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Introduction

Given a group scheme G over a base scheme S , we introduce the problem of the universal extension of G by a vector group: assuming $\mathcal{H}om_S(G, V) = 0$ for all vector groups V over S , we want to find an extension of S -group schemes:

$$0 \longrightarrow V(G) \longrightarrow E(G) \longrightarrow G \longrightarrow 0$$

such that $V(G)$ is a vector group and such that this extension is universal for all extensions of G by a vector group over S , i.e. any other extension is a push-out of this one. This extension will be called the universal extension of G .

The aim of this work is to study the problem of universal extensions for group schemes (in particular, our main interest will be abelian varieties over a field of characteristic zero, notably \mathbb{C}). We shall give necessary and sufficient conditions for the universal extension of a group scheme to exist and characterize it in the case of abelian schemes, in order to apply these results and constructions in the setting of Deligne's 1-motives and their de Rham realizations.

In the first chapter we shall introduce the concepts of algebraic de Rham cohomology for a smooth scheme X defined over a base S and then, when S is the spectrum of a field of characteristic 0, we shall employ the means of hypercoverings defined in [9] to extend such definitions also to singular algebraic varieties.

The second chapter, which relies heavily on the first four chapters of [19] and on the formalism of Grothendieck, will revolve around the problem of the existence of universal extensions for a group scheme ([19, §1]). In particular, we shall characterize the universal extension of an abelian scheme over any base S as the set of isomorphism classes of rigidified extensions ([19, §2]) and then as the set of isomorphism classes of \mathfrak{h} -extensions ([19, §3]). In the end we will show that the first de Rham cohomology vector space of an abelian scheme is naturally isomorphic to the

Lie algebra of its universal extension ([19, §4]).

In the last chapter we shall establish an equivalence of categories between mixed Hodge structures and 1–motives over \mathbb{C} (following Deligne’s work, in [9, Chapter 10]). We shall then introduce the Hodge and de Rham realizations for 1–motives. As an application, we show that for any algebraic variety over a field of characteristic 0 there exists a 1–motive (that is, the cohomological Picard 1–motive) whose de Rham realization equals to the first de Rham cohomology vector space of such algebraic variety. In this way, we shall provide a direct (algebraic) proof, in all generality, of the comparison isomorphism between the first de Rham cohomology vector space and the first singular cohomology group of a (possibly singular) algebraic variety over \mathbb{C} . The key reference for this chapter is [3]).

At the end of this dissertation, there are featured three appendixes.

The first one collects some basic definitions about geometry of schemes (mainly to fix our notations) and especially about group schemes and abelian schemes, with a brief survey of the functorial and categorial perspective (that will be employed extensively in this work). The references are classical ones in algebraic geometry literature, from Grothendieck’s work ([13], [11], [14]), to Hartshorne’s ([15]), Milne’s ([20]), Mumford’s ([22]) and Oort’s ([25]).

The second appendix collects some definitions and properties of torsors and connections over torsors in the case of group schemes, used extensively in chapter 2, generalizing the concepts of torsors for line bundles in differential geometry. The notations and results here are based on [19, §2].

The last appendix features a brief recap of Hodge theory, which follows the work of Deligne in [8] and [9] as well as the account of mixed Hodge theory in [26]. We shall define pure and mixed Hodge structures, explaining the motivations and showing how (polarized) pure and mixed Hodge structures arise naturally in studying the cohomology groups of projective varieties. In this appendix the reader can find the elementary definitions of cohomological descent and hypercoverings together with the main results which are used in defining the algebraic de Rham cohomology for any scheme over a base field of characteristic 0.

Chapter 1

Algebraic de Rham cohomology

1.1 Algebraic de Rham complex

We shall now introduce some definitions that generalize some concepts of classical differential geometry to the geometry of schemes (see [15, Chapter 8]).

Definition 1.1 ([15]). Let $f: X \rightarrow S$ be a morphism of schemes, and let \mathcal{F} be an \mathcal{O}_X -module.

- An S -derivation into \mathcal{F} is a morphism of abelian sheaves $D: f_*\mathcal{O}_X \rightarrow \mathcal{F}$ such that for every open subset $U \subseteq X$ the map $D_U: \mathcal{O}_X(U) \rightarrow \mathcal{F}(U)$ is an $\mathcal{O}_S(U)$ -derivation (in the sense of modules) of $\mathcal{O}_X(U)$ into $\mathcal{F}(U)$. We denote with $\text{Der}_S(\mathcal{O}_X, \mathcal{F})$ the set of such derivations;
- Consider the diagonal morphism $\Delta: X \rightarrow X \times_S X$. This map is always an immersion and $\Delta(X)$ is locally closed in $X \times_S X$, that is $\Delta(X)$ is a closed subscheme of an open subset W of $X \times_S X$, so $\Delta(X)$ is defined (in W) by a sheaf of ideals \mathcal{I} of \mathcal{O}_X . We have an exact sequence of \mathcal{O}_X -modules:

$$0 \rightarrow \Delta^*\mathcal{I}^2 \rightarrow \Delta^*\mathcal{I} \rightarrow \Delta^*\left(\frac{\mathcal{I}}{\mathcal{I}^2}\right) \rightarrow 0$$

The sheaf of differentials $\Omega_{X/S}^1$ of X over S is the conormal sheaf $\Delta^*\left(\frac{\mathcal{I}}{\mathcal{I}^2}\right)$.

The sheaf of differentials $\Omega_{X/S}^1$ is naturally a quasi-coherent \mathcal{O}_X -module,. It is the module of differentials $\Omega_{\mathcal{O}_X/f^{-1}\mathcal{O}_S}^1$ endowed with its universal S -derivation $d_{X/S}: \mathcal{O}_X \rightarrow \Omega_{X/S}^1$, and it represents the functor from the category $\mathcal{O}_X\text{-Mod}$ of \mathcal{O}_X -modules to the category Set of sets sending an \mathcal{O}_X -module \mathcal{F} to the set $\text{Der}_{\mathcal{O}_S}(\mathcal{O}_X, \mathcal{F})$. We denote this functor as $\mathcal{D}er_{\mathcal{O}_S}(\mathcal{O}_X, -)$.

Let us focus on the case when X is a smooth and proper scheme over S . Then $\Omega_{X/S}^1$ is finitely generated and locally free, so in this case it makes sense to take the i -th exterior power $\Omega_{X/S}^i := \bigwedge^i \Omega_{X/S}^1$. This is the sheaf of the differential forms of the i -th order of X over S . Thus, we have the algebraic de Rham complex of sheaves $\Omega_{X/S}^*$:

$$0 \rightarrow \mathcal{O}_X \rightarrow \Omega_{X/S}^1 \rightarrow \Omega_{X/S}^2 \rightarrow \Omega_{X/S}^3 \rightarrow \dots$$

where the differential operators are the exterior differentials.

Definition 1.2. The tangent bundle of a smooth and proper scheme X over a base S is the sheaf $\mathcal{T}_{X/S} := \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/S}^1, \mathcal{O}_X)$.

This is a locally free \mathcal{O}_S -module, which is isomorphic to the sheaf $\mathcal{D}er_{\mathcal{O}_S}(\mathcal{O}_X, \mathcal{O}_X)$ of the derivations of \mathcal{O}_X into itself killing \mathcal{O}_S . Moreover, by the universal property of the sheaf of differential forms $\Omega_{X/S}^1$, it follows that $\Omega_{X/S}^1 = (\mathcal{T}_{X/S})^\vee$.

1.2 de Rham cohomology of a smooth scheme

The sheaves of differentials $\Omega_{X/S}^i$ of X over S are quasi-coherent \mathcal{O}_X -modules. In the abelian category $\mathcal{Q}Coh_{/\mathcal{O}_X}$ of quasi-coherent \mathcal{O}_X -modules there are enough injectives (i.e. every object can be injected in an injective object), so there exists a Cartan-Eilenberg resolution, that is:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \Omega_{X/S}^1 & \longrightarrow & \Omega_{X/S}^2 \longrightarrow \dots \\
 & & \downarrow & \searrow & \downarrow & \searrow & \downarrow \\
 & & & B^0(\Omega_{X/S}^*) & & B^1(\Omega_{X/S}^*) & \\
 0 & \longrightarrow & I^{0,0} & \longrightarrow & I^{1,0} & \longrightarrow & I^{2,0} \longrightarrow \dots \\
 & & \downarrow & \searrow & \downarrow & \searrow & \downarrow \\
 & & & B_h^0(I^{*,0}) & & B_h^1(I^{*,0}) & \\
 0 & \longrightarrow & I^{0,1} & \longrightarrow & I^{1,1} & \longrightarrow & I^{2,1} \longrightarrow \dots \\
 & & \downarrow & \searrow & \downarrow & \searrow & \downarrow \\
 & & & B_h^0(I^{*,1}) & & B_h^1(I^{*,1}) & \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \dots & & \dots & & \dots
 \end{array}$$

where I^{**} is a double complex such that for all couples of indexes (i, j) $I^{i,j}$ is an injective \mathcal{O}_X -module and such that for all i the complex of the horizontal cobound-

aries $B_h^i(I^{**})$ and the complex of the horizontal cohomologies $H_h^i(I^{**})$ are injective resolutions of $B^i(\Omega_{X/S}^*)$ and $H^i(\Omega_{X/S}^*)$ respectively.

Remark 1.3. This conditions imply that $\Omega_{X/S}^* \longrightarrow I^{0,*}$ and $Z_h^i(I^{**}) \longrightarrow Z^i(\Omega_{X/S}^*)$ are injective resolutions too.

Definition 1.4 (Algebraic de Rham cohomology). The i -th de Rham cohomology vector space $H_{\text{dR}}^i(X)$ of a smooth and proper scheme X over S is the hypercohomology vector space

$$\mathbb{H}^i(X, \Omega_{X/S}^*) := \mathbb{R}^i \Gamma(X, \Omega_{X/S}^*)$$

1.3 Algebraic de Rham cohomology of an abelian variety

An abelian scheme A over a base S (see Definition A.1 and in general Appendix A for definitions and properties) is smooth and proper, so we can consider its algebraic de Rham complex to compute the de Rham cohomology. It turns out that the cohomology of an abelian scheme is extremely well-behaved, as the following proposition shows.

Proposition 1.5.

1. The \mathcal{O}_A -modules $H_{\text{dR}}^i(A)$ and $H^q(A, \Omega_{A/S}^p)$ are finite dimensional, locally free and their formation commutes with base change;
2. We have a spectral sequence (called the Hodge-de Rham spectral sequence) defined in page 1 by $E_1^{p,q} := H^q(A, \Omega_{A/S}^p) \implies H_{\text{dR}}^{p+q}(A)$. When the characteristic of the base field is 0, the spectral sequence degenerates in page 1;
3. $H_{\text{dR}}^*(A) \cong \bigwedge^* H_{\text{dR}}^1(A)$.

Proof. [4, Proposition 2.5.2]. □

Remark 1.6. By the part 1 of the previous proposition, we can consider the sheaf over $\mathcal{S}ch/S$ defined by the association $T \mapsto H_{\text{dR}}^i(A_T)$. We shall denote it with the same symbol $H_{\text{dR}}^i(A)$ as the \mathcal{O}_A -module.

1.4 Long exact sequence of hypercohomology

Let A be an abelian scheme over S , and consider the Deligne complex:

$$\Omega_{A/S,m}^* : \mathcal{O}_A^* \xrightarrow{\text{dlog}} \Omega_{A/S}^1 \xrightarrow{\text{d}} \Omega_{A/S}^2 \xrightarrow{\text{d}} \Omega_{A/S}^3 \longrightarrow \dots$$

where differential operator $\mathcal{O}_A \xrightarrow{\text{dlog}} \Omega_{A/S}^1$ is defined by $f \mapsto \frac{df}{f}$ (see [8, Definition 2.2.4.1]). What we shall do now is give a geometric interpretation to a portion of the long exact sequence of hypercohomology associated to the short exact sequence of complexes:

$$0 \longrightarrow \tau_1(\Omega_{A/S}^*) \longrightarrow \Omega_{A/S,m}^* \longrightarrow \mathcal{O}_A^*[0] \longrightarrow 0$$

defined by:

$$\begin{array}{ccccccc} 0 & \longrightarrow & 0 & \longrightarrow & \mathcal{O}_A^* & \xlongequal{\quad} & \mathcal{O}_A^* \longrightarrow 0 \\ & & \downarrow & \circlearrowleft & \downarrow \text{dlog} & \circlearrowleft & \downarrow \\ 0 & \longrightarrow & \Omega_{A/S}^1 & \xlongequal{\quad} & \Omega_{A/S}^1 & \longrightarrow & 0 \longrightarrow 0 \\ & & \downarrow \text{d} & \circlearrowleft & \downarrow \text{d} & \circlearrowleft & \downarrow \\ 0 & \longrightarrow & \Omega_{A/S}^2 & \xlongequal{\quad} & \Omega_{A/S}^2 & \longrightarrow & 0 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \dots & & \dots & & \dots \end{array}$$

We define a group functor $\text{Pic}_{A/S}^{\natural}$ on $\mathcal{S}ch/S$ by the law:

$$T \mapsto \{\text{isomorphism classes of line bundles on } A \times_S T \text{ with an integrable connection}^1\}$$

and we denote by $\mathcal{P}ic_{A/S}^{\natural}$ the associated Zariski sheaf.

For any T over S , we have the forgetting map:

$$\mathcal{P}ic_{A/S}^{\natural}(T) \longrightarrow H^1(A_S, \mathcal{O}_{A_T}^*)$$

which by passage to the associated sheaves yields, since A is an abelian variety, a homomorphism:

$$\mathcal{P}ic_{A/S}^{\natural} \xrightarrow{\pi} \mathcal{P}ic_{A/S}$$

Since global 1-forms on an abelian scheme are closed, and since the map:

$$H^0(A, \mathcal{O}_A^*) \xrightarrow{\text{dlog}} H^1(A, \Omega_{A/S}^1)$$

is the zero map, the indeterminacy in putting an integrable connection on the trivial bundle \mathcal{O}_A is precisely $\Gamma(A, \Omega_{A/S}^1) = \Gamma(S, \underline{\omega}_A)$. Passing to the associated sheaves we find the kernel of the map π to be precisely $\underline{\omega}_A$.

The obstruction to putting any connection on a line bundle \mathcal{L} over A is furnished by the cocycle arising as the logarithmic derivative of the transition function defining

¹See Appendix B.

\mathcal{L} . That is, the morphism:

$$H^1(A, \mathcal{O}_A^*) \longrightarrow H^1(A, \Omega_{A/S}^1)$$

defined by $(f_{i,j}) \mapsto \frac{df_{i,j}}{f_{i,j}}$.

There is an obvious map:

$$H^1(A, \mathcal{O}_A^*) \longrightarrow \mathbb{H}^2(A, \tau_1(\Omega_{A/S}^*))$$

given in terms of Čech cocycles (for some affine open cover \mathcal{U} of A) by:

$$(f_{i,j}) \mapsto \left(\left(\frac{df_{i,j}}{f_{i,j}} \right), 0 \right) \in \mathcal{C}^1(\mathcal{U}, \Omega_{A/S}^1) \oplus \mathcal{C}^0(\mathcal{U}, \Omega_{A/S}^2)$$

If this cocycle is a coboundary there are closed forms ω_i such that $\frac{df_{i,j}}{f_{i,j}} = \omega_i - \omega_j$, and hence \mathcal{L} will admit an integrable connection. The converse equally holds.

Proposition 1.7. *There is an isomorphism $\text{Pic}^{\flat}(A) := \text{Pic}_{A/S}^{\flat}(S) \longrightarrow \mathbb{H}^1(A, \Omega_{A/S,m}^*)$.*

Sketch of proof. To any line bundle \mathcal{L} with an integrable connection (\mathcal{L}, ∇) we associate the cohomology class of the Čech cocycle $((f_{i,j}), (\omega_i)) \in \mathcal{C}^1(\mathcal{U}, \mathcal{O}_A^*) \oplus \mathcal{C}^0(\mathcal{U}, \Omega_{A/S}^1)$, where $f_{i,j}$ are the transition functions of the line bundle and ω_i is the connection form for the induced connection on $\mathcal{L}|_{U_i}$. \square

We have finally arrived at the geometrical description of a portion of the above mentioned cohomology sequence.

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(A, \Omega_{A/S}^*) & \longrightarrow & \text{Pic}^{\flat}(A) & \longrightarrow & \text{Pic}(A) \longrightarrow \mathbb{H}^2(A, \tau_1(\Omega_{A/S}^*)) \\ & & \parallel & \circlearrowleft & \downarrow \mathbb{R} & \circlearrowleft & \parallel & \circlearrowleft & \parallel \\ 0 & \longrightarrow & \mathbb{H}^1(A, \tau_1(\Omega_{A/S}^*)) & \longrightarrow & \mathbb{H}^1(A, \tau_1(\Omega_{A/S,m}^*)) & \longrightarrow & H^1(A, \mathcal{O}_A^*) \longrightarrow \mathbb{H}^2(A, \tau_1(\Omega_{A/S}^*)) \end{array}$$

1.5 Algebraic de Rham cohomology for arbitrary varieties in characteristic 0

We recall this fundamental result in algebraic geometry that will be pivotal in defining a reasonable concept of algebraic de Rham cohomology for (possibly singular) algebraic varieties over a field k of characteristic 0 (not necessarily algebraically closed).

Theorem 1.8 (Hironaka's resolution of singularities in characteristic 0). *Let X be a reduced singular scheme defined over a field k of characteristic 0. Then there exists a closed subscheme $D \subseteq X$ defined by a sheaf of ideals \mathcal{I} of \mathcal{O}_X such that:*

1. *The points of D are exactly the singular locus of X ;*
2. *There exists \tilde{X} a non-singular scheme over k and a morphism $f: \tilde{X} \rightarrow X$ such that the sheaf of ideals $f^{-1}\mathcal{I}$ is an invertible sheaf and f is universal among all the morphisms $g: \tilde{X} \rightarrow X$ which make $g^{-1}\mathcal{I}$ into an invertible sheaf, that is \tilde{X} is the blowing-up of X in its singular locus D .*

In particular, it follows that $U := X \setminus D$ is a dense Zariski open subset of X and f induces an isomorphism $f|_{f^{-1}(U)}: f^{-1}(U) \xrightarrow{\cong} U$.

Proof. [16]. □

Consider now an algebraic variety X defined over k (of characteristic 0). We introduce some further notation (see also [9, Section 3.1]):

Definition 1.9. A normal crossing divisor Y of a smooth algebraic variety X is a divisor (that is, a closed subscheme of codimension 1) such that the inclusion $Y \hookrightarrow X$ is locally isomorphic to the inclusion of an intersection of hyperplane coordinates in k^n .

Consider a normal crossing divisor Y in an algebraic variety X . Denote with $j: X^* \hookrightarrow X$, where $X^* := X \setminus Y \subseteq X$.

Definition 1.10. With the previous notations, we denote with $\Omega_{X/k}^1(\log Y)$ the locally free \mathcal{O}_X -submodule of $j_*\Omega_{X^*/k}^1$ generated by $\Omega_{X/k}^1$ and by $\frac{dz_i}{z_i}$, where z_i is a local equation for a local irreducible component of Y .

The complex $\Omega_{X/k}^*(\log Y)$ is by definition the sheaf:

$$\mathcal{O}_X \xrightarrow{d} \Omega_{X/k}^1(\log Y) \xrightarrow{d} \Omega_{X/k}^2(\log Y) \rightarrow \dots$$

where $\Omega_{X/k}^p(\log Y)$ is the p -th exterior power $\bigwedge^p \Omega_{X/k}^1(\log Y)$ (which is a locally free \mathcal{O}_X -submodule of $j_*\Omega_{X^*/k}^p$), and is called the sheaf of the p -differential forms on X with logarithmic pole along Y .

The complex of sheaves $\Omega_{X/k}^*(\log Y)$ is called the logarithmic de Rham complex of X along Y (see [9, Definition 3.1.2]).

These notations generalize directly to the case of simplicial schemes.

Definition 1.11.

1. A simplicial scheme X_* over a base X is smooth if X_n is smooth for all n .
2. A simplicial scheme X_* over a base X is proper if X_n is proper for all n .
3. A simplicial normal crossing divisor D_* of X_* is a family of normal crossing divisors $D_n \subseteq X_n$ such that $U_n = X_n \setminus D_n$ yields an open simplicial subscheme of X_* .

If D_* is a simplicial normal crossing divisor, then the logarithmic de Rham complexes $\Omega_{X_n/x}^*(\log(D_n))$ endowed with their filtration of weights form a filtered complex $\Omega_{X_*/X}^*(\log(D_*))$ over X_* (see [9, Lemma 6.2.7]).

Let us now consider the case in which X is an arbitrary algebraic variety over k of characteristic 0. By Theorem 1.8, we can consider the diagram:

$$\begin{array}{ccc} X_* & \hookrightarrow & \overline{X}_* \\ \downarrow & & \\ X & & \end{array}$$

where X_* is a smooth proper hypercovering of X and \overline{X}_* is a smooth compactification of X_* with normal crossing boundary divisor $Y_* := \overline{X}_* \setminus X_*$ (see Appendix C, Section C.3 for the construction of such proper hypercovering X_*).

Definition 1.12. The algebraic de Rham cohomology for an algebraic variety over k in characteristic 0 is defined as:

$$H_{\text{dR}}^*(X) := \mathbb{H}^*(\overline{X}_*, \Omega_{\overline{X}_*/k}^*(\log Y_*))$$

Remark 1.13. If X is a smooth scheme over k , then we can consider its smooth compactification \overline{X} and the normal crossing boundary Y as constant simplicial schemes. In particular, we obtain that:

$$H_{\text{dR}}^*(X) = \mathbb{H}^*(\overline{X}, \Omega_{\overline{X}/k}^*(\log Y))$$

If moreover X is smooth and proper over k (hence compact), it follows that $Y = \emptyset$ and this definition coincides with the one we gave in Definition 1.4 for nonsingular varieties over k .

In Deligne (see [9, Chapter 8], see also [26, Chapters 4 and 5] and Appendix C) it is proved that the singular cohomology of a hypercovering X_* of an algebraic variety X over \mathbb{C} carries a natural Hodge structure.

In fact, for any integer n , we have that $\Omega_{\overline{X}_n/\mathbb{C}}^*(\log Y_n)$ has an ascending filtration of

weights W and a Hodge filtration F (the trivial filtration), which induce filtrations W and F on the singular cohomology of X_n ([8, Section 3.2], see also Section C.2.3 in C). So we have that $H^*(X_*, \mathbb{Z})$, together with the filtration of weights W on the rational cohomology vector space and the Hodge filtration F on the complex cohomology vector space, gives rise to a mixed Hodge structure ([9, Scholium 8.1.9] and [9, Example 8.1.12], the construction is analougue to the one in [8, Chapter 3.2] that is briefly presented in Section C.2.3).

Finally, using the fact $H^*(X, \mathbb{C}) := \mathbb{H}^*(X_*, \mathbb{C}) \cong \mathbb{H}^*(\overline{X}_*, \Omega_{\overline{X}_*/\mathbb{C}}^*)$ ([8, 3.2.2]) we can transfer such Hodge structure to $H^*(X, \mathbb{C})$. Such Hodge structure is independent from X_* and \overline{X}_* and for any morphism of \mathbb{C} -varieties $f: X \rightarrow Y$ induces a morphism of mixed Hodge structures $f^*: H^*(Y) \rightarrow H^*(X)$ ([9, Proposition 8.2.2]). By comparison with the case $k = \mathbb{C}$ and cohomological descent, it follows that these facts hold also for any algebraic variety defined over a field k of characteristic 0 (see [3, Remark 2.18]).

Chapter 2

Universal extensions of abelian schemes

2.1 Universal extensions of group schemes by a vector group

2.1.1 The universal extension problem

From now on, all group schemes G over a base scheme S are supposed to be commutative, flat, separated and locally of finite presentation. Even if the object of main interest are abelian varieties over a field, in the following sections we shall consider generic abelian schemes (with the properties listed above) over a base scheme S , since the same results hold.

Definition 2.1. A sequence of morphisms between group schemes over S :

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is exact if f is an isomorphism onto the kernel of g and g is an epimorphism (in the categorical sense). Such an exact sequence is said to be an extension of C by A .

Remark 2.2.

1. The category of commutative group schemes over a base S in general is not abelian. Notably, while the kernel of a group scheme morphism always exists, we do not have in general a reasonable concept of quotient group scheme. So, we shall see any group scheme G as a fppf (or fpqc, or étale) sheaf on $\mathcal{S}ch/S$ (by the Yoneda embedding $G \mapsto \text{Hom}_S(-, G)$). In the category of group sheaves on the category $\mathcal{S}ch/S$ equipped with the fppf (or fpqc, or étale) topology, we

have an obvious definition of exact sequences, and thus we say that a sequence of group schemes over S is exact if it is exact in the category above (see also [21, Chapter IV, §3]).

2. When we work with commutative group varieties over a field k in any characteristic, a result of Grothendieck (see [14, SGA 3, Exp. VIA, theorem 5.2]) assures that such category is abelian, thus there always exists the cokernel of a group scheme morphism and we have a clearer concept of exact sequence of commutative group varieties over k .

We are now ready to introduce the main definition of this chapter. Assume that $\mathcal{H}om_S(G, V) = 0$ for all vector groups V (that is a group scheme which is locally isomorphic to a finite product of $\mathbb{G}_{a,S}$'s).

Definition 2.3. A universal extension of G is an extension of group schemes over S :

$$0 \longrightarrow V(G) \longrightarrow E(G) \longrightarrow G \longrightarrow 0$$

such that $V(G)$ is a vector group and such that the extension is universal among all the extensions of G by a vector group over S , that is for any extension by a vector group M of G :

$$0 \longrightarrow M \longrightarrow F \longrightarrow G \longrightarrow 0$$

there exist unique morphisms $V(G) \longrightarrow M$ and $E \longrightarrow F$ making the following diagram commute:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V(G) & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \parallel & & \\ 0 & \longrightarrow & M & \longrightarrow & F & \longrightarrow & G & \longrightarrow & 0 \end{array}$$

\circlearrowleft \circlearrowleft

Clearly, if it exists such a universal extension, $V(G)$ and $E(G)$ are determined up to a canonical isomorphism (by the usual argument for objects with universal property).

The universal property can be translated as follows: the canonical connecting morphism from $\mathrm{Hom}_{\mathcal{O}_S}(V(G), M)$ to $\mathrm{Ext}_{\mathcal{O}_S}^1(G, M)$ in the long exact sequence induced by the above extension by deriving the contravariant functor $\mathrm{Hom}_{\mathcal{O}_S}(-, M)$ must be an isomorphism.

2.1.2 Existence of the universal extension

Recall that for any \mathcal{O}_S -module locally free of finite rank E , the dual \mathcal{O}_S -module E^\vee is defined by:

$$E^\vee = \mathcal{H}om_{\mathcal{O}_S}(E, \mathcal{O}_S)$$

$(E^\vee)^\vee \cong E$ and for any \mathcal{O}_S -module F , one has $\mathcal{H}om_{\mathcal{O}_S}(E, F) \cong E^\vee \otimes_{\mathcal{O}_S} F$.

Proposition 2.4. *Let G an S -group scheme satisfying the previous conditions. Suppose that:*

1. $\mathcal{H}om_{\mathcal{O}_S}(G, \mathbb{G}_{a,S}) = 0$;
2. $\mathcal{E}xt_{\mathcal{O}_S}^1(G, \mathbb{G}_{a,S})$ is a locally free \mathcal{O}_S -module of finite rank.

as sheaves for the Zariski global topology over S . Set

$$V(G) := \mathcal{E}xt_{\mathcal{O}_S}^1(G, \mathbb{G}_{a,S})^\vee$$

Then there exists a universal extension of G by $V(G)$.

Proof. The assertion follows from the fact that for any \mathcal{O}_S -module locally free of finite rank M , $\mathcal{E}xt_{\mathcal{O}_S}^1(G, M) \cong \mathcal{E}xt_{\mathcal{O}_S}^1(G, \mathbb{G}_{a,S}) \otimes_{\mathcal{O}_S} M$. Thus, one has:

$$\begin{aligned} \mathcal{H}om_{\mathcal{O}_S}(V(G), M) &\cong \mathcal{H}om_{\mathcal{O}_S}(\mathcal{H}om_{\mathcal{O}_S}(\mathcal{E}xt_{\mathcal{O}_S}^1(G, \mathbb{G}_{a,S}), \mathcal{O}_S), M) \\ &\cong \mathcal{E}xt_{\mathcal{O}_S}^1(G, \mathbb{G}_{a,S}) \otimes_{\mathcal{O}_S} M \\ &\cong \mathcal{E}xt_{\mathcal{O}_S}^1(G, M) = \mathcal{E}xt_{\mathcal{O}_S}^1(G, M) \end{aligned}$$

□

We shall now show that the hypothesis of this proposition are satisfied for any abelian scheme A over S .

Proposition 2.5. *Let $\pi: A \longrightarrow S$ be an abelian scheme over S of dimension d . Then for any T scheme over S :*

1. Any morphism of sheaves of sets over T $\psi: A_T \longrightarrow Q_T$, where Q_T is a quasi-coherent sheaf over T , is a constant map. In particular, ϕ admits a factorization:

$$\begin{array}{ccc} A_T & \xrightarrow{\psi} & Q_T \\ \pi_T \searrow & \circlearrowleft & \nearrow s \\ & T & \end{array}$$

where s is a section of Q_T ;

$$2. \pi_* \mathcal{O}_{A_T} \cong \mathcal{O}_T;$$

$$3. R^1 \Gamma(T, \pi_* \mathcal{O}_{A_T}) = R^1 \Gamma(T, \pi_* \mathcal{O}_A) \otimes_{\mathcal{O}_S} \mathcal{O}_T \text{ is a locally free } \mathcal{O}_T\text{-module of rank } d.$$

Proof. The proof of 1. is featured in Mazur's book ([19]), while facts 2. and 3. are standard results of abelian schemes ([11, Page 232 – 12, Remark 5.2] for 2. and [4, Lemma 2.5.3] for 3.). \square

Now we shall prove that any quasi-compact group scheme which satisfies these conditions admit a universal extension.

Theorem 2.6. *Assume $\pi: G \longrightarrow S$ quasi-compact such that conditions 1., 2. and 3. of Proposition 2.5 are satisfied. Then G possesses a universal extension:*

$$0 \longrightarrow (R^1 f_* \mathcal{O}_G)^\vee \longrightarrow E(G) \longrightarrow G \longrightarrow 0$$

Proof. Let M denote a quasi-coherent sheaf. The presheaf $\mathcal{E}xt_S^1(G, M)$ for the flat topology described by $T \mapsto \text{Ext}^1(G_T, M_T)$ is a sheaf by descent theory (see [12] and 'Notes on Grothendieck topologies, fibered categories, and descent theory' in [10]). First, we want to prove that the composition:

$$\lambda: \text{Ext}_{\mathcal{O}_S}^1(G, M) \longrightarrow H^1(G, f^* M) \longrightarrow \Gamma(S, R^1 \pi_* \pi^* M)$$

is an isomorphism. Since $\mathcal{E}xt_S^1(G, M)$ is a sheaf, we can assume S is affine.

- λ is injective: let E be an extension of G by M , and assume $\varphi: G \longrightarrow E$ is a section as sheaves of sets. We can assume $\varphi(0) = 0$ (just subtracting $\varphi(0)$ to this morphism). Let the map $G \times_S G \longrightarrow M$ be the obstruction to ϕ being a homomorphism, i.e. the map that sends an ordered couple (g_1, g_2) to $\varphi(g_2)^{-1} \cdot \varphi(g_1)^{-1} \cdot \varphi(g_1 \cdot g_2)$. This map sends $(0, 0)$ to 0, but by 1. this map has to be constant, so the obstruction is 0;
- λ is surjective: let E be a principal homogeneous space for M over the base G . S is affine, thus a result of Serre yields that E admits a section e lying over the identity section of G . We shall now impose a group structure on E : this yields a group extension structure on E . To show this, it is sufficient to know that the cohomology class representing E in $H^1(G, \pi^* M)$ is primitive, that is it is a generator of $H^1(G, \pi^* M)$, but by the Künneth formula whenever G is

an abelian scheme $H^1(G, \pi^*M)$ consists only of primitive elements ([28, III, 4.2]).

Now we establish the isomorphism $\mathrm{Hom}_{\mathcal{O}_S} \left((R^1 f_* \mathcal{O}_G)^\vee, M \right) \cong \mathrm{Ext}_S^1(G, M)$, that is we want to represent the functor $\mathcal{E}xt_S^1(G, -)$.

To prove this, we only need to show that $\Gamma(S, R^1 \pi_* \pi^* M) \cong \Gamma(S, R^1 \pi_* \mathcal{O}_G \otimes_{\mathcal{O}_S} M)$, since the left hand side is isomorphic to $\mathrm{Ext}_{\mathcal{O}_S}^1(G, M)$ by the previous proposition and the right hand side is isomorphic to $\mathrm{Hom}_{\mathcal{O}_S} \left((R^1 f_* \mathcal{O}_G)^\vee, M \right)$, but this of course holds because of the condition 3. in proposition 2.5. \square

2.2 Universal extension as the rigidification of $\mathcal{E}xt$

2.2.1 Rigidification of $\mathcal{E}xt$

Fix an S -group scheme G and an exact sequence (ε) of fppf sheaves of abelian groups over S :

$$0 \longrightarrow G \longrightarrow E \longrightarrow F \longrightarrow 0$$

Let $F_1 := \mathrm{Inf}_S^1(F) \subseteq F$ denote the first infinitesimal neighborhood of the identity section of F over S (cfr. Appendix A, Section A.2.4). We have already seen that this is an S -pointed sheaf.

We introduce the following notation (due to Grothendieck, cfr. [14, SGA 7]).

Definition 2.7.

1. A rigidification r of the extension (ε) is a homomorphism $r: F_1 \longrightarrow E$ of S -pointed S -schemes such that:

$$\begin{array}{ccc} F_1 & \xrightarrow{\quad r \quad} & E \\ & \searrow \quad \circlearrowright \quad \nearrow & \\ & F & \end{array}$$

2. A rigidified extension of F by G is an exact sequence (ε) together with a rigidification r of it;
3. If H is an S -group scheme, an (ε) -rigidified homomorphism $\varphi: G \longrightarrow H$ is a homomorphism of S -group $\varphi: G \longrightarrow H$ together with a rigidification of the induced push-out exact sequence $(\varphi_* \varepsilon)$:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & G & \longrightarrow & E & \longrightarrow & F \longrightarrow 0 \\
 & & \downarrow \varphi & \circlearrowleft & \downarrow & \circlearrowleft & \parallel \\
 0 & \longrightarrow & H & \longrightarrow & E' & \longrightarrow & F \longrightarrow 0
 \end{array}$$

Denote by $\text{Extrig}_S(F, G)$ the set of isomorphism classes of rigidified extensions of F by G .

We shall now give an abelian group structure on $\text{Extrig}_S(F, G)$: given two exact sequences (ε) and (ε') :

$$\begin{array}{ccccccc}
 0 & \longrightarrow & G & \longrightarrow & E & \longrightarrow & F \longrightarrow 0 \\
 0 & \longrightarrow & G & \longrightarrow & E' & \longrightarrow & F \longrightarrow 0
 \end{array}$$

one has the Baer sum $(\bar{\varepsilon}) := (\varepsilon + \varepsilon')$ given by the last row of the following diagram:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & G \times_S G & \longrightarrow & E \times_S E' & \longrightarrow & F \times_S F \longrightarrow 0 \\
 & & \downarrow \nabla & \circlearrowleft & \downarrow & \circlearrowleft & \parallel \\
 0 & \longrightarrow & G & \longrightarrow & P & \longrightarrow & F \times_S F \longrightarrow 0 \\
 & & \parallel & \circlearrowleft & \uparrow & \circlearrowleft & \uparrow \Delta \\
 0 & \longrightarrow & G & \longrightarrow & \bar{E} & \longrightarrow & F \longrightarrow 0
 \end{array}$$

If moreover the two extensions (ε) and (ε') are endowed with rigidification r and r' respectively, then $(\bar{\varepsilon})$ has a rigidification obtained by the natural rigidification on the external product:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & G \times_S G & \longrightarrow & E \times_S E' & \longrightarrow & F \times_S F \longrightarrow 0 \\
 & & & & \uparrow \bar{r} & \nwarrow r \times r' & \\
 & & & & (F \times_S F)_1 & \longrightarrow & F_1 \times_S F_1
 \end{array}$$

The Baer sum induces an abelian group structure on $\text{Extrig}_S(F, G)$ which is bifunctorial in F and G .

2.2.2 Universal extension of an abelian scheme

In this section, we shall express the universal extension of an abelian scheme as an $\mathcal{E}xtrig_S$.

Let S be a scheme, A an abelian scheme over S , and consider an extension (ε) of A by $\mathbb{G}_{m,S}$:

$$0 \longrightarrow \mathbb{G}_{m,S} \longrightarrow E \longrightarrow A \longrightarrow 0$$

This extension makes E a principal homogeneous space over A under the group $\mathbb{G}_{m,S}$, so by descent ([14, SGA I, Exp. XI, 4.3] and [13, EGA IV, 17.7.3]) E is a smooth A -scheme. In particular, if S is affine, one can lift the identity section in the following way:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathbb{G}_{m,S} & \longrightarrow & E & \longrightarrow & A \longrightarrow 0 \\
 & & & & \nwarrow \scriptstyle r & & \uparrow \\
 & & & & & & A_1 \\
 & & & & \nearrow \scriptstyle e & & \uparrow \\
 & & & & & & S
 \end{array}$$

thus we have a rigidification of the extension (ε) .

Denoting by $\mathcal{E}xtrig_S(A, \mathbb{G}_{m,S})$ the Zariski sheaf associated to the presheaf $T \mapsto \text{Extrig}_T(A_T, \mathbb{G}_{m,T})$ we have a surjective morphism:

$$\mathcal{E}xtrig_S(A, \mathbb{G}_{m,S}) \longrightarrow \mathcal{E}xt_S^1(A, \mathbb{G}_{m,S})$$

Let us study the kernel of this map. It consists of the rigidifications τ of the trivial extension:

$$0 \longrightarrow \mathbb{G}_{m,S} \longrightarrow \mathbb{G}_{m,S} \times_S A \longrightarrow A \longrightarrow 0$$

To give a morphism $A_1 \longrightarrow \mathbb{G}_{m,S} \times_S A$ of S -pointed S -schemes which projects to the inclusion $A_1 \hookrightarrow A$ is equivalent to giving a morphism of S -pointed S -schemes $A_1 \longrightarrow \mathbb{G}_{m,S}$, which is equivalent to giving an element in $\Gamma(S, \underline{\omega}_{A/S})$. The inclusion $\Gamma(S, \underline{\omega}_{A/S}) \hookrightarrow \mathcal{E}xtrig_S(A, \mathbb{G}_{m,S})$ is clearly additive, so we have an exact sequence of Zariski (fppf, étale...) sheaves:

$$0 \longrightarrow \Gamma(S, \underline{\omega}_{A/S}) \longrightarrow \mathcal{E}xtrig_S(A, \mathbb{G}_{m,S}) \longrightarrow \mathcal{E}xt_S^1(A, \mathbb{G}_{m,S}) \longrightarrow 0$$

The dual abelian scheme A^* exists and is isomorphic to $\mathcal{E}xt_S^1(A, \mathbb{G}_{m,S})$ ([25]). Thus by descent theory it follows that $\mathcal{E}xtrig_S(A, \mathbb{G}_{m,S})$ is representable and is a smooth group scheme over S .

Proposition 2.8. *Let S be a scheme, A abelian scheme over S , and denote with $E(A^*)$ the universal extension of A^* by a vector group. The canonical morphism:*

$$E(A^*) \longrightarrow \mathcal{E}xtrig_S(A, \mathbb{G}_{m,S})$$

is an isomorphism that is functorial in A .

Proof. Both $E(A^*)$ and $\mathcal{E}xtrig_S(A, \mathbb{G}_{m,S})$ commute with base change, since $E(A^*)$ is built via objects which are compatible with arbitrary base change.

To show that $E(A^*) \rightarrow \mathcal{E}xtrig_S(A, \mathbb{G}_{m,S})$ is an isomorphism is equivalent to showing that the morphism $\underline{\omega}_{A/S} \rightarrow \underline{\omega}_{A/S}$ giving rise to it is an isomorphism. Since this problem is local on S we can safely suppose that S is affine.

A is proper and smooth over S , thus finitely presented, so we can assume that $S = \text{Spec}(R)$ with R of finite type over \mathbb{Z} ([13, EGA IV, 8.9.1, 8.10.5, ...]).

Given any $\mathfrak{m} \subseteq R$ maximal ideal, it follows from [19, 2.6.2] that the morphism:

$$\frac{\underline{\omega}_{A/S}}{\mathfrak{m}^n \cdot \underline{\omega}_{A/S}} \longrightarrow \frac{\underline{\omega}_{A/S}}{\mathfrak{m}^n \cdot \underline{\omega}_{A/S}}$$

is an isomorphism for all $n \geq 1$, thus the determinant of the correspondent endomorphism of $\underline{\omega}_{A/S} \otimes \hat{R}_{\mathfrak{m}}$ is invertible in $\hat{R}_{\mathfrak{m}}$, thus invertible in $R_{\mathfrak{m}}$. So the endomorphism of $\underline{\omega}_{A/S}$ is an automorphism.

Next we shall prove the functoriality: let $u: A \rightarrow B$ a morphism:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \underline{\omega}_{B/S} & \longrightarrow & E(B^*) & \longrightarrow & B^* \longrightarrow 0 \\ & & \swarrow id & \downarrow & \swarrow & \downarrow & \swarrow id \downarrow \\ 0 & \longrightarrow & \underline{\omega}_{B/S} & \longrightarrow & \mathcal{E}xtrig_S(B, \mathbb{G}_{m,S}) & \longrightarrow & B^* \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \underline{\omega}_{A/S} & \longrightarrow & E(A^*) & \longrightarrow & A^* \longrightarrow 0 \\ & & \swarrow id & \downarrow & \swarrow & \downarrow & \swarrow id \downarrow \\ 0 & \longrightarrow & \underline{\omega}_{A/S} & \longrightarrow & \mathcal{E}xtrig_S(A, \mathbb{G}_{m,S}) & \longrightarrow & A^* \longrightarrow 0 \end{array}$$

The two maps:

$$E(B^*) \rightarrow E(A^*) \rightarrow \mathcal{E}xtrig_S(A, \mathbb{G}_{m,S})$$

and

$$E(B^*) \rightarrow \mathcal{E}xtrig_S(B, \mathbb{G}_{m,S}) \rightarrow \mathcal{E}xtrig_S(A, \mathbb{G}_{m,S})$$

coincide, since their difference is a map $E(B^*) \rightarrow \underline{\omega}_{A/S}$ which vanishes on $\underline{\omega}_B$ (by the commutativity of the diagrams), so this yields a map $B^* \rightarrow \underline{\omega}_{A/S}$ which is 0 since for any abelian scheme A and M quasi-coherent \mathcal{O}_S -module one has $\text{Hom}_{\mathcal{G}_p\text{Sch}}(G, M) = 0$. \square

2.3 Rigidified extensions and \natural -extensions

Let $0 \longrightarrow \mathbb{G}_{m,S} \longrightarrow E \longrightarrow A \longrightarrow 0$ be an extension (ε) of an abelian scheme A over an affine base scheme S . The aim of this section is to show how the following structures on (ε) are equivalent:

- A rigidification r of (ε) ;
- An integrable connection, compatible with the group structure, on E regarded as a $\mathbb{G}_{m,S}$ -torsor over A (see Appendix B).

This discussion will yield another explicit description of the universal extension of an abelian scheme.

Denote with $\mathcal{E}xt_S^{\natural}(X, G)$ the category whose objects are \natural -extensions of a smooth group scheme G by X and whose arrows are horizontal morphisms between extensions (see Definition B.5 in Appendix B). Since G is commutative, the category $\mathcal{E}xt_S(X, G)$ of all extensions of X by G is endowed with a composition law which corresponds to taking the contracted product of the underlying torsors. Upon passing to the set of isomorphism classes of objects the induced composition law gives the standard group structure to $\text{Ext}_S^1(X, G)$.

We replicate this idea with $\mathcal{E}xt_S^{\natural}(X, G)$: it is clear that we can define the Baer sum of two \natural -extension and that by passing to isomorphism classes we obtain a group $\text{Ext}_S^{\natural}(X, G)$.

We shall now construct a homomorphism $\text{Ext}_S^{\natural}(A, G) \longrightarrow \text{Extrig}_S(A, G)$, with A and G commutative group schemes defined over S . Then we shall prove that if $G = \mathbb{G}_{m,S}$ and A is an abelian scheme, such a homomorphism is actually an isomorphism.

Consider a given \natural -extension (ε) of A by G :

$$0 \longrightarrow G \longrightarrow E \longrightarrow A \longrightarrow 0$$

Let $i: \text{Inf}_S^1(A) \hookrightarrow A$ be the inclusion of the first infinitesimal neighborhood, $\pi: \text{Inf}_S^1(A) \longrightarrow S$ be the structural morphism, and denote by $\tau: \text{Inf}_S^1(A) \longrightarrow \Delta^1(A)$ the morphism determined by the conditions $p_1 \circ \tau = e_A \circ \pi$ and $p_2 \circ \tau = i$, where e_A is the identity section of A . Since the \natural -structure on E is given by an isomorphism $\nabla: p_1^*(E) \longrightarrow p_2^*(E)$, we can pull-back ∇ via τ to obtain $\tau^*(\nabla): \pi^* \circ e_A^*(E) \longrightarrow$

$i^*(E)$.

E is a group scheme, thus $e_A^*(E)$ and hence $\pi^* \circ e_A^*(E)$ are equipped with an obvious choice of section (that is the identity section). Via $\tau^*(\nabla)$ we transfer such section in $i^*(E)$ and then by composition with $i^*(E) \rightarrow E$ we obtain a morphism:

$$\sigma: \text{Inf}_S^1(A) \rightarrow E$$

Such σ will be the rigidification of (ε) .

Lemma 2.9. *The morphism $\sigma: \text{Inf}_S^1(A) \rightarrow E$ is a rigidification of (ε) , that is:*

- i. σ is a morphism of S -schemes;
- ii. The following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\quad} & A \\ & \swarrow \sigma \quad \circlearrowright \quad \searrow & \\ & \text{Inf}_S^1(A) & \end{array}$$

- iii. It is a morphism of S -pointed schemes.

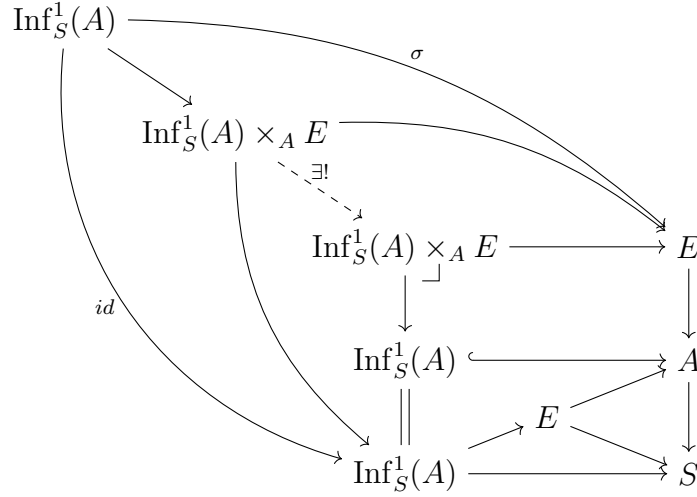
Proof.

- i. Consider the morphisms $\varphi: \text{Inf}_S^1(A) \rightarrow E \rightarrow A$ and the morphism $E \rightarrow A$. We have the following pull-back diagram:

$$\begin{array}{ccccc} \text{Inf}_S^1(A) & & \xrightarrow{\quad \sigma \quad} & & E \\ & \searrow \exists! & & \searrow & \downarrow \\ & \text{Inf}_S^1(A) \times_A E & \xrightarrow{\quad} & & E \\ & \downarrow \lrcorner & & \circlearrowright & \downarrow \\ & \text{Inf}_S^1(A) & \xrightarrow{\quad \varphi \quad} & & A \end{array}$$

(Note: A curved arrow labeled id goes from $\text{Inf}_S^1(A)$ to $\text{Inf}_S^1(A)$ in the bottom row.)

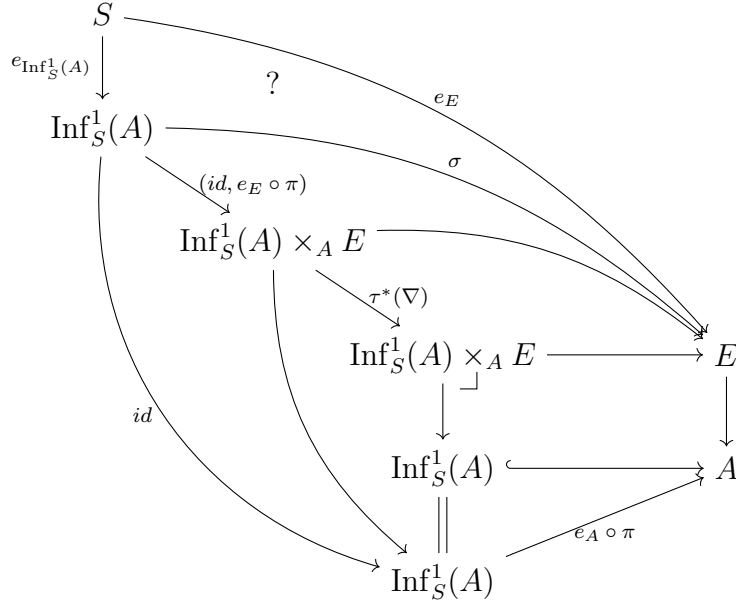
Considering now $\text{Inf}_S^1(A)$ as an A -scheme via the inclusion i , we have another pull-back diagram that together with the previous one yields:



This diagram commutes by simple arguments about universal properties of the pull-back;

ii. It is sufficient to observe that $i^*(E) = \text{Inf}_S^1(A) \times_A E$, thus the result follows from the diagram in i.;

iii. Let us consider the pull-back diagram:



We have to check the commutativity of the diagram $?$. It commutes if $\tau^*(\nabla)$ preserves the second component of the morphism $u = (e_{\text{Inf}_S^1(A)}, e_E): S \rightarrow \text{Inf}_S^1(A) \times_A E$.

Let us return to the connection $\nabla: p_1^*(E) \xrightarrow{\cong} p_2^*(E)$. By composing $e_A: S \rightarrow A$ with $\Delta: A \rightarrow \Delta^1(A)$, S can be viewed as a $\Delta^1(A)$ -scheme. So $p_1^*(E)$ and $p_2^*(E)$ have both obvious sections with values in the $\Delta^1(A)$ -scheme S (namely δ_1 and δ_2 , respectively). They are the sections with components $S \hookrightarrow \Delta^1(A)$ and $S \xrightarrow{e_E} E$.

Under the identification of $\Delta^* p_i^*(E)$ with E , such unit sections are identified with the identity section $e_E: S \rightarrow E$; but by definition of a connection, $\Delta^*(\nabla) = id_E$, thus ∇ must map $\delta_1: S \rightarrow p_1^*(E)$ in the corresponding section $\delta_2: S \rightarrow p_2^*(E)$. This means that the second component remains $S \xrightarrow{e_E} E$.

Let us now consider the first factor $e_{\text{Inf}_S^1(A)}: S \rightarrow \text{Inf}_S^1(A)$. Since $\tau \circ e_{\text{Inf}_S^1(A)} = \Delta \circ e_A$, it follows from the definitions that $\tau^*(\delta_1) = u$. This implies that $\tau^*(\nabla) \circ u$ has as its second component the unit section $e_E: S \rightarrow E$, and this completes the proof. \square

We shall now prove the main theorem of this section, that yields the connection between rigidified extensions and \natural -extensions of an abelian scheme:

Theorem 2.10. *If A is an abelian scheme, the homomorphism*

$$\text{Ext}_S^{\natural}(A, \mathbb{G}_{m,S}) \longrightarrow \text{Extrig}_S(A, \mathbb{G}_{m,S})$$

is an isomorphism.

Proof. To prove our claim, we shall construct an inverse of this morphism. Assume given a rigidified extension:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{G}_{m,S} & \longrightarrow & E & \xrightarrow{j} & A \longrightarrow 0 \\ & & & & \nwarrow \sigma & & \uparrow \\ & & & & & & \text{Inf}_S^1(A) \end{array}$$

σ defines a section of $i^*(E)$, and hence a trivialization $\rho: (e_A \circ \pi)^*(E) \xrightarrow{\cong} i^*(E)$ via $e \mapsto (id_{\text{Inf}_S^1(A)}, \sigma)$.

By definition of $\text{Inf}_S^1(A)$, the map $p_2 - p_1: \Delta^1(A) \rightarrow A$ factors as:

$$\Delta^1(A) \xrightarrow{\eta} \text{Inf}_S^1(A) \xhookrightarrow{i} A$$

Thus, denoting by $\pi_{\Delta^1(A)}: \Delta^1(A) \rightarrow S$ the structural morphism, we have that $\eta^*(\rho): (e_A \circ \pi_{\Delta^1(A)})^*(E) \xrightarrow{\cong} (p_2 - p_1)^*(E)$ is an isomorphism. Multiplying both source and target of this arrow by $p_1^*(E)$ and using the fact that E is a group scheme we obtain the following diagram:

$$\begin{array}{ccc} (e_A \circ \pi_{\Delta^1(A)})^*(E) \wedge^{\mathbb{G}_{m,S}} p_1^*(E) & \xrightarrow{\eta^*(\rho) \wedge^{\mathbb{G}_{m,S}} p_1^*(E)} & (p_2 - p_1)^*(E) \wedge^{\mathbb{G}_{m,S}} p_1^*(E) \\ \downarrow \text{IR} & \circlearrowleft & \downarrow \text{IR} \\ p_1^*(E) & \xrightarrow{\nabla'} & p_2^*(E) \end{array}$$

Such ∇' defines a \mathfrak{h} -structure on the extension, so the inverse mapping is defined by associating to a rigidified extension the \mathfrak{h} -extension with the same underlying extension endowed with the \mathfrak{h} -structure defined by ∇' . We have to prove that this is a good definition and that is an inverse of the previous morphism, so we have to prove:

1. $\Delta^*(\nabla') = id_E$;
2. The map $\text{Extrig}_S(A, \mathbb{G}_{m,S}) \longrightarrow \text{Ext}_S^{\mathfrak{h}}(A, \mathbb{G}_{m,S}) \longrightarrow \text{Extrig}_S(A, \mathbb{G}_{m,S})$ is the identity;
3. The map $\text{Ext}_S^{\mathfrak{h}}(A, \mathbb{G}_{m,S}) \longrightarrow \text{Extrig}_S(A, \mathbb{G}_{m,S})$ is injective;
4. Δ' is integrable;
5. The isomorphism $\pi_1^*(E) \overset{\mathbb{G}_{m,S}}{\wedge} \pi_2^*(E) \xrightarrow{\cong} s^*(E)$ is horizontal.¹

The proof of the first two statements hold for any A and for any G , while those of the other three rely on the assumptions that A is an abelian scheme and $G = \mathbb{G}_{m,S}$.

1. Since $\Delta^*(\nabla')$ is a morphism over A , it suffices to show that is the identity when E is viewed as a sheaf on $\mathcal{S}ch/S$. Since our situation commutes with base change it suffices to show the mapping it induces on the S -valued points $E(S) \longrightarrow E(S)$ is the identity.

Let $\xi: S \longrightarrow E$ be given so that ξ defines morphisms $\xi_1: S \longrightarrow p_1^*(E)$ and $\xi_2: S \longrightarrow p_2^*(E)$. Since $\Delta: A \longrightarrow \Delta^1(A)$ is a monomorphism it suffices to show that $\nabla' \circ \xi_1 = \xi_2$. To check that it is true let us recall the vertical isomorphism of the previous diagram.

Let $\alpha, \beta: T \longrightarrow A$ be given and consider E_α, E_β and $E_{\alpha+\beta}$ be the torsors deduced from E by the corresponding base change. By definition, $E_\alpha \overset{\mathbb{G}_{m,S}}{\wedge} E_\beta$ is a sheaf associated to the quotient of $E_\alpha \times_T E_\beta$ by the action of $\mathbb{G}_{m,S}$, thus if T' is any S -scheme the elements of $\Gamma(T', E_\alpha \overset{\mathbb{G}_{m,S}}{\wedge} E_\beta)$ are given locally by triples of S -morphisms $x: T' \longrightarrow E, y: T' \longrightarrow E$, and $t': T' \longrightarrow T$ such that:

$$\begin{array}{ccc} T' & \xrightarrow{x} & E \\ t' \downarrow & \circlearrowleft & \downarrow \\ T & \xrightarrow{\alpha} & A \end{array} \qquad \begin{array}{ccc} T' & \xrightarrow{y} & E \\ t' \downarrow & \circlearrowleft & \downarrow \\ T & \xrightarrow{\beta} & A \end{array}$$

The isomorphism in question is determined by associating to (t', x, y) the pair $(t', x + y) \in \Gamma(T', E_{\alpha+\beta})$.

Return now to the starting rigidified extension: $\xi_1 = (\Delta \circ j \circ \xi, \xi)$ and $\xi_2 = (\Delta \circ j \circ$

¹We follow the notations introduced in Section B, Definition B.5.

$\xi, \xi)$, and after the above explication of the vertical isomorphism it is obvious that ξ_1 corresponds to the class of $(\Delta \circ j \circ \xi, e_E, \xi)$.

On the other hand, projection of $(e_A \circ \pi_{\Delta^1(A)})^*(E)$ assigns to $(\Delta \circ j \circ \xi, e_E)$ the pair $(\eta \circ \Delta \circ j \circ \xi, e_E)$ which it transforms via ρ into $(\eta \circ \Delta \circ j \circ \xi, \sigma \circ \eta \circ \Delta \circ j \circ \xi)$. Therefore $\eta^*(\rho) \overset{\mathbb{G}_{m,S}}{\wedge} p_1^*(E)$ will transform the class of $(\Delta \circ j \circ \xi, e_E, \xi)$ to the class of $(\Delta \circ j \circ \xi, \sigma \circ \eta \circ \Delta \circ j \circ \xi, \xi)$. Since $e_A \circ \pi_A = (p_2 - p_1) \circ \Delta = i \circ \eta \circ \Delta$ and also $e_A \circ \pi_A = i \circ e_{\text{Inf}_S^1(A)} \circ \pi_A$, it follows that $\eta \circ \Delta = e_{\text{Inf}_S^1(A)} \circ \pi_A$. Hence $\sigma \circ \eta \circ \Delta \circ j \circ \xi = \sigma \circ e_{\text{Inf}_S^1(A)} \circ \pi_A \circ j \circ \xi = e_E \circ \pi_A \circ j \circ \xi = e_E$. Thus under the isomorphism:

$$(p_2 - p_1)^*(E) \overset{\mathbb{G}_{m,S}}{\wedge} p_1^*(E) \xrightarrow{\cong} p_2^*(E)$$

we have that $(\Delta \circ j \circ \xi, \sigma \circ \eta \circ \Delta \circ j \circ \xi, \xi)$ corresponds to $(\Delta \circ j \circ \xi, \xi)$ which shows finally that $\Delta^*(\nabla) = id_E$.

2. Consider again the starting rigidified extension. We associate a connection ∇' on E to σ and then a rigidification σ' is associated to ∇' . It is to be shown that $\sigma' = \sigma$. σ' is the projection onto E of $\tau^*(\nabla')(id_{\text{Inf}_S^1(A)}, e_E \circ \pi_{\text{Inf}_S^1(A)})$, hence it is the projection onto E of $\nabla'(\tau, e_E \circ \pi_{\text{Inf}_S^1(A)})$. But as it follows from the definition of ∇' in terms of the above diagram defining ∇' , this projection, denoting with p_E the projection onto E , is simply the sum:

$$\begin{aligned} p_E \left((\eta^*(\rho))(\tau, e_E \circ \pi_{\text{Inf}_S^1(A)}) \right) + e_E \circ \pi_{\text{Inf}_S^1(A)} &= p_E \left((\eta^*(\rho))(\tau, e_E \circ \pi_{\text{Inf}_S^1(A)}) \right) \\ &= p_E \left(\rho(\eta \circ \tau, e_E \circ \pi_{\text{Inf}_S^1(A)}) \right) \end{aligned}$$

But since $i \circ \eta \circ \tau = (p_2 - p_1) \circ \tau = p_2 \circ \tau - p_1 \circ \tau = i - e_A \circ \pi_{\text{Inf}_S^1(A)} = i$, which is a monomorphism, it follows that $\eta \circ \tau = id_{\text{Inf}_S^1(A)}$, which implies by the definition of ρ , that $\sigma' = \sigma$.

3. To show that the map $\text{Ext}_S^1(A, \mathbb{G}_{m,S}) \longrightarrow \text{Extrig}_S(A, \mathbb{G}_{m,S})$ is injective, we must show that if ∇ defines a \mathfrak{h} -structure on the trivial extension:

$$0 \longrightarrow \mathbb{G}_{m,S} \longrightarrow \mathbb{G}_{m,S} \times_S A \longrightarrow A \longrightarrow 0$$

whose associated rigidification σ is trivial, then ∇ itself is trivial. ∇ is determined by giving a section of $\Gamma(\Delta^1(A), \mathcal{O}_{\Delta^1(A)}^*)$ of the form $1 + \omega$, with $\omega \in \Gamma(A, \Omega_{A/S}^1)$. The corresponding ρ associated to the rigidification σ is determined by a unit in $\Gamma(\Delta^1(A), \mathcal{O}_{\Delta^1(A)}^*)$ of the form $1 + \omega'$ with $\omega' \in \Gamma(S, \underline{\omega}_{A/S})$, since it is an automorphism of $\mathbb{G}_{m, \text{Inf}_S^1(A)}$.

One has that $\omega' = \tau^*(\omega)$, but since A is an abelian scheme the map $\Gamma(A, \Omega_{A/S}^1) \longrightarrow$

$\Gamma(S, \underline{\omega}_{A/S})$ is an isomorphism, and so our result follows.

4. The curvature tensor $e(\nabla')$ is an element of $\Gamma\left(S, \pi_{A*}(\Omega_{A/S}^2)\right)$. As mentioned before, since in our case $G = \mathbb{G}_{m,S}$ we have that E corresponds to a line bundle \mathcal{L}_E and ∇' to a connection on it. Since A is an abelian scheme any global 1-form is closed, thus the curvature $e(\nabla')$ is actually independent of the connection on E . This allows us to define a morphism:

$$\mathcal{E}xt_S^1(A, \mathbb{G}_{m,S}) \longrightarrow \pi_{A*}(\Omega_{A/S}^2)$$

in the following way: given an absolutely affine scheme T over S , and an extension:

$$0 \longrightarrow \mathbb{G}_{m,T} \longrightarrow E' \longrightarrow A_T \longrightarrow 0$$

we can take any structure of rigidified extension on it and by the above procedure we put a connection on E' , hence finally obtaining the curvature tensor which lies in $\Gamma\left(T, \pi_{A_T*}(\Omega_{A_T/S}^2)\right) = \Gamma\left(T, \left(\pi_{A*}(\Omega_{A/S}^2)\right)_T\right)$. Passing to the associated sheaves gives the morphism above.

Since $\mathcal{E}xt_S^1(A, \mathbb{G}_{m,S})$ is an abelian scheme and $\pi_{A*}(\Omega_{A/S}^2)$ is a vector group, this morphism is constant. The image of the trivial extension is clearly 0, so the map is identically zero and thus ∇' is integrable.

5. Again, we start by replacing E by the corresponding line bundle \mathcal{L}_E . We are to show that the isomorphism $s^*(\mathcal{L}_E) \xrightarrow{\cong} \pi_1^*(\mathcal{L}_E) \otimes_{\mathcal{O}_S} \pi_2^*(\mathcal{L}_E)$ is horizontal. By using this isomorphism, the problem can be interpreted as that of showing that two connections on $s^*(\mathcal{L}_E)$ are the same. Taking their difference we obtain a section $\delta(\nabla') \in \Gamma(S, \underline{\omega}_{A \times_S A})$. Now, to mimic the strategy employed in the proof of 4., we will use the following result.

Lemma 2.11. *Let X be a scheme over S , let \mathcal{L}_1 and \mathcal{L}_2 be line bundles on X and $\nabla_1, \nabla_2, \nabla'_1, \nabla'_2$ connections on \mathcal{L}_i . Consider an isomorphism $\varphi: \mathcal{L}_1 \xrightarrow{\cong} \mathcal{L}_2$. Denote with δ and δ' the difference between $\varphi^*(\nabla_2)$ and ∇_1 and the difference between $\varphi^*(\nabla'_2)$ and ∇'_1 respectively. Then we have:*

$$\delta - \delta' = \nabla_2 - \nabla'_2 - (\nabla_1 - \nabla'_1)$$

Proof. The assertion is local, hence we can assume $X = \text{Spec}(B)$, $S = \text{Spec}(A)$ affine, and \mathcal{L}_1 and \mathcal{L}_2 trivial. Translating, the connections ∇_i and ∇'_i correspond to differential forms ω_i and ω'_i (respectively) in $\Omega_{B/A}^1$ and φ corresponds to the mapping multiplication by a unit $b \in B^*$.

Thus $\varphi^*(\nabla_2)$ correspond to $\frac{db}{b} + \underline{\omega}_2$ and $\varphi^*(\nabla_2) - \nabla_1 = \frac{db}{b} + (\underline{\omega}_2 - \underline{\omega}_1)$ and analougously $\varphi^*(\nabla'_2) - \nabla'_1 = \frac{db}{b} + (\underline{\omega}'_2 - \underline{\omega}'_1)$. To find our result we only need to subtract these quantities. \square

We apply this lemma to $\mathcal{L}_1 = s^*(\mathcal{L})$, $\mathcal{L}_2 = \pi_1^*(\mathcal{L}) \otimes_{\mathcal{O}_S} \pi_2^*(\mathcal{L})$, and for any two connections $\bar{\nabla}, \tilde{\nabla}$ on \mathcal{L} let:

$$\begin{aligned}\nabla_1 &= s^*(\bar{\nabla}) \\ \nabla'_1 &= s^*(\tilde{\nabla}) \\ \nabla_2 &= \pi_1^*(\bar{\nabla}) \otimes_{\mathcal{O}_S} \pi_2^*(\bar{\nabla}) \\ \nabla'_2 &= \pi_1^*(\tilde{\nabla}) \otimes_{\mathcal{O}_S} \pi_2^*(\tilde{\nabla})\end{aligned}$$

Then if $\bar{\nabla} - \tilde{\nabla} = \psi \in \Gamma(A, \Omega_{A/S}^1)$ the lemma asserts that:

$$\delta(\bar{\nabla}) - \delta(\tilde{\nabla}) = \pi_1^*(\psi) + \pi_2^*(\psi) - s^*(\psi)$$

Since A is an abelian scheme, ψ is a primitive element and thus $\delta(\bar{\nabla}) = \delta(\tilde{\nabla})$. So, $\delta(\nabla)$ does not depend on the connection put on the line bundle \mathcal{L} , so we can define (as above) a morphism:

$$\mathcal{E}xt_S^1(A, \mathbb{G}_{m,S}) \longrightarrow \underline{\omega}_{A \times_S A}$$

that is constantly 0 (by the same arguments). The trivial connection on the trivial extension is compatible with the group structure, and so (the morphism being constantly 0) any connection placed on any extension is similarly compatible. \square

2.4 Relation between one dimensional de Rham cohomology and the Lie algebra of the universal extension

The aim of this section is to establish an isomorphism between $H_{\text{dR}}^1(A)$ and the Lie algebra of $\mathcal{E}xtrig_S(A, \mathbb{G}_{m,S})$.

2.4.1 Lie algebra of $\mathcal{P}ic^{\natural}$

We shall now emply the definitions and remarks of Section 1.4, in particular we want to study the Lie algebra of the group functor $\mathcal{P}ic_{A/S}^{\natural}$.

For any group functor G on $\mathcal{S}ch/S$ the formation of $\mathcal{L}ie(G)$ commutes with taking of

the associated Zariski sheaf. Thus to calculate the Lie algebra of $\mathcal{P}ic_{A/S}^{\flat}$ it suffices to calculate the Lie algebra of $\text{Pic}_{A/S}^{\flat}$ and then sheafify it.

Proposition 2.12. $H_{\text{dR}}^1(A)$ is canonically isomorphic to $\mathcal{L}ie\left(\mathcal{P}ic_{A/S}^{\flat}\right)$.

Proof. We must examine $\ker(\text{Pic}^{\flat}(S[\varepsilon]) \rightarrow \text{Pic}^{\flat}(S))$, which by the Proposition 1.7 can be regarded as $\ker(\mathbb{H}^1(A_{k[\varepsilon]}, \Omega_{A_{S[\varepsilon]}/S[\varepsilon], m}^*) \rightarrow \mathbb{H}^1(A, \Omega_{A/S, m}^*))$. We have a split exact sequence of complexes of sheaves of abelian groups on A :

$$0 \rightarrow \Omega_{A/S}^* \rightarrow \Omega_{A_{S[\varepsilon]}/S[\varepsilon], m}^* \rightarrow \Omega_{A/S, m}^* \rightarrow 0$$

and hence at least as abelian groups we have that $\mathbb{H}^1(A, \Omega_{A/S}^*) \xrightarrow{\cong} \mathcal{L}ie(\text{Pic}_{A/S}^{\flat})(S)$. By straightforward computation, it follows that the module structure coincide too, and passing to the associated sheaves we find that $H_{\text{dR}}^1(A) \xrightarrow{\cong} \mathcal{L}ie\left(\mathcal{P}ic_{A/S}^{\flat}\right)$ as desired. \square

Lemma 2.13. $\mathbb{H}^*(A, \tau_1(\Omega_{A/S}^*))$ is locally free, and hence commutes with arbitrary base change.

Proof. From the exact sequence:

$$0 \rightarrow \tau_1(\Omega_{A/S}^*) \rightarrow \Omega_{A/S}^* \rightarrow \mathcal{O}_A \rightarrow 0$$

using the local freeness of $H_{\text{dR}}^*(A)$, $\mathbb{H}^*(A, \mathcal{O}_A)$ and the degeneration of the Hodge-de Rham spectral sequence, we read the result from the short exact sequence:

$$0 \rightarrow \mathbb{H}^i(A, \tau_1(\Omega_{A/S}^*)) \rightarrow \mathbb{H}^i(A, \Omega_{A/S}^*) \rightarrow \mathbb{H}^i(A, \mathcal{O}_A) \rightarrow 0$$

\square

Knowing that $\mathbb{H}^2\left(A, \tau_1\left(\Omega_{A/S}^*\right)\right)$ is a locally free module commuting with base change, we obtain the exact sequence of Zariski sheaves on $\mathcal{S}ch/S$:

$$0 \rightarrow \underline{\omega}_A \rightarrow \mathcal{P}ic_{A/S}^{\flat} \rightarrow \mathcal{P}ic_{A/S} \rightarrow \mathbb{H}^2(A, \tau_1(\Omega_{A/S}^*))$$

Let us consider the dual abelian variety $A^* = \mathcal{P}ic_{A/S}^0$ and the composite of its inclusion into $\mathcal{P}ic_{A/S}$ with the map $\mathcal{P}ic_{A/S} \rightarrow \mathbb{H}^2\left(A, \tau_1\left(\Omega_{A/S}^1\right)\right)$. This composite is 0 because there are no non-trivial homomorphism from an abelian variety to a locally free quasi-coherent module, hence the image of $\mathcal{P}ic_{A/S}^{\flat}$ in $\mathcal{P}ic_{A/S}$ contains A^* and there is an exact sequence:

$$0 \rightarrow \underline{\omega}_{A/S} \rightarrow \mathcal{P}ic_{A/S}^{\flat} \times_{\mathcal{P}ic_{A/S}} A^* \rightarrow A^* \rightarrow 0$$

Definition 2.14. We shall denote the identity component $\mathcal{P}ic_{A/S}^{\natural} \times_{\mathcal{P}ic_{A/S}} A^*$ of $\mathcal{P}ic_{A/S}^{\natural}$ by $\left(\mathcal{P}ic_{A/S}^{\natural}\right)^0$.

By the above construction, $\left(\mathcal{P}ic_{A/S}^{\natural}\right)^0$ is a smooth group variety which is obtained by considering the Zariski sheaf associated to the presheaf assigning to a scheme S defined over k the set of isomorphism classes of (\mathcal{L}, ∇) where the cohomology class of \mathcal{L} is primitive, or equivalently the \mathbb{G}_{m,A_S} -torsor corresponding to \mathcal{L} is an extension of A_S by $\mathbb{G}_{m,S}$.

Proposition 2.15. $H_{\text{dR}}^1(A)$ is canonically isomorphic to $\mathcal{L}ie\left(\left(\mathcal{P}ic_{A/S}^{\natural}\right)^0\right)$.

Proof. $\mathcal{L}ie\left(\mathcal{P}ic_{A/S}^{\natural} \times_{\mathcal{P}ic_{A/S}} A^*\right) \cong \mathcal{L}ie\left(\mathcal{P}ic_{A/S}^{\natural}\right) \times_{\mathcal{L}ie(\mathcal{P}ic_{A/S})} \mathcal{L}ie(A^*)$. Since $\mathcal{L}ie(A^*) \cong \mathcal{L}ie(\mathcal{P}ic_{A/S})$, the result follows from the Proposition 2.12. \square

2.4.2 The isomorphism between $\mathcal{E}xt_S^{\natural}$ and $\left(\mathcal{P}ic^{\natural}\right)^0$

For any abelian scheme A defined over S , define a homomorphism:

$$\mathcal{E}xt_S^{\natural}(A, \mathbb{G}_m) \xrightarrow{\xi} \left(\mathcal{P}ic_{A/S}^{\natural}\right)^0 = \mathcal{P}ic_{A/S}^{\natural} \times_{\mathcal{P}ic_{A/S}} \mathcal{P}ic_{A/S}^0$$

in the following way: any element $e \in \mathcal{E}xt_S^{\natural}(A, \mathbb{G}_m)$ may be regarded as an isomorphism class \mathcal{L} of invertible sheaves on A endowed with an integrable connection and with a horizontal isomorphism $\epsilon: s^*(G) \xrightarrow{\cong} p_1^*(\mathcal{L}) \otimes_S p_2^*(\mathcal{L})$, where $p_1, p_2: A \times_S A \rightarrow A$ are the projections and $s = p_1 + p_2$ is the sum morphism.

By forgetting ϵ (respectively the connection), we obtain an element of $\mathcal{P}ic_{A/S}^{\natural}$ (respectively of $\mathcal{P}ic_{A/S}^0$).

Theorem 2.16. The morphism $\xi: \mathcal{E}xt_S^{\natural}(A, \mathbb{G}_m) \rightarrow \left(\mathcal{P}ic_{A/S}^{\natural}\right)^0$ is an isomorphism.

Proof. It is injective, since any two horizontal isomorphisms between line bundles differ by multiplication by a unit in \mathcal{O}_S . Thus if there is a horizontal isomorphism, an isomorphism compatible with the morphism ϵ is also horizontal.

To show that it is surjective, we shall define a morphism of schemes over S :

$$\eta: A^* \rightarrow \underline{\omega}_{A \times_S A}$$

which expresses the obstruction to surjectivity of ξ . Let $\mathcal{L} \in \mathcal{E}xt(A, \mathbb{G}_m)$, and choose any integrable connection ∇ on \mathcal{L} . This induces connections on $s^*(\mathcal{L})$, $p_1^*(\mathcal{L})$, $p_2^*(\mathcal{L})$

and $p_1^*(\mathcal{L}) \otimes_S p_2^*(\mathcal{L})$ as well. The extension structure of \mathcal{L} gives us an explicit isomorphism:

$$\epsilon: s^*(\mathcal{L}) \xrightarrow{\cong} p_1^*(\mathcal{L}) \otimes_S p_2^*(\mathcal{L})$$

Consider the difference between the connection on $s^*(\mathcal{L})$ and the pull-back of the connection on $p_1^*(\mathcal{L}) \otimes_S p_2^*(\mathcal{L})$ via the above morphism. This difference $i(\nabla)$ is a section of $\underline{\omega}_{A \times_S A}$. By Lemma 2.11, we have that $i(\nabla)$ depends only on \mathcal{L} and not on the chosen integrable connection ∇ .

We define $\eta(\mathcal{L}) = i(\nabla)$. Since A^* is an abelian variety and $\underline{\omega}_{A \times_S A}$ is a locally free module, η is a constant map. Since $\eta(0) = 0$, η is identically zero. So ϵ is horizontal and ξ is surjective. \square

2.5 The universal extension on an abelian variety in the analytic category over \mathbb{C}

Let A be an abelian scheme over S , where S is a locally of finite type scheme defined over \mathbb{C} . We can consider A as a family of complex analytic spaces.

Without significant changes, we can carry over the theory of $\mathcal{E}xtrig_S$ also in the analytic category, and thus obtain the analytic versions and the natural maps:

$$\begin{aligned} \mathcal{E}xtrig_S^{an}(A, \mathbb{G}_m) &\longrightarrow \mathcal{E}xtrig_S(A, \mathbb{G}_m) \\ \mathcal{E}xtrig_S^{an}(A, \mathbb{G}_a) &\longrightarrow \mathcal{E}xtrig_S(A, \mathbb{G}_a) \end{aligned}$$

Proposition 2.17. *The two maps defined above are isomorphisms.*

Proof. For each fiber over S , this follows by GAGA theory. Consequently, the morphisms are analytic morphisms bijective on the underlying point sets, and considering vertical and horizontal tangent vectors, it is satisfied the Jacobian criterion. \square

As a direct consequence, the exponential sequence of analytic groups over \mathbb{C} :

$$0 \longrightarrow 2\pi i\mathbb{Z} \longrightarrow \mathbb{G}_a \xrightarrow{\exp} \mathbb{G}_m \longrightarrow 0$$

gives rise to the following diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & \underline{\omega}_{A/S} & \xlongequal{\quad\quad\quad} & \underline{\omega}_{A/S} & & \\
 & & \downarrow & & \downarrow & & \\
 & & \mathcal{E}xtrig_S(A^{an}, \mathbb{G}_a) & \rightarrow & \mathcal{E}xtrig_S(A^{an}, \mathbb{G}_m) & & \\
 & & \downarrow & & \downarrow & & \\
 & & \mathcal{O}_{A^{an}} & \xrightarrow{\quad\quad\quad} & \mathcal{P}ic^0_{A/S} & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

and thus by the snake lemma we get the exact sequence:

$$0 \longrightarrow H^1(A^{an}, \mathbb{Z}) \longrightarrow \mathcal{E}xtrig_S(A^{an}, \mathbb{G}_a) \longrightarrow \mathcal{E}xtrig_S(A^{an}, \mathbb{G}_m) \longrightarrow 0$$

over any affine base S .

Corollary 2.18. *There exists an exact sequence of analytic groups over S :*

$$0 \longrightarrow R^1 f_* \mathbb{Z} \longrightarrow H^1_{\text{dR}}(A^{\text{an}}_S) \longrightarrow E(A^*)^{\text{an}} \longrightarrow 0$$

where $R^1 f_* \mathbb{Z}$ denotes the locally constant sheaf of abelian groups and H^1_{dR} denotes the relative de Rham cohomology over the base S .

Proof. The corollary follows from the identification $\mathcal{E}xtrig_S(A, \mathbb{G}_m) = E(A^*)$ and $\mathcal{E}xtrig_S(A, \mathbb{G}_a) = H^1_{\text{dR}}(A/S)$. \square

Chapter 3

1–motives and their realizations

Motivations and main result

While there are several cohomology theories, arising from considering different structures (topological, differentiable, analytic...) on an algebraic variety, they are not independent one from the other. For example, the de Rham cohomology of a differentiable manifold is naturally isomorphic to the complexification of the singular (Betti) cohomology.

One can ask if the various information gathered by different cohomology functors can be recovered by the same object. More precisely, an important conjecture of Beilinson, Grothendieck and Deligne revolves around the existence of a triangulated category endowed with a t –structure whose heart is an abelian category (the category of mixed motives). Then we would have a natural functor from the category of schemes to the category of motives, with the universal property that every cohomology functor from the category of schemes factorizes through such functor.

There have been many attempts to find such a category, notably 'Triangulated categories of motives over a field' in [30] for motives over a field and [6] for motives over a finite dimensional Noetherian scheme. The theory is highly sophisticated and lies outside of the aim of this dissertation, but it is important in order to contextualize and justify the definitions of 1–motives over a base scheme S (following and generalizing [9, Chapter 10]) that we shall now introduce.

The main result of this chapter (and of all this dissertation) is the de Rham realization for 1–motives defined over \mathbb{C} and the comparison isomorphism between the de Rham realization and the complexification of the Hodge realization of a 1–motive over \mathbb{C} . As a direct application, we shall recover in all generality (and without

assumptions on the smoothness of our variety) the comparison isomorphism:

$$H^1(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong H_{\text{dR}}^1(X)$$

for any algebraic variety X over \mathbb{C} .

3.1 1-motives

Definition 3.1 (1-motive). A 1-motive over a base scheme S consists of:

1. An S -group scheme X which is locally isomorphic to a constant \mathbb{Z} -module of finite type, an abelian scheme A and an algebraic torus T over S ;
2. A semi-abelian group G , that is an extension of A by T over S ;
3. An S -homomorphism $u: X \rightarrow G$.

A 1-motive is free if the \mathbb{Z} -module X is free.

It is convenient to consider a 1-motive to be a bounded complex of group schemes in degree 0 and 1, that is $M = [X \xrightarrow{u} G]$.

Remark 3.2. The notion of "motive" in this definition is justified by the following fact: given the category $\mathcal{M}_1(k)$ of 1-motives over k (not necessarily free, that is X may have non trivial torsion part), this is an abelian category (while the category \mathcal{M}_1^{fr} of free motives is not). Denote with $\mathcal{M}_1(k) \otimes \mathbb{Q}$ the abelian category of 1-motives over k up to isogeny and with $D^b(\mathcal{M}_1(k) \otimes \mathbb{Q})$ the category of bounded complexes over $\mathcal{M}_1(k) \otimes \mathbb{Q}$. When k is perfect, in [30] Voevodsky proved that there exists a fully faithful embedding:

$$D^b(\mathcal{M}_1(k) \otimes \mathbb{Q}) \hookrightarrow \text{DM}_{-, \text{ét}}^{\text{eff}}(k) \otimes \mathbb{Q}$$

whose image is the thick subcategory $d_{\leq 1} \text{DM}_{gm}^{\text{eff}}(k) \otimes \mathbb{Q}$ of the effective geometrical motives generated by the motives of smooth curves. Deligne's 1-motives represent mixed motives of dimension ≤ 1 (see also [1]).

Let $M = [X \xrightarrow{u} G]$ be a 1-motive over S . The exponential map:

$$\exp: \mathcal{L}ie(G) \rightarrow G$$

realizes $\mathcal{L}ie(G)$ as the universal covering space of G , thus its kernel is the fundamental group $\pi_1(G)$. Since G is a topological group, $\pi_1(G)$ is abelian and thus

isomorphic to the first cohomology group $H_1(G, \mathbb{Z})$. Thus we have a short exact sequence of sheaves over \mathcal{O}_S :

$$0 \longrightarrow H_1(G, \mathbb{Z}) \hookrightarrow \mathcal{L}ie(G) \xrightarrow{\exp} G \longrightarrow 0$$

We can thus consider the pull-back exact sequence:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_1(G, \mathbb{Z}) & \longrightarrow & T_{\mathbb{Z}}(M) & \xrightarrow{\beta} & X \longrightarrow 0 \\ & & \parallel & \circlearrowleft & \alpha \downarrow & \circlearrowleft & \downarrow u \\ 0 & \longrightarrow & H_1(G, \mathbb{Z}) & \longrightarrow & \mathcal{L}ie(G) & \xrightarrow{\exp} & G \longrightarrow 0 \end{array}$$

Definition 3.3. The Hodge realization $T_{\mathbb{Z}}(M)$ of the 1-motive M is the fiber product of $\mathcal{L}ie(G)$ and X over G .

3.2 Equivalence of categories between $\underline{\text{MHS}}_1^{fr}$ and $\mathcal{M}_1^{fr}(\mathbb{C})$

Free 1-motives as torsion free graded polarizable mixed Hodge structures

We shall briefly focus on the case $S = \text{Spec}(\mathbb{C})$. We set

1. $W_{-1}(T_{\mathbb{Z}}(M)) := \ker(\beta) = H_1(G, \mathbb{Z})$;
2. $W_{-2}(T_{\mathbb{Z}}(M)) = H_1(T, \mathbb{Z}) = \ker(H_1(G, \mathbb{Z}) \longrightarrow H_1(A, \mathbb{Z}))$

This defines a filtration with weights. On the other hand, α can be extended to $\alpha_{\mathbb{C}}: T_{\mathbb{Z}}(M) \otimes \mathbb{C} \longrightarrow \mathcal{L}ie(G)$. Setting $F^0(T_{\mathbb{Z}}(M) \otimes \mathbb{C}) = \ker(\alpha_{\mathbb{C}})$, we have a Hodge filtration of $T_{\mathbb{C}}(M) := T_{\mathbb{Z}}(M) \otimes \mathbb{C}$.

Lemma 3.4. *The triple $T(M) = (T_{\mathbb{Z}}(M), W, F)$ is a torsion free mixed Hodge structure of type $\{(0, 0), (0, -1), (-1, 0), (-1, -1)\}$ and $\text{Gr}_{-1}^W(T(M))$ is polarizable.*

Proof. First, we consider the following diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & W_{-1}(\mathrm{T}_{\mathbb{C}}(M)) \cap F^0 & \xrightarrow{\cong} & \ker(\alpha_{A,\mathbb{C}}) & & \\
 & & \downarrow & \circlearrowleft & \downarrow & & \\
 0 & \longrightarrow & \mathrm{H}_1(T, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} & \longrightarrow & \mathrm{H}_1(G, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} & \xrightarrow{\beta} & \mathrm{H}_1(A, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \longrightarrow 0 \\
 & & \downarrow \mathrm{IR} & \circlearrowleft & \downarrow \alpha_{G,\mathbb{C}} & \circlearrowleft & \downarrow \alpha_{A,\mathbb{C}} \\
 0 & \longrightarrow & \mathcal{L}ie(T) & \longrightarrow & \mathcal{L}ie(G) & \longrightarrow & \mathcal{L}ie(A) \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

The first singular cohomology of T is a finitely generated free abelian group of rank d , while the Lie algebra $\mathcal{L}ie(T)$ is isomorphic to d copies of \mathbb{C} . Thus, we have that $\mathrm{H}_1(T, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\cong} \mathcal{L}ie(T)$ is an isomorphism, and in particular injective, so one has that $W_{-2}(\mathrm{T}_{\mathbb{C}}(M)) \cap F^0 = 0$. Thus, $(W_{-2}(\mathrm{T}_{\mathbb{C}}(M)), F)$ is a Hodge structure of type $(-1, -1)$.

Moreover, since $W_{-1}(\mathrm{T}_{\mathbb{C}}(M)) \cap F^0 \xrightarrow{\cong} \ker(\alpha_{A,\mathbb{C}})$ is an isomorphism (by the snake lemma), and in particular surjective, F induces on $\mathrm{Gr}_{-1}^W(\mathrm{T}_{\mathbb{Z}}(M)) \otimes \mathbb{C} \cong \mathrm{H}_1(A, \mathbb{Z}) \otimes \mathbb{C}$ the Hodge filtration of $\mathrm{H}_1(A, \mathbb{Z}) \otimes \mathbb{C}$: this makes $\mathrm{H}_1(A, \mathbb{Z})$ a Hodge structure of type $\{(-1, 0), (0, -1)\}$, since for any abelian scheme $X \xrightarrow{\pi_X} S$ defined over \mathbb{C} , the exponential map defines the exact sequence of sheaves over S^{an} :

$$0 \longrightarrow R_1 \pi_{X*} \mathbb{Z} \longrightarrow \mathcal{L}ie(X) \longrightarrow X \longrightarrow 0$$

whose Hodge filtration is given by the exact short sequence:

$$0 \longrightarrow \ker(\alpha) \longrightarrow R_1 \pi_{X*} \mathbb{Z} \otimes_{\mathbb{Z}} \mathcal{O}_X \xrightarrow{\alpha} \mathcal{L}ie(X) \longrightarrow 0$$

and this establishes an equivalence of categories between the abelian schemes over S and the category of the families of continuously polarizable torsion free Hodge structures, that is a local system $H_{\mathbb{Z}}$ of \mathbb{Z} -modules of finite type over S such that for all $s \in S$ there exists a Hodge structure on the fiber $(H_{\mathbb{Z}})_s$ which varies continuously with respect to s , of type $\{(-1, 0), (0, -1)\}$ such that the Hodge filtration varies holomorphically on S (see [9, 4.4.3]).

Now, let us consider the diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & W_{-1}(T_{\mathbb{C}}(M)) \cap F^0 & \longrightarrow & F^0 & \longrightarrow & V \longrightarrow 0 \\
 & & \downarrow & \circlearrowleft & \downarrow \alpha_{G,\mathbb{C}} & \circlearrowleft & \downarrow \mathbb{R} \\
 0 & \longrightarrow & H_1(G, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} & \longrightarrow & T_{\mathbb{Z}}(M) \otimes_{\mathbb{Z}} \mathbb{C} & \longrightarrow & X \otimes \mathbb{C} \longrightarrow 0 \\
 & & \downarrow & \circlearrowleft & \downarrow & & \\
 & & \mathcal{L}ie(G) & \xlongequal{\quad} & \mathcal{L}ie(G) & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

F^0 is sent to $\mathrm{Gr}_0^W(T_{\mathbb{C}}(M))$, which is thus of type $(0, 0)$.

This concludes the construction of $T(M)$, which is obviously functorial in M . \square

Torsion free graded polarizable mixed Hodge structures as free 1-motives

Let now H be a torsion free mixed Hodge structure of type $\{(0, 0), (0, -1), (-1, 0), (-1, -1)\}$ and suppose $\mathrm{Gr}_{-1}^W(H)$ to be polarized. By the remark in the previous proof, we know that the complex torus: e

$$A = \frac{H_{\mathbb{Z}} \setminus \mathrm{Gr}_{-1}^W(H_{\mathbb{C}})}{F^0 \mathrm{Gr}_{-1}^W(H_{\mathbb{C}})}$$

corresponds to an abelian variety. Let us consider now T the torus of the group of characters dual of $\mathrm{Gr}_{-2}^W(H_{\mathbb{Z}})$: $H_1(T, \mathbb{Z}) = \mathrm{Gr}_{-2}^W(H_{\mathbb{Z}})$. Then, the analytic complex group:

$$G = \frac{W_{-1}(H_{\mathbb{Z}}) \setminus W_{-1}(H_{\mathbb{C}})}{F^0 \cap W_{-1}(H_{\mathbb{C}})}$$

is an extension of A by T .

By the correspondence between the extensions of an abelian variety A by a torus T as algebraic complex groups and the extensions of an abelian variety A by a torus T as analytic complex groups we highlighted in Section 2.5, we obtain, from the analytic complex group G defined above, an extension of A by T (as algebraic groups). Set $X = \mathrm{Gr}_0^W(H_{\mathbb{Z}})$; we define the morphism $u: X \longrightarrow G$ (which yields a 1-motive) in the following way:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & W_{-1}(H_{\mathbb{Z}}) & \longrightarrow & \frac{W_{-1}(H_{\mathbb{C}})}{F^0 \cap W_{-1}(H_{\mathbb{C}})} & \longrightarrow & G \longrightarrow 0 \\
 & & \parallel & & \downarrow \mathbb{R} & & \uparrow \exists! \\
 & & & \circlearrowleft & \frac{H_{\mathbb{C}}}{F^0} & \circlearrowright & \\
 & & & & \uparrow & & \\
 0 & \longrightarrow & W_{-1}(H_{\mathbb{Z}}) & \longrightarrow & H_{\mathbb{Z}} & \longrightarrow & X \longrightarrow 0
 \end{array}$$

We can now state the main theorem of this section:

Theorem 3.5. *The association:*

$$(H, W, F) \mapsto \left[\mathrm{Gr}_0^W(H_{\mathbb{Z}}) \xrightarrow{u} \frac{W_{-1}(H_{\mathbb{Z}}) \setminus W_{-1}(H_{\mathbb{C}})}{F^0 \cap W_{-1}(H_{\mathbb{C}})} \right]$$

is functorial and is a quasi-inverse of the functor $T: [X \xrightarrow{u} G] \mapsto (T_{\mathbb{Z}}(M), W, F)$ defined above.

In particular, this construction establishes an equivalence of categories between $\underline{\mathrm{MHS}}_1^{fr}$ and $\mathcal{M}_1^{fr}(\mathbb{C})$.

Proof. It follows immediately by the definitions and the constructions of this sections. \square

This construction and proof can be extended to all graded polarizable mixed Hodge structures of type $\{(0, 0), (0, -1), (-1, 0), (-1, -1)\}$ and all 1-motives over \mathbb{C} , yielding an equivalence of categories between $\underline{\mathrm{MHS}}_1$ and $\mathcal{M}_1(\mathbb{C})$. The detailed proof of this fact (which is essentially the same presented here) can be found in [2, Proposition 1.5].

3.3 Étale and de Rham realizations of 1-motives

3.3.1 Étale realization

Let $M = [X \xrightarrow{u} G]$ be a 1-motive over S . For any graded mixed Hodge structure H of type $\{(0, 0), (0, -1), (-1, 0), (-1, -1)\}$ the filtration W of $H_{\mathbb{Z}}$ defines a filtration W on H of mixed Hodge substructures. Thus, one has that if $M = (X, A, T, G, u)$ is given by an extension G of an abelian variety A and an algebraic torus T together with a morphism of group schemes $X \xrightarrow{u} G$ (seen as a complex of

group schemes in degrees 0 and 1), the filtration W corresponds to:

$$\begin{cases} W_i(M) = M & \text{if } i \geq 0 \\ W_i(M) = G \text{ (i.e. } W_i(M) = (\{0\}, A, T, G, 0)) & \text{if } i = -1 \\ W_i(M) = T \text{ (i.e. } W_i(M) = (\{0\}, T, T, \{0\}, 0)) & \text{if } i = -2 \\ W_i(M) = 0 & \text{if } i < -2 \end{cases}$$

and one has trivially:

$$\begin{cases} \mathrm{Gr}_i(W) = X & \text{if } i = 0 \\ \mathrm{Gr}_i(W) = A & \text{if } i = -1 \\ \mathrm{Gr}_i(W) = T & \text{if } i = -2 \end{cases}$$

What we want to prove in this section is that for any 1-motive M defined over \mathbb{C}^1 , we have an étale realization $\hat{T}(M)$, and we shall prove via purely algebraic tools that:

$$\hat{T}(M) = T_{\mathbb{Z}}(M) \otimes_{\mathbb{Z}} \hat{\mathbb{Z}} = \prod_{\ell} T_{\mathbb{Z}}(M) \otimes_{\mathbb{Z}} \mathbb{Z}_{\ell}$$

For all integers $n > 0$, consider the complex of abelian groups $[\mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z}]$ (in degrees -1 and 0) and set $T_{\frac{\mathbb{Z}}{n\mathbb{Z}}}(M)$ to be the 0-th cohomology group of the complex:

$$[X \xrightarrow{u} G] \otimes_{\mathbb{Z}} [\mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z}]$$

that is, the 0-th cohomology group of the complex:

$$\begin{bmatrix} X & \xrightarrow{u} & G \\ \downarrow n & & \downarrow -n \\ X & \xrightarrow{u} & G \end{bmatrix}$$

By definition, we obtain that:

$$T_{\frac{\mathbb{Z}}{n\mathbb{Z}}}(M) = \frac{\{(x, g \text{ such that } u(x) = ng)\}}{\{(nx, u(x) \text{ such that } x \in X)\}}$$

In the derived category, one has:

$$T_{\frac{\mathbb{Z}}{n\mathbb{Z}}}(M) = H^0 \left(M \otimes \frac{\mathbb{Z}}{n\mathbb{Z}} \right)$$

¹In fact, defined over any algebraically closed field k of characteristic 0.

For $n = md$, we define the transition morphisms $\phi_{m,n}: T_{\frac{\mathbb{Z}}{n\mathbb{Z}}}(M) \longrightarrow T_{\frac{\mathbb{Z}}{m\mathbb{Z}}}(M)$ by $(x, g) \mapsto (x, dg)$. In this way, the $T_{\frac{\mathbb{Z}}{n\mathbb{Z}}}(M)$'s form a direct system and we can take the direct limit:

$$\hat{T}(M) := \varinjlim_n T_{\frac{\mathbb{Z}}{n\mathbb{Z}}}(M)$$

The filtration W on $T_{\mathbb{Z}}(M)$ described above induces naturally a filtration W on both $T_{\frac{\mathbb{Z}}{n\mathbb{Z}}}(M)$ and $\hat{T}(M)$. In particular:

$$\begin{aligned} \mathrm{Gr}_0^W(\hat{T}(M)) &= X \otimes_{\mathbb{Z}} \hat{\mathbb{Z}} \\ \mathrm{Gr}_{-1}^W(\hat{T}(M)) &= \varinjlim_n A_n = \hat{A} \\ \mathrm{Gr}_{-2}^W(\hat{T}(M)) &= \varinjlim_n T_n = Y \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}(1) \end{aligned}$$

where Y denotes the dual of the group of characters of $T = W_{-2}(M)$.

Proposition 3.6. *Let be $M = [X \xrightarrow{u} G]$ a 1-motive defined over \mathbb{C} . Then:*

$$\hat{T}(M) \cong T_{\mathbb{Z}}(M) \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$$

Proof. In fact, the natural morphism $[T_{\mathbb{Z}}(M) \longrightarrow \mathcal{L}ie(G)] \longrightarrow [X \longrightarrow G]$ is a quasi-isomorphism, and the quasi isomorphisms:

$$\left[\begin{array}{ccc} X & \xrightarrow{u} & G \\ \downarrow n & & \downarrow -n \\ X & \xrightarrow{u} & G \end{array} \right] \longleftarrow \left[\begin{array}{ccc} T_{\mathbb{Z}} & \xrightarrow{u} & \mathcal{L}ie(G) \\ \downarrow n & & \downarrow -n \downarrow \mathbb{R} \\ T_{\mathbb{Z}} & \xrightarrow{u} & \mathcal{L}ie(G) \end{array} \right] \longrightarrow \left[\begin{array}{ccc} T_{\mathbb{Z}} & \xrightarrow{e} & 0 \\ \downarrow n & & \downarrow 0 \\ T_{\mathbb{Z}} & \xrightarrow{e} & 0 \end{array} \right]$$

yield the isomorphisms $T_{\frac{\mathbb{Z}}{n\mathbb{Z}}}(M) \cong \frac{T_{\mathbb{Z}}}{n T_{\mathbb{Z}}}$ which maps an element $t \in W_{-1}(T_{\mathbb{Z}})$ to $\exp(\frac{t}{n}) \in G_n$. By passing to the limit, from this isomorphism it follows our claim. \square

3.3.2 De Rham realization

Consider now a 1-motive $M = [X \xrightarrow{u} G]$ defined over a base scheme S . We want to construct a locally free \mathcal{O}_S -module $T_{\mathrm{dR}}(M)$ endowed with filtrations W and F . In this section we shall use (and generalize) some results we have proved in Chapter 2.

Again, we consider our 1-motive as a complex of group schemes in degrees 0 and -1 .

Definition 3.7. A \mathbb{G}_a -extension of M is an extension (as complexes of group

schemes) of M by a vector group (considered as a complex concentrated in degree 0) which is universal (in the sense of Definition 2.3) among all the extensions of M by vector groups.

We want to prove that any semi-abelian group G admits a universal extension. This will allow us to prove the remarkable result that any 1-motive (over any base S) admits a universal \mathbb{G}_a -extension.

Proposition 3.8. *For any semi-abelian group G over any base S , the universal extension $E(G)$ of G exists and is isomorphic to $E(A) \times_A G$.*

Proof. By Proposition 2.4, any semi-abelian group G admits a universal extension by the vector group $\mathcal{E}xt_S^1(G, \mathbb{G}_a)^\vee$ if G is such that $\mathcal{H}om_S(G, \mathbb{G}_a) = 0$ and $\mathcal{E}xt_S^1(G, \mathbb{G}_a)$ is a vector space of locally finite rank. Consider the exact sequence:

$$0 \longrightarrow T \longrightarrow G \longrightarrow A \longrightarrow 0$$

defining G as an extension of an abelian variety A by an algebraic torus T , and apply to this sequence the functor $\mathcal{H}om_S(-, \mathbb{G}_{a,S})$ to obtain the following long exact sequence:

$$\begin{array}{ccccccc} 0 \longrightarrow \mathcal{H}om_S(A, \mathbb{G}_{a,S}) & \longrightarrow & \mathcal{H}om_S(G, \mathbb{G}_{a,S}) & \longrightarrow & \mathcal{H}om_S(T, \mathbb{G}_{a,S}) & \longrightarrow & \\ & & & & \downarrow & & \\ & & & & \mathcal{E}xt_S^1(A, \mathbb{G}_{a,S}) & \longrightarrow & \mathcal{E}xt_S^1(G, \mathbb{G}_{a,S}) \longrightarrow \mathcal{E}xt_S^1(T, \mathbb{G}_{a,S}) \longrightarrow \cdots \end{array}$$

By what we have seen in Section 2, we know that $\mathcal{H}om_S(A, \mathbb{G}_{a,S}) = 0$ since $\mathbb{G}_{a,S}$ is a quasi-coherent locally free \mathcal{O}_S -module, but also $\mathcal{H}om_S(T, \mathbb{G}_{a,S}) = 0$ (since $\mathcal{H}om_S(\mathbb{G}_{m,S}, \mathbb{G}_{a,S})$ is trivial), so $\mathcal{H}om_S(G, \mathbb{G}_{a,S}) = 0$. Moreover, $\mathcal{E}xt_S^1(T, \mathbb{G}_{a,S}) = 0$ (cfr. [9, 10.1.7.b]), and in particular we have that $\mathcal{E}xt_S^1(A, \mathbb{G}_{a,S})$ (which is locally free of finite rank) is isomorphic to $\mathcal{E}xt_S^1(G, \mathbb{G}_{a,S})$ by the map that sends an extension E of A by $\mathbb{G}_{a,S}$ to the extension $E \times_A G$ of G by $\mathbb{G}_{a,S}$. In particular, we have that the universal extension $E(G)$ of G by $\mathbb{G}_{a,S}$ exists and is isomorphic to $E(A) \times_A G$. \square

Let us remark that the extensions of M by $\mathbb{G}_{a,S}$ do not have automorphisms, so $\mathcal{H}om_S(M, \mathbb{G}_{a,S})$ is trivial (since $\mathcal{H}om_S(X, \mathbb{G}_{a,S})$ and $\mathcal{H}om_S(G, \mathbb{G}_{a,S})$ are) and $\mathcal{E}xt_S^1(M, \mathbb{G}_{a,S})$ is a finite dimensional locally free \mathcal{O}_S -module. since $\mathcal{E}xt_S^1(X, \mathbb{G}_{a,S}) = 0$. This allows us to say that, over any base S , there exists a universal extension (which we shall denote $M^\natural = [X \xrightarrow{u^\natural} G^\natural]$) of M by the vector group $\mathcal{E}xt_S^1(M, \mathbb{G}_a)^\vee$, considered as a complex concentrated in degree 0:

$$\begin{array}{ccccccc}
 & & 0 & \longrightarrow & X & \xlongequal{\quad} & X \\
 & & \downarrow & & \downarrow & \circlearrowleft & \downarrow \\
 0 & \longrightarrow & \mathcal{E}xt_S^1(M, \mathbb{G}_a)^\vee & \longrightarrow & G^\natural & \longrightarrow & G \longrightarrow 0
 \end{array}$$

In particular we have that the extension of G by $\mathcal{E}xt_S^1(M, \mathbb{G}_a)^\vee$ is the extension of G obtained via pull-back along the inclusion $\mathcal{E}xt_S^1(G, \mathbb{G}_a)^\vee \hookrightarrow \mathcal{E}xt_S^1(M, \mathbb{G}_a)^\vee$ by the universal extension of G :

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathcal{E}xt_S^1(G, \mathbb{G}_a)^\vee & \longrightarrow & E(G) & \longrightarrow & G \longrightarrow 0 \\
 & & \downarrow & & \downarrow & \circlearrowleft & \parallel \\
 0 & \longrightarrow & \mathcal{E}xt_S^1(M, \mathbb{G}_a)^\vee & \longrightarrow & G^\natural & \longrightarrow & G \longrightarrow 0
 \end{array}$$

Remark 3.9. In general we do not have that G^\natural is the same universal extension $E(G)$ of G , unless $X = 0$.

Definition 3.10. The de Rham realization of the 1-motive M over S is $T_{\text{dR}}(M) := \mathcal{L}ie(G^\natural)$ with the Hodge filtration given by $F^0 T_{\text{dR}}(M) = \ker(\mathcal{L}ie(G^\natural) \rightarrow \mathcal{L}ie(G)) \cong \mathcal{E}xt_k^1(M, \mathbb{G}_a)^\vee$.

T_{dR} and M^\natural are constructions which are functorial in M , and we define the filtration W of $T_{\text{dR}}(M)$ by taking the filtration induced by the filtration W of M .

What we shall prove now is the comparison isomorphism that relates the Hodge realization $T_{\mathbb{Z}}(M)$ and the de Rham realization $T_{\text{dR}}(M)$ of a 1-motive M , that holds in all generality for any 1-motive M over \mathbb{C} .

Proposition 3.11. *Let M be a 1-motive over \mathbb{C} . Then:*

$$(T_{\text{dR}}(M), W, F) \cong (T_{\mathbb{C}}(M), W, F)$$

where $T_{\mathbb{C}}(M) := T_{\mathbb{Z}}(M) \otimes_{\mathbb{Z}} \mathbb{C}$.

Proof. $\mathcal{E}xt_{\mathbb{C}}^i(X, \mathbb{G}_a)$, $\mathcal{E}xt_{\mathbb{C}}^i(A, \mathbb{G}_a)$ and $\mathcal{E}xt_{\mathbb{C}}^i(T, \mathbb{G}_a)$, by what we have seen in Section 2.5, can be considered in both an analytic and an algebraic context which are equivalent. Consider the morphism $T_{\mathbb{C}}(M) = T_{\mathbb{Z}}(M) \otimes_{\mathbb{Z}} \mathbb{C} \rightarrow \mathcal{L}ie(G) \rightarrow G$. We have the diagram:

$$\begin{array}{ccc}
 X & \xrightarrow{\quad} & \frac{T_{\mathbb{C}}(M)}{H_1(G)} \\
 \parallel & \circlearrowleft & \downarrow \\
 X & \longrightarrow & G
 \end{array}$$

which defines an extension of M^{an} by the vector group $F^0 T_{\mathbb{C}}(M)$, and thus an extension of M by that algebraic vector group. We only need to prove that this extension is universal: it suffices to show that the category of extensions of $\left[X \longrightarrow \frac{T_{\mathbb{C}}(M)}{H_1(G)} \right]$ by \mathbb{G}_a is trivial, that is it consists of one object (up to isomorphism). This is equivalent to the category of the extensions of $\frac{T_{\mathbb{C}}(M)}{T_{\mathbb{Z}}(M)}$ by \mathbb{G}_a and in fact we have:

$$\mathcal{E}xt_{\mathbb{C}}^i(T_{\mathbb{C}}(M), \mathbb{G}_a) = 0$$

for $i = 0, 1$. □

Remark 3.12. Of course, an abelian variety over a field k is naturally identified with the 1-motive defined by $M = (0, A, 0, A, 0)$. In this setting then $\mathcal{E}xt_k^1(M, \mathbb{G}_a) \cong \mathcal{E}xt_k^1(A, \mathbb{G}_a)$ and then $\mathcal{E}xt_k^1(A, \mathbb{G}_a)^{\vee}$ is precisely the vector group $V(A)$ of Proposition 2.4. So $\mathcal{L}ie(A^{\natural}) \cong \mathcal{L}ie(E^{\natural})$ where E^{\natural} is the universal extension (in the sense of Chapter 2) of the dual abelian variety A^* , and we know that this is isomorphic to the first de Rham cohomology vector space of the abelian variety. Moreover, by the above construction, $T_{\mathbb{Z}}(A)$ is precisely the first cohomology group $H^1(A, \mathbb{Z})$.

In this way, we can immediately conclude that for abelian varieties the usual first de Rham cohomology vector space is precisely the vector space detected by the de Rham realization of the associated 1-motive, and by the isomorphism $T_{dR}(M) \cong T_{\mathbb{Z}}(M) \otimes_{\mathbb{Z}} \mathbb{C}$, we have the usual comparison isomorphism:

$$H_{dR}^1(A) \cong H^1(A, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$$

3.4 Picard 1-motive

In this section, we shall focus on 1-motives over a field k of characteristic 0, in order to define the generalization in the 1-motivic setting of the relative Picard variety of an arbitrary algebraic variety X over k . Its de Rham realization will be strongly related to the first de Rham cohomology vector space of the starting algebraic variety X (see also [3]).

3.4.1 Simplicial Picard functor

Let $\pi: V_{*} \longrightarrow S$ be a simplicial scheme over S (that is, a simplicial object in the category of schemes, in the sense of Definition C.26), with morphisms $d_k^i: V_i \longrightarrow$

V_{i-1} . Denote with $\text{Pic}(V_*)$ the group of isomorphism classes of simplicial line bundles on V_* (that is, invertible \mathcal{O}_{V_*} -modules).

Proposition 3.13. *The elements of $\text{Pic}(V_*)$ isomorphically correspond to isomorphism classes of line bundles \mathcal{L} on V_0 , together with an isomorphism $\alpha: (d_0^1)^*\mathcal{L} \xrightarrow{\cong} (d_1^1)^*\mathcal{L}$ satisfying the following cocycle condition: the composite:*

$$((d_1^2)^*(\alpha))^{-1} \circ ((d_2^2)^*(\alpha)) \circ ((d_0^2)^*(\alpha))$$

yields the identity $1 \in \Gamma(V_2, \mathbb{G}_{m,S})$.

Moreover, there is a functorial isomorphism:

$$\text{Pic}(V_*) \cong \mathbb{H}^1(V_*, \mathcal{O}_{V_*}^*)$$

Proof. [3, Proposition 4.1.1]. □

Definition 3.14. The simplicial relative Picard functor $\mathcal{P}ic_{V_*/S}$ on the category of schemes over S is the sheafification of the functor defined by $T \mapsto \text{Pic}(V_* \times_S T)$ with respect to the fpqc topology.

This means in particular that if $\pi: V_* \times_S T \rightarrow T$ is the structural morphism, then:

$$\mathcal{P}ic_{V_*/S}(T) \cong H_{fpqc}^0(T, R^1 \pi_* (\mathcal{O}_{V_* \times_S T}^*))$$

Now let us focus on the case in which the base scheme S is the spectrum of a field k .

Lemma 3.15. *If X_* is a simplicial scheme which is proper and smooth over k then the functor $\mathcal{P}ic_{V_*/k}$ is representable by a group scheme locally of finite type over k (that is, a k -group variety).*

Proof. [3, section A.3]. □

3.4.2 Cohomological Picard 1-motive of an algebraic variety

Let X_* be a smooth simplicial k -variety, regarded as an open Zariski subset of a proper smooth simplicial scheme \overline{X}_* such that the complement $Y_* = \overline{X}_* \setminus X_*$ has components Y_i which are normal crossing divisors in \overline{X}_i .

We have a spectral sequence:

$$E_1^{p,q} := H_{Y_p}^q(\overline{X}_p, \mathcal{O}_{\overline{X}_p}^*) \implies \mathbb{H}_{Y_*}^{p+q}(\overline{X}_*, \mathcal{O}_{\overline{X}_*}^*)$$

In the above formula, $H_{Y_p}^q(\overline{X}_p, \mathcal{O}_{\overline{X}_p}^*) = R^q(\Gamma_{Y_p}(\overline{X}_p, \mathcal{O}_{\overline{X}_p}^*))$, where $\Gamma_{Y_p}(\overline{X}_p, \mathcal{O}_{\overline{X}_p}^*)$ is the abelian sheaf $\left\{s \in \Gamma(\overline{X}_p, \mathcal{O}_{\overline{X}_p}^*) \text{ such that } \text{supp}(s) \in Y_p\right\}$, denotes the cohomology with support in Y_p .

By our assumption, every component of \overline{X}_* is smooth, and thus $H_{Y_p}^q(\overline{X}_p, \mathcal{O}_{\overline{X}_p}^*) \neq 0$ if and only if $q = 1$, in which case one has:

$$H_{Y_p}^1(\overline{X}_p, \mathcal{O}_{\overline{X}_p}^*) \cong \text{Div}_{Y_p}(\overline{X}_p)$$

where $\text{Div}_{Y_p}(\overline{X}_p)$ is the group of divisors D on \overline{X} supported on Y_p . So our spectral sequence yields that:

$$\mathbb{H}_{Y_*}^1(\overline{X}_*, \mathcal{O}_{\overline{X}_*}^*) \cong \ker\left(\text{Div}_{Y_0}(\overline{X}_0) \xrightarrow{d_0^* - d_1^*} \text{Div}_{Y_1}(\overline{X}_1)\right)$$

We denote with $\mathcal{D}iv_{Y_*}(\overline{X}_*)$ the sheafification of the group functor on $\mathcal{S}ch/k$ that associates to any k -variety T the subgroup of divisors on $\overline{X}_{0,T}$ given by:

$$\ker\left(\text{Div}_{Y_{0,T}}(\overline{X}_{0,T}) \xrightarrow{d_0^* - d_1^*} \text{Div}_{Y_{1,T}}(\overline{X}_{1,T})\right)$$

We have a canonical morphism:

$$\mathcal{D}iv_{Y_*}(\overline{X}_*) = \mathbb{H}_{Y_*}^1(\overline{X}_*, \mathcal{O}_{\overline{X}_*}^*) \longrightarrow \mathbb{H}^1(\overline{X}_*, \mathcal{O}_{\overline{X}_*}^*) \cong \mathcal{P}ic_{X_*/k}$$

Denote with $\mathcal{P}ic^0(\overline{X}_*)$ the identity component $\mathcal{P}ic_{X_*/k}^0 \subseteq \mathcal{P}ic_{X_*/k}$, and denote with $\mathcal{D}iv_{Y_*}^0(\overline{X}_*)$ the inverse image of $\mathcal{P}ic^0(\overline{X}_*)$ via the canonical morphism above.

Definition 3.16 (Cohomological Picard 1-motive). Let X be an algebraic variety over k , and consider X_* a proper smooth hypercovering such that \overline{X}_* is in every component the complement in a proper smooth compact simplicial scheme \overline{X}_* of Y_* which is in every component a normal crossing divisor. Then the cohomological Picard 1-motive of X is:

$$\text{Pic}^+(X) := \left[\mathcal{D}iv_{Y_*}^0(\overline{X}_*) \longrightarrow \mathcal{P}ic_{X_*/k}^0\right]$$

Remark 3.17. This definition makes sense, since $\mathcal{D}iv_{Y_*}^0(\overline{X}_*)$ is a \mathbb{Z} -module of finite type (being abelian and finitely generated) and $\mathcal{P}ic_{X_*/k}^0$ is a semi-abelian variety ([3, Proposition 4.1.3]).

Theorem 3.18. Let X be an algebraic variety defined over \mathbb{C} . Then the Hodge realization $T_{\mathbb{Z}}(\text{Pic}^+(X))$ is isomorphic to $H^1(X)(1)$, where the twist is the Tate

twist.

Sketch of proof. We have an exact sequence:

$$0 \longrightarrow \mathbb{H}^1(X_*, \mathbb{Z}(1)) \longrightarrow H^1(X, \mathbb{Z}(1)) \longrightarrow \mathcal{D}iv_{Y_*}^0(\overline{X}_*) \longrightarrow 0$$

By universal cohomological descent (see Section C.3 in Appendix C) we have that $H^1(X, \mathbb{Z}(1)) \cong \mathbb{H}^1(X_*, \mathbb{Z}(1)_*)$, and thus the result follows from [3, Lemma 4.3.1]. \square

3.4.3 de Rham realization of Pic^+

Let k be a field of characteristic 0. Following the notations introduced in 1.4, for any simplicial k -scheme X_* we shall denote by $\mathrm{Pic}^{\natural}(X_*)$ the group of isomorphism classes of pairs $(\mathcal{L}_*, \nabla_*)$ where \mathcal{L}_* is a simplicial line bundle over X and ∇_* is a simplicial integrable connection (that is, an integrable connection in the sense of Section B for the simplicial k -scheme X_*):

$$\nabla_*: \mathcal{L}_* \longrightarrow \mathcal{L}_* \otimes_{\mathcal{O}_{X_*}} \Omega_{X_*/k}^1$$

Definition 3.19. The simplicial relative \natural -Picard scheme $\mathcal{P}ic_{X_*/k}^{\natural}$ is the sheafification (with respect to the fpqc topology) of the functor on $\mathcal{S}ch/k$ defined by:

$$T \mapsto \mathrm{Pic}^{\natural}(X_* \times_k T)$$

For a given pair $(\mathcal{L}_*, \nabla_*)$ on X_* one has clearly a line bundle \mathcal{L} and an integrable connection ∇ on X_0 , together with an isomorphism:

$$\alpha: d_0^*(\mathcal{L}, \nabla) \xrightarrow{\cong} d_1^*(\mathcal{L}, \nabla)$$

such that α satisfies the cocycle condition described in Proposition 3.13. From that proposition, together to a simplicial revisit of the construction in Section 1.4 (see also [9, Construction 10.3.10]), we have the following result:

Proposition 3.20. *For any simplicial scheme X_* over a field of characteristic 0, the elements in $\mathcal{P}ic_{X_*/k}^{\natural}(k) = \mathrm{Pic}^{\natural}(X_*)$ are in natural bijection with isomorphism classes of triples $(\mathcal{L}, \nabla, \alpha)$, with \mathcal{L} invertible sheaf on X_0 , ∇ an integrable connection on \mathcal{L} and α an isomorphism between $d_0^*(\mathcal{L}, \nabla)$ and $d_1^*(\mathcal{L}, \nabla)$ satisfying the cocycle condition of Proposition 3.13. We have a functorial isomorphism:*

$$\mathrm{Pic}^{\natural}(X_*) \cong \mathbb{H}^1\left(X_*, \left[\mathcal{O}_{X_*}^* \xrightarrow{\mathrm{dlog}} \Omega_{X_*/k}^1\right]\right)$$

From the above proposition, together with the exact sequence of complexes of simplicial sheaves:

$$0 \longrightarrow \Omega_{X_*/k}^1[-1] \longrightarrow \left[\mathcal{O}_{X_*}^* \xrightarrow{\text{dlog}} \Omega_{X_*/k}^1 \right] \longrightarrow \mathcal{O}_{X_*}^*[0] \longrightarrow 0$$

by analogous arguments to those we did in 2.4.1 we obtain the following exact sequence of fpqc sheaves:

$$0 \longrightarrow \mathbb{H}^0(X_*, \Omega_{X_*/k}^1) \longrightarrow \mathcal{P}ic_{X_*/k}^{\natural} \longrightarrow \mathcal{P}ic_{X_*/k} \longrightarrow \mathbb{H}^1(X_*, \Omega_{X_*/k}^1)$$

Again, we have that the semi-abelian variety $\mathcal{P}ic_{X_*/k}^0$ is mapped to 0 in $\mathbb{H}^1(X_*, \Omega_{X_*/k}^1)$ using the fact that X_* is smooth and proper over k . By pulling back along the inclusion $\mathcal{P}ic^0 \hookrightarrow \mathcal{P}ic$ we obtain an extension:

$$0 \longrightarrow \mathbb{H}^0(X_*, \Omega_{X_*/k}^1) \longrightarrow \left(\mathcal{P}ic_{X_*/k}^{\natural} \right)^0 \longrightarrow \mathcal{P}ic_{X_*/k}^0 \longrightarrow 0$$

that is the \mathbb{G}_a -extension of the semi-abelian scheme $\mathcal{P}ic_{X_*/k}^0$ since $\left(\mathcal{P}ic_{A/S}^{\natural} \right)^0 \cong \left(\mathcal{P}ic_{A/S}^0 \right)^{\natural}$ (see [3, Lemma 4.5.2]).

More generally, let X_* be a smooth simplicial k -variety in characteristic 0, and consider a smooth compactification \overline{X}_* with normal crossing boundary divisor Y_* . We define $\text{Pic}^{\natural-\log}(X_*)$ as the group of isomorphism classes $(\mathcal{L}_*, \nabla_*^{\log})$ with \mathcal{L}_* simplicial line bundle on \overline{X}_* and ∇_*^{\log} a simplicial integrable connection on \mathcal{L} with logarithmic poles along Y_* . Equivalently, ∇_*^{\log} is an isomorphism:

$$\mathcal{L}_* \xrightarrow{\cong} \mathcal{L}_* \otimes_{\mathcal{O}_{X_*}} \Omega_{X_*/k}^1(\log Y_*)$$

satisfying the Leibniz product rule. We sheafify the functor:

$$T \mapsto \text{Pic}^{\natural-\log}(X_* \times_k T)$$

to obtain $\mathcal{P}ic_{X_*/k}^{\natural-\log}$, which obviously injects into $\mathcal{P}ic_{X_*/k}^{\natural}$.

It is clear, by a natural generalization of Proposition 3.20, that we have the following result:

Proposition 3.21. *For any k -simplicial smooth variety X_* with smooth compactification \overline{X}_* with normal crossing boundary divisor Y_* , the elements in $\mathcal{P}ic_{X_*/k}^{\natural-\log}(k) = \text{Pic}^{\natural-\log}(X_*)$ are in natural bijection with isomorphism classes of triples $(\mathcal{L}, \nabla^{\log}, \alpha)$, with \mathcal{L} invertible sheaf on \overline{X}_0 , ∇^{\log} an integrable connection on \mathcal{L} and α an iso-*

morphism between $d_0^*(\mathcal{L}, \nabla^{\log})$ and $d_1^*(\mathcal{L}, \nabla^{\log})$ satisfying the cocycle condition of Proposition 3.13. We have a functorial isomorphism:

$$\mathrm{Pic}^{\natural-\log}(X_*) \cong \mathbb{H}^1 \left(X_*, \left[\mathcal{O}_{\overline{X}_*}^* \xrightarrow{\mathrm{dlog}} \Omega_{\overline{X}_*/k}^1(\log Y_*) \right] \right)$$

We are now ready the main theorem of this section:

Theorem 3.22. *Let X be a k -variety, with k field of characteristic 0. Then one has:*

$$\mathrm{T}_{\mathrm{dR}}(\mathrm{Pic}^+(X)) \cong \mathrm{H}_{\mathrm{dR}}^1(X)(1)$$

Proof. Choose a smooth hypercovering X_* of X and a smooth compactification \overline{X}_* of it with normal crossing boundary divisor Y_* and consider the cohomological Picard 1-motive $\mathrm{Pic}^+(X) = [\mathcal{D}iv_{Y_*}^0(\overline{X}_*) \rightarrow \mathcal{P}ic_{\overline{X}_*/k}^0]$. We have the following exact diagram of complexes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_{\overline{X}_*}^* & \xlongequal{\quad} & \mathcal{O}_{\overline{X}_*}^* & \longrightarrow & 0 \\ & & \mathrm{dlog} \downarrow & \circlearrowleft & \downarrow \mathrm{dlog} & \circlearrowleft & \downarrow \\ 0 & \longrightarrow & \Omega_{\overline{X}_*/k}^1 & \longrightarrow & \Omega_{\overline{X}_*/k}^1(\log Y_*) & \longrightarrow & \mathcal{Q}_* \longrightarrow 0 \end{array}$$

where \mathcal{Q}_* denotes the cokernel of the inclusion of $\Omega_{\overline{X}_*/k}^1$ in $\Omega_{\overline{X}_*/k}^1(\log Y_*)$. Thus, we have the following diagram obtained by the push-out along the morphism induced by the dlog map in cohomology of the \mathbb{G}_a -extension of $\mathcal{P}ic_{X_*/k}^0$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{H}^0(\overline{X}_*, \Omega_{\overline{X}_*/k}^1) & \longrightarrow & (\mathcal{P}ic_{X_*/k}^{\natural})^0 & \longrightarrow & \mathcal{P}ic_{X_*/k}^0 \longrightarrow 0 \\ & & \downarrow & \circlearrowleft & \downarrow & \circlearrowleft & \parallel \\ 0 & \longrightarrow & \mathbb{H}^0(\overline{X}_*, \Omega_{\overline{X}_*/k}^1(\log Y_*)) & \longrightarrow & (\mathcal{P}ic_{X_*/k}^{\natural-\log})^0 & \longrightarrow & \mathcal{P}ic_{X_*/k}^0 \longrightarrow 0 \end{array}$$

Proving that $\mathbb{H}^0(\overline{X}_*, \Omega_{\overline{X}_*/k}^1(\log Y_*))$ is isomorphic to $\mathrm{Ext}_k^1(\mathrm{Pic}^+(X), \mathbb{G}_a)^\vee$ then completes the proof. In fact, $\mathrm{T}_{\mathrm{dR}}(\mathrm{Pic}^+(X))$ is the Lie algebra of the semi-abelian variety in degree 0 of the universal extension of $\mathrm{Pic}^+(X)$, and the isomorphism above would prove that the universal extension of $\mathrm{Pic}^+(X)$ is the 1-motive:

$$\mathrm{Pic}^+(X)^{\natural} := \left[\mathcal{D}iv_{Y_*}^0(\overline{X}_*) \xrightarrow{u^{\natural}} (\mathcal{P}ic_{X_*/k}^{\natural-\log})^0 \right]$$

The Lie algebra of $(\mathcal{P}ic_{X_*/k}^{\natural-\log})^0$ is canonically isomorphic to $\mathbb{H}^1(X_*, \Omega_{\overline{X}_*/k}^*(\log Y_*))$ since $(\mathcal{P}ic_{X_*/k}^{\natural-\log})^0$ is representable by the pull-back of the universal extension of the

abelian quotient of $\mathcal{P}ic_{X_*/k}^0$.

Its Lie algebra is canonically isomorphic to $\mathbb{H}^1(\overline{X}_*, \mathcal{O}_{\overline{X}_*} \rightarrow \Omega_{\overline{X}_*/k}^1(\log Y_*))$. Such isomorphism is compatible with the Hodge structures if we shift the index of the filtration on $H_{\text{dR}}^1(X)$ by 1, and this proves our claim (see also [3, Lemma 4.3.1]).

Denote by \mathcal{K} the kernel sheaf $\ker(\mathbb{H}^0(\overline{X}_*, \mathcal{Q}_*) \rightarrow \mathbb{H}^1(\overline{X}_*, \mathcal{Q}_*))$ and consider the exact sequence (where the vertical maps are induced by universal properties):

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathcal{E}xt_k^1(\mathcal{P}ic_{X_*/k}^0, \mathbb{G}_a)^\vee & \longrightarrow & \mathcal{E}xt_k^1(\text{Pic}^+(X), \mathbb{G}_a)^\vee & \longrightarrow & \mathcal{H}om_k(\mathcal{D}iv_{Y_*}^0(\overline{X}), \mathbb{G}_a)^\vee & \longrightarrow 0 \\ & \downarrow \cong & & \downarrow & & \downarrow & \\ 0 \longrightarrow & \mathbb{H}^0(\overline{X}_*, \Omega_{X_*/k}^1) & \longrightarrow & \mathbb{H}^0(\overline{X}_*, \Omega_{\overline{X}_*/k}^1(\log Y_*)) & \longrightarrow & \mathcal{K} & \longrightarrow 0 \end{array}$$

The first vertical map is an isomorphism, so by proving that also the third one is an isomorphism we can conclude by the five lemma. For any index i , Y_i in X_i is a normal crossing divisor, and we have an exact sequence (compatible with the face and degeneracy morphisms of the simplicial scheme \overline{X}_*):

$$0 \longrightarrow \Omega_{\overline{X}_i/k}^1 \longrightarrow \Omega_{\overline{X}_i/k}^1(\log Y_i) \longrightarrow \oplus_{i,j} \mathcal{O}_{Y_{i,j}} \longrightarrow 0$$

where $Y_{i,j}$ is the j -th smooth component of Y_i .

By this construction we have a canonical identification:

$$\mathbb{H}^0(\overline{X}_*, \mathcal{Q}_*) = \ker \left(\bigoplus_{0,j} \mathbb{H}^0(Y_{0,j}, \mathcal{O}_{Y_{0,j}}) \rightarrow \bigoplus_{1,j} \mathbb{H}^0(Y_{1,j}, \mathcal{O}_{Y_{1,j}}) \right) \cong \mathcal{D}iv_{Y_*}(\overline{X}_*) \otimes k$$

Then, $\mathcal{D}iv_{Y_*}^0(\overline{X}_*) \otimes k = \ker(\mathcal{D}iv_{Y_*}(\overline{X}_*) \otimes k \rightarrow \mathbb{H}^1(\overline{X}_*, \Omega_{\overline{X}_*/k}^1))$, and since we have that $\mathcal{D}iv_{Y_*}^0(\overline{X}_*) \otimes k \cong \mathcal{H}om(\mathcal{D}iv_{Y_*}^0(\overline{X}), \mathbb{G}_a)^\vee$, the proof is complete. \square

Putting together the results in Proposition 3.11 and Theorems 3.18 and 3.22, we easily obtain the following result.

Corollary 3.23. *For any complex algebraic variety X , there is a comparison isomorphism:*

$$H_{\text{dR}}^1(X) \cong H^1(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$$

Remark 3.24. By proving Theorem 3.22, we have in a purely algebraic way that the Hodge filtration of $H_{\text{dR}}^1(X)$ does not depend on the choice of the hypercovering X_* and of its compactification \overline{X}_* , since $\text{Pic}^+(X)$ does not depend on such choices. This is proved by showing that $T_{\text{ét}}(\text{Pic}^+(X)) \cong H_{\text{ét}}^1(X, \mathbb{Z}(1))$ ([3, Theorem 4.4.3]) and the fact that $T_{\text{ét}}$ is a faithful functor from $\mathcal{S}ch/k$ that reflects isomorphisms ([3,

Proposition 1.3.1]), together with the fact that pull-backing cycles and simplicial line bundles of two different hypercoverings and compactifications in a third hypercovering and compactification which maps to them, one obtains an isomorphism of the étale realizations of the two $\mathrm{Pic}^+(X)$ motives, and thus the isomorphism of the two motives themselves.

Appendix A

Fundamentals of algebraic geometry

This chapter collects some useful tools of algebraic geometry which are employed extensively in our dissertation, such as the definitions and main properties of group and abelian schemes or the main motivations and results of descent theory.

The most elementary definitions and constructions of the geometry of schemes will be omitted, but we follow the standard notations of algebraic geometry literature (see [15] or [18] for example).

A.1 Group and abelian schemes over an arbitrary base

A.1.1 Definitions

In this section, we shall introduce the concepts of group scheme over a base scheme S .

Definition A.1. A group scheme G over a base scheme S is a group object in the category $\mathcal{S}ch/S$, that is an S -scheme with:

1. an S -morphism $m: G \times G \longrightarrow G$;
2. an S -morphism $i: G \longrightarrow G$;
3. an S -point $e: S \longrightarrow G$;

satisfying the axioms:

1. (associativity) the following diagram commutes:

$$\begin{array}{ccc}
G \times_S G \times_S G & \xrightarrow{m \times_S id_G} & G \times_S G \\
id_G \times_S m \downarrow & \circlearrowleft & \downarrow m \\
G \times_S G & \xrightarrow{m} & G
\end{array}$$

2. (identity) the following diagrams commute:

$$\begin{array}{ccc}
G \times_S S & \xrightarrow{id_G \times_S e} & G \times_S G \\
\searrow p_1 & \circlearrowleft & \swarrow m \\
& G &
\end{array}
\qquad
\begin{array}{ccc}
S \times_S G & \xrightarrow{e \times_S id_G} & G \times_S G \\
\searrow p_2 & \circlearrowleft & \swarrow m \\
& G &
\end{array}$$

3. (inverse) the following diagram commutes:

$$\begin{array}{ccccccc}
G & \xrightarrow{\Delta} & G \times_S G & \xrightleftharpoons[id_G \times_S i]{i \times_S id_G} & G \times_S G & \xrightarrow{m} & G \\
& \searrow \pi_G & & \circlearrowleft & & \swarrow e & \\
& & S & & & &
\end{array}$$

An abelian scheme A over S is an S -group scheme of finite type with connected fibers such that A is smooth and proper over S .

Grothendieck ([13, EGA IV]) proved that a group scheme has connected fibers if and only if it has geometrically connected fibers, thus the non-ambiguity of the notation. An abelian scheme is commutative, that is, denoting with $s: G \times_S G \rightarrow G \times_S G$ the obvious switching isomorphism, then the following diagram commutes:

$$\begin{array}{ccc}
G \times_S S & \xrightarrow{s} & G \times_S G \\
\searrow m & \circlearrowleft & \swarrow m \\
& G &
\end{array}$$

Remark A.2. It is important to remark that if S is a normal scheme (e.g., S is the spectrum of a field) then an abelian scheme A is projective over S ([28]).

Our definition is not so practical. To define a group scheme over S we need several morphisms over S and then we have to verify they respect the axioms above. To avoid this problem, we use the following fundamental result of category theory in our discussion:

Theorem (Yoneda lemma). Let $F: \mathcal{C}^{op} \rightarrow \mathcal{Set}$ a contravariant functor from a category \mathcal{C} to \mathcal{Set} , and let H_A be a representable functor (that is there exists a natural isomorphisms between H_A and the functor $\text{Hom}_{\mathcal{C}}(-, A)$ for some object A of \mathcal{C}). Then there exists a natural isomorphism (in \mathcal{Set} , this means a bijection) between the set $\text{Hom}(H_A, F)$ of the natural transformations of functors between H_A and F , and the set $F(A)$, which is functorial in A and F .

Let us return to the discussion of group schemes. Any S -scheme X can be canonically interpreted as the representant of the contravariant functor H_X from the category \mathcal{Sch}_S of schemes over S to the category \mathcal{Set} of sets, and this association defines a covariant functor from \mathcal{Sch}_S to the category of contravariant functors from \mathcal{Sch}_S to \mathcal{Set} (that we denote with $\mathcal{Set}^{\mathcal{Sch}_S^{op}}$). This means that the maps defining the structure of an S -group scheme G become natural transformations between functors satisfying analogous axioms.

Moreover, using Yoneda lemma, we have that for any S -scheme X , T there is a natural bijection:

$$\text{Hom}(H_T, H_X) \cong H_X(T) = \text{Hom}_S(T, X)$$

where $\text{Hom}_S(T, X)$, by the definition of T -valued points of an S -scheme X , is nothing more than the set of T -valued points of X .

It follows that giving a structure of S -group schemes to G is equivalent to give a group structure to the set $G(T)$ of its T -valued points, which is functorial in T ([21, Prop. 3.6]). Moreover, if G is an abelian group over S , it is easily checked that the group structure on $G(T)$ is that of a commutative group.

In this way, we can identify an S -scheme X with the contravariant functor from \mathcal{Sch}_S to \mathcal{Set} it represents (that will be said the functor of points of X). This result has important theoretical implications: to convert the usual constructions of group theory (e.g. kernels and quotients of group homomorphisms) in the theory of group schemes, it is more convenient to think them as the category theoretical functors satisfying their universal property, and then to study their representability.

A.1.2 Elementary examples

We shall now discuss some notable examples of group schemes. Following the above discussion, we shall first describe them via their functor of points and then show their representant, if it exists.

A.1.2.1 The additive group scheme

We denote with $\mathbb{G}_{a,S}$ the contravariant functor from $\mathcal{S}ch/S$ to $\mathcal{S}et$ which associates to an S -scheme T the additive group $(\mathcal{O}_T(T), +)$ of its global sections. This functor is representable, and its representant is a group scheme (denoted with $\mathbb{G}_{a,S}$ as well and called the additive group scheme over S). When $S = \text{Spec}(R)$ is affine, $\mathbb{G}_{a,S}$ is the affine line $\text{Spec}(R[t]) = \mathbb{A}_R^1$ over R .

A group scheme that is locally isomorphic to a product of d copies of the S -additive group scheme $\mathbb{G}_{a,S}$ will be called a vector group of rank d over S .

A.1.2.2 The multiplicative group scheme

We denote with $\mathbb{G}_{m,S}$ the contravariant functor from $\mathcal{S}ch/S$ to $\mathcal{S}et$ which associates to an S -scheme T the multiplicative group $(\mathcal{O}_T^*(T), \times)$ of invertible elements of its global sections. This functor is representable, and its representant is a group scheme (denoted with $\mathbb{G}_{m,S}$ as well and called the multiplicative group scheme over S). When $S = \text{Spec}(R)$ is affine, $\mathbb{G}_{m,S}$ is the affine line without one point $\text{Spec}(R[t, \frac{1}{t}]) = \mathbb{A}_R^1 \setminus \{0\}$ over R .

A group scheme that is locally isomorphic to a product of d copies of the S -multiplicative group scheme $\mathbb{G}_{m,S}$ will be called an algebraic torus of rank d over S .

A.1.2.3 The kernel of a group scheme morphism

Let F, G be two contravariant group functors in $\mathcal{S}et^{\mathcal{S}ch/S^{op}}$, represented respectively by two group schemes F and G , and let $f: F \rightarrow G$ be a natural transformation of functors. Denote with $\ker(f)$ the contravariant functor that associates to an S -scheme T the group $\ker(f_T: F(T) \rightarrow G(T))$. This functor is representable, and its representant is the group scheme of the kernel of the morphism of group schemes $f: F \rightarrow G$, which is a closed subscheme of F ([27]).

Suppose that $S = \text{Spec}(R)$, $F = \text{Spec}(A)$ and $G = \text{Spec}(B)$ are all affine group schemes; by the equivalence of categories between affine schemes and the opposite category of commutative unitary rings, a morphism of S -schemes $f: F \rightarrow G$ corresponds to a morphism of commutative rings $\phi: B \rightarrow A$. Then the kernel group scheme $\ker(f)$ is the pull-back of the diagram:

$$\begin{array}{ccc} \ker(f) & \xrightarrow{\quad} & S \\ \downarrow \lrcorner & \circlearrowleft & \downarrow e_G \\ F & \xrightarrow{\quad f \quad} & G \end{array}$$

Translating this in the language of commutative unitary rings, it follows that $\ker(f)$ is the affine group scheme $\mathrm{Spec}(A \otimes_B R)$ where R is seen as a B -algebra via the identity section $e: B \rightarrow R$.

This morphism is surjective, so $R \cong \frac{B}{\ker(e)}$ and thus the above tensor product becomes $\frac{A}{\ker(e) \cdot A} \cong \frac{A}{(\phi(\ker(e)))}$, thus $\ker(f) = \mathrm{Spec}\left(\frac{A}{(\phi(\ker(e)))}\right)$.

A.1.2.4 The $\mathcal{H}om$ scheme

Given two schemes X and Y defined over a base scheme S , consider the functor $\mathcal{H}om_S(X, Y)$ which maps an S -scheme T to the set $\mathrm{Hom}_T(X_T, Y_T)$. This functor is called the $\mathcal{H}om$ functor.

The representability of the $\mathcal{H}om$ functor, under certain hypothesis, is guaranteed by the following result of Grothendieck ([11, FGA II, exp. 221]).

Theorem A.3. *Let S be a locally Noetherian scheme, let X be an S -scheme that is projective and flat over S , while Y is an S -scheme that is quasi-projective over S . Then the functor $\mathcal{H}om_S(X, Y)$ is representable by a locally Noetherian S -scheme.*

Let us remark that in our dissertation we are interested in base schemes which are the spectrum of a field k (in particular, they are Noetherian, and all schemes over $\mathrm{Spec}(k)$ are trivially flat). So if X and Y are projective varieties, $\mathcal{H}om_k(X, Y)$ is representable.

A.1.2.5 The Cartier dual

Let G be a group scheme over S with structural morphism $\pi: G \rightarrow S$. The Cartier dual G^D of G is nothing more than the group of characters of G , that is the functor $\mathcal{H}om_S(G, \mathbb{G}_{m,S})$ (in the sense of the previous definition). If G is a commutative group scheme finite and locally free over S , then considering the \mathcal{O}_S -module $A := \pi_* \mathcal{O}_G$ and its dual $A^D := \mathrm{Hom}_{\mathcal{O}_S}(A, \mathcal{O}_S)$, we have that $G^D := \mathrm{Spec}(A^D)$ is a group scheme which is commutative, finite and locally free over S and it represents the functor G^D . Moreover, the homomorphism $(G^D)^D \rightarrow G$ induced by $A \rightarrow (A^D)^D$ is an isomorphism ([27]).

A.1.3 The relative Picard scheme

We recall that the Picard group $\mathrm{Pic}(X)$ of a scheme is the group of isomorphism classes of invertible sheaves (i.e. coherent \mathcal{O}_X -modules \mathcal{S} such that there exists a coherent \mathcal{O}_X -module \mathcal{T} such that the tensor product $\mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{T}$ is isomorphic to \mathcal{O}_X), endowed with the tensor product. A standard result in algebraic geometry yields

that Picard group is isomorphic to the first sheaf cohomology group $H^1(X, \mathcal{O}_X^*)$. We would be tempted to define a functor that maps an S -scheme X to its Picard group, but in general this presheaf over $\mathcal{S}ch/S$ is not representable. Thus, given an S -scheme $\pi: X \longrightarrow S$, we define the relative Picard functor $\mathcal{P}ic_{X/S}$ as the functor:

$$T \mapsto \frac{\text{Pic}(X_T)}{\pi_{X,T}^* \text{Pic}(T)}$$

where $\pi_{X,T}: X_T \longrightarrow T$ is the base change structural morphism. Equivalently, this is the sheafification of the functor on $\mathcal{S}ch/S$:

$$T \mapsto \text{coker}(\text{Pic}(T) \longrightarrow \text{Pic}(X_T))$$

or again the functor that associates to an S -scheme T the group $H^1(X_T, \mathcal{O}_{X_T}^*)$.

The representability of this functor is a major issue in algebraic geometry. Grothendieck ([11, FGA II, exp. 232]) proved that if X is projective, flat and finitely presented over S , with geometric fibers that are reduced and irreducible, then $\mathcal{P}ic_{X/S}$ is representable by a separated S -group scheme which is locally of finite type over S . When S is the spectrum of a field k , another important result by Murre ([23]) states that whenever X is a proper scheme over k then $\mathcal{P}ic_{X/k}$ is representable by a group scheme which is locally of finite type over k .

A.1.3.1 The dual of an abelian scheme

Let A be a projective abelian scheme defined over S . We have seen that in this case there exists the Picard scheme $\mathcal{P}ic_{A/S}$ and it is a group scheme over S . We define $\mathcal{P}ic_{A/S}^0$ as the connected component of the identity of $\mathcal{P}ic_{A/S}$, which is as well a group scheme over S , separated and locally of finite type over S . Moreover, if $S = \text{Spec}(k)$ with k field, then $\mathcal{P}ic_{A/k}^0$ is also proper over k .

We have the following result:

Proposition A.4. *Let A be a projective abelian scheme defined over S , and let $n: \mathcal{P}ic_{A/S} \longrightarrow \mathcal{P}ic_{A/S}$ denote the multiplication by n . Then the scheme:*

$$\mathcal{P}ic_{A/S}^\tau := \bigcup_{n>0} n^{-1}(\mathcal{P}ic_{A/S}^0)$$

is a projective abelian scheme which coincides with the identity component $\mathcal{P}ic_{A/S}^0$. It is denoted with A^ and is called the dual abelian scheme of A .*

*Moreover, A is canonically isomorphic to A^{**} .*

Proof. [22, Corollary 6.8]. □

A.2 Group and abelian varieties over a field

A.2.1 Definitions and first properties

If the base scheme S is the spectrum of a field k and G is a k -algebraic variety, then the definitions of group and abelian schemes collapse to the more common notions of group and abelian varieties respectively.

Definition A.5.

- A group variety over k is an algebraic variety over k that is a group scheme over $\text{Spec}(k)$.
- An abelian variety over k is an algebraic variety over k that is an abelian scheme over $\text{Spec}(k)$.

The conditions for a group variety to be an abelian variety are not as strict as the ones for a group scheme over an arbitrary base scheme S . In fact algebraic varieties are by definition of finite type, and every group variety is separated, as we easily show with the following proposition and its corollary.

Proposition A.6. *An S -group scheme is separated if and only if the identity section $e: S \rightarrow G$ is a closed immersion.*

Proof.

\Rightarrow) In general, given any morphism of schemes $f: X \rightarrow Y$, a section $s: Y \rightarrow X$ of f is an embedding and it is closed when f is separated;

\Leftarrow) The image of the diagonal $\Delta: G \rightarrow G \times_S G$ is the inverse image of $e(S)$ under the map:

$$G \times_S G \xrightarrow{id_G \times_S i} G \times_S G \xrightarrow{m} G$$

and therefore is closed.

□

Corollary A.7. *Every group variety G over $\text{Spec}(k)$ with k arbitrary field is separated.*

Proof. Of course, the section $e: \text{Spec}(k) \rightarrow G$ is a closed embedding, so our result follows from the above proposition. □

A.2.2 Smoothness and differential structure of group varieties

Again, let X be a group variety over k , and let $x \in X(k)$ be a k -rational point. We have two obvious morphisms of k -schemes (the left and right translations by x):

$$\begin{aligned} t_x^r: X &\xrightarrow{\cong} X \times_k \operatorname{Spec}(k) \xrightarrow{id_X \times x} X \times_k X \xrightarrow{m} X \\ t_x^l: X &\xrightarrow{\cong} \operatorname{Spec}(k) \times_k X \xrightarrow{x \times id_X} X \times_k X \xrightarrow{m} X \end{aligned}$$

which on points are given by $t_x^r(y) = m(y, x)$ and $t_x^l(y) = m(x, y)$.

More generally, for any k -scheme T and for any $x \in X(T)$ there are T -morphisms:

$$\begin{aligned} t_x^r: X_T &\xrightarrow{\cong} X_T \times_T T \xrightarrow{id_{X_T} \times x} X_T \times_T X_T \xrightarrow{m} X_T \\ t_x^l: X_T &\xrightarrow{\cong} T \times_T X_T \xrightarrow{x \times id_{X_T}} X_T \times_T X_T \xrightarrow{m} X_T \end{aligned}$$

Geometrically, this implies that any group variety X is a principal homogeneous space over itself, i.e. there is an action of X over X that is regular.

In particular, since the set of points in which X is smooth is a dense Zariski open subset of X which is stable under translations, this implies that X is smooth with trivial tangent bundle.

Proposition A.8. *Let X be a group variety over k which is geometrically integral. X is smooth over k and denoting with $T_{X,e}$ the tangent space at the identity e , one has $\mathcal{T}_{X/k} \cong T_{X,e} \otimes_k \mathcal{O}_X$. In particular, it follows that $\Omega_{X/k}^n \cong (\bigwedge^n (T_{X,e})^\vee) \otimes_k \mathcal{O}_X$ and if $n = \dim(X)$ then $\Omega_{X/k}^n \cong \mathcal{O}_X$.*

Proof. [21, Proposition 1.5]. □

If the characteristic of the field is 0, we can rely on a more powerful result (that is, we can omit the hypothesis of geometrically integrality of our group variety).

Theorem A.9 (Cartier). *Let G be a group variety over a field k of characteristic 0. Then G is reduced, and hence smooth over k .*

Proof. The proof is a standard result of Oort ([24]). □

So, any group variety over a field k of characteristic 0 is smooth, separated and of finite type; thus we can reformulate the definition of abelian variety in the following terms:

Definition A.10. An abelian variety A over a field k of characteristic 0 is an algebraic variety with connected fibers which is a proper group scheme over $\text{Spec}(k)$.

Remark A.11. It is interesting to remark that in general separated group varieties over a field k are smooth only if $\text{char}(k) = 0$. Some counterexamples in $\text{char} = p > 0$ are the group scheme $\mu_{p,k}$ (which represents the functor that associates to a k -scheme the p -th roots of unity in the ring of its global sections) and $\alpha_{p,k}$ (which represents the functor that associates to a k -scheme the p -nilpotent elements in the ring of its global sections), which are reduced but not geometrically reduced. Since we are interested in algebraic varieties defined over a field k of characteristic 0 (notably, \mathbb{C}) this counterexamples are not particularly debilitating.

A.2.3 Lie algebra of a group scheme

In our discussion, we need to define the Lie algebra of a group scheme. The main references for this section will be [20] (for group varieties) and [14, SGA III] for a more general approach.

Remember that given a Lie group G (that is, in our notation, a group object in the category of the differentiable manifolds), there is a canonical way to associate to it a Lie algebra \mathfrak{g} (that is, a vector space with a non-commutative associative operation $[-, -]$) by considering the tangent space at the identity. The operation is given by the Lie bracket $[x, y] := xy - yx$.

We want to generalize this construction. We address this issue in two steps: first we start to define the Lie algebra of a group scheme (not necessarily a variety) G over a field k , and then we construct the Lie algebra functor of a group functor $G: \mathcal{S}ch_S \rightarrow \mathcal{S}et$ (that, when G is representable, will be representable and its representant will yield the desired Lie algebra of G).

Let us study the tangent space at the identity of a group scheme G over k . Consider the spectrum of the dual numbers $\text{Spec} \left(\frac{k[\varepsilon]}{(\varepsilon^2)} \right)$. This is the tangent space at the identity, since considering the morphism $k[\varepsilon] \rightarrow k$ defined by $\varepsilon \mapsto 0$, then in the sets of $k[\varepsilon]$ - and k -valued points respectively of G we have $G(k[\varepsilon]) \rightarrow G(k)$. So we have:

$$\begin{array}{ccc} f^{-1}(e) & \longrightarrow & \{e\} \\ \downarrow & \circlearrowleft & \downarrow \\ G(k[\varepsilon]) & \longrightarrow & G(k) \end{array}$$

$f^{-1}(e)$ is the tangent space $T_e(G)$ but also (by definition) the kernel of the map $G(k[\varepsilon]) \longrightarrow G(k)$. So we define $\text{Lie}(G) := \ker(G(k[\varepsilon]) \longrightarrow G(k))$.

Thus an element of $\text{Lie}(G)$ is a homomorphism $\varphi: \mathcal{O}_G \longrightarrow k[\varepsilon]$ such that:

$$(\varepsilon \mapsto 0) \circ \varphi: \mathcal{O}_G \longrightarrow k$$

is the coidentity map.

In particular, denoting with I the augmentation ideal (that is, the kernel of the coidentity map), $I \mapsto (\varepsilon)$ and since $\varepsilon^2 = 0$ φ factors through:

$$\frac{\mathcal{O}_G}{I^2} \cong k \oplus \frac{I}{I^2}$$

and $\varphi: (a, b) \mapsto a + D(b) \cdot \varepsilon$, with $D(b) \in k$. Thus we get a bijection $\varphi \mapsto D$ between $\text{Lie}(G)$ and $\text{Hom}_k(\frac{I}{I^2}, k)$.

This gives the correct suggestion in order to define the Lie algebra functor $\mathcal{L}ie$ for a group functor defined on $\mathcal{S}ch/S$. We define the dual numbers scheme $I_S(M) := \text{Spec}(\mathcal{O}_S \oplus M)$, where M is a quasi-coherent \mathcal{O}_S -module (regarded as a sheaf of ideals such that $M^2 = 0$). I_S is a contravariant functor on the category $\mathcal{Q}Coh/S$, that associates to the morphisms $0 \rightarrow M$ and $M \rightarrow 0$ the structure morphism $I_S(M) \rightarrow I_S(0) = S$ and a section $S \rightarrow I_S(M)$ respectively.

Next, given a free \mathcal{O}_S -module M of finite type and X a group functor over $\mathcal{S}ch/S$, we define the tangent bundle $\mathcal{T}_{X/S}(M)$ of X over S with respect to M as the group functor $\mathcal{H}om_S(I_S(M), X)$.

Remark A.12. When $M = 0$ and $X = S$, since $\mathcal{H}om_S(I_S, X)$ is canonically isomorphic to the functor $V(\Omega_{X/S})$ defined on $\mathcal{S}ch/S$ that to an S -scheme T associates $\mathcal{H}om_{\mathcal{O}_T}(\Omega_{X/S} \otimes_{\mathcal{O}_S} \mathcal{O}_T, \mathcal{O}_T)$, we have the concept of tangent bundle defined in Definition 1.2. In general, the group functor $\mathcal{H}om_S(I_S(M), X)$ is representable by the vector bundle $\underline{\Omega}_{X/S} := \text{Spec}(\text{Sym}(\Omega_{X/S}))$, where $\text{Sym}(\Omega_{X/S})$ is the symmetric algebra of the \mathcal{O}_S -module $\Omega_{X/S}$.

We have the usual pull-back diagram (where $e: S \longrightarrow X$ is the identity section):

$$\begin{array}{ccc} P & \longrightarrow & \mathcal{T}_{X/S}(M) \\ \downarrow \lrcorner & \circlearrowleft & \downarrow \\ S & \xrightarrow{e} & X \end{array}$$

and we define $\mathcal{L}ie_{X/S}(M)$ to be the pull-back P . When $M = \mathcal{O}_S$, we denote $\mathcal{L}ie_{X/S}(M)$ as $\mathcal{L}ie(X)$.

Theorem. If X is a representable group functor over $\mathcal{S}ch/S$, then the functor $\mathcal{L}ie_{X/S}(M)$ is representable. In particular, $\mathcal{L}ie(X)$ is represented by the vector bundle $e^*\Omega_{X/S}$.

Proof. [14, Exp. III, Prop 3.3.]. □

A.2.4 First infinitesimal neighborhood of the identity section and group schemes morphisms

Given a separated group scheme G over S with identity section e , the first infinitesimal neighborhood G_1 of the identity section is defined the closed group subscheme of G with the same underlying topological space cut out by \mathcal{I}^2 , where \mathcal{I} is the sheaf of ideals that defines $e(S)$ in G . In other words, $G_1 := V(\mathcal{I}^2)$.

Denote with $e_1: S \rightarrow G_1$ the identity section of this group scheme. The diagram:

$$\begin{array}{ccc} G_1 & \xhookrightarrow{\quad} & G \\ & \swarrow e \quad \circlearrowright \quad \searrow e_1 & \\ & S & \end{array}$$

is commutative, so this is a morphism of S -pointed S -schemes.

There is a natural isomorphism of functors on $\mathcal{S}ch/S$:

$$\mathcal{H}om_{S\text{-pointed } S\text{-scheme}}(G_1, \mathbb{G}_{m,S}) \xrightarrow{\cong} e^*\Omega_{G/S}^1$$

By the notation $\underline{\omega}_{G/S}$, we shall denote the quasi-coherent \mathcal{O}_S -module defined by either side of this formula. There is a natural isomorphism:

$$G_1 \xrightarrow{\cong} \text{Spec}(\mathcal{O}_S \oplus \underline{\omega}_{G/S})$$

Remark A.13. When $S = \text{Spec}(k)$, $\underline{\omega}_{G/S}$ is just the cotangent space of G at the identity.

Proposition A.14. When the Cartier dual G^D of G is representable, then $\underline{\omega}_{G^D/S}$ represents the functor $\mathcal{H}om_{\mathcal{G}pSch}(G, -)$ on the category $\mathcal{QC}oh/S$ of the quasi-coherent modules over S which sends M to the group $\text{Hom}_{\mathcal{G}pSch}(G, M)$ of the group scheme morphisms between G and M (see [19, Proposition 1.4]).

Appendix B

Torsors and connections

In this appendix, following the work of Mazur and Messing in [19, §2], we shall introduce the concepts and properties of G -torsors and connections over G -torsors, used extensively in Chapter 2.

B.1 G -torsors and connections on \mathcal{O}_X -modules

Definition B.1. Let G be a commutative smooth S -group scheme. A torsor P for G is a principal homogeneous space which is locally trivial for the étale topology, that is for an étale base change $S' \rightarrow S$ the torsor $S' \times_S P$ becomes the trivial torsor $S' \times_S G$ (where G acts only on the second component).

Consider now X an arbitrary S -scheme, G a commutative smooth S -group and let P be a torsor on X under the group G_X . Denote with $\Delta^1(X) = \Delta^1(X/S)$ denote the first infinitesimal neighborhood of the image of X via the diagonal morphism $\Delta: X \rightarrow X \times_S X$.

The projections $p_j: X \times_S X \rightarrow X$ ($j = 1, 2$) induce obviously the projections (which we shall denote with the same symbol) $p_j: \Delta^1(X) \rightarrow X$.

Definition B.2. A connection ∇ on the G_X -torsor P is an isomorphism (as $G_{\Delta^1(X)}$ -torsors) $\nabla: p_1^*(P) \rightarrow p_2^*(P)$ which restricts to the identity on X (equivalently: $\Delta^*(\nabla) = id_P$).

Given an \mathcal{O}_X -module E , a connection on E is an $\mathcal{O}_{\Delta^1(X)}$ -isomorphism $\nabla: p_1^*(E) \rightarrow p_2^*(E)$ which restricts to the identity on X .

Given (E, ∇) an \mathcal{O}_X -module with a connection, we obtain an \mathcal{O}_S -linear homomorphism $\nabla': E \rightarrow E \otimes_{\mathcal{O}_S} \Omega_{X/S}^1$ in the following way: let j_1, j_2 be the homo-

morphisms $\mathcal{O}_X \longrightarrow \mathcal{O}_{\Delta^1(X)}$ corresponding to p_1 and p_2 respectively. Then, denoting with $j_i(E)$ the morphisms $E \longrightarrow p_i^*(E)$, we get $\nabla' = (\nabla^{-1} \circ j_2(E)) - j_1(E)$.

Example B.1.

1. If $G = \mathbb{G}_{m,S}$, then the connections on a $\mathbb{G}_{m,S}$ -torsor P are in bijection with the connections on the line bundle corresponding to P ;
2. If $G = \mathbb{G}_{a,S}$, then the $\mathbb{G}_{a,S}$ -torsors P are in bijection with the extensions (ε) of \mathcal{O}_X by \mathcal{O}_X . The connections on P correspond to isomorphisms of extensions $p_1^*(\varepsilon) \xrightarrow{\cong} p_2^*(\varepsilon)$ which restrict to $\text{id}_{(\varepsilon)}$ on X .

We have a category having as objects the pairs (P, ∇) , with P G -torsor and ∇ connection on P . The homomorphism sets $\text{Hom}((P, \nabla), (Q, \nabla'))$ are given by morphisms $\eta: P \longrightarrow Q$ such that:

$$\begin{array}{ccc} p_1^*(P) & \xrightarrow{p_1^*\eta} & p_1^*(Q) \\ \nabla \downarrow \mathbb{I}\mathbb{R} & \circlearrowleft & \nabla' \downarrow \mathbb{I}\mathbb{R} \\ p_2^*(P) & \xrightarrow{p_2^*\eta} & p_2^*(Q) \end{array}$$

Such morphism $\eta: P \longrightarrow Q$ is said to be horizontal when the connections on P and Q are understood as being given.

B.2 The curvature of a connection

What we want to do next is to pick an element in $\Gamma(X, \Omega_{X/S}^2 \otimes_{\mathcal{O}_S} \mathcal{L}ie(G))$ in a suitable way to define the curvature tensor of a connection, according to the standard differential geometry theory in which the curvature of a connection on a principal G -bundle E (with G a Lie group) can be seen as a 2-form ω over $E \times_G \mathfrak{g}$, where \mathfrak{g} is the Lie algebra of G .

We shall first define the curvature of a connection on the trivial bundle G_X and then show that these tensors can be patched together to give a definition for an arbitrary torsor P .

B.2.1 Connections on G_X

A connection on G_X is simply an automorphism of $G_{\Delta^1(X)}$ which restricts to the identity. This is completely determined by telling what it does to the unit section,

and hence is determined by giving an arbitrary element ξ in $\ker(\Gamma(\Delta^1(X), G) \rightarrow \Gamma(X, G))$. We have:

$$\begin{aligned} \ker(\Gamma(\Delta^1(X), G) \rightarrow \Gamma(X, G)) &\cong \operatorname{Hom}_{\mathcal{O}_X}(\underline{\omega}_G \otimes_{\mathcal{O}_S} \mathcal{O}_X, \Omega_{X/S}^1) \\ &\cong \Gamma(\Delta^1(X), \Omega_{X/S}^1 \otimes_{\mathcal{O}_S} \mathcal{L}ie(G)) \end{aligned}$$

Thus, we define the curvature form of the connection to be the image of ξ in $\Gamma(X, \Omega_{X/S}^2 \otimes_{\mathcal{O}_S} \mathcal{L}ie(G))$ under the morphism:

$$d \otimes_{\mathcal{O}_S} id_{\mathcal{L}ie(G)}: \Omega_{X/S}^1 \otimes_{\mathcal{O}_S} \mathcal{L}ie(G) \rightarrow \Omega_{X/S}^2 \otimes_{\mathcal{O}_S} \mathcal{L}ie(G)$$

B.2.2 Connections on an arbitrary G -torsor

Now let P be an arbitrary G -torsor on X endowed with a connection ∇ . P becomes trivial after an étale base change $X' \rightarrow X$, by our definition of torsor. We have an induced connection on $P_{X'}$: choosing a trivialization of $P_{X'}$, we construct the curvature of the induced connection which lies in $\Gamma(X', \Omega_{X'/S}^2 \otimes_{\mathcal{O}_S} \mathcal{L}ie(G)) \cong \Gamma(X', f_* \Omega_{X'/S}^2 \otimes_{\mathcal{O}_S} \mathcal{L}ie(G))$. Note that this equality follows from the fact that X' is étale over X .

In order to show that this local construction descends to define a section in $\Gamma(X, \Omega_{X/S}^2 \otimes_{\mathcal{O}_S} \mathcal{L}ie(G))$ (that will be the curvature of the connection ∇), we only need to show that the curvature of $P_{X'}$ is independent of the choice of trivialization, since then the application of p_1^* and p_2^* to our section in $\Gamma(X', f_* \Omega_{X'/S}^2 \otimes_{\mathcal{O}_S} \mathcal{L}ie(G))$ yields the same section of $\Gamma(X' \times_X X', f_* \Omega_{X' \times_X X'/S}^2 \otimes_{\mathcal{O}_S} \mathcal{L}ie(G))$ and thus we can apply descent.

To do this, take two trivializations $\varphi: P \rightarrow G$ and $\psi: P \rightarrow G$, and express the comparison $\psi \circ \varphi^{-1}$ as an S -morphism $g: X \rightarrow G$.

The difference between the two curvatures obtained by the above process is then $d\alpha \in \Gamma(X, \Omega_{X/S}^2 \otimes_{\mathcal{O}_S} \mathcal{L}ie(G))$, where $\alpha = p_2^*(g) - p_1^*(g)$ is interpreted as an element in $\Gamma(X, \Omega_{X/S}^1 \otimes_{\mathcal{O}_S} \mathcal{L}ie(G))$ via the isomorphism:

$$\ker(\operatorname{Hom}_S(\Delta^1(X), G)) \rightarrow \operatorname{Hom}_S(X, G) \xrightarrow{\cong} \Gamma(X, \Omega_{X/S}^1 \otimes_{\mathcal{O}_S} \mathcal{L}ie(G))$$

and d is induced from the exterior differential:

$$d: \Omega_{X/S}^1 \rightarrow \Omega_{X/S}^2$$

Lemma B.3. $d\alpha = 0$.

Proof. Let $\pi_G: G \rightarrow S$ and $\pi_X: X \rightarrow S$ the structure morphisms of G and X respectively. α can be viewed as a homomorphism $\alpha: \underline{\omega}_G \rightarrow \pi_*\Omega_{X/S}$ via the isomorphism:

$$\Gamma(X, \Omega_{X/S}^1 \otimes_{\mathcal{O}_S} \mathcal{L}ie(G)) \cong \text{Hom}_{\mathcal{O}_S}(\underline{\omega}_G, \pi_*\Omega_{X/S}^1)$$

Let us study the diagram:

$$\begin{array}{ccccc} & & G \times_S G & \xrightarrow{g} & G \\ & \nearrow (p_1 \circ g, p_2 \circ g) & \uparrow \circlearrowleft & \uparrow \circlearrowleft & \uparrow \circlearrowleft \\ \Delta^1(X) & \longrightarrow & \Delta^1(G) & \longrightarrow & \text{Inf}_S^1(G) \\ \uparrow \circlearrowleft & \circlearrowleft & \uparrow \circlearrowleft & \circlearrowleft & \uparrow \circlearrowleft \\ X & \longrightarrow & G & \longrightarrow & S \end{array}$$

By the isomorphisms above we have that α is the composition of the two top horizontal arrows in the following diagram:

$$\begin{array}{ccccc} \underline{\omega}_G & \longrightarrow & \pi_*\Omega_{G/S}^1 & \xrightarrow{dg} & \pi_*\Omega_{X/S}^1 \\ & & \downarrow d & \circlearrowleft & \downarrow d \\ & & \pi_*\Omega_{G/S}^2 & \longrightarrow & \pi_*\Omega_{X/S}^2 \end{array}$$

The image of $\underline{\omega}_G$ in $\pi_*\Omega_{G/S}^1$ is killed by d , thus by the commutativity of the diagram it follows that $d\alpha = 0$. \square

This ends the discussion on the construction of the global curvature of the G -torsor P .

B.3 \natural -torsors

Definition B.4.

1. A connection ∇ on P is integrable if the curvature associated to (P, ∇) is 0.
2. A G -torsor endowed with an integrable connection is a \natural -torsor.
 \natural -torsors form a full subcategory of the category of G -torsors with connection (P, ∇) that we denote with $\mathcal{Tors}^\natural(X, G)$.

Since by our hypothesis G is commutative, we can define the contracted product $P \overset{G}{\wedge} Q$ of two G -torsors P, Q in the following way: it is the sheafification of the presheaf given by the quotient of $P \times_S Q$ by the action of G : $g(p, q) = (g \cdot p, g^{-1} \cdot q)$.

$P \overset{G}{\wedge} Q$ is a G -torsor by letting G act on either of the factors.

If P and Q are endowed with connections ∇_P and ∇_Q respectively, then we have a connection defined on $P \overset{G}{\wedge} Q$ in the following way:

$$\begin{array}{ccc} p_2^*(P) \overset{G}{\wedge} p_2^*(Q) & & p_2^*(P \overset{G}{\wedge} Q) \\ \parallel & & \parallel \\ \nabla_P \overset{G}{\wedge} \nabla_Q: p_1^*(P) \overset{G}{\wedge} p_1^*(Q) & \xrightarrow{\cong} & p_1^*(P \overset{G}{\wedge} Q) \end{array}$$

The curvature tensor associated to $\nabla_P \overset{G}{\wedge} \nabla_Q$ is the sum of the curvature tensors associated to ∇_P and to ∇_Q . In particular, it is obvious that if P and Q are both \mathfrak{g} -torsors then also $P \overset{G}{\wedge} Q$ is a \mathfrak{g} -torsor.

If X is an S -group, then it is possible to impose additional structures on a G_X -torsor P requiring that P is an S -group. Thus, we obtain a central extension of S by G :

$$e \longrightarrow G \longrightarrow P \longrightarrow S \longrightarrow e$$

To do so, the most convenient way, denoting with $\pi_j: X \times_S X \longrightarrow X$ the projection maps and with $s: X \times_S X \rightarrow X$ the addition law, is to give an isomorphism $\beta: \pi_1^*(P) \overset{G}{\wedge} \pi_2^*(P) \xrightarrow{\cong} s^*(P)$, and requiring the diagrams that express associativity and commutativity to commute.

Definition B.5. A \mathfrak{g} -extension of the smooth group G by the commutative group X is a triple (P, ∇, β) , with (P, ∇) a \mathfrak{g} -torsor on X under G and where β is a horizontal morphism $\pi_1^*(P) \overset{G}{\wedge} \pi_2^*(P) \xrightarrow{\cong} s^*(P)$ such that (P, β) defines a group structure on P making it an extension of X by G .

Appendix C

Hodge theory

Motivations

The cohomology vector spaces $H^n(X, \mathbb{C})$ of a compact Kähler variety X over \mathbb{C} are endowed with a Hodge structure of weight n , that is a natural bigraduation:

$$H^n(X, \mathbb{C}) \cong \bigoplus_{p+q=n} H^{p,q}$$

such that $H^{p,q} \cong \overline{H^{q,p}}$. In this appendix, following the work of Deligne ([8] and [9]) and the account on Hodge theory of Steenbrink ([26]), we shall define the main concepts and results of Hodge theory, and show that the complex cohomology of a nonsingular algebraic variety (not necessarily compact) is endowed with a more general structure which makes $H^n(X, \mathbb{C})$ into a "sequence of extensions" of Hodge structures with descending weights $2n \leq p \leq n$, whose Hodge numbers $h^{p,q} = \dim H^{p,q}$ are 0 for all $p, q \geq n$. In the end we give the definition of hypercoverings, which are used in Chapter 1 in order to define the algebraic de Rham cohomology for arbitrary varieties in characteristic 0.

C.1 Filtrations

C.1.1 Filtered objects

Let \mathcal{A} be an abelian category.

Definition C.1. A descending filtration F of type \mathbb{Z} of an object A of \mathcal{A} is a family $F^n(A)_{n \in \mathbb{Z}}$ of subobjects of A such that for all $n \leq m$ $F^m(A) \subseteq F^n(A)$. If F is a descending filtration on A , set $F^\infty(A) = 0$ and $F^{-\infty}(A) = A$.

A filtered object is an object endowed with such a filtration.

Remark C.2. There is the obvious, dual definition of ascending filtration of an object. It is straightforward to see that if F is a descending filtration, then by the law:

$$F_n(A) := F^{-n}(A)$$

we have an ascending filtration of A . The converse trivially holds.

Thus we can restrict ourselves to consider only descending filtrations.

Definition C.3. The shifted filtration of a descending filtration W is by definition the filtration given by:

$$W[n]^p(A) := W^{n+p}(A)$$

Definition C.4. A filtration is finite if there exist integers n, m such that $F^n(A) = A$ and $F^m(A) = 0$.

Definition C.5. A morphism of filtered objects $(A, F) \longrightarrow (B, F)^1$ is a morphism $f: A \longrightarrow B$ in the category \mathcal{A} such that $f(F^n(A)) \subseteq F^n(B)$.

The filtered objects (and the finitely filtered objects) of an abelian category form an additive category in which there exist direct and inverse limits (and thus kernels, cokernels, images and coimages).

Let $j: X \hookrightarrow A$ be a subobject of a filtered object (A, F) . It inherits a filtered structure in the following way:

$$F^n(X) := F^n(A) \cap X$$

Analogously, if we consider the quotient $q: A \twoheadrightarrow \frac{A}{X}$ it has a filtered structure in the following way:

$$F^n\left(\frac{A}{X}\right) := q(F^n(A)) \cong \frac{(F^n(A) + X)}{X}$$

Definition C.6. A morphism of filtered objects $f: (A, F) \longrightarrow (B, F)$ is strict (or strictly compatible) if the canonical morphism $\text{CoIm}(f) \hookrightarrow \text{Im}(f)$ is an isomorphism of filtered objects.

In particular, in the category $\mathcal{M}od_R$ of modules over a commutative unitary ring R , a morphism of filtered objects $f: A \longrightarrow B$ is strict if and only if $f(F^n(A)) = f(A) \cap F^n(B)$.

¹We use the same symbol for the filtrations on A and B , since there will be hardly any confusion.

Definition C.7. If (A, F) is a filtered object in \mathcal{A} , we have a graded object $\text{Gr}_F(A)$ in $\mathcal{A}^{\mathbb{Z}}$ defined by:

$$\text{Gr}_F^n(A) := \frac{F^n(A)}{F^{n+1}(A)}$$

Definition C.8. Let $\otimes: \mathcal{A}_1 \times \cdots \times \mathcal{A}_n \rightarrow \mathcal{B}$ be a right exact multiadditive functor, and consider A_i a filtered object with finite filtration in \mathcal{A}_i . We define a filtration on $\otimes_{i=1}^n(A_i)$ by:

$$F^k \left(\bigotimes_{i=1}^n (A_i) \right) := \sum_{\sum k_i = k} \left(\text{Im} \left(\bigotimes_{i=1}^n F^{k_i}(A_i) \rightarrow \bigotimes_{i=1}^n (A_i) \right) \right)^2$$

Dually, if H is a left exact multiadditive functor, we set:

$$F^k(H(A_i)) := \bigcap_{\sum k_i = k} \left(\ker \left(H(A_i) \rightarrow H \left(\frac{A_i}{F^{k_i}(A_i)} \right) \right) \right)$$

If F is an exact functor, the two definitions are equivalent.

In this setting, we have two obvious morphisms:

$$\bigotimes_{i=1}^n (\text{Gr}_F(A_i)) \rightarrow \text{Gr}_F \left(\bigotimes_{i=1}^n (A_i) \right)$$

and:

$$\text{Gr}_F(H(A_i)) \rightarrow H(\text{Gr}_F(A_i))$$

If F is an exact functor, these two morphisms are isomorphisms and one is the inverse to the other.

Let $^\circ: \mathcal{A} \rightarrow \mathcal{A}^{op}$ be the identity contravariant functor, and let (A, F) be a filtered object of \mathcal{A} . Then, the $\left(\frac{A}{F^n(A)} \right)^\circ$'s are subobjects of A° .

Definition C.9. The dual filtration of F of A° is given by:

$$F^n(A^\circ) := \left(\frac{A}{F^{1-n}(A)} \right)^\circ$$

We have that the bidual $(A^\circ)^\circ$ is isomorphic to A as filtered objects.

From this definition, we extend the notion of a filtration to contravariant functors in certain variables. In particular, for the contravariant left exact functor Hom , we

²We mean sum of subobjects.

set:

$$F^k(\mathrm{Hom}_{\mathcal{A}}(A, B)) := \{f: A \longrightarrow B \text{ such that } f(F^n(A)) \subseteq F^{n+k}(B) \text{ for all } n\}$$

In particular, $F^0(\mathrm{Hom}_{\mathcal{A}}(A, B)) = \mathrm{Hom}((A, F), (B, F))$.

C.1.2 Opposed filtrations

Let A be an object in \mathcal{A} endowed with two filtrations F and G . We have $(\mathrm{Gr}_G^n \mathrm{Gr}_F^m(A))_{n,m \in \mathbb{Z}}$. Moreover we have a canonical isomorphism:

$$\mathrm{Gr}_G^n \mathrm{Gr}_F^m(A) \xrightarrow{\cong} \mathrm{Gr}_F^m \mathrm{Gr}_G^n(A)$$

Consider now H a third filtration on A . It induces a filtration on $\mathrm{Gr}_F(A)$, $\mathrm{Gr}_G \mathrm{Gr}_F(A)$ and $\mathrm{Gr}_F \mathrm{Gr}_G(A)$. In general they are not the same: in the formula:

$$\mathrm{Gr}_H \mathrm{Gr}_G \mathrm{Gr}_F(A)$$

H and G play a symmetric role, but not G and F .

Definition C.10. Two filtrations F and \overline{F} on A are said to be n -opposed if $\mathrm{Gr}_F^p \mathrm{Gr}_{\overline{F}}^q(A) = 0$ for $p + q \neq n$.

Given a bigraded object $A^{p,q}$ in \mathcal{A} such that:

1. $A^{p,q} = 0$ for all but a finite number of couples (p, q) ;
2. $A^{p,q} = 0$ for all p and q such that $p + q \neq n$;

then one can define two n -opposed filtrations of A in the following way:

- $F^p(A) := \bigoplus_{p' \geq p} A^{p',q'}$;
- $\overline{F}^q(A) := \bigoplus_{q' \geq q} A^{p',q'}$.

$A^{p,q}$ is then given by $\mathrm{Gr}_F^p \mathrm{Gr}_{\overline{F}}^q(A)$.

Conversely:

Proposition C.11. Let F and \overline{F} be two finite filtrations on A . F and \overline{F} are n -opposed if and only if for all p, q such that $p + q = n$ one has:

$$F^p(A) \oplus \overline{F}^q(A) \xrightarrow{\cong} A$$

Moreover, if F and \overline{F} are two n -opposed filtrations, by setting:

$$\begin{cases} A^{p,q} = 0 & \text{if } p + q \neq n \\ A^{p,q} = F^p(A) \cap \overline{F}^q(A) & \text{if } p + q = n \end{cases}$$

then $A = \bigoplus A^{p,q}$ and F and \overline{F} are deduced by the bigraduation $A^{p,q}$ of A in the way described above.

Proof. It is a trivial result that follows immediately by applying downward induction to our definitions. \square

The previous proposition establishes an equivalence between the category of the objects of \mathcal{A} endowed with two n -opposed finite filtrations and the category of the objects of \mathcal{A} endowed with a bigraduation of the type described in C.1.2.

Definition C.12. Three finite filtrations W , F and \overline{F} on A in \mathcal{A} are said to be opposed if $\text{Gr}_F^p \text{Gr}_{\overline{F}}^q \text{Gr}_W^n(A) = 0$ for all $p + q + n \neq 0$.

This condition is symmetric in F and \overline{F} . This means that F and \overline{F} induce two n -opposed filtrations on $\frac{W^n(A)}{W^{n+1}(A)}$, by setting $A^{p,q} := \text{Gr}_F^p \text{Gr}_{\overline{F}}^q \text{Gr}_W^{-p-q}(A)$, from which follows the decomposition:

$$\frac{W^n(A)}{W^{n+1}(A)} = \bigoplus_{p+q=-n} A^{p,q}$$

which makes $\text{Gr}_W(A)$ a bigraded object.

Lemma C.13. Let W , F and \overline{F} three finite opposed filtrations on A , let $\sigma = \{(p_i, q_i)\}_{i \in I_{\geq 0}}$ with $I \subseteq \mathbb{Z} \times \mathbb{Z}$ such that:

- a. $p_i \leq p_j$ and $q_i \leq q_j$ for all $i \geq j$;
- b. for all $i > 0$, $p_i + q_i = p_0 + q_0 + 1 - i$.

Setting $p = p_0$, $q = q_0$, $n = -p - q$ and:

$$A_\sigma := \left(\sum_{0 \leq i} (W^{n+i}(A) \cap F^{p_i}(A)) \right) \cap \left(\sum_{0 \leq i} (W^{n+i}(A) \cap \overline{F}^{q_i}(A)) \right)$$

then the quotient $\pi: W^n(A) \twoheadrightarrow \text{Gr}_W^n(A)$ induces an isomorphism $A_\sigma \xrightarrow{\cong} A^{p,q} \subseteq \text{Gr}_W^n(A)$.

We can state the main theorem of this section (for details, see [9, Theorem 1.2.10]):

Theorem C.14. *Let \mathcal{A} be an abelian category, and denote with \mathcal{A}' the category of the objects of \mathcal{A} endowed with three opposed filtrations W , F and \overline{F} , with morphisms given by the morphisms in \mathcal{A} compatible with the three filtrations. Then:*

- \mathcal{A}' is an abelian category;
- The kernel (respectively the cokernel) of a morphism $f: A \longrightarrow B$ in \mathcal{A}' is the kernel of the morphism f in \mathcal{A} endowed with the filtration induced as a subobject of A (respectively is the cokernel of the morphism f in \mathcal{A} endowed with the filtration induced as a quotient of B);
- All the morphisms $f: A \longrightarrow B$ in \mathcal{A}' are strictly compatible with the filtrations W , F and \overline{F} . The morphism $\mathrm{Gr}_W(f)$ is compatible with the bigraduations of $\mathrm{Gr}_W(A)$ and $\mathrm{Gr}_W(B)$. The morphisms $\mathrm{Gr}_F(f)$ and $\mathrm{Gr}_{\overline{F}}(f)$ are strictly compatible with the filtration induced by W ;
- The forgetful functor, and the functors Gr_W , Gr_F , $\mathrm{Gr}_{\overline{F}}$ and $\mathrm{Gr}_W \mathrm{Gr}_F \cong \mathrm{Gr}_F \mathrm{Gr}_W \cong \mathrm{Gr}_{\overline{F}} \mathrm{Gr}_F \mathrm{Gr}_W \cong \mathrm{Gr}_{\overline{F}} \mathrm{Gr}_W \cong \mathrm{Gr}_W \mathrm{Gr}_{\overline{F}}$ from \mathcal{A}' to \mathcal{A} are all exact.

C.2 Hodge structures

C.2.1 Pure structures

In this section, we shall work with \mathbb{R} and its algebraic closure \mathbb{C} . We shall denote by \mathbb{S} the real algebraic group deduced by Weil restriction of scalars from the group \mathbb{G}_m over \mathbb{C} , that is, defining the contravariant functor $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)$ from $\mathcal{Sch}_{/\mathbb{R}}$ to \mathcal{Set} given by:

$$T \mapsto \mathbb{G}_m(T \times_{\mathbb{R}} \mathbb{C})$$

the evaluation of $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)$ in \mathbb{R} (see also [5, Section 7.6]).

It is clear that the \mathbb{R} -rational points of $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)$ are the \mathbb{C} -rational points of \mathbb{G}_m , that is: $\mathbb{S}(\mathbb{R}) = \mathbb{C}^*$.

This is a connected group of multiplicative type, so it is an algebraic torus. Thus, it is described by the abelian group of finite type:

$$X(\mathbb{S}) := \mathrm{Hom}_{\mathbb{C}}(\mathbb{S}_{\mathbb{C}}, \mathbb{G}_m) = \mathcal{H}om_{\mathbb{R}}(\mathbb{S}, \mathbb{G}_m)(\mathbb{C})$$

of its complex characters, endowed with the action of $\mathrm{Gal}_{\mathbb{C}/\mathbb{R}} \cong \frac{\mathbb{Z}}{2\mathbb{Z}}$.

The group $X(\mathbb{S})$ is generated by z and \bar{z} which induce respectively the identity

and the complex conjugation:

$$\mathbb{C}^* = \mathbb{S}(\mathbb{R}) \longrightarrow \mathbb{S}(\mathbb{C}) \longrightarrow \mathbb{G}_m(\mathbb{C}) = \mathbb{C}^*$$

The complex conjugation switches z and \bar{z} .

One has a canonical morphism $w: \mathbb{G}_m \longrightarrow \mathbb{S}$ that, on the real points, induces the inclusion $\mathbb{R} \hookrightarrow \mathbb{C}$. So $z \circ w = \bar{z} \circ w = id$.

Definition C.15. A real Hodge structure is a finitely generated vector space V endowed with an action of the algebraic real group \mathbb{S} .

To give a real vector space V with an action of \mathbb{S} is equivalent to giving a bifiltration $V^{p,q}$ on $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ such that $V^{p,q} = \overline{V^{q,p}}$ for all indexes p and q ([8, 2.1.5]). The action of \mathbb{S} and the bigraduation are mutually determined by the fact that, on $V^{p,q}$, \mathbb{S} acts by multiplication of $z^p \cdot \bar{z}^q$.

Given the commutative monoid (\mathbb{C}, \times) let $\bar{\mathbb{S}}$ be the Weil restriction $\text{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{C}, \times)$. If V is a real vector space, it is verified that to give an action of $\bar{\mathbb{S}}$ on V is equivalent to giving a bifiltration $V^{p,q}$ on $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ such that $p, q \ V^{p,q} = \overline{V^{q,p}}$ for all indexes p and q and such that $V^{p,q} = 0$ if $p \cdot q < 0$ ([8, 2.1.6]).

Definition C.16. Let V be a real Hodge structure defined by a representation σ of \mathbb{S} and by a bigraduation $V^{p,q}$. The graduation of $V_{\mathbb{C}}$ by $V_{\mathbb{C}}^n := \bigoplus_{p+q=n} V^{p,q}$ is defined over \mathbb{R} and it is said to be a graduation of weights.

V is said to be a Hodge structure of weight n if $V^{p,q} = 0$ for all $p+q \neq n$.

Let V be a real Hodge structure. The Hodge filtration on $V_{\mathbb{C}}$ is defined by:

$$F^p(V_{\mathbb{C}}) := \bigoplus_{p' \geq p} V^{p',q'}$$

It follows from the previous discussion that:

Proposition C.17. *Let n be an integer. We have an equivalence between the category of real Hodge structures of weight n and the category of the couples formed by a real vector space V and a filtration on $V_{\mathbb{C}}$ which is n -opposed to its complex conjugate \bar{F} .*

Definition C.18. A Hodge structure H of weight n consists of:

1. A \mathbb{Z} -module of finite type $H_{\mathbb{Z}}$ (the integer lattice);

2. A real Hodge structure of weight n on $H_{\mathbb{R}} = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$.

A Hodge structure of weight n is said to be of type $S \subseteq \mathbb{Z} \times \mathbb{Z}$ if $H_{\mathbb{C}}^{p,q} = 0$ for all $(p, q) \notin S$.

A morphism of Hodge structure $f: H \rightarrow H'$ is a homomorphism $f: H_{\mathbb{Z}} \rightarrow H'_{\mathbb{Z}}$ such that $f_{\mathbb{R}}: H_{\mathbb{R}} \rightarrow H'_{\mathbb{R}}$ is compatible with the action of \mathbb{S} (equivalently, such that $f_{\mathbb{C}}$ is compatible with the bigraduation of $H_{\mathbb{C}}$ or, again, such that $f_{\mathbb{C}}$ is compatible with the Hodge filtration).

With these notations, the Hodge structures of weight n form an abelian category. If H is a Hodge structure of weight n and H' is a Hodge structure of weight n' , one defines the Hodge structure $H \otimes H'$ of weight $n + n'$ in the following way:

1. $(H \otimes H')_{\mathbb{Z}} := H_{\mathbb{Z}} \otimes_{\mathbb{Z}} H'_{\mathbb{Z}}$;
2. The action of \mathbb{S} on $(H \otimes H')_{\mathbb{R}} = H_{\mathbb{R}} \otimes_{\mathbb{R}} H'_{\mathbb{R}}$ is the tensor product of the action on $H_{\mathbb{R}}$ with the action on $H'_{\mathbb{R}}$.

The bigraduation (respectively, the Hodge filtration) on $(H \otimes H')_{\mathbb{C}} = H_{\mathbb{C}} \otimes_{\mathbb{C}} H'_{\mathbb{C}}$ is the tensor product of the bigraduations (respectively of the Hodge filtrations) of $H_{\mathbb{C}}$ and $H'_{\mathbb{C}}$.

In a similar way, one defines the Hodge structure $\mathcal{H}om(H, H')$ of weight $n - n'$, the hodge structure $\bigwedge^p H$ of weight $p \cdot n$ and the dual Hodge structure H^* of H .

Remark C.19. The Hodge structure $\mathcal{H}om(H, H')$ and the group of the homomorphisms $\text{Hom}(H, H')$ are linked. In fact, the latter is the subgroup of $\mathcal{H}om(H, H')_{\mathbb{Z}}$ formed by the homomorphisms of type $(0, 0)$.

Definition C.20. The Tate Hodge structure $\mathbb{Z}(1)$ is the Hodge structure of weight -2 , of rank 1, purely in bidegree $(-1, -1)$, with integer lattice given by $2\pi i\mathbb{Z} \subsetneq \mathbb{C}$. The action of \mathbb{S} on $\mathbb{Z}(1)$ is the multiplication by the inverse of the norm:

$$N: S \rightarrow \mathbb{G}_m$$

which on the real points is identified by the norm $\text{Norm}_{\mathbb{C}/\mathbb{R}}: \mathbb{C}^* \rightarrow \mathbb{R}^*$.

For $n \in \mathbb{Z}$, one defines the twisted Tate Hodge structure $\mathbb{Z}(n)$ as the n -th tensor power of $\mathbb{Z}(1)$. As such, $\mathbb{Z}(n)$ is a Hodge structure of weight $-2n$, rank 1, purely in bidegree $(-n, -n)$, with integer lattice given by $(2\pi i)^n \mathbb{Z} \subsetneq \mathbb{C}$. The action of \mathbb{S} is given by the n -th iteration of the inverse of the norm N .

We define the real Tate Hodge structure $\mathbb{R}(n)$ (and analogously the real Tate Hodge structure $\mathbb{R}(n)$) to be the underlying real Hodge structure of $\mathbb{Z}(1)$ (analogously of $\mathbb{Z}(n)$).

Definition C.21. A polarization of a Hodge structure H of weight n is a homomorphism:

$$(x, y): H \otimes H' \longrightarrow \mathbb{Z}(-n)$$

such that the real bilinear form $(2\pi i)^n(x, iy)$ is symmetric and positive definite.

Analogously, one defines a polarization of a real Hodge structure H of weight n to be a homomorphism:

$$(x, y): H \otimes H \longrightarrow \mathbb{R}(n)$$

such that the real bilinear form $(2\pi i)^n(x, iy)$ defined over $H_{\mathbb{R}}$ is symmetric and positive definite.

C.2.2 Mixed structures

Definition C.22. A mixed Hodge structure H consists of:

1. A \mathbb{Z} -module $H_{\mathbb{Z}}$ of finite type (the integer lattice);
2. An ascending finite filtration W_n of $H_{\mathbb{Q}} = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ (the filtration of weights);
3. A descending finite filtration F^p of $H_{\mathbb{C}} = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ (the Hodge filtration).

such that the filtrations $W_{\mathbb{C}}$ (deduced from W by extension of scalars), F and its complex conjugate \bar{F} form a system of three opposed filtrations (in the sense of Definition C.12).

Denoting again with W the filtration on $H_{\mathbb{Z}}$ deduced by W by taking its inverse image with respect to the natural inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$, the axioms of mixed Hodge structure means that for all n the filtrations F and its complex conjugate \bar{F} induce on $\mathbb{C} \otimes_{\mathbb{Z}} \text{Gr}_n^W(H_{\mathbb{Z}})$ a system of n -opposed filtrations. Moreover, $\text{Gr}_n^W(H_{\mathbb{Z}})$ is endowed with a Hodge structure of weight n with the Hodge filtration induced by F .

Definition C.23. A pure Hodge structure H of weight n is a mixed Hodge structure with integer lattice $H_{\mathbb{Z}}$ and Hodge filtration F on $H_{\mathbb{C}}$ such that the filtration of weights W on $H_{\mathbb{Q}}$ is given by:

$$\begin{cases} W_i(H_{\mathbb{Q}}) = 0 & \text{if } i \neq n \\ W_i(H_{\mathbb{Q}}) = H_{\mathbb{Q}} & \text{if } i = n \end{cases}$$

We have the category MHS whose objects are mixed Hodge structures and whose arrows $f: H \longrightarrow H'$ are homomorphisms $H_{\mathbb{Z}} \longrightarrow H'_{\mathbb{Z}}$ compatible with filtrations W

and F (note that this condition assures that f is compatible with \overline{F} too).

Then, as an immediate corollary of Theorem C.14, we obtain the following result.

Theorem C.24.

1. MHS is an abelian category;
2. The kernel (respectively the cokernel) of a morphism $f: H \rightarrow H'$ is the mixed Hodge structure whose underlying integer lattice is $K := \ker(f: H_{\mathbb{Z}} \rightarrow H'_{\mathbb{Z}})$ (respectively, $C := \operatorname{coker}(f: H_{\mathbb{Z}} \rightarrow H'_{\mathbb{Z}})$). Thus, $K \otimes_{\mathbb{Z}} \mathbb{Q}$ and $K \otimes_{\mathbb{Z}} \mathbb{C}$ are endowed with the subobject filtrations (respectively, $C \otimes_{\mathbb{Z}} \mathbb{Q}$ and $C \otimes_{\mathbb{Z}} \mathbb{C}$ are endowed with the quotient filtrations) of the filtrations W and F of $H_{\mathbb{Q}}$ and $H_{\mathbb{C}}$;
3. All the morphisms $f: H \rightarrow H'$ are strictly compatible with the filtrations W of $H_{\mathbb{Q}}$ and $H'_{\mathbb{Q}}$ and F of $H_{\mathbb{C}}$ and $H'_{\mathbb{C}}$. This induces morphisms of Hodge structures:

$$\operatorname{Gr}_n^W(f): \operatorname{Gr}_n^W(H_{\mathbb{Q}}) \rightarrow \operatorname{Gr}_n^W(H'_{\mathbb{Q}})$$

and morphisms which are strictly compatible with the induced by $W_{\mathbb{C}}$

$$\operatorname{Gr}_F^p(f): \operatorname{Gr}_F^p(H_{\mathbb{C}}) \rightarrow \operatorname{Gr}_F^p(H'_{\mathbb{C}})$$

4. The functor Gr_n^W is an exact functor from MHS to the category of rational Hodge structures of weight n ;
5. The functor Gr_F^p is an exact functor.

Let H be a mixed Hodge structure. The $W_n(H_{\mathbb{Z}})$ endowed with the filtrations induced by W and F form a system of mixed Hodge substructures $W_n(H)$ of H . The quotient $\frac{W_n(H)}{W_{n-1}(H)}$ is identified with $\operatorname{Gr}_W^n(H_{\mathbb{Z}})$ endowed with the Hodge structure described in Definition C.23.

We shall denote such Hodge structure with $\operatorname{Gr}_n^W(H)$.

Definition C.25. Denote with $H^{p,q} := \operatorname{Gr}_F^p \operatorname{Gr}_{\overline{F}}^q \operatorname{Gr}_{p+q}^W(H_{\mathbb{C}})$. The Hodge numbers are the integers $h^{p,q} = \dim_{\mathbb{C}}(H^{p,q})$.

The Hodge numbers of a mixed Hodge structure H coincide thus with the Hodge numbers of Hodge structure $\operatorname{Gr}_{p+q}^W(H)$.

C.2.3 Hodge structure on the singular cohomology of a smooth complex algebraic variety

The most important result of this discussion is that for any smooth complex algebraic variety X , the singular cohomology group $H^*(X, \mathbb{Z})$ is the underlying integer lattice of a mixed Hodge structure. This is obtained by considering X as an open dense Zariski subset of a smooth compact variety \overline{X} with normal crossing boundary divisor $Y = \overline{X} \setminus X$. We have that for any good compactification \overline{X} with boundary normal crossing divisor Y one has that $H^*(X, \mathbb{C})$ is isomorphic to $\mathbb{H}^*\left(\overline{X}, \Omega_{\overline{X}/\mathbb{C}}^*(\log Y)\right)$ ([26, Theorem 4.2]). So, we consider the ascending filtration of weights on $\Omega_{\overline{X}/\mathbb{C}}^p(\log Y)$ given by the submodules $W_n \Omega_{\overline{X}/\mathbb{C}}^p(\log Y)$ generated by elements of the form:

$$\frac{dt_{i(1)}}{t_{i(1)}} \wedge \dots \frac{dt_{i(m)}}{t_{i(m)}}$$

for $m \leq n$, with α holomorphic and $t_{i(j)}$ local equation of the distinct local component Y_j of Y ([8, 3.1.5]). Together with the trivial filtration by truncations, this construction gives a structure of bifiltered complex on $\Omega_{\overline{X}/\mathbb{C}}^p(\log Y)$.

We have thus two spectral sequences ${}_W E^{p,q}$ and ${}_F E^{p,q}$. The first, changing the indexes ${}_W E_1^{p,q} \mapsto E_2^{2p+q, -p}$ is nothing more than the Leray spectral sequence of the inclusion $j: X \hookrightarrow \overline{X}$ ([8, 3.2.4]), while the second is described by:

$${}_F E_1^{p,q} = H^q\left(\overline{X}, \Omega_{\overline{X}/\mathbb{C}}^p(\log Y)\right) \implies H^{p+q}(X, \mathbb{C})$$

These two sequences converge and provide two filtrations W and F on $H^*(X, \mathbb{C})$. In particular W is deduced by an ascending filtration W on $H^*(X, \mathbb{Q}) := H^*(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$, and thus we have that:

$$H^*(X) := (H^*(X, \mathbb{Z}), H^*(X, \mathbb{C}), W, F)$$

is a mixed Hodge structure, whose construction is functorial in X and which does not depend on the choice of the smooth compactification \overline{X} ([8, Theorem 3.2.5]).

C.3 Cohomological descent and hypercoverings

Definition C.26.

1. The simplicial indexing category Δ is the category whose objects are the totally ordered sets $\Delta_n = \{0, 1, \dots, n\}$, with $n+1$ face arrows $\delta_i: \Delta_n \longrightarrow \Delta_{n+1}$ such that $\{i\} \notin \text{Im } \delta_i$ and n degeneracy arrows $\sigma_i: \Delta_n \longrightarrow \Delta_{n-1}$ such that

$$\sigma_i(i) = \sigma_i(i+1) = i;$$

2. A simplicial object in a category \mathcal{C} is a contravariant functor from Δ to \mathcal{C} .
3. A simplicial topological space X_* is a simplicial object in the category \mathcal{Top} of topological spaces. A sheaf \mathcal{F}_* on a simplicial topological space X_* is a collection of sheaves \mathcal{F}_n on X_n such that for any $f: \Delta_n \rightarrow \Delta_m$ there is a map $\mathcal{F}^*(f): \mathcal{F}_n \rightarrow \mathcal{F}_m$.

Given a simplicial topological space X_* , one can consider the global sections functor $\Gamma(X, -)$ on the category of sheaves on X_* . This functor is left exact, thus one can derive it to obtain:

$$H^i(X_*, \mathcal{F}_*) := R^i \Gamma(X_*, \mathcal{F}_*)$$

Consider now a simplicial topological space X_* with an augmentation morphism $a: X_* \rightarrow S$. For any sheaf \mathcal{F} over S , we have the morphism:

$$\varphi: \mathcal{F} \rightarrow a_* a^* \mathcal{F}$$

Definition C.27. $a: X_* \rightarrow S$ is a morphism of cohomological descent if for all abelian sheaves \mathcal{F} over S one has:

$$\mathcal{F} \xrightarrow{\cong} \ker(a_{0*} a_0^* \mathcal{F} \rightarrow a_{1*} a_1^* \mathcal{F})$$

and $R^i \Gamma(S, a_* a^* \mathcal{F}) = 0$ for $i > 0$.

If $a: X_* \rightarrow S$ is of cohomological descent, then for any complex K_* of quasi-coherent sheaves on S such that $K_i = 0$ for all $i \leq 0$ we have that the canonical morphism:

$$R\Gamma(S, K_*) \rightarrow R\Gamma(S, R a_* a^* K_*) \cong R\Gamma(S, a^* K_*)$$

is an isomorphism ([9, 5.3.3]). In particular, for any abelian sheaf \mathcal{F} on S we have a spectral sequence:

$$E_1^{p,q} = H^q(X_p, a_p^* \mathcal{F}) \implies H^{p+q}(S, \mathcal{F})$$

and for any complex of abelian sheaves K_* on S we have a spectral sequence:

$$E_1^{p,q} = H^q(X_p, a_p^* K_*) \implies H^{p+q}(S, K_*)$$

Given a morphism $X \rightarrow S$ of topological spaces, we can obtain a simplicial topo-

logical space X_* with augmentation $X_* \rightarrow S$ in the following way: set $X_0 := X$, and consider $X_1 := X \times_S X$. Then the two face maps $\sigma_i: X_1 \rightarrow X_0$ correspond to the projection and the degeneracy map $\delta_0: X_0 \rightarrow X_1$ corresponds to the diagonal morphism. For $n = 2$, X_2 is the subspace of $X \times_S X \times_S X$ formed of the triples (x_0, x_1, x_2) such that $\delta_0(x_0) = \delta_0(x_1)$, $\delta_1(x_0) = \delta_0(x_2)$ and $\delta_1(x_1) = \delta_1(x_2)$. Inductively, X_n is the subspace of the $(n+1)$ -fold fiber product of X over S and the map $\varphi: \Delta_n \rightarrow \Delta_m$ corresponds to the morphism described in coordinates by $(x_0, \dots, x_m) \mapsto (x_{\varphi(0)}, \dots, x_{\varphi(n)})$.

This construction yields in all generality whenever \mathcal{C} has all finite limits, and gives rise to a right adjoint functor to the natural n -truncation functor sk_n from the category Δ to the category $\Delta_{\leq n}$. The space X_n is also called the n -coskeleton of X_* , and is denoted with $\mathrm{cosk}(\mathrm{sk}_n(X_*))$ (see [9, 5.1.1]).

Definition C.28. A continuous function $X \rightarrow S$ is of cohomological descent if the augmentation morphism of $X_* \rightarrow S$ is of cohomological descent. A continuous function $X \rightarrow S$ is universally of cohomological descent if for any base change $T \rightarrow S$ the continuous function $X_T \rightarrow T$ is of cohomological descent.

We can now state the fundamental result of this section, which due to Grothendieck ([14, SGA 4, V bis]).

Theorem C.29.

1. *The continuous morphisms of universal cohomological descent form a Grothendieck topology on the category of topological spaces, called the universally cohomological descent topology.*
2. *Any proper and surjective morphism is universally of cohomological descent.*
3. *Any morphism $X \rightarrow S$ admitting local sections over S is universally of cohomological descent.*
4. *Given a k -truncated simplicial topological space $X_* \rightarrow S$, for any $k \geq n \geq -1$ we have a natural map:*

$$\varphi_n: \mathrm{cosk}(X_*) \rightarrow \mathrm{cosk}(\mathrm{sk}_n(X_*))$$

Then X_ is a k -truncated hypercovering of S for the universally cohomological descent topology if the morphisms:*

$$(\varphi_n)_{n+1}: X_{n+1} \rightarrow \mathrm{cosk}(\mathrm{sk}_n(X_*))_{n+1}$$

is universally of cohomological descent.

We can explicitly construct a proper hypercovering of a topological space S . First, we choose a morphism $X_0 \rightarrow S$ which is proper and surjective. Then, consider $\text{cosk}(\{X_0\}) = X_0 \times_S X_0$ and take a proper surjective morphism $N_1 \rightarrow \text{cosk}(\{X_0\})$. In general we do not have a morphism $X_0 \rightarrow N_1$, so we consider $X_1 := N_1 \amalg X_0$. Inductively, we construct topological spaces X_n and n -truncated simplicial topological spaces ${}_nX_* \rightarrow S$ which are successive n -skeletons of a proper hypercovering of S .

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