frequency (ω_p) of the pulses is shifted until some equilibrium is reached where the Doppler shift of the modulator is canceled out by an equal and opposite frequency shift from the active medium. The pulse now has a frequency ω_p, such that $\omega_a = \omega_p + \nu$ and ν is the frequency shift of the pulse.

In all cases, we will consider that the detuning is small enough that the laser remains mode locked. For larger detuning, the output of the laser changes to a FM laser type of signal [4], [6], [18], which we will not consider in this paper. FM laser operation has been observed for the Nd: YAG laser, and we will report this in another paper [19].

In Section III., we showed that the line shape could be expanded about any frequency ω_p and that the line shape was the same as before with g replaced by $g/(1 + 2jn)$ and $\Delta\omega$ replaced by $\Delta\omega(1 + 2jn)$ where $\eta = \nu/\Delta\omega$. Hence the pulse coming out of the active medium is given by

$$E_2(t) = E_0G'/(4\sqrt{\gamma A'}) \exp[-(t - B')^2/4A'] \exp(j\omega_p t), \tag{46}$$

where

$$A' = 1/4\gamma + 4g/\Delta\omega^2(1 + 2j\eta)^3 \tag{47a}$$

$$B' = 2g/\Delta\omega(1 + 2j\eta)^2 \tag{47b}$$

$$G' = \exp[g/(1 + 2j\eta)]. \tag{47c}$$

One important difference between the pulse as given by (46) and the pulse without detuning is that B' is now complex. Now consider the term $(t - B')^2/4A'$. If we split B' in its real and imaginary part, substitute it in the above expression and multiply out, we get

$$\begin{aligned}(t - B')^2/4A' &= \left[t - \frac{2g(1 - 4\eta^2)}{\Delta\omega(1 + 4\eta^2)}\right]^2 \Big/ 4A' + \frac{16jg\eta}{\Delta\omega(1 + 4\eta^2)^2} \\ &\cdot \left[t - \frac{2g(1 - 4\eta^2)}{\Delta\omega(1 + 4\eta^2)}\right] \Big/ 4A' - \frac{64g^2\eta^2}{\Delta\omega^2(1 + 4\eta^2)^4 4A'}.\end{aligned} \tag{48}$$

Now consider the following expansion

$$\frac{16jg\eta}{\Delta\omega(1 + 4\eta^2)^2 4A'} = \frac{K_1}{4A'} + jK_2, \tag{49}$$

where K_1 and K_2 are real. After some algebraic manipulations, it can be shown that

$$K_2 = \frac{16g\nu(\alpha^2 + \beta^2)(1 + 4\eta^2)}{16g(\alpha^2 + \beta^2)(1 - 12\eta^2) + \alpha\,\Delta\omega^2(1 + 4\eta^2)^3} \tag{50a}$$

$$K_1 = K_2\left\{\frac{\beta}{\alpha^2 + \beta^2} - \frac{16g}{\Delta\omega^2}\left[\frac{6\eta - 8\eta^3}{(1 + 4\eta^2)^3}\right]\right\}. \tag{50b}$$

We can now substitute (49) back in (48), complete the square, and thus finally the pulse can be written as

$$\begin{aligned}E_2(t) &= \frac{E_0G'}{4\sqrt{\gamma A'}} \\ &\cdot \exp - \left[t - \frac{2g(1 - 4\eta^2)}{\Delta\omega(1 + 4\eta^2)^2} + \frac{K_1}{2}\right]^2 \Big/ 4A' \\ &\cdot \exp[j(\omega_0 - K_2)t] \exp\left[K_1^2/16A' + jK_2\left(\frac{2g(1 - 4\eta^2)}{\Delta\omega(1 + 4\eta^2)^2}\right)\right. \\ &\left. + \frac{64g^2\eta^2}{\Delta\omega^2(1 + 4\eta^2)^4 4A'}\right].\end{aligned} \tag{51}$$

The expression we have obtained here reveals some very interesting effects of the active medium on the pulses.

1) The delay of the pulse envelope has changed from $2g/\Delta\omega$ to

$$2g(1 - 4\eta^2)/\Delta\omega(1 + 4\eta^2)^2 + K_1/2,$$

and this change will compensate exactly for the change in modulation frequency.

2) A frequency shift K_2 has been introduced and this will exactly cancel out the Doppler shift from the modulator.

3) The last exponential shows the additional attenuation and phase shift that has been introduced.

The self-consistency equation can be obtained as before, and with the transmission through the modulator given by (13), the following conditions will result:

$$T_m = \frac{2L}{c} \mp 2\left(\frac{\delta_c}{\pi}\right)\frac{\lambda_a}{c}\cos\theta + \frac{2g(1 - 4\eta^2)}{\Delta\omega(1 + 4\eta^2)^2} - \frac{K_1}{2} \tag{52a}$$

$$K_2 = \pm 2\delta_c\omega_m \sin\theta \tag{52b}$$

$$\gamma = 1/(4A') \mp j\delta_c \cos\theta\omega_m^2 \tag{52c}$$

$$|rG/2\sqrt{\gamma A'}| \cdot |\exp[K_1^2/16A' + 16g^2\eta^2/(\Delta\omega^2(1 + 4\eta^2)^4A')]| = 1 \tag{52d}$$

In the last condition, we have only considered the magnitude, since the phase angle ϕ is of no consequence as discussed before.

From (52c) we can solve for γ, and with the same approximation as before we get

$$\begin{aligned}\gamma &\simeq (\omega_m\,\Delta\omega/4)\sqrt{\mp j\delta_c \cos\theta(1 + 2j\eta)^3/g} \\ &= (\omega_m\,\Delta\omega/4)\sqrt{\delta_c\cos\theta/g}\,(1 + 4\eta^2)^{3/4}(\cos\psi + j\sin\psi)\end{aligned} \tag{53}$$

where

$$\psi = \mp \pi/4 + 3/2\tan^{-1}(2\eta). \tag{54}$$

From (53) we get α_{FM} and β_{FM} and hence the pulsewidth and bandwidth are given by

$$\tau_p(\text{FM}) = \frac{\sqrt{2\ln 2}}{\pi}\left(\frac{g}{\delta_c\cos\theta}\right)^{1/4} \cdot \left(\frac{1}{1 + 4\eta^2}\right)^{3/8}\left(\frac{1}{f_m\,\Delta f\cos\psi}\right)^{1/2} \tag{55}$$

$$\Delta f_p(\text{FM}) = \sqrt{2\ln 2}\left(\frac{\delta_c\cos\theta}{g}\right)^{1/4}(1 + 4\eta^2)^{3/8}\left(\frac{f_m\,\Delta f}{\cos\psi}\right)^{1/2}. \tag{56}$$

Note that the pulsewidth–bandwidth product is now

$$\tau_p \cdot \Delta f_p = \frac{2\ln 2}{\pi\cos\psi}.$$

With the assumption that $\Delta\omega/\omega_m \gg 1$, it can be shown that K_1 and K_2 are given by the following approximate expressions:

$$K_2 \simeq \frac{16 g \eta \alpha}{\Delta\omega(1+4\eta^2)^2 \cos^2 \psi} \quad (57a)$$

$$K_1 \simeq -\frac{16 g \eta \tan \psi}{\Delta\omega(1+4\eta^2)^2}. \quad (57b)$$

From (52a) we now get the modulation frequency

$$f_m(\mathrm{FM}) = 1 \Bigg/ \left[2L/c \pm 2\left(\frac{\delta_c}{\pi}\right)\frac{\lambda_a}{c}\cos\theta + \frac{2g(1-4\eta^2+4\eta\tan\psi)}{\Delta\omega(1+4\eta^2)^2} \right]. \quad (58)$$

Substituting K_2 in (52b) it can be shown that

$$\frac{\sin\theta}{\sqrt{\cos\theta}} = \pm 2\sqrt{\frac{g}{\delta_c}} \frac{\eta}{(1+4\eta^2)^{5/4}\cos\psi},$$

and from this equation it follows that

$$\cos\theta = \sqrt{4\left(\frac{g}{\delta_c}\right)^2\left[\frac{\eta^4}{(1+4\eta^2)^5\cos^4\psi}\right] + 1} - 2\left(\frac{g}{\delta_c}\right)\left(\frac{\eta^2}{(1+4\eta^2)^{5/2}\cos^2\psi}\right). \quad (59)$$

Finally, we can consider (52d), and substituting from (49), this equation can be simplified to

$$|rG/2\sqrt{\gamma A'}| \cdot |\exp K_2^2 A'| = 1$$

and g is given by

$$g = (1+4\eta^2)[\tfrac{1}{2}\ln(1/R) + \tfrac{1}{2}\ln 4\gamma A' - K_2^2 A'] \quad (60a)$$

and with the same approximations as before, it can be shown that g is approximately given by

$$g \simeq \tfrac{1}{2}(1+4\eta^2)\ln(1/R). \quad (60b)$$

We should note here that the ideal mode-locking frequencies for the positive and negative modes are different. From (58) this frequency difference is

$$4(\delta_c \lambda a/\pi c) f_{m0}^2$$

and for $f_{m0} = 250$ MHz, the frequency difference is 0.28 δ_c kHz, and hence this difference is small, but it is significant because it splits the degeneracy of the two possible modes of the FM mode-locked laser.

We have now obtained all the equations to describe the behavior of the FM mode-locked laser with detuning. In obtaining these equations, several approximations were made in going from the self-consistency conditions given by (52a)–(52d) to the final equations. However, starting with these approximate values of η, g, γ, etc., one can get the exact solutions of (52a)–(52d) by a suitable iterative procedure. This was done for some typical cases of the Nd:YAG laser, and in all cases it was found that the approximations given by (55)–(60) were within a few percent of the exact solutions of the self-consistency conditions.

In particular, we can again consider the case of the Nd:YAG laser with 10 percent total round-trip loss, a 60-cm-long cavity (optical length), and a 120-GHz linewidth. The results are shown in Figs. 2–7. These results clearly illustrate some of the interesting peculiarities of the theory. We will consider these results in some detail and try to obtain some physical insight of what happens in the mode-locked laser.

Fig. 2 shows the phase shift of the pulse with respect to the modulation signal versus detuning. Note the distinct asymmetry in these curves. Fig. 3 shows the frequency shift of the optical frequency of the pulse. We note that the frequency shifts of the positive and negative modes are in opposite directions. This is to be expected, because when the pulse goes through the modulator at a phase angle θ from the extremes of the phase variation, the Doppler frequency shift is in opposite directions for the two modes [see (13)]. We also note the asymmetry of the curves. The reason for this we will see later.

Fig. 4 shows the variation in pulsewidth with detuning for several δ_c. The most surprising observation here is that the pulses continue to get shorter for negative detuning. Fig. 5 shows the variation of the bandwidth with detuning, and we notice that the bandwidth keeps on increasing for positive detuning, even though the pulses get longer as shown in Fig. 4, which is somewhat surprising. Fig. 6 shows the variation of β with detuning, and this figure provides the clue to what is happening in the laser. Consider, say, the positive mode and negative detuning. We notice that β decreases from its value for ideal mode locking. At the same time, we notice from Fig. 4 that the pulses get shorter. What is happening is that the pulses are being compressed as described by Giordmaine *et al.* [18]. They show that when a pulse with a linear frequency chirp is passed through a dispersive medium, and the frequency chirp has the correct sign so that the leading edge of the pulse is retarded, the pulse is compressed. This is precisely what happens in the mode-locked laser. When the modulation signal is detuned, the pulse shifts towards the side of the line. The so-called "anomalous dispersion" of the active medium now provides the dispersion to compress the pulses. It turns out that we get the correct sign for β for pulse compression when the detuning is negative. It can be seen from Fig. 6 that β actually goes through zero. When $\beta \simeq 0$ we get optimum compression, and we actually see from Fig. 4 that the pulse length has a minimum when $\beta \simeq 0$. For large enough negative detuning, β changes sign. It is interesting that the mechanism of mode locking, as described earlier, is now completely changed around. When the pulse now passes through the modulator, the linear frequency chirp induced by the modulator substracts from the frequency chirp on the pulse, and the spectral width of the pulse is decreased. The pulse then passes through the active medium in such a way that the dispersion increases β, and hence the active medium increases the spectral width. The

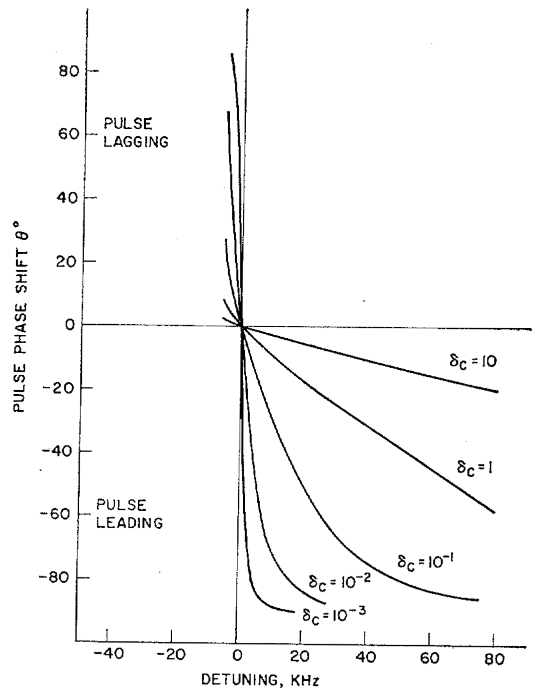

Fig. 2. Phase shift of pulse with respect to modulation signal versus detuning for laser with internal FM modulation. Conditions are for a typical Nd:YAG laser with a 60-cm cavity, 10 percent round-trip loss, and 120-GHz linewidth.

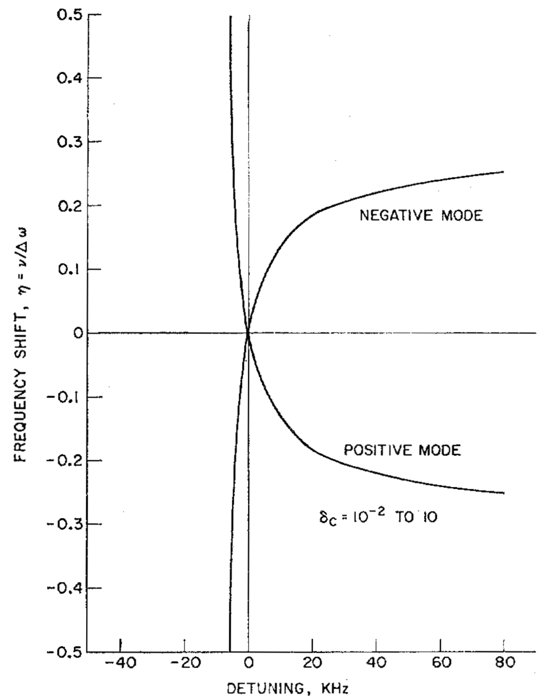

Fig. 3. Frequency shift of pulse off line center versus detuning for laser with internal FM modulation. Same conditions as for Fig. 2. Dependence on δ_c too small to show on the scale of this figure.

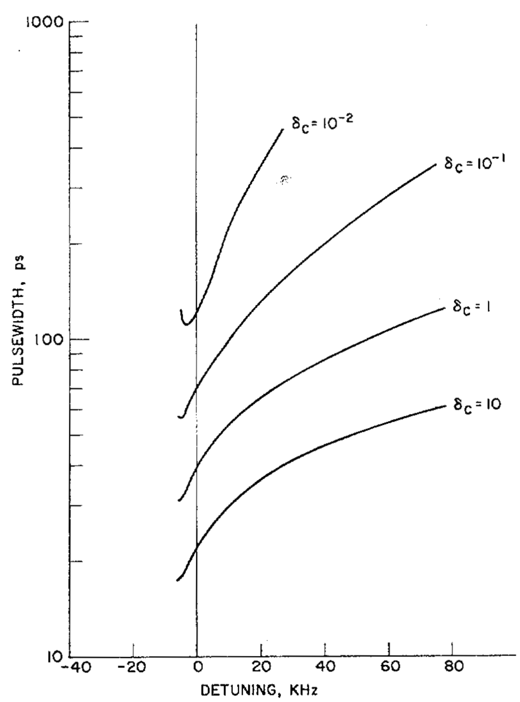

Fig. 4. Pulsewidth versus detuning (FM modulation).

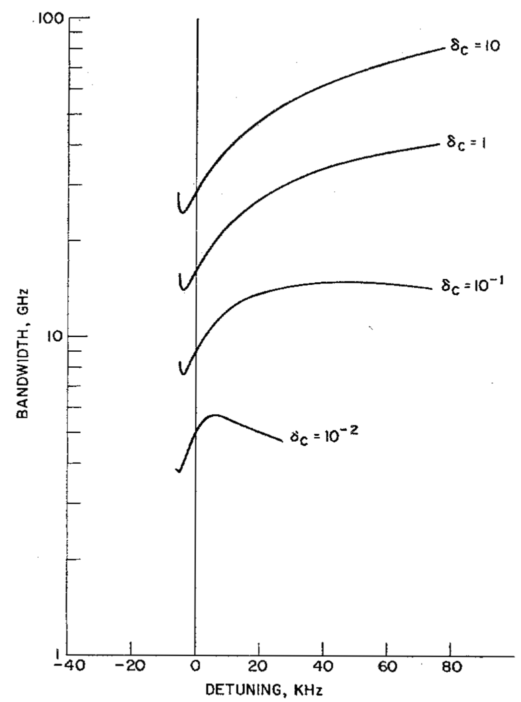

Fig. 5. Bandwidth of pulse versus detuning (FM modulation).

roles of the modulator and active medium have thus been switched around.

For positive detuning, we get pulse stretching, and the pulses keep on getting longer as the detuning increases. From (55) we note that the pulse length approaches infinity as $|\psi| \to 90°$, and from (54) we see that this condition is satisfied when $\eta \to \mp 0.288$ for the positive and negative mode, respectively. When this condition is approached, mode locking becomes impossible. We note from (58), that as $|\psi| \to 90°$ the detuning will also approach infinity, and hence as we keep on increasing the positive detuning, the optical frequency of the pulse will not shift beyond $\eta = \mp 0.288$. For negative detuning, however, the pulse rapidly shifts off line center as shown

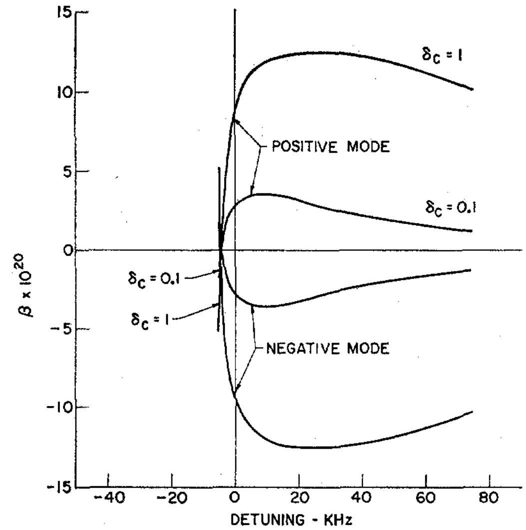

Fig. 6. β (frequency chirp) versus detuning (FM modulation).

in Fig. 3. For large enough positive or negative detuning the laser will go into the FM-laser mode of operation [18], [19].

We noted previously that the bandwidth kept increasing for positive detuning, even though the pulse length got longer. To explain this, we consider the expression for bandwidth (5),

$$\Delta f_p = \frac{1}{\pi}\sqrt{2 \ln 2[(\alpha^2 + \beta)^2/\alpha]}.$$

For no detuning, $\alpha = \beta$, and for positive detuning, $\beta > \alpha$ and hence

$$\Delta f_p \simeq \frac{1}{\pi}\sqrt{2 \ln 2\beta^2/\alpha} = \tau_p(\beta/\pi).$$

We see that for a pulse with a linear frequency chirp, the bandwidth keeps on increasing even though the pulse gets longer.

We can also obtain the change in output power with detuning from (116) in Appendix I. If $P_{\mathrm{ml}}(\Delta f_m)$ is the power output with detuning Δf_m, we obtain

$$\frac{P_{\mathrm{ml}}(\Delta f_m)}{P_{\mathrm{ml}}(0)} = \frac{[R_p(g_f/g) - 1]\left[1 - \frac{1}{2\ln 2}(\Delta f_{p0}/\Delta f)^2\right]}{\left[R_p\left(\frac{g_f}{g_0}\right) - 1\right]\left\{\left(\frac{1}{1+4\eta^2}\right)\left[1 - \frac{1}{2\ln 2(1+4\eta^2)}\left(\frac{\Delta f_p}{\Delta f}\right)^2\right]\right\}}, \quad (61)$$

where R_p is the normalized pump power for the free-running laser, and g_f, g_0, and g are the saturated single-pass gains through the active medium for the free-running laser, mode-locked laser with no detuning, and mode-locked laser with detuning, respectively.

Note that we should use the complete expressions for g and g_0 here. It will be difficult to get the value of g_f due to the complicated nature of the etalon effects, but we can define R_p', where $R_p' = R_p(g_f/g_0)$, and then R_p' is the normalized pump power of the mode-locked laser with no detuning that can be determined. If we assume that $\Delta f \gg \Delta f_p$, and use the approximate expression for g and g_0, (62) simplifies to

$$\frac{P_{\mathrm{ml}}(\Delta f_m)}{P_{\mathrm{ml}}(0)} = \frac{R_p - (1 + 4\eta^2)}{R_p - 1}. \quad (62)$$

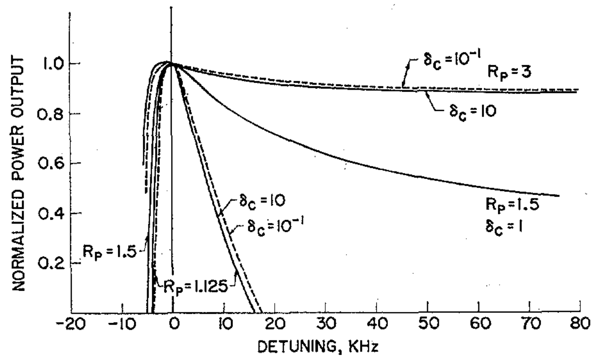

Fig. 7. Power output from laser versus detuning for various depths of modulation and normalized pump powers (FM modulation).

The variation of power output with detuning is shown in Fig. 7. For negative detuning, we found that the pulse rapidly moves off line center, and hence the power output drops rapidly. For positive detuning, the power output changes much slower, particularly far above threshold. It can be seen that the power output does not depend very much on δ_c.

We note that for $\delta_c = 10$ and $R_p = 3$, there is an initial power increase for negative detuning. To understand this, we should note that there are mainly two effects contributing to the change in output power. First, there is the shift off line center that causes a decrease in output power but this effect becomes smaller as we get further above threshold. The second effect is the change in spectral width of the pulse. A narrow spectrum more effectively saturates the line, and hence the power output can increase as the spectrum becomes narrower. For the above-mentioned condition, the laser is far above threshold and the spectral width with no detuning is large. Hence, the second effect can dominate for negative detuning, and give rise to the increase in power output.

B. AM Modulation

Consider now what happens when we detune the modulation frequency with an amplitude modulator. We saw previously that when the pulse frequency shifts off line center, there is a small frequency pulling by the active medium and that a FM modulator could compensate for this frequency change. However, an amplitude modulator can not introduce a frequency shift, and hence we can conclude that with an amplitude modulator, the pulse frequency remains on line center, i.e., $\nu = 0$.

Also, with an amplitude modulator there is no frequency chirp during the pulse, and hence $\beta = 0$. Thus the pulse coming out of the active medium is the same as we obtained before with no detuning (21). When this pulse passes through the amplitude modulator with a general modulation characteristic as given by (18), the pulse we obtain is given by

$$E_3(t) = E_0G/(4\sqrt{\alpha A}) \exp[-(t - B^2/4A] \exp(j\omega_a t) \cdot \exp - [2\delta_0 + 2\delta_1\omega_m(t - B) + 2\delta_2\omega_m^2(t - B)^2]. \quad (63)$$

If we let $K = [1/4A + 2\delta_2\omega_m^2]$ and include the round-trip time $(2L_0/c)$ and effective reflectivity, the pulse after one round trip can be written as

$$E_4(t) = \frac{rE_0G}{4\sqrt{\alpha A}} \exp - K\left(t - B + \frac{\delta_1\omega_m}{K} - \frac{2L_0}{c}\right)^2 \cdot \exp j\omega_a\left(t - \frac{2L_0}{c}\right) \exp[+(\delta_1\omega_m)^2/K] \exp(-2\delta_0). \quad (64)$$

The self-consistency conditions now become

$$K = 1/(4A) + \delta_2\omega_m^2 = \alpha \quad (65a)$$

$$T_m = 2[2L_0/c + B - \delta_1\omega_m/K] \quad (65b)$$

$$rG/(2\sqrt{\alpha A}) \exp[(\delta_1\omega_m)^2/K] \exp - \delta_0 = 1. \quad (65c)$$

From the first equation, we can solve for α, and from this we can get the pulsewidth and bandwidth, which will be the same as the case with no detuning δ_l replaced by δ_2 [(36) and (37)]. For a second equation, we can substitute for K and B, and show that the modulation frequency is given by

$$f_m(\text{AM}) = \frac{1}{2}\left[1 \Big/ \left(\frac{2L_0}{c} + \frac{2g}{\Delta\omega} - \frac{4\delta_1}{\Delta\omega}\sqrt{\frac{g}{\delta_2}}\right)\right]. \quad (66)$$

For the last equation, we can show that $(\delta_1\omega_m)^2/K \ll g$ assuming $\omega_m/\Delta\omega \ll 1$, and hence we find

$$g \simeq \delta_0 + \tfrac{1}{2}\ln(1/R). \quad (67)$$

For any particular amplitude modulator, we can evaluate δ_0, δ_1, and δ_2 as a function of θ and get the output characteristics of the laser from the above equations. One will usually find that the pulses get longer with detuning, and the output power decreases, and is the same for positive and negative detuning.

VIII. Higher Order Modulation

We have so far only considered the case where the modulator goes through only one cycle per round-trip time of the pulse. It is, however, possible that the modulator goes through several cycles per round-trip time. In general, we can have a modulation frequency p times the fundamental modulation frequency or axial mode spacing. We will call p the order of the modulation. All the previous theory we have developed is still good, with the only modification being that the modulation frequencies we have obtained are multiplied by p. For the particular example of the FM mode-locked Nd:YAG laser we considered, the detuning is multiplied by p.

The advantage in going to higher order modulation is that we can use much higher modulation frequencies and consequently obtain much shorter pulses.

The theory we have developed does not tell us whether there will only be one pulse traveling around inside the laser cavity, or whether there will be p pulses, spaced at the period of the modulator. This will depend on the detailed saturation mechanism of the active medium, and where the pulses cross in the active medium and hence the position of the active medium will be of importance. We will not attempt to find an answer to this question that can best be answered experimentally.

IX. Etalon Effects

Fabry–Perot etalons are commonly used inside laser cavities for axial mode selection [21], [22]. The effect of the etalon is to reduce the bandwidth of the system to produce the required mode selection.

It is possible to use an etalon inside a mode-locked laser to reduce the system bandwidth and obtain longer pulses. This etalon will usually consist of a parallel uncoated glass flat. It is shown in Appendix II that the transmission can be expanded about the peak, to give

$$t_e = \exp\left[j\sqrt{8R}\left(\frac{\omega - \omega_e}{\Delta\omega_e}\right) - 4\left(\frac{\omega - \omega_e}{\Delta\omega_e}\right)^2\right], \quad (68)$$

where $\Delta\omega_e$ is the bandwidth of the etalon and

$$\Delta\omega_e = (c/2h')\sqrt{8/R}\,(1 - R).$$

The one important requirement is that the bandwidth is large compared to the axial mode spacing so that there will be several axial modes under the transmission peak.

If we multiply the gain of the active medium and t_e, we can obtain an effective gain of the system, given by

$$g_e(\omega) = G \exp - \left(\frac{2jg}{\Delta\omega} - \frac{\sqrt{8R}}{\Delta\omega_e}\right)(\omega - \omega_a) - 4\left(\frac{g}{\Delta\omega^2} + \frac{1}{\Delta\omega_e^2}\right)(\omega - \omega_a)^2 \quad (69)$$

and we assume that the transmission peak of the etalon is on line center.

The two effects of the etalon are to change the bandwidth of the system, and to introduce some phase shift $\sqrt{8}\ R/\Delta\omega_e$, which slightly changes the modulation frequency. This is in addition to the change in modulation frequency due to the change in optical length of the cavity when the etalon is introduced.

In particular, we can consider the FM mode-locked laser with no detuning. The pulsewidth is now given by

$$\tau_{p0}(\text{FM}) = \frac{\sqrt{2\sqrt{2}\ln 2}}{\pi}\left(\frac{1}{\delta_c}\right)^{1/4}\left(\frac{1}{f_m}\right)^{1/2}\left(\frac{g_0}{\Delta f^2} + \frac{1}{\Delta f_e^2}\right)^{1/4}. \quad (70)$$

The case of particular interest is where the bandwidth of the etalon is much less than the linewidth, i.e.,

$$1/\Delta f_e^2 > g_0/\Delta f^2.$$

The pulsewidth now becomes:

$$\tau_{p0}(\text{FM}) \simeq \frac{\sqrt{2\sqrt{2}\ln 2}}{\pi}\left(\frac{1}{\delta_c}\right)^{1/4}\left(\frac{1}{f_m\,\Delta f_e}\right)^{1/2}, \quad (71)$$

and we note that the pulsewidth is now entirely determined by the etalon and modulator.

We can consider the same Nd:YAG laser as before with an uncoated quartz etalon of thickness h. The pulsewidths that can be obtained from this system as a function of h are shown in Fig. 8. Note that the bandwidth of the etalon can be written in terms of the index of refraction and the thickness

$$\Delta f_e = (\sqrt{8}c/\pi h)n/(n^2 - 1). \tag{72}$$

Substituting in (71) and evaluating the constants, we get

$$\tau_p = 150\sqrt{h}/(\delta_c)^{1/4} \text{ ps}.$$

From Fig. 8 we note that for etalon thicknesses between 1 mm and 1 cm, we obtain good control of the pulsewidth. For etalon thickness less than 1 mm, the linewidth of the active medium begins to take over, and thus there is no advantage in using an etalon less than 1 mm thick inside the Nd:YAG laser. Another advantage of the etalon is that it usually improves the stability of the mode-locked laser.

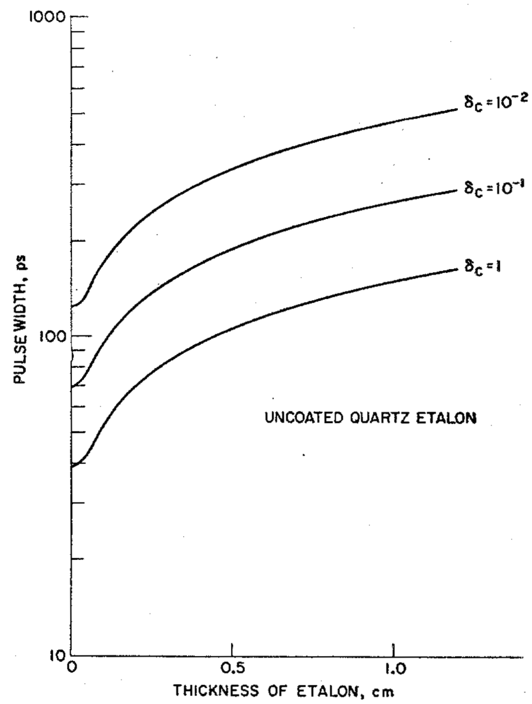

Fig. 8. Pulsewidth in FM mode-locked laser with uncoated quartz etalon.

X. Effects of Host Medium Dispersion and Dispersion of Other Components

The index of refraction of the host medium or any other optical components in the cavity such as the modulator crystal can be written as

$$n = n + n'(\omega - \omega_a) + (n''/2)(\omega - \omega_a)^2 \cdots. \tag{73}$$

For YAG at 1.064 μ, $n' = 12 \times 10^{-17}$ and $n'' \simeq 2 \times 10^{-33}$ [23] and for $LiNbO_3$ (modulator crystal) $n_e' = 3.1 \times 10^{-17}$ and $n_e'' \simeq 0$, $n_0' = 3.8 \times 10^{-17}$ and $n_0'' \simeq 0$ [24].

The total optical length now becomes

$$L_{opt} = L_0 + L_0'(\omega - \omega_a) + L_0''(\omega - \omega_a)^2 \cdots, \tag{74}$$

where

$$L_0 = l_{cav} + (n_1 - 1)l_{c1} + (n_2 - 1)l_{c2} + \cdots$$
$$L_0' = n_1' l_{c1} + n_2 l_{c2} + \cdots$$
$$L_0'' = (n_1''/2)l_{c1} + (n_2''/2)l_{c2} + \cdots.$$

L_0 is just the optical length we considered before. The other two terms now introduce an additional phase shift per round trip given by

$$(\omega/c)[2L_0'(\omega - \omega_a) + 2L_0''(\omega - \omega_a)^2] \simeq (\omega_a/c)[2L_0'(\omega - \omega_a) + 2L_0''(\omega - \omega_a)^2].$$

We can add this to the active medium and obtain an effective gain

$$g_e(\omega) = g \exp - \left[2jg'\left(\frac{\omega - \omega_a}{\Delta\omega}\right) - 4g''\left(\frac{\omega - \omega_a}{\Delta\omega}\right)^2\right], \tag{75}$$

where

$$g' = g + (L_0'\omega_a \Delta\omega/c)$$
$$g'' = g + j(L_0''\omega_a \Delta\omega^2/2c).$$

We can now consider the mode-locked laser without detuning. The results we obtained previously are still valid, with g' and g'' replacing g in the appropriate places. In particular, the FM modulation frequency becomes

$$f_{m0}(\text{FM}) = 1\bigg/\left[2L_0/c \pm 2\left(\frac{\delta_c}{\pi}\right)\frac{\lambda_a}{c} + \frac{2g_0}{\Delta\omega} + \frac{2L_0'\omega_a}{c}\right], \tag{76}$$

and a similar result is obtained for AM. This is not a very important change; however, the factor γ as given by (29) becomes

$$\gamma = (\omega_m \Delta\omega/4)\sqrt{\delta_g\bigg/\left(g_0 + j\frac{L_0''\omega_a \Delta\omega^2}{2c}\right)}. \tag{77}$$

We will usually have that

$$g_0 > L_0''\omega_a\Delta\omega^2/2c$$

and γ can be written as

$$\gamma \simeq \frac{\omega_m \Delta\omega}{4}\sqrt{\frac{\delta_g}{g_0}}\left(1 - j\frac{L_0''\omega_a \Delta\omega^2}{4g_0 c}\right). \tag{78}$$

For the FM case, $\delta_g = \mp j\delta_c$ and for γ_{FM} we get

$$\gamma_{FM} = \frac{\omega_m \Delta\omega}{4}\sqrt{\frac{\delta_c}{2g_0}}\left[\left(1 \mp \frac{L_0''\omega_a \Delta\omega^2}{4g_0 c}\right) \mp j\left(1 \pm \frac{L_0''\omega_a \Delta\omega^2}{4g_0 c}\right)\right]. \tag{79}$$

We note that α_{FM} and β_{FM} now have different values for the positive and negative modes and that for the positive mode the dispersion has caused some pulse stretching, while for the negative mode there is some pulse compression. Thus the dispersion is another effect that tends to lift the degeneracy of the two modes.

For the AM case $\delta_g = 2\delta_l$ and γ_{AM} is essentially given by (77) and we see that the dispersion has introduced some frequency chirp in the pulse.

One can now consider the whole problem of detuning again, including the effects of dispersion. However, the results will probably differ very little from those we have obtained already. For the FM case with a slight shift off line center, the dispersion of the active medium will be much larger than the host dispersion.

In cases where it becomes necessary, the whole problem can be solved by the methods described in this paper.

XI. Limitations of the Theory

The two main limitations of the theory are that we assume the line shape is Gaussian, and the pulse is Gaussian with a linear frequency chirp under all conditions.

The first assumption will break down when the bandwidth of the pulses becomes comparable to the linewidth. This will happen as one goes to higher modulation frequencies to generate shorter pulses. Exactly where the theory will break down is hard to say and can best be determined experimentally.

The second assumption may seriously affect some of the results, particularly the behavior of the FM mode-locked laser with detuning. Small distortions in the pulse shape may change the results considerably. However, many of the characteristics described here have been observed in a Nd:YAG laser with a FM modulator, and these results will be presented in Part II of this paper.

Appendix I

Gain and Saturation of Homogeneous Line

In this appendix we consider the basic equations for a homogeneously broadened line, and from these equations we derive an expression for gain of the medium. We also investigate how the line saturates for some general signal. In particular, we consider how the line is saturated by Gaussian pulses with a frequency not necessarily at line center.

The equations for a homogeneously broadened line are given by [25]

$$\ddot{\mathbf{P}} + \frac{2}{T_2}\dot{\mathbf{P}} + \omega_a^2\mathbf{P} = -\frac{2\omega_a L\,|\mathbf{\mu}|^2}{3\hbar} N\mathbf{E} \tag{80}$$

$$\dot{N} + \frac{N - N_e}{T_1} = \frac{2}{\hbar\omega_a}\dot{\mathbf{P}}\cdot\mathbf{E}, \tag{81}$$

where $\mathbf{P}$ is the polarization of the medium, $\mathbf{E}$ the electric field of the applied signal, ω_a the center frequency of the line, $|\mathbf{\mu}|$ the dipole matrix element, L the Lorentz correction factor to relate local fields to macroscopic fields, and N the population inversion $(N_2 - N_1)$. The full atomic linewidth is given by $\Delta\omega = 2/T_2$. N_e is the population inversion with no applied signal and is proportional to the pump rate. Since $\mathbf{P}$ and $\mathbf{E}$ are polarized in the same direction, we can drop the vector notation.

When a laser is above threshold, the population inversion has a constant value N_0 and a time-dependent component $\Delta N(t)$ that depends on the changes in E and P. We will assume that this component $\Delta N(t)$ is negligible compared to N_0, and hence that N is constant. With this approximation, we neglect changes in the pulse shape of a mode-locked laser due to saturation during the pulse, and also such effects as π pulses [26].

We now consider an electric field $E(t)$ with a Fourier transform $E(\omega)$. Hence we get

$$E(t) = \int_{-\infty}^{\infty} E(\omega)\exp(j\omega t)\,d\omega. \tag{82}$$

Similarly for the polarization

$$P(t) = \int_{-\infty}^{\infty} P(\omega)\exp(j\omega t)\,d\omega. \tag{83}$$

We can substitute in (80), and since N is constant, (80) is linear, and we can get an expression for the susceptibility $\chi(\omega)$, where $P(\omega) = \epsilon_0\chi(\omega)E(\omega)$

$$\chi(\omega) = -\frac{2\omega_a L\,|\mathbf{\mu}|^2 N}{3\hbar\epsilon_0}\left[\frac{1}{(\omega_a^2 - \omega^2) + j\,\Delta\omega\,\omega}\right]. \tag{84}$$

We now consider the propagation of an electromagnetic wave through a medium with a susceptibility given by (84). If we neglect losses in the medium, the propagation is governed by Maxwell's equation in the following form

$$\frac{\partial^2 E}{\partial z^2} + \frac{n^2}{c^2}\frac{\partial^2 E}{\partial t^2} = -\mu_0\frac{\partial^2 P}{\partial t^2}, \tag{85}$$

where n is the index of refraction of the medium.

The propagation constant β is defined such that the total field $E(t)$ at point z is given by

$$E(t) = \int_{-\infty}^{\infty} E(\omega)\exp[i(\omega t - \beta z)]\,d\omega. \tag{86}$$

It can be shown [25] that the propagation constant β is given by

$$\beta = \frac{n\omega}{c} + \frac{\chi(\omega)\omega}{2nc}. \tag{87}$$

Considering only the effects of the active medium, i.e., the second term in (87), the gain due to the active medium is given by

$$g_c(\omega) = \exp\left\{-i\left[\frac{\chi(\omega)\omega}{2nc}\right]L_c\right\}, \tag{88}$$

where L_c is the length of the active medium.

If we further make the approximation for $\chi(\omega)$ that $(\omega^2 - \omega_a^2) \simeq 2\omega(\omega - \omega_a)$, it follows that g in (7) is given by

$$g = \left[\frac{L\,|\mathbf{\mu}|^2\,\omega_a}{3\hbar\epsilon_0 nc\,\Delta\omega}\right]NL_c. \tag{89}$$

We next consider how the homogeneously broadened line is saturated by a repetitive train of mode-locked pulses. The spectrum of the mode-locked pulses is now no longer narrow compared to the linewidth, and hence we must consider how this changes the expression for the saturation of a homogeneous line obtained by several others [25], [27], [28].

Since we have assumed that N is constant, (81) becomes

$$\frac{N - N_e}{T_1} = \frac{2}{\hbar\omega_a}\overline{\dot{P}(t)E(t)}, \tag{90}$$

and we have taken the average over $\dot{P}(t)E(t)$. When we have a periodic signal, such as a repetitive train of pulses with a periodicity T_m, then (90) becomes

$$\frac{N - N_e}{T_1} = \frac{2}{\hbar\omega_a}\frac{1}{T_m}\int_{-\infty}^{\infty}\dot{P}(t)E(t)\,dt, \tag{91}$$

where $P(t)$ and $E(t)$ are now for a single pulse.

Using the power theorem [16] for Fourier transforms, and noting that $F[\dot{P}(t)] = i\omega F[P(t)]$ where $F[\ ]$ indicates the Fourier transform of the function in parentheses, (90) can be written as

$$\frac{N - N_e}{T_1} = \frac{2}{\hbar\omega_a T_m}\frac{1}{2\pi}\int_{-\infty}^{\infty} i\omega F[P(t)]\cdot F[E(-t)]\,d\omega. \tag{92}$$

Substituting (81) and (82) we get

$$\frac{N - N_e}{T_1} = \frac{2}{\hbar\omega_a T_m}\frac{1}{2\pi}\int_{-\infty}^{\infty} i\omega\chi(\omega)E(\omega)E(-\omega)\,d\omega. \tag{93}$$

From (84) we note that the real part of $\chi(\omega)$, $\chi'(\omega)$ is even in ω, while the imaginary part $\chi''(\omega)$ is odd. Hence (93) can finally be written in the form

$$\frac{N - N_e}{T_1} = -\frac{2}{\hbar\omega_a T_m}\frac{1}{\pi}\int_0^{\infty}\omega\chi''(\omega)E(\omega)E(-\omega)\,d\omega. \tag{94}$$

We can now substitute for $\chi''(\omega)$ and solve for N

$$N = N_e\Big/\left[1 + \frac{4L\,|u|^2\,T_1}{3\hbar^2\epsilon_0\,\Delta\omega T_m}\frac{1}{\pi}\int_0^{\infty}\frac{E(\omega)E(-\omega)\,d\omega}{1 + 4[(\omega - \omega_a)/\Delta\omega]^2}\right] \tag{95}$$

and we have again made the approximation that $(\omega^2 - \omega_a^2) \simeq 2\omega(\omega - \omega_a)$.

The average power density inside the laser cavity is given by

$$I = n\epsilon_0 c\frac{1}{T_m}\int_{-\infty}^{\infty}|E(t)|^2\,dt = \frac{n\epsilon_0 c}{T_m\pi}\int_0^{\infty}E(\omega)E(-\omega)\,d\omega \tag{96}$$

and if we define the saturation power density I_s [25] by

$$I_s = \frac{3n\epsilon_0^2 c\hbar^2\,\Delta\omega}{4L\,|\mu|^2\,T_1}, \tag{97}$$

then (95) can finally be written as

$$N = N_e\Big/\left[1 + \frac{n\epsilon_0 c}{I_s}\frac{1}{T_m\pi}\int_0^{\infty}\frac{E(\omega)E(-\omega)}{1 + 4[(\omega - \omega_a)/\Delta\omega]^2}\right]. \tag{98}$$

We can now consider the special case of a Gaussian pulse given by

$$E(t) = E_0\exp(-\alpha t^2)\cos(\omega_p t + \beta t^2). \tag{99}$$

It can be shown that

$$E(\omega) = \frac{E_0}{2}\sqrt{\frac{\pi}{\gamma}}\exp[-(\omega - \omega_p)^2/4\gamma] + \frac{E_0^*}{2}\sqrt{\frac{\pi}{\gamma^*}}\exp[-(\omega + \omega_p)^2/4\gamma^*] \tag{100}$$

and substituting in (98) and (97), we get

$$N = N_e\Big/\left[1 + \frac{I}{I_s}\sqrt{\frac{\alpha}{2(\alpha^2 + \beta^2)}}\frac{1}{\sqrt{\pi}}\cdot\int_0^{\infty}\frac{\exp[-(\omega - \omega_p)^2\alpha/(\alpha^2 + \beta^2)]}{1 + 4[(\omega - \omega_a)/\Delta\omega]^2}\,d\omega\right]. \tag{101}$$

The integral has no exact analytical solution. However, with the assumption that the spectral width of the pulse is small compared to the full linewidth, we can expand the Lorentzian line shape about ω_p. If we let $\omega_a = \omega_p + \nu$ and $\nu/\Delta\omega = \eta$, it follows that

$$\frac{1}{1 + 4[(\omega - \omega_a)/\Delta\omega]^2} \simeq \frac{1}{(1 + 4\eta^2)}\left[1 + \frac{8\eta}{1 + 4\eta^2}\cdot\left(\frac{\omega - \omega_p}{\Delta\omega}\right) - \frac{4}{1 + 4\eta^2}\left(\frac{\omega - \omega_p}{\Delta\omega}\right)^2\right]. \tag{102}$$

From (111) we now get

$$N \simeq N_e\Big/\left\{1 + \frac{I}{I_s}\left[\left(\frac{1}{1 + 4\eta^2}\right)\left(1 - \left(\frac{\Delta f_p}{\Delta f}\right)^2\cdot\frac{1}{(1 + 4\eta^2)(2\ln 2)}\right)\right]\right\}, \tag{103}$$

where Δf_p is the spectral width of the pulse. The function in the square brackets shows how the line saturates, and we will call this the saturation function $S(\eta, \Delta f_p)$.

From (113) it follows that the laser output power (P) is given by

$$P = P_s\left[\frac{N_e}{N} - 1\right]\frac{1}{S(\eta, \Delta f_p)}. \tag{104}$$

When the laser is free running, $S(\eta, \Delta f_p) = 1$ and N_e/N is equal to the ratio of the pump power to the pump power at threshold R_p. We have shown that the gain through the active medium is proportional to the population inversion N, and if the gain of the free-running laser is g_f, then (114) becomes

$$P = P_s\left[R_p\left(\frac{g_f}{g}\right) - 1\right]\left[\frac{1}{S(\eta, \Delta f_p)}\right]. \tag{105}$$

Finally, the ratio of the output power for any two conditions is given by

$$\frac{P_1}{P_2} = \frac{[R_p(g_f/g_1) - 1]S(\eta_2, \Delta f_{p2})}{[R_p(g_f/g_2) - 1]S(\eta_1, \Delta f_{p1})}. \tag{106}$$

This equation will give the variation in output power for any condition of the mode-locked laser.

Appendix II

Etalon Effects

The amplitude transmission through a lossless Fabry–Perot etalon is given by [29]

$$\tau_e = (1 - R)/(1 - Re^{i\delta}), \tag{107}$$

where

$$\delta = (4\pi/\lambda)nh \cos\theta \tag{108}$$

and R is the reflection of both reflecting surfaces. The effective length of the etalon is given by $h' = nh\cos\theta$. We will consider the transmission of the etalon near a particular transmission peak at ω_e. From (118) we get

$$\delta = \omega_e(2h'/c) + (\omega - \omega_e)(2h'/c). \tag{109}$$

Since ω_e is at the transmission peak, $\omega_e(2h'/c) = 2m\pi$, and hence (117) can be written as

$$\tau_e = \frac{1 - R}{1 - R\exp[i(\omega - \omega_e)(2h'/c)]}. \tag{110}$$

We want to approximate the transmission of the etalon near ω_e by a Gaussian of the form $\exp - [ib(\omega - \omega_e) + a(\omega - \omega_e)^2]$. Expanding the exponential in (120), we get

$$\tau_e \simeq 1 \Big/ \left[1 - i\left(\frac{R}{1-R}\right)\left(\frac{2h'}{c}\right)(\omega - \omega_e) + \frac{1}{2}\left(\frac{R}{1-R}\right)\left(\frac{2h'}{c}\right)^2(\omega - \omega_e)^2 + \cdots\right]. \tag{111}$$

Expanding the Gaussian

$$\tau_e \simeq 1/[1 + ib(\omega - \omega_e) + [a - (b^2/2)](\omega - \omega_e)^2] + \cdots. \tag{112}$$

Equating coefficients, we get

$$b = -R/(1 - R)(2h'/c)$$
$$a = \tfrac{1}{2}R/(1 - R)^2(2h'/c)^2. \tag{113}$$

Finally, we can write the transmission of the etalon as

$$\tau_e \simeq \exp\left[i\sqrt{8}\,R\left(\frac{\omega - \omega_e}{\Delta\omega_e}\right) - 4\left(\frac{\omega - \omega_e}{\Delta\omega_e}\right)^2\right], \tag{114}$$

where $\Delta\omega_e$ is the effective bandwidth of the etalon, given by

$$\Delta\omega_e = (c/2h)(\sqrt{8/R})(1 - R). \tag{115}$$

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