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Grothendieck duality via Brown's representability

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Introduction

An interesting result in algebraic geometry is Serre duality for a Cohen-Macaulay projective scheme X of dimension n over an algebraically closed field

$$\mathrm{Ext}^i(\mathcal{F}, \omega_X) \simeq \mathrm{H}^{n-i}(X, \mathcal{F})'$$

where \mathcal{F} is a coherent sheaf on X and ω_X is a dualizing sheaf on X .

Serre duality is a particular case of Grothendieck duality, which is usually expressed in the context of the derived categories of bounded below complexes of quasi-coherent sheaves $\mathcal{D}^+(\mathcal{Qcoh}(-))$ over noetherian, separated schemes. If $f : X \rightarrow Y$ is a proper morphism between noetherian, separated schemes, \mathcal{F} is an object in $\mathcal{D}^+(\mathcal{Qcoh}(X))$ and \mathcal{G} an object in $\mathcal{D}^+(\mathcal{Qcoh}(Y))$, then Grothendieck duality amounts to a natural isomorphism

$$Rf_* R\mathrm{Hom}_X(\mathcal{F}, f^! \mathcal{G}) \simeq R\mathrm{Hom}_Y(Rf_* \mathcal{F}, \mathcal{G})$$

where $f^!$ is a right adjoint to Rf_* . In the classical proof of Grothendieck duality the functor $f^!$ is obtained constructively through local computation. Following the work of A. Neeman, we generalize Grothendieck duality to arbitrary complexes of quasi-coherent sheaves, deducing the existence of $f^!$ by the property of $\mathcal{D}(\mathcal{Qcoh}(-))$ and of the functor Rf_* and applying Brown's representability theorem.

In the first chapter we give the definition of triangulated category and deduce some key properties of such categories, for instance the unicity up to isomorphisms of the mapping cone and the closure of the class of triangles with respect to coproduct.

In the second chapter we follow the construction the Verdier quotient of a triangulated category, verifying some of its properties, also showing that it is a triangulated category.

In the third chapter we define compactly generated triangulated categories and give the two key theorems for our approach: Thomason's localisation theorem and Brown's representability theorem.

Thomason's localisation theorem guarantees that the Verdier quotient of a compactly generated triangulated category is compactly generated and it gives some relations between the category of compact objects in the localisation and the category of compact objects in the original category.

Brown's representability theorem states that a cohomological functor from a compactly generated triangulated category, and that sends coproducts in products, is representable. An obvious corollary to this result is that every triangulated functor from a compactly generated triangulated category, and that respects coproducts, has a right adjoint.

In the last chapter we apply the previous theory in order to obtain Grothendieck duality. We define derived categories and show that they have a natural structure of triangulated categories. Then we focus on derived categories of complexes of quasi-coherent sheaves on quasi-compact, separated schemes, proving that the functor Rf_* respects coproducts and that $\mathcal{D}(\mathcal{Qcoh}(-))$ is a compactly generated triangulated category. By Thomason's localisation theorem we also identify the category of compact objects in $\mathcal{D}(\mathcal{Qcoh}(-))$ with the category of perfect complexes on X . So Brown's representability theorem says that Rf_* has a right adjoint, $f^!$, and Grothendieck duality can be easily deduced.

Chapter 1

Triangulated categories

In this chapter we will define triangulated categories and state some of their basic properties. In this chapter we will follow mostly closely [9].

1.1 Assumptions and terminology

In order to avoid the set-theoretic problems that would arise when we consider quotient categories, we will use a broader definition of category, as in [5]. Essentially, we will not require the collection of morphisms between two objects to be a set. To use this different notion we will not need to modify most of the usual definitions in category theory, and most of the results still hold, but we will need to tweak some of them.

In particular, for additive category we will require the existence of a commutative and associative binary operation on the collection of morphisms between two objects, with the zero morphism as the neutral element and such that every morphism has an opposite. Obviously, if the collection of morphisms between two objects is a set, this is equivalent to require an abelian group structure on it, with the zero morphism as the unit.

We say that a category \mathcal{C} *contains countable coproducts* if, for any countable set Λ and any collection $\{x_\lambda | \lambda \in \Lambda\}$ of objects x_λ of \mathcal{C} indexed by Λ , the categorical coproduct

$$\coprod_{\lambda \in \Lambda} x_\lambda$$

exists in \mathcal{C} .

Let \mathcal{C} be an additive category and suppose we are given an additive and invertible endofunctor Σ of \mathcal{C} , which we will call *suspension*. We call *candidate triangle* with respect to Σ a diagram of the form

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

where $v \circ u$, $w \circ v$ and $\Sigma u \circ w$ are zero morphisms.

If we apply Σ^n , $n \in \mathbb{Z}$ to the diagram we can extend it in both directions indefinitely in order to obtain a complex associated to the candidate triangle.

The suspension functor Σ is an invertible endofunctor on \mathcal{T} , so it admits an inverse Σ^{-1} , that is also a right and left adjoint of Σ . Since Σ admits right and left adjoint, it commutes with the formation of products and coproducts.

A morphism of candidate triangles is a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow f & & \downarrow g & & \downarrow h & & \downarrow \Sigma f \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array}$$

whose rows are candidate triangles. Clearly we can make compositions of morphisms of candidate triangles. A morphism of candidate triangles is an isomorphism of candidate triangles if the vertical morphisms are isomorphisms. An identity of candidate triangles is an isomorphism of candidate triangles where the vertical morphisms are identity morphisms.

Applying Σ^n , $n \in \mathbb{Z}$, to the vertical morphisms we can extend the morphism between two candidate triangles to a morphism between the respective associated complexes.

Two morphisms of candidate triangles

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow f & & \downarrow g & & \downarrow h & & \downarrow \Sigma f \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array}$$

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow f' & & \downarrow g' & & \downarrow h' & & \downarrow \Sigma f' \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array}$$

are said to be homotopic if there are morphisms $\alpha : Y \rightarrow X'$, $\beta : Z \rightarrow Y'$, $\gamma : \Sigma X \rightarrow Z'$ such that $f - f' = \alpha \circ u + \Sigma^{-1}(w' \circ \gamma)$, $g - g' = \beta \circ v + u' \circ \alpha$, $h - h' = \gamma \circ w + v' \circ \beta$.

The morphisms $\Sigma^n \alpha$, $\Sigma^n \beta$ and $\Sigma^n \gamma$, $n \in \mathbb{Z}$, define an homotopy between the morphisms of the associated complexes.

Lemma 1.1.1. *If the morphisms of triangles*

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow f & & \downarrow g & & \downarrow h & & \downarrow \Sigma f \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array}$$

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow f' & & \downarrow g' & & \downarrow h' & & \downarrow \Sigma f' \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array}$$

are homotopic and the diagrams

$$\begin{array}{ccccccc} A & \xrightarrow{l} & B & \xrightarrow{m} & C & \xrightarrow{n} & \Sigma A \\ \downarrow a & & \downarrow b & & \downarrow c & & \downarrow \Sigma a \\ X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \end{array} \quad (1.1)$$

$$\begin{array}{ccccccc} X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \\ \downarrow a' & & \downarrow b' & & \downarrow c' & & \downarrow \Sigma a' \\ A' & \xrightarrow{l'} & B' & \xrightarrow{m'} & C' & \xrightarrow{n'} & \Sigma A' \end{array} \quad (1.2)$$

are morphism of triangles, then the two possible compositions of these morphisms are homotopic.

Proof. Let $\alpha : Y \rightarrow X'$, $\beta : Z \rightarrow Y'$, $\gamma : \Sigma X \rightarrow Z'$ be the morphisms such that $f - f' = \alpha \circ u + \Sigma^{-1}(w' \circ \gamma)$, $g - g' = \beta \circ v + u' \circ \alpha$, $h - h' = \gamma \circ w + v' \circ \beta$. If we define $\alpha' = a' \circ \alpha \circ b$, $\beta' = b' \circ \beta \circ c$ and $\gamma' = c' \circ \gamma \circ \Sigma a$ then

$$\begin{aligned} \alpha' \circ l + \Sigma^{-1}(n' \circ \gamma') &= a' \circ \alpha \circ b \circ l + \Sigma^{-1}(n' \circ c' \circ \gamma \circ \Sigma a) = \\ &= a' \circ \alpha \circ u \circ a + \Sigma^{-1}(\Sigma a' \circ w' \circ \gamma \circ \Sigma a) = a' \circ (\alpha \circ u + \Sigma^{-1}(w' \circ \gamma)) \circ a = \\ &= a' \circ f \circ a - a' \circ f' \circ a \\ \beta' \circ m + l' \circ \alpha' &= b' \circ \beta \circ c \circ m + l' \circ a' \circ \alpha \circ b = \\ &= b' \circ \beta \circ v \circ b + b' \circ u \circ \alpha \circ b = b' \circ (\beta \circ v + u' \circ \alpha) \circ b = \\ &= b' \circ g \circ b - b' \circ g' \circ b \\ \gamma' \circ n + m' \circ \beta' &= c' \circ \gamma \circ \Sigma a \circ n + m' \circ b' \circ \beta \circ c = \\ &= c' \circ \gamma \circ w \circ c + c' \circ v' \circ \beta \circ c = c' \circ (\gamma \circ w + v' \circ \beta) \circ c = \\ &= c' \circ h \circ c - c' \circ h' \circ c \end{aligned}$$

Taking either (1.1) or (1.2) to be the identity morphism of triangles, we have the thesis. \square

The homotopy gives an equivalence relation on morphisms of triangles, and by the previous lemma this relation is compatible with the composition.

Given a candidate triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

we say that the diagram

$$Y \xrightarrow{-v} Z \xrightarrow{-w} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y$$

is obtained rotating forward the candidate triangle, and that the diagram

$$\Sigma^{-1} Z \xrightarrow{-\Sigma^{-1} w} X \xrightarrow{-u} Y \xrightarrow{-v} Z$$

is obtained rotating back the candidate triangle. Note that these two diagrams are candidate triangles, so rotating back or forward an arbitrary number of times a candidate triangle we obtain another candidate triangle.

The complexes associated to the candidate triangles obtained by rotation are obtained by shifting the complex associated to the original triangle and substituting every morphism with the opposite if needed.

If we have a morphism of candidate triangles

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow f & & \downarrow g & & \downarrow h & & \downarrow \Sigma f \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array}$$

we call *mapping cone* on this map of candidate triangles the candidate triangle

$$Y \oplus X' \xrightarrow{\begin{pmatrix} -v & 0 \\ g & u' \end{pmatrix}} Z \oplus Y' \xrightarrow{\begin{pmatrix} -w & 0 \\ h & v' \end{pmatrix}} \Sigma X \oplus Z' \xrightarrow{\begin{pmatrix} -\Sigma u & 0 \\ \Sigma f & w' \end{pmatrix}} \Sigma Y \oplus \Sigma X'$$

Note that if the morphisms of candidate triangles

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow f & & \downarrow g & & \downarrow h & & \downarrow \Sigma f \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array}$$

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow f' & & \downarrow g' & & \downarrow h' & & \downarrow \Sigma f' \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array}$$

are homotopic, then we have a commutative diagram

$$\begin{array}{ccccccc} Y \oplus X' & \xrightarrow{\begin{pmatrix} -v & 0 \\ g & u' \end{pmatrix}} & Z \oplus Y' & \xrightarrow{\begin{pmatrix} -w & 0 \\ h & v' \end{pmatrix}} & \Sigma X \oplus Z' & \xrightarrow{\begin{pmatrix} -\Sigma u & 0 \\ \Sigma f & w' \end{pmatrix}} & \Sigma Y \oplus \Sigma X' \\ \downarrow \begin{pmatrix} 1_Y & 0 \\ \alpha & 1_{X'} \end{pmatrix} & & \downarrow \begin{pmatrix} 1_Z & 0 \\ \beta & 1_{Y'} \end{pmatrix} & & \downarrow \begin{pmatrix} 1_{\Sigma X} & 0 \\ \gamma & 1_{Z'} \end{pmatrix} & & \downarrow \begin{pmatrix} 1_{\Sigma Y} & 0 \\ \Sigma \alpha & 1_{\Sigma X'} \end{pmatrix} \\ Y \oplus X' & \xrightarrow{\begin{pmatrix} -v & 0 \\ g' & u' \end{pmatrix}} & Z \oplus Y' & \xrightarrow{\begin{pmatrix} -w & 0 \\ h' & v' \end{pmatrix}} & \Sigma X \oplus Z' & \xrightarrow{\begin{pmatrix} -\Sigma u & 0 \\ \Sigma f' & w' \end{pmatrix}} & \Sigma Y \oplus \Sigma X' \end{array}$$

where the vertical morphisms are isomorphisms. This means that the mapping cones on homotopic morphisms of candidate triangles are isomorphic.

1.2 Definition and first properties

Definition 1.2.1. A triangulated category \mathcal{T} is an additive category, together with an additive and invertible endofunctor Σ called suspension functor, and a class of candidate triangles with respect to Σ called triangles that satisfy the following conditions:

[T1] Any candidate triangle which is isomorphic to a triangle is a triangle.

[T2] For any object X in \mathcal{T} the candidate triangle

$$X \xrightarrow{1} X \longrightarrow 0 \longrightarrow \Sigma X$$

is a triangle.

[T3] For any morphism $f : X \rightarrow Y$ in \mathcal{T} there exists a triangle of the form

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow \Sigma X$$

The object Z is called a mapping cone on the morphism f .

[T4] The candidate triangle obtained rotating back or forward a triangle is a triangle.

[T5] For any commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow f & & \downarrow g & & & & \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array}$$

whose rows are triangles there exists a morphism $h : Z \rightarrow Z'$ such that

(a) the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow f & & \downarrow g & & \downarrow h & & \downarrow \Sigma f \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array}$$

is commutative;

(b) the mapping cone

$$Y \oplus X' \xrightarrow{\begin{pmatrix} -v & 0 \\ g & u' \end{pmatrix}} Z \oplus Y' \xrightarrow{\begin{pmatrix} -w & 0 \\ h & v' \end{pmatrix}} \Sigma X \oplus Z' \xrightarrow{\begin{pmatrix} -\Sigma u & 0 \\ \Sigma f & w' \end{pmatrix}} \Sigma Y \oplus \Sigma X'$$

is a triangle.

Definition 1.2.2. Let \mathcal{T} be a triangulated category. A contravariant functor H from \mathcal{T} to some abelian category \mathcal{A} is called cohomological if, for every triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

the sequence

$$H(Z) \xrightarrow{H(v)} H(Y) \xrightarrow{H(u)} H(X)$$

is exact in the abelian category \mathcal{A} .

A covariant functor \tilde{H} from \mathcal{T} to \mathcal{A} is called homological if, for every triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

the sequence

$$\tilde{H}(X) \xrightarrow{\tilde{H}(u)} \tilde{H}(Y) \xrightarrow{\tilde{H}(v)} H(Z)$$

is exact in the abelian category \mathcal{A} .

Note that by Definition 1.2.1[T4] we have an exact sequence for every rotation of the triangle, so we can extend the exact sequence in both directions indefinitely. This means that we can say that a cohomological functor H takes a triangle to the long exact sequence in \mathcal{A}

$$\cdots \longrightarrow H(\Sigma X) \xrightarrow{H(w)} H(Z) \xrightarrow{H(v)} H(Y) \xrightarrow{H(u)} H(X) \xrightarrow{H(\Sigma^{-1}w)} H(\Sigma^{-1}Z) \longrightarrow \cdots$$

Dually, the same holds for an homological functor.

Lemma 1.2.3. *Let \mathcal{T} be a triangulated category and U an object of \mathcal{T} , then the representable functor $\text{Hom}(-, U)$ is cohomological and the representable functor $\text{Hom}(U, -)$ is homological.*

Proof. Let

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

be a triangle, we have to prove that the sequence

$$\text{Hom}(Z, U) \xrightarrow{\circ v} \text{Hom}(Y, U) \xrightarrow{\circ u} \text{Hom}(X, U)$$

is exact.

Since we have obtained the sequence from a triangle, we know that the composite $v \circ u$ is zero.

Let $f \in \text{Hom}(Y, U)$ such that $f \circ u = 0$, then we have the commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow & & \downarrow f & & & & \downarrow \\ 0 & \longrightarrow & U & \xrightarrow{-1_U} & U & \longrightarrow & 0 \end{array}$$

The first row is a triangle by our hypothesis, while the second row is a triangle by the axioms [T2] and [T4] of Definition 1.2.1. Then by axiom [T5] of Definition 1.2.1 there exists a morphism $h : Z \rightarrow U$ that completes the commutative diagram above. In particular the square

$$\begin{array}{ccc} Y & \xrightarrow{v} & Z \\ \downarrow f & & \downarrow h \\ U & \xrightarrow{-1} & U \end{array}$$

commutes, so $f = (-h) \circ v$. Then $-h \in \text{Hom}(Z, U)$ maps to f via the composition with v .

Hence the sequence of the groups of morphisms is exact, so $\text{Hom}(-, U)$ is a cohomological functor.

Dually, we can prove that $\text{Hom}(U, -)$ is homological.

□

Definition 1.2.4. A cohomological functor H from a triangulated category \mathcal{T} to an abelian category \mathcal{A} is called decent if

- coproducts exist in \mathcal{A} and coproducts of exact sequences are exact in \mathcal{A} ;
- the functor H commutes with coproducts.

The functor $\text{Hom}(-, U) : \mathcal{T} \rightarrow \mathcal{A}b$ is cohomological, commutes with coproducts and $\mathcal{A}b$ satisfies the conditions above, so it is decent.

Definition 1.2.5. Let \mathcal{T} a triangulated category, a candidate triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

is called a pre-triangle if, for every decent cohomological functor $H : \mathcal{T} \rightarrow \mathcal{A}$ the long sequence

$$\cdots \longrightarrow H(\Sigma X) \xrightarrow{H(w)} H(Z) \xrightarrow{H(v)} H(Y) \xrightarrow{H(u)} H(X) \xrightarrow{H(\Sigma^{-1}w)} H(\Sigma^{-1}Z) \longrightarrow \cdots$$

is exact in \mathcal{A} .

By definition of cohomological functors it is obvious that a triangle is also a pre-triangle.

Since decent cohomological functors commute with coproducts, a direct summand of a pre-triangle is a pre-triangle.

Lemma 1.2.6. Let \mathcal{T} be a triangulated category that contains countable coproducts. Then a countable coproduct of pre-triangles is a pre-triangle.

Proof. Let Λ be a countable set and let

$$X_\lambda \longrightarrow Y_\lambda \longrightarrow Z_\lambda \longrightarrow \Sigma X_\lambda$$

be a pre-triangle for every $\lambda \in \Lambda$. Then

$$\coprod_{\lambda \in \Lambda} X_\lambda \longrightarrow \coprod_{\lambda \in \Lambda} Y_\lambda \longrightarrow \coprod_{\lambda \in \Lambda} Z_\lambda \longrightarrow \coprod_{\lambda \in \Lambda} \Sigma X_\lambda \simeq \Sigma \left(\coprod_{\lambda \in \Lambda} X_\lambda \right)$$

is a candidate triangle.

Let H be a decent cohomological functor. Then for every $\lambda \in \Lambda$ the sequence

$$\cdots \longrightarrow H(\Sigma X_\lambda) \longrightarrow H(Z_\lambda) \longrightarrow H(Y_\lambda) \longrightarrow H(X_\lambda) \longrightarrow H(\Sigma^{-1}Z_\lambda) \longrightarrow \cdots$$

is exact in \mathcal{A} , so the coproduct of these sequences is exact. Since H is decent the maps

$$\begin{aligned} H \left(\coprod_{\lambda \in \Lambda} X_\lambda \right) &\rightarrow \coprod_{\lambda \in \Lambda} H(X_\lambda) \\ H \left(\coprod_{\lambda \in \Lambda} Y_\lambda \right) &\rightarrow \coprod_{\lambda \in \Lambda} H(Y_\lambda) \\ H \left(\coprod_{\lambda \in \Lambda} Z_\lambda \right) &\rightarrow \coprod_{\lambda \in \Lambda} H(Z_\lambda) \end{aligned}$$

are isomorphisms.

We conclude that if we apply a decent functor H to the product of the pre-triangles we obtain a long exact sequence in \mathcal{A} , so it is a pre-triangle. \square

Lemma 1.2.7. *Consider a morphism of pre-triangles*

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow f & & \downarrow g & & \downarrow h & & \downarrow \Sigma f \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array}$$

with f and g isomorphism. Then for any decent cohomological functor H , the morphism $H(h)$ is an isomorphism.

Proof. Since f and g are isomorphisms, $H(f)$ and $H(g)$ are isomorphisms. We have the commutative diagram in \mathcal{A}

$$\begin{array}{ccccccccc} H(\Sigma Y') & \xrightarrow{H(\Sigma u')} & H(\Sigma X') & \xrightarrow{H(w')} & H(Z') & \xrightarrow{H(v')} & H(Y') & \xrightarrow{H(u')} & H(X') \\ \downarrow H(\Sigma g) & & \downarrow H(\Sigma f) & & \downarrow H(h) & & \downarrow H(g) & & \downarrow H(f) \\ H(\Sigma Y) & \xrightarrow{H(\Sigma u)} & H(\Sigma X) & \xrightarrow{H(w)} & H(Z) & \xrightarrow{H(v)} & H(Y) & \xrightarrow{H(u)} & H(X) \end{array}$$

where the rows are exact. By the 5-lemma, it follows that $H(g)$ is an isomorphism. \square

Proposition 1.2.8. *Consider a morphism of pre-triangles*

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow f & & \downarrow g & & \downarrow h & & \downarrow \Sigma f \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array}$$

If f and g are isomorphisms, then h is an isomorphism.

Proof. $\text{Hom}(-, U)$ is a decent cohomological functor, so by the previous lemma the natural map

$$\text{Hom}(h, U) : \text{Hom}(Z', U) \rightarrow \text{Hom}(Z, U)$$

is an isomorphism for every U . Then the map

$$\text{Hom}(h, -) : \text{Hom}(Z', -) \rightarrow \text{Hom}(Z, -)$$

is an isomorphism. By Yoneda's Lemma it follows that h is an isomorphism. \square

In particular this means that the mapping cone Z in axiom [T3] of Definition 1.2.1 is unique up to isomorphisms. Note that the isomorphism between two mapping cones on the same morphism isn't canonical, so obtaining the mapping cone is not functorial.

Proposition 1.2.9. *Let \mathcal{T} be a triangulated category that contains countable coproducts. Then a countable coproduct of triangles is a triangle.*

Proof. Let Λ be a countable set, and consider a collection of triangles in \mathcal{T}

$$X_\lambda \longrightarrow Y_\lambda \longrightarrow Z_\lambda \longrightarrow \Sigma X_\lambda$$

indexed by $\lambda \in \Lambda$.

By axiom [T3] of Definition 1.2.1 from the natural morphism

$$\coprod_{\lambda \in \Lambda} X_\lambda \longrightarrow \coprod_{\lambda \in \Lambda} Y_\lambda$$

we obtain a triangle

$$\coprod_{\lambda \in \Lambda} X_\lambda \longrightarrow \coprod_{\lambda \in \Lambda} Y_\lambda \longrightarrow Q \longrightarrow \Sigma(\coprod_{\lambda \in \Lambda} X_\lambda)$$

Since Σ commutes with coproducts, for any $\lambda \in \Lambda$ we have a commutative diagram

$$\begin{array}{ccccccc} \coprod_{\lambda \in \Lambda} X_\lambda & \longrightarrow & \coprod_{\lambda \in \Lambda} Y_\lambda & \longrightarrow & Q & \longrightarrow & \Sigma(\coprod_{\lambda \in \Lambda} X_\lambda) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X_\lambda & \longrightarrow & Y_\lambda & \longrightarrow & Z_\lambda & \longrightarrow & \Sigma X_\lambda \end{array}$$

and by axiom [T5] of Definition 1.2.1 we can complete this diagram to a morphism of triangles

$$\begin{array}{ccccccc} \coprod_{\lambda \in \Lambda} X_\lambda & \longrightarrow & \coprod_{\lambda \in \Lambda} Y_\lambda & \longrightarrow & Q & \longrightarrow & \Sigma(\coprod_{\lambda \in \Lambda} X_\lambda) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X_\lambda & \longrightarrow & Y_\lambda & \longrightarrow & Z_\lambda & \longrightarrow & \Sigma X_\lambda \end{array}$$

Taking the coproduct of these maps we obtain a commutative diagram

$$\begin{array}{ccccccc} \coprod_{\lambda \in \Lambda} X_\lambda & \longrightarrow & \coprod_{\lambda \in \Lambda} Y_\lambda & \longrightarrow & Q & \longrightarrow & \Sigma(\coprod_{\lambda \in \Lambda} X_\lambda) \\ \downarrow 1 & & \downarrow 1 & & \downarrow & & \downarrow 1 \\ \coprod_{\lambda \in \Lambda} X_\lambda & \longrightarrow & \coprod_{\lambda \in \Lambda} Y_\lambda & \longrightarrow & \coprod_{\lambda \in \Lambda} Z_\lambda & \longrightarrow & \Sigma(\coprod_{\lambda \in \Lambda} X_\lambda) \end{array}$$

The first row is a triangle, the second row is a pre-triangle by Proposition 1.2.6, hence this is a morphism of pre-triangles. Then by Proposition 1.2.8 the morphism from Q to $\coprod_{\lambda \in \Lambda} Z_\lambda$ is an isomorphism, so this diagram is an isomorphism of candidate triangles. Since the first row is a triangle, by axiom [T1] of Definition 1.2.1 it follows that the second row is also a triangle. \square

Lemma 1.2.10. *Consider two candidate triangles*

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

$$X' \longrightarrow Y' \longrightarrow Z' \longrightarrow \Sigma X'$$

If the direct sum

$$X \oplus X' \longrightarrow Y \oplus Y' \longrightarrow Z \oplus Z' \longrightarrow \Sigma X \oplus \Sigma X'$$

is a triangle, then the summands are also triangles.

Proof. The candidate triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

is a pre-triangle because it is a summand of a triangle. By axiom [T3] of Definition 1.2.1 there exists a triangle

$$X \longrightarrow Y \longrightarrow Q \longrightarrow \Sigma X$$

We have the commutative diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Q & \longrightarrow & \Sigma X \\ \downarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} & & \downarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} & & & & \downarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ X \oplus X' & \longrightarrow & Y \oplus Y' & \longrightarrow & Z \oplus Z' & \longrightarrow & \Sigma X \oplus \Sigma X' \end{array}$$

By axiom [T5] of Definition 1.2.1 it can be completed to a morphism of triangles

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Q & \longrightarrow & \Sigma X \\ \downarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} & & \downarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} & & \downarrow & & \downarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ X \oplus X' & \longrightarrow & Y \oplus Y' & \longrightarrow & Z \oplus Z' & \longrightarrow & \Sigma X \oplus \Sigma X' \end{array}$$

We can compose it with the projection

$$\begin{array}{ccccccc} X \oplus X' & \longrightarrow & Y \oplus Y' & \longrightarrow & Z \oplus Z' & \longrightarrow & \Sigma X \oplus \Sigma X' \\ \downarrow \begin{pmatrix} 1 & 0 \end{pmatrix} & & \downarrow \begin{pmatrix} 1 & 0 \end{pmatrix} & & \downarrow \begin{pmatrix} 1 & 0 \end{pmatrix} & & \downarrow \begin{pmatrix} 1 & 0 \end{pmatrix} \\ X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \end{array}$$

to obtain the morphism of pre-triangles

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Q & \longrightarrow & \Sigma X \\ \downarrow 1 & & \downarrow 1 & & \downarrow h & & \downarrow 1 \\ X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \end{array}$$

Since the first two vertical morphisms are isomorphisms, by Proposition 1.2.8 it follows that h is an isomorphism. Since the first row is a triangle, by axiom [T1] of Definition 1.2.1 also the second row is a triangle.

Simmetrically, the other summand is also a triangle.

□

Lemma 1.2.11. *Let the following diagram be a candidate triangle*

$$X \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} A \oplus Y \xrightarrow{\begin{pmatrix} 1 & \alpha \\ \beta & \gamma \end{pmatrix}} A \oplus Z \xrightarrow{(f' \ g')} \Sigma X$$

Then it is isomorphic to the direct sum of the candidate triangles

$$0 \longrightarrow A \xrightarrow{1} A \longrightarrow 0$$

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

Proof. The identity on A can be factorized as

$$A \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} A \oplus Y \xrightarrow{\begin{pmatrix} 1 & \alpha \\ \beta & \gamma \end{pmatrix}} A \oplus Z \xrightarrow{\begin{pmatrix} 1 & 0 \end{pmatrix}} A$$

Then the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{1} & A & \longrightarrow & 0 \\ \downarrow & & \downarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} & & \downarrow \begin{pmatrix} 1 \\ \beta \end{pmatrix} & & \downarrow \\ X & \longrightarrow & A \oplus Y & \xrightarrow{\begin{pmatrix} 1 & \alpha \\ \beta & \gamma \end{pmatrix}} & A \oplus Z & \longrightarrow & \Sigma X \\ \downarrow & & \downarrow \begin{pmatrix} 1 & \alpha \end{pmatrix} & & \downarrow \begin{pmatrix} 1 & 0 \end{pmatrix} & & \downarrow \\ 0 & \longrightarrow & A & \xrightarrow{1} & A & \longrightarrow & 0 \end{array}$$

is a factorization of the identity on the candidate triangle

$$0 \longrightarrow A \xrightarrow{1} A \longrightarrow 0$$

So it is a direct summand of

$$X \longrightarrow A \oplus Y \xrightarrow{\begin{pmatrix} 1 & \alpha \\ \beta & \gamma \end{pmatrix}} A \oplus Z \longrightarrow \Sigma X$$

and the other summand is given by the kernel of the projection

$$\begin{array}{ccccccc} X & \longrightarrow & A \oplus Y & \xrightarrow{\begin{pmatrix} 1 & \alpha \\ \beta & \gamma \end{pmatrix}} & A \oplus Z & \longrightarrow & \Sigma X \\ \downarrow & & \downarrow & & \downarrow \begin{pmatrix} 1 & 0 \end{pmatrix} & & \downarrow \\ 0 & \longrightarrow & A & \xrightarrow{1} & A & \longrightarrow & 0 \end{array}$$

With a simple computation we obtain

$$\begin{array}{ccccccc} X & \xrightarrow{g} & Y & \xrightarrow{-\beta \circ \alpha + \gamma} & Z & \xrightarrow{g'} & \Sigma X \\ \downarrow 1 & & \downarrow \begin{pmatrix} -\alpha \\ 1 \end{pmatrix} & & \downarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} & & \downarrow 1 \\ X & \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} & A \oplus Y & \xrightarrow{\begin{pmatrix} 1 & \alpha \\ \beta & \gamma \end{pmatrix}} & A \oplus Z & \xrightarrow{\begin{pmatrix} f' & g' \end{pmatrix}} & \Sigma X \\ \downarrow & & \downarrow \begin{pmatrix} 1 & \alpha \end{pmatrix} & & \downarrow \begin{pmatrix} 1 & 0 \end{pmatrix} & & \downarrow \\ 0 & \longrightarrow & A & \xrightarrow{1} & A & \longrightarrow & 0 \end{array}$$

so the second direct summand is

$$X \xrightarrow{g} Y \xrightarrow{-\beta \circ \alpha + \gamma} Z \xrightarrow{g'} \Sigma X$$

□

To conclude the section, we define homotopy colimits and establish a result about idempotent morphisms in a triangulated category, following [2].

Definition 1.2.12. *Let \mathcal{T} be a triangulated category that contains countable coproducts and*

$$X_0 \xrightarrow{j_1} X_1 \xrightarrow{j_2} X_2 \xrightarrow{j_3} \cdots$$

be a sequence of objects and morphisms in \mathcal{T} .

Let $\tau : \coprod_{i=0}^{\infty} X_i \rightarrow \coprod_{i=0}^{\infty} X_i$ be the direct sum of $j_{i+1} : X_i \rightarrow X_{i+1}$, $i \geq 0$. By axiom [T3] of Definition 1.2.1 there exists a triangle

$$\coprod_{i=0}^{\infty} X_i \xrightarrow{\tau} \coprod_{i=0}^{\infty} X_i \longrightarrow Z \longrightarrow \Sigma(\coprod_{i=0}^{\infty} X_i)$$

We call Z the homotopy colimit of the sequence and denote it with $\text{Hocolim } X_i$. Since $\text{Hocolim } X_i$ is a mapping cone, it is unique up to isomorphism.

Let \mathcal{T} be a triangulated category that contains countable coproducts and

$$\cdots \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_0 = 0$$

be a complex in \mathcal{T} . We can complete $X_2 \rightarrow X_1$ to a triangle

$$X_2 \longrightarrow X_1 \xrightarrow{h_1} Y_1 \xrightarrow{g_1} \Sigma X_2$$

By axioms [T2] and [T3] of Definition 1.2.1 the diagram

$$X_3 \longrightarrow 0 \longrightarrow \Sigma X_3 \xrightarrow{-\Sigma^1 X_3} \Sigma X_3$$

is a triangle. Since the composite $X_3 \rightarrow X_2 \rightarrow X_1$ is 0, we have a commutative diagram

$$\begin{array}{ccccccc} X_3 & \longrightarrow & 0 & \longrightarrow & \Sigma X_3 & \longrightarrow & \Sigma X_3 \\ \downarrow & & \downarrow & & & & \downarrow \\ X_2 & \longrightarrow & X_1 & \xrightarrow{h_1} & Y_1 & \xrightarrow{g_1} & \Sigma X_2 \end{array}$$

whose rows are triangles, so by axiom [T5] of Definition 1.2.1 there is a morphism $\alpha_1 : \Sigma X_3 \rightarrow Y_1$ that completes the diagram to a morphism of triangles.

If the composite $\Sigma X_4 \rightarrow \Sigma X_3 \rightarrow Y_1$ is zero, we can repeat the process in order to obtain an element Y_2 and a morphism $\alpha_2 : \Sigma^2 X_4 \rightarrow Y_2$, and so on.

If we can iterate the process indefinitely, we obtain a sequence of objects and morphisms in \mathcal{T} ,

$$Y_1 \longrightarrow Y_2 \longrightarrow Y_3 \longrightarrow \cdots$$

and we call *totalization* $|\{X_n\}|$ of the complex $\{X_n\}$ the object

$$|\{X_n\}| = \text{Hocolim } Y_n$$

Lemma 1.2.13. *Let \mathcal{T} be a triangulated category that contains countable coproducts and*

$$\cdots \longrightarrow X_n \xrightarrow{i_{n-1}} X_{n-1} \longrightarrow \cdots \xrightarrow{i_1} X_1 \xrightarrow{i_0} X_0 \longrightarrow 0$$

be a complex in \mathcal{T} and $j_k : X_k \rightarrow X_{k+1}$, $k \geq 1$, be morphisms such that $i_k \circ j_k \circ i_k = i_k$ for any $k \leq 1$. Then the totalization of the complex exists and it is functorial for the maps of complexes that commutes with both the maps i_k and the maps j_k .

Proof. Let Y_1 be the mapping cone on $X_1 \rightarrow X_0$, then, following the construction above, we have a morphism $\alpha_1 : \Sigma X_2 \rightarrow Y_1$ such that $g_1 \circ \alpha_1 = \Sigma i_1$. By our assumption on the i_k and j_k , we can replace α_1 with $\tilde{\alpha}_1 = \alpha_1 \circ \Sigma j_1 \circ \Sigma i_1$. Then $g_1 \circ \tilde{\alpha}_1 = \Sigma i_1$ and $\tilde{\alpha}_1 \circ \Sigma i_2 = (\alpha_1 \circ \Sigma j_1 \circ \Sigma i_1) \circ \Sigma i_2 = 0$, so we can always iterate the process up to a substitution of α_k with $\alpha_k \circ \Sigma^k j_k \circ \Sigma^k i_k$.

If we have a map of complexes $\{f_k\}$ between $\{X'_k, i'_k, j'_k\}$ and $\{X_k, i_k, j_k\}$ that commutes with both the i_k and the j_k , then by axiom [T5] of Definiton 1.2.1 we can complete the commutative diagram

$$\begin{array}{ccccccc} X'_1 & \xrightarrow{i'_0} & X'_0 & \xrightarrow{h'_1} & Y'_1 & \xrightarrow{g'_1} & \Sigma X'_1 \\ \downarrow f_1 & & \downarrow f_0 & & \downarrow \beta_1 & & \downarrow \Sigma f_1 \\ X_1 & \xrightarrow{i_0} & X_0 & \xrightarrow{h_1} & Y_1 & \xrightarrow{g_1} & \Sigma X_1 \end{array}$$

to a morphism of triangles through a morphism $\beta_1 : Y'_1 \rightarrow Y_1$.

Consider the diagram

$$\begin{array}{ccc} \Sigma X'_2 & \xrightarrow{\alpha'_1} & Y'_1 \\ \downarrow \Sigma f_2 & & \downarrow \beta_1 \\ \Sigma X_2 & \xrightarrow{\alpha_1} & Y_1 \end{array}$$

It is not necessarily commutative, but if we compose $\alpha_1 \circ \Sigma f_2$ or $\beta_1 \circ \alpha'_1$ with g_1 we must obtain the same result, so $(\beta_1 \circ \alpha'_1 - \alpha_1 \circ \Sigma f_2) \circ g_1 = 0$. Let $s_1 = \beta_1 \circ \alpha'_1 - \alpha_1 \circ \Sigma f_2$. Then by Definiton 1.2.1[T5] the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Sigma X'_2 & \xrightarrow{1_{\Sigma X'_2}} & \Sigma X'_2 & \longrightarrow & 0 \\ \downarrow & & & & \downarrow s_1 & & \downarrow \\ X_1 & \longrightarrow & X_0 & \xrightarrow{h_1} & Y_1 & \xrightarrow{g_1} & \Sigma X_1 \end{array}$$

can be completed to a morphism of triangle through a morphism $p_1 : X'_2 \rightarrow X_0$, and $h_1 \circ p_1 = s_1$.

By construction it is $\alpha'_1 = \alpha'_1 \circ \Sigma j'_1 \circ \Sigma i'_1$ and $\alpha = \alpha \circ \Sigma j_1 \circ \Sigma i_1$, so we have

$$\begin{aligned} (h_1 \circ p_1) \circ (\Sigma j'_1 \circ \Sigma i'_1) &= (\beta_1 \circ \alpha'_1 - \alpha_1 \circ \Sigma f_2) \circ (\Sigma j'_1 \circ \Sigma i'_1) = \\ &= \beta_1 \circ \alpha'_1 \circ \Sigma j'_1 \circ \Sigma i'_1 - \alpha_1 \circ \Sigma f_2 \circ \Sigma j'_1 \circ \Sigma i'_1 = \beta_1 \circ \alpha'_1 - \alpha_1 \circ \Sigma j_1 \circ \Sigma f_2 \circ \Sigma i'_1 = \\ &= \beta_1 \circ \alpha'_1 - \alpha_1 \circ \Sigma j_1 \circ \Sigma i_1 \circ \Sigma f_2 = \beta_1 \circ \alpha'_1 - \alpha_1 \circ \Sigma f_2 = \\ &= h_1 \circ p_1 \end{aligned}$$

Then we can replace p_1 with $p_1 \circ \Sigma j'_1 \circ \Sigma i'_1$ and β_1 with $\tilde{\beta}_1 = \beta_1 - h_1 \circ (p_1 \circ \Sigma j'_1) \circ g'_1$. With this convention we have

$$\begin{aligned} \tilde{\beta}_1 \circ \alpha'_1 &= \beta_1 \circ \alpha'_1 - h_1 \circ p_1 \circ \Sigma j'_1 \circ g'_1 \circ \alpha'_1 = \\ &= \beta_1 \circ \alpha'_1 - h_1 \circ p_1 \circ \Sigma j'_1 \circ \Sigma i'_1 = \beta_1 \circ \alpha'_1 - h_1 \circ p_1 = \beta_1 \circ \alpha'_1 - s = \\ &= \beta_1 \circ \alpha'_1 - (\beta_1 \circ \alpha'_1 - \alpha_1 \circ \Sigma f_2) = \alpha_1 \circ \Sigma f_2 \end{aligned}$$

so the square

$$\begin{array}{ccc} \Sigma X'_2 & \xrightarrow{\alpha'_1} & Y'_1 \\ \downarrow \Sigma f_2 & & \downarrow \tilde{\beta}_1 \\ \Sigma X_2 & \xrightarrow{\alpha_1} & Y_1 \end{array}$$

is commutative, so we can iterate the process to get a collection of morphisms $\{\beta_k\}$. This yields a morphism between the sequence of the Y'_k and the sequence of the Y_k , and so a morphism from the homotopy colimit of the first sequence to the homotopy colimit of the second. \square

Proposition 1.2.14. *Let \mathcal{T} be a triangulated category that contains countable coproducts, then if $e : X \rightarrow X$ is an idempotent in \mathcal{T} , it is also split in \mathcal{T} .*

Proof. By Lemma 1.2.13 each of the three complexes

$$\begin{aligned} \cdots &\xrightarrow{1-e} X \xrightarrow{e} X \xrightarrow{1-e} X \xrightarrow{e} X \\ \cdots &\xrightarrow{e} X \xrightarrow{1-e} X \xrightarrow{e} X \xrightarrow{1-e} X \\ \cdots &\xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} X \oplus X \xrightarrow{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}} X \oplus X \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} X \oplus X \end{aligned}$$

(with the $j_k = 1$) admits totalization. Since the third complex is isomorphic to the direct sum of the first two, also the totalization of the third complex is isomorphic to the direct sum of the totalization of the first two. Let Y be the totalization of the first complex and Z the totalization of the second one, the totalization of the third is X , so $X = Y \oplus Z$. Since e is zero on Y and $1 - e$ is zero on Z , e is split. \square

1.3 Homotopy cartesian squares

Let \mathcal{T} be a triangulated category. We say that a commutative square

$$\begin{array}{ccc} Y & \xrightarrow{f} & Z \\ \downarrow g & & \downarrow g' \\ Y' & \xrightarrow{f'} & Z' \end{array}$$

is *homotopy cartesian* if there exists a morphism $\delta : Z' \rightarrow \Sigma Y$, called *differential*, such that the diagram

$$Y \xrightarrow{\begin{pmatrix} g \\ -f \end{pmatrix}} Y' \oplus Z \xrightarrow{\begin{pmatrix} f' & g' \end{pmatrix}} Z' \xrightarrow{\delta} \Sigma Y$$

is a triangle.

In this case we say that Y is the *homotopy pullback* of

$$\begin{array}{ccc} & & Z \\ & & \downarrow g' \\ Y' & \xrightarrow{f'} & Z' \end{array}$$

and that Z' is the *homotopy pushout* of

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow g & & \\ Y' & & \end{array}$$

Suppose we are given the first of the two diagrams above, then we can complete the morphism $\begin{pmatrix} -f' & -g' \end{pmatrix}$ from $Y' \oplus Z$ to Z' to obtain a triangle

$$Y' \oplus Z \longrightarrow Z' \longrightarrow \tilde{Y} \longrightarrow \Sigma T' \oplus \Sigma Z$$

Since rotating back this triangle gives another triangle, taking $Y = \Sigma^{-1}\tilde{Y}$ we obtain that the commutative square

$$\begin{array}{ccc} Y & \xrightarrow{f} & Z \\ \downarrow g & & \downarrow g' \\ Y' & \xrightarrow{f'} & Z' \end{array}$$

is homotopy cartesian. Moreover, the homotopy pullback is unique up to isomorphisms.

Furthermore, if we have a commutative square

$$\begin{array}{ccc} T & \xrightarrow{f''} & Z \\ \downarrow g'' & & \downarrow g' \\ Y' & \xrightarrow{f'} & Z' \end{array}$$

the composite map

$$T \xrightarrow{\begin{pmatrix} g'' \\ -f'' \end{pmatrix}} Y' \oplus Z \xrightarrow{\begin{pmatrix} f' & g' \end{pmatrix}} Z'$$

is zero. Since the homological functor $\text{Hom}(T, -)$ takes the triangle

$$Y \xrightarrow{\begin{pmatrix} g \\ -f \end{pmatrix}} Y' \oplus Z \xrightarrow{\begin{pmatrix} f' & g' \end{pmatrix}} Z' \xrightarrow{\delta} \Sigma Y$$

to a long exact sequence, if we are given a morphism in $\text{Hom}(T, Y' \oplus Z)$ whose image in $\text{Hom}(T, Z')$ is zero, it must be the image of a morphism in $\text{Hom}(T, Y)$.

This means that there is a map $T \rightarrow Y$ such that the diagram

$$\begin{array}{ccccc}
 & & & & \\
 & & & & \\
 T & & & & \\
 & \searrow^{g''} & & & \\
 & & Y & \xrightarrow{f} & Z \\
 & \searrow^{f'''} & \downarrow g & & \downarrow g' \\
 & & Y' & \xrightarrow{f'} & Z'
 \end{array}$$

is commutative. This justify the terminology of the homotopy pullback.

Lemma 1.3.1. *Consider a commutative diagram*

$$\begin{array}{ccccccc}
 X & \xrightarrow{u} & Y & \longrightarrow & Z & \xrightarrow{w} & \Sigma X \\
 \downarrow 1 & & \downarrow f & & & & \downarrow 1 \\
 X & \xrightarrow{f \circ u} & Y' & \longrightarrow & Z' & \xrightarrow{w'} & \Sigma X
 \end{array}$$

whose rows are triangles. It can be completed to a morphism of triangles

$$\begin{array}{ccccccc}
 X & \xrightarrow{u} & Y & \longrightarrow & Z & \xrightarrow{w} & \Sigma X \\
 \downarrow 1 & & \downarrow f & & \downarrow & & \downarrow 1 \\
 X & \xrightarrow{f \circ u} & Y' & \longrightarrow & Z' & \xrightarrow{w'} & \Sigma X
 \end{array}$$

where the middle commutative square is homotopy cartesian. Furthermore, we can choose the composite $\Sigma u \circ w'$ to be the differential $\delta : Z' \rightarrow \Sigma Y$.

Proof. The commutative diagram can be completed to a morphism of triangles

$$\begin{array}{ccccccc}
 X & \xrightarrow{u} & Y & \longrightarrow & Z & \xrightarrow{w} & \Sigma X \\
 \downarrow 1 & & \downarrow f & & \downarrow & & \downarrow 1 \\
 X & \xrightarrow{f \circ u} & Y' & \longrightarrow & Z' & \xrightarrow{w'} & \Sigma X
 \end{array}$$

as in Definition 1.2.1[T5], so the mapping cone on it

$$X \oplus Y \longrightarrow Y' \oplus Z \longrightarrow \Sigma X \oplus Z' \longrightarrow \Sigma X \oplus \Sigma Y$$

is a triangle. By Lemma 1.2.11 this is isomorphic to the direct sum of the candidate triangles

$$X \longrightarrow 0 \longrightarrow \Sigma X \longrightarrow \Sigma X$$

$$Y \longrightarrow Y' \oplus Z \longrightarrow Z' \longrightarrow \Sigma Y$$

Then by Lemma 1.2.10 the latter is a triangle, so the middle square is homotopy cartesian.

Furthermore, the characterization of the summand triangles in the proof of Lemma 1.2.11 shows that we can choose $\Sigma u \circ w'$ as the differential. \square

Lemma 1.3.2. *Consider an homotopy cartesian square*

$$\begin{array}{ccc} Y & \longrightarrow & Z \\ \downarrow g & & \downarrow h \\ Y' & \longrightarrow & Z' \end{array}$$

If we are given a triangle

$$Y \xrightarrow{g} Y' \longrightarrow Y'' \longrightarrow \Sigma Y$$

then there exists a triangle

$$Z \xrightarrow{h} Z' \longrightarrow Y'' \longrightarrow \Sigma Z$$

such that it completes the homotopy cartesian square to the morphism of triangles

$$\begin{array}{ccccccc} Y & \xrightarrow{g} & Y' & \longrightarrow & Y'' & \longrightarrow & \Sigma Y \\ \downarrow & & \downarrow & & \downarrow 1 & & \downarrow \\ Z & \xrightarrow{h} & Z' & \longrightarrow & Y'' & \longrightarrow & \Sigma Z \end{array}$$

Proof. By definition of homotopy cartesian square, we have a triangle

$$Y \longrightarrow Y' \oplus Z \longrightarrow Z' \longrightarrow \Sigma Y$$

By Definiton 5[T5] the commutative diagram

$$\begin{array}{ccccccc} Y & \longrightarrow & Y' \oplus Z & \longrightarrow & Z' & \longrightarrow & \Sigma Y \\ \downarrow 1 & & \downarrow & & & & \downarrow 1 \\ Y & \xrightarrow{g} & Y' & \longrightarrow & Y'' & \longrightarrow & \Sigma Y \end{array}$$

can be completed to a morphism of triangles

$$\begin{array}{ccccccc} Y & \longrightarrow & Y' \oplus Z & \longrightarrow & Z' & \longrightarrow & \Sigma Y \\ \downarrow 1 & & \downarrow & & & & \downarrow 1 \\ Y & \xrightarrow{g} & Y' & \longrightarrow & Y'' & \longrightarrow & \Sigma Y \end{array}$$

such that the mapping cone on it

$$Y \oplus Y' \oplus Z \longrightarrow Y' \oplus Z' \longrightarrow \Sigma Y \oplus Y'' \longrightarrow \Sigma Y \oplus \Sigma Y' \oplus \Sigma Z$$

is a triangle. Then by Lemma 1.2.10 its direct summand

$$Z \longrightarrow Z' \longrightarrow Y'' \longrightarrow \Sigma Z$$

is also a triangle. Hence we have the morphism of triangles

$$\begin{array}{ccccccc} Y & \xrightarrow{g} & Y' & \longrightarrow & Y'' & \longrightarrow & \Sigma Y \\ \downarrow & & \downarrow & & \downarrow 1 & & \downarrow \\ Z & \xrightarrow{h} & Z' & \longrightarrow & Y'' & \longrightarrow & \Sigma Z \end{array}$$

□

Proposition 1.3.3. *Let $f : X \rightarrow Y$ and $g : Y \rightarrow Y'$ be two morphisms, consider three triangles*

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow \Sigma X$$

$$X \xrightarrow{g \circ f} Y' \longrightarrow Z' \longrightarrow \Sigma X$$

$$Y \xrightarrow{g} Y' \longrightarrow Y'' \longrightarrow \Sigma Y$$

Then there is a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ \downarrow 1 & & \downarrow g & & \downarrow & & \downarrow 1 \\ X & \xrightarrow{g \circ f} & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & Y'' & \xrightarrow{1} & Y'' & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Sigma X & \xrightarrow{\Sigma f} & \Sigma Y & \longrightarrow & \Sigma Z & \longrightarrow & \Sigma^2 X \end{array}$$

where the first two rows and the second column are the initial triangles, every row and column of the diagram is a triangle and the square

$$\begin{array}{ccc} Y & \longrightarrow & Z \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Z' \end{array}$$

is homotopy cartesian, and the differential is given by the composition

$$Z' \longrightarrow \Sigma X \longrightarrow \Sigma Y$$

or

$$Z' \longrightarrow Y'' \longrightarrow \Sigma Y$$

Proof. We have the commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ \downarrow 1 & & \downarrow g & & & & \downarrow \\ X & \xrightarrow{g \circ f} & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X \end{array}$$

and by Lemma 1.3.1 it can be completed to a morphism of triangles

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ \downarrow 1 & & \downarrow g & & \downarrow & & \downarrow \\ X & \xrightarrow{g \circ f} & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X \end{array}$$

such that the square

$$\begin{array}{ccc} Y & \longrightarrow & Z \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Z' \end{array}$$

is homotopy cartesian. Then by Lemma 1.3.2 and since

$$Y \xrightarrow{g} Y' \longrightarrow Y'' \longrightarrow \Sigma Y$$

is a triangle, we have a morphism of triangles

$$\begin{array}{ccccccc} Y & \xrightarrow{g} & Y' & \longrightarrow & Y'' & \longrightarrow & \Sigma Y \\ \downarrow & & \downarrow & & \downarrow 1 & & \downarrow \\ Z & \xrightarrow{h} & Z' & \longrightarrow & Y'' & \longrightarrow & \Sigma Z \end{array}$$

Assembling these morphisms of triangles we obtain the commutative diagram required. \square

Definition 1.3.4. *A candidate triangle*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

is called *contractible* if the identity morphism on the candidate triangle

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow 1_X & & \downarrow 1_Y & & \downarrow 1_Z & & \downarrow 1_{\Sigma X} \\ X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \end{array}$$

is homotopic to the zero morphism

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow 0 & & \downarrow 0 & & \downarrow 0 & & \downarrow 0 \\ X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \end{array}$$

Lemma 1.3.5. *A contractible candidate triangle is a pre-triangle.*

Proof. Let H be a decent cohomological functor and

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

a contractible candidate triangle. Then there are morphisms $\alpha : Y \rightarrow X$, $\beta : Z \rightarrow Y$, $\gamma : \Sigma X \rightarrow Z$ such that $1_X = \alpha \circ f + \Sigma^{-1}(h \circ \Gamma)$, $1_Y = \beta \circ g + f \circ \alpha$, $1_Z = \gamma \circ w + g \circ \beta$.

Then the identity on the complex

$$\cdots \longrightarrow H(\Sigma X) \xrightarrow{H(w)} H(Z) \xrightarrow{H(v)} H(Y) \xrightarrow{H(u)} H(X) \longrightarrow \cdots$$

is homotopic via the morphisms $H(\Sigma^n \alpha)$, $H(\Sigma^n \beta)$, $H(\Sigma^n \gamma)$ to the zero morphism, so the long exact sequence given by H is exact. \square

Proposition 1.3.6. *A contractible candidate triangle is a triangle.*

Proof. Suppose we are given a contractible candidate triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

By Definition 1.2.1[T3] there is a triangle

$$X \xrightarrow{u} Y \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X$$

By definition of contractible candidate triangles, there are morphisms $\alpha : Y \rightarrow X$, $\beta : Z \rightarrow Y$, $\gamma : \Sigma X \rightarrow Z$ that define the homotopy of the identity on the candidate triangle to the zero morphism. Since $v \circ u = 0$ it is $\beta \circ v \circ u = 0$. Applying the cohomological functor $\text{Hom}(-, Y)$ to the pre-triangle

$$X \xrightarrow{u} Y \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X'$$

we obtain an exact sequence, since $\beta \circ v \in \text{Hom}(Y, Y)$ is sent to 0 in $\text{Hom}(X, Y)$ there is a morphism $\beta' \in \text{Hom}(Y', Y)$ such that $\beta' \circ v' = \beta \circ v$.

Since $1_X = \alpha \circ u + \Sigma^{-1}(w \circ \Gamma)$, $1_Y = \beta \circ v + u \circ \alpha$, $1_Z = \gamma \circ w + v \circ \beta$ it is $(\gamma \circ w' + v \circ \beta') \circ v' = \gamma \circ w' \circ v' + v \circ \beta' \circ v' = 0 + v \circ \beta \circ v = v$ and $w \circ (\gamma \circ w' + v \circ \beta') = w \circ \gamma \circ w' + w \circ v \circ \beta' = w \circ \gamma \circ w' + 0 = w'$, so if we set $\theta = \gamma \circ w' + v \circ \beta'$ we have a morphism of pre-triangles

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X \\ \downarrow 1 & & \downarrow 1 & & \downarrow \theta & & \downarrow 1 \\ X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \end{array}$$

and by Lemma 1.2.8 θ is an isomorphism. Then the diagram above is an isomorphism of pre-triangles, since the first row is a triangle, also the second row is a triangle. □

Lemma 1.3.7. *The mapping cone on the zero morphism between triangles is a triangle.*

Proof. Let the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow 0 & & \downarrow 0 & & \downarrow 0 & & \downarrow 0 \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array}$$

be a zero morphism of triangles. Then the mapping cone on it

$$Y \oplus X' \xrightarrow{\begin{pmatrix} -v & 0 \\ 0 & u' \end{pmatrix}} Z \oplus Y' \xrightarrow{\begin{pmatrix} -w & 0 \\ 0 & v' \end{pmatrix}} \Sigma X \oplus Z' \xrightarrow{\begin{pmatrix} -\Sigma u & 0 \\ 0 & w' \end{pmatrix}} \Sigma Y \oplus \Sigma X' \quad (1.3)$$

is the direct sum of the triangles

$$X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X'$$

$$Y \xrightarrow{-v} Z \xrightarrow{-w} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y$$

so by Proposition 1.2.9 the candidate triangle 1.3 is a triangle. □

Lemma 1.3.8. *Let the diagram*

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow f & & \downarrow g & & \downarrow h & & \downarrow \Sigma f \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array}$$

be a morphism of triangles where one of the row is a contractible triangle. Then the mapping cone on it is a triangle.

Proof. By Lemma 1.1.1 the morphism above is homotopic to the zero morphism. Since the mapping cones on homotopic morphisms of triangles are isomorphic, by Lemma 1.3.7 the mapping cone on the first morphism of triangles is a triangle. □

Chapter 2

Verdier localisation

2.1 Definiton

Let \mathcal{T}_1 and \mathcal{T}_2 be triangulated categories. An additive functor $F : \mathcal{T}_1 \rightarrow \mathcal{T}_2$ together with a natural isomorphisms $\phi_X : \Sigma F(X) \rightarrow F(\Sigma X)$ is called *triangulated* if it sends triangles in \mathcal{T}_1 to triangles in \mathcal{T}_2 . We call *kernel* of F the full subcategory of the objects T of \mathcal{T}_1 such that $F(T)$ is isomorphic to zero in \mathcal{T}_2 . Unless it is necessary, we will simply say that a functor F is triangulated without specifying the isomorphisms ϕ_X .

A *triangulated subcategory* \mathcal{D} of a triangulated category \mathcal{T} is a full subcategory of \mathcal{T} such that:

1. every object of \mathcal{T} isomorphic to an object of \mathcal{D} is in \mathcal{D} ;
2. $\Sigma \mathcal{D} = \mathcal{D}$;
3. the mapping cone on any morphism between two objects in \mathcal{D} lies in \mathcal{D} .

Obviously a triangulated subcategory is a triangulated category.

Let $F : \mathcal{T}_1 \rightarrow \mathcal{T}_2$ be a triangulated functor and \mathcal{D} be the kernel of F . Then

1. any object T isomorphic to an object in \mathcal{D} is sent to 0 by F , so T is in \mathcal{D} ;
2. since Σ and F commute, $\Sigma \mathcal{D} = \mathcal{D}$;
3. if

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

is a triangle in \mathcal{T}_1 with X and Y in \mathcal{D} , then by Proposition 1.2.8 $F(Z)$ is isomorphic to zero, so Z is in \mathcal{D} .

Then the kernel of a triangulated functor is a triangulated subcategory.

A triangulated subcategory is called *thick* if it contains every direct summand of its objects.

Since a triangulated functor is additive, its kernel is a thick subcategory.

Lemma 2.1.1. *Let $F : \mathcal{T}_1 \rightarrow \mathcal{T}_2$ be a triangulated category. If F has a right or left adjoint G , then G is also a triangulated functor.*

Proof. We assume that G is a right adjoint of F , the proof for a left adjoint is dual. We have that $F\Sigma = \Sigma F$, so by adjunction, since Σ^{-1} is the right adjoint of Σ , we have $\Sigma^{-1}G = G\Sigma^{-1}$. Since G commutes with Σ^{-1} , it also commutes with Σ .

To show that G is triangulated, we need to show that, given a triangle in \mathcal{T}_2

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

then the diagram

$$G(X) \xrightarrow{G(u)} G(Y) \xrightarrow{G(v)} G(Z) \xrightarrow{G(w)} G(\Sigma X) = \Sigma G(X)$$

is a triangle in \mathcal{T}_1 .

Let K be the mapping cone on $G(u)$ in \mathcal{T}_1 , since F is a triangulated functor, the diagram

$$FG(X) \xrightarrow{FG(u)} FG(Y) \longrightarrow F(K) \longrightarrow \Sigma FG(X)$$

is a triangle in \mathcal{T}_2 . Let $\epsilon_X : FG(X) \rightarrow X$ and $\epsilon_Y : FG(Y) \rightarrow Y$ be the counits of adjunction, since ϵ_X and ϵ_Y are natural we obtain a commutative square

$$\begin{array}{ccc} FG(X) & \xrightarrow{FG(u)} & FG(Y) \\ \downarrow \epsilon_X & & \downarrow \epsilon_Y \\ X & \xrightarrow{u} & Y \end{array}$$

and by axiom [T5] of Definition 1.2.1 we have a morphism of triangles

$$\begin{array}{ccccccc} FG(X) & \xrightarrow{FG(u)} & FG(Y) & \longrightarrow & F(K) & \longrightarrow & \Sigma FG(X) \\ \downarrow \epsilon_X & & \downarrow \epsilon_Y & & \downarrow \alpha & & \downarrow \Sigma \epsilon_X \\ X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \end{array}$$

For any object T in \mathcal{T}_∞ we have a map $\beta_T = \alpha \circ F(-) : \text{Hom}_{\mathcal{T}_\infty}(T, K) \rightarrow \text{Hom}_{\mathcal{T}_\infty}(F(T), Z)$.

Applying $\text{Hom}_{\mathcal{T}_1}(T, -)$ to the triangle

$$G(X) \xrightarrow{G(u)} G(Y) \longrightarrow K \longrightarrow \Sigma G(X)$$

we obtain an exact sequence

$$\begin{aligned} \text{Hom}_{\mathcal{T}_1}(T, G(X)) &\longrightarrow \text{Hom}_{\mathcal{T}_1}(T, G(Y)) \longrightarrow \text{Hom}_{\mathcal{T}_1}(T, K) \longrightarrow \\ &\longrightarrow \text{Hom}_{\mathcal{T}_1}(T, \Sigma G(X)) \longrightarrow \text{Hom}_{\mathcal{T}_1}(T, \Sigma G(Y)) \end{aligned}$$

and applying $\text{Hom}_{\mathcal{T}_2}(F(T), -)$ to the triangle 2.1 we get the exact sequence

$$\begin{aligned} \text{Hom}_{\mathcal{T}_2}(F(T), X) &\longrightarrow \text{Hom}_{\mathcal{T}_2}(F(T), Y) \longrightarrow \text{Hom}_{\mathcal{T}_2}(F(T), Z) \longrightarrow \\ &\longrightarrow \text{Hom}_{\mathcal{T}_2}(F(T), \Sigma X) \longrightarrow \text{Hom}_{\mathcal{T}_2}(F(T), \Sigma Y) \end{aligned}$$

By adjunction we have a commutative diagram where the first two and the last two morphisms are isomorphism,

$$\begin{array}{ccccccc}
\mathrm{Hom}_{\mathcal{T}_1}(T, G(X)) & \longrightarrow & \mathrm{Hom}_{\mathcal{T}_1}(T, G(Y)) & \longrightarrow & \mathrm{Hom}_{\mathcal{T}_1}(T, K) & \longrightarrow & \\
\downarrow & & \downarrow & & \downarrow \beta_T & & \\
\mathrm{Hom}_{\mathcal{T}_2}(F(T), X) & \longrightarrow & \mathrm{Hom}_{\mathcal{T}_2}(F(T), Y) & \longrightarrow & \mathrm{Hom}_{\mathcal{T}_2}(F(T), Z) & \longrightarrow & \\
\\
& \longrightarrow & \mathrm{Hom}_{\mathcal{T}_1}(T, \Sigma G(X)) & \longrightarrow & \mathrm{Hom}_{\mathcal{T}_1}(T, \Sigma G(Y)) & & \\
& & \downarrow & & \downarrow & & \\
& \longrightarrow & \mathrm{Hom}_{\mathcal{T}_2}(F(T), \Sigma X) & \longrightarrow & \mathrm{Hom}_{\mathcal{T}_2}(F(T), \Sigma Y) & &
\end{array}$$

and for the five lemma we have that β_T is an isomorphism. We conclude that α is the counit of the adjunction, so K is isomorphic to $G(Z)$. Then

$$G(X) \xrightarrow{G(u)} G(Y) \xrightarrow{G(v)} G(Z) \xrightarrow{G(w)} G(\Sigma Z) = \Sigma G(Z)$$

is a triangle in \mathcal{T}_1 . □

Let \mathcal{D} be a triangulated subcategory of the triangulated category \mathcal{T} . We say that a morphism $f : X \rightarrow Y$ is a \mathcal{D} -morphism if its mapping cone is in \mathcal{D} .

The identity on any object is a \mathcal{D} -morphism because its mapping cone is 0 up to isomorphisms.

Lemma 2.1.2. *Let $u : X \rightarrow Y$ and $g : Y \rightarrow Z$ be morphisms in \mathcal{T} . If any two of f , g and $g \circ u$ are \mathcal{D} -morphisms, then the third is also a \mathcal{D} -morphism.*

Proof. By Proposition 1.3.3 we have a diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \longrightarrow & W & \longrightarrow & \Sigma X \\
\downarrow 1 & & \downarrow g & & \downarrow & & \downarrow 1 \\
X & \xrightarrow{g \circ u} & Z & \longrightarrow & W' & \longrightarrow & \Sigma X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & W'' & \xrightarrow{1} & W'' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma X & \xrightarrow{\Sigma u} & \Sigma Y & \longrightarrow & \Sigma W & \longrightarrow & \Sigma^2 X
\end{array}$$

where the third column, in particular, is a triangle. By assumption, two objects between W , W' and W'' are in \mathcal{D} , so the third one is also in \mathcal{D} . Then all of the three morphisms are \mathcal{D} -morphisms. □

In particular, this means that compositions of \mathcal{D} -morphisms are \mathcal{D} -morphisms.

Lemma 2.1.3. *The homotopy pullback of a \mathcal{D} -morphism is a \mathcal{D} -morphism.
The homotopy pushout of a \mathcal{D} -morphism is a \mathcal{D} -morphism.*

Proof. Suppose we are given the diagram

$$\begin{array}{ccc} & & Z \\ & & \downarrow g' \\ Y' & \xrightarrow{f'} & Z' \end{array}$$

where g' is a \mathcal{D} -morphism. Via the pullback we have an homotopy cartesian square

$$\begin{array}{ccc} Y & \xrightarrow{f} & Z \\ \downarrow g & & \downarrow g' \\ Y' & \xrightarrow{f'} & Z' \end{array}$$

By Lemma 1.3.2 there is a morphism of triangles

$$\begin{array}{ccccccc} Y & \xrightarrow{f} & Z & \longrightarrow & Y'' & \longrightarrow & \Sigma Y \\ \downarrow g & & \downarrow g' & & \downarrow 1 & & \downarrow \Sigma f \\ Y' & \xrightarrow{f'} & Z' & \longrightarrow & Y'' & \longrightarrow & \Sigma Y' \end{array}$$

Since f' is a \mathcal{D} -morphism, Y'' is an object in \mathcal{D} , so f is a \mathcal{D} -morphism.

Simmetrically we can prove that if g' is a \mathcal{D} -morphism, g is also a \mathcal{D} -morphism.

A dual argument shows that the same holds for the pushout. □

Consider the class of diagrams of the form

$$\begin{array}{ccc} & Z & \\ f \swarrow & & \searrow g \\ X & & Y \end{array}$$

where f is a \mathcal{D} -morphism, for brevity sake we will write $[Z, f, g]$ for such a diagram. We say that two diagrams $[Z, f, g]$ and $[Z, f', g']$ are equivalent if there exist a diagram $[Z'', f'', g'']$ and morphisms u and v such that the diagram

$$\begin{array}{ccccc} & & Z & & \\ & f \swarrow & \uparrow u & \searrow g & \\ X & & Z'' & \xrightarrow{g''} & Y \\ & \nwarrow f'' & \downarrow v & \nearrow g' & \\ & & Z' & & \end{array}$$

is commutative.

Proposition 2.1.4. *The relation defined above is an equivalence relation.*

Proof. Reflexivity and simmetry are obvious, so we only need to prove transitivity.

Suppose $[Z, f, g]$ is equivalent to $[Z', f', g']$ and $[Z', f', g']$ to $[Z'', f'', g'']$. Then we have the commutative diagrams

$$\begin{array}{ccccc}
 & & Z & & \\
 & f \swarrow & \uparrow \tilde{u} & \searrow g & \\
 X & \xleftarrow{\tilde{f}} & \tilde{Z} & \xrightarrow{\tilde{g}} & Y \\
 & f' \swarrow & \downarrow \tilde{v} & \searrow g' & \\
 & & Z' & &
 \end{array}$$

and

$$\begin{array}{ccccc}
 & & Z' & & \\
 & f' \swarrow & \uparrow \tilde{u} & \searrow g' & \\
 X & \xleftarrow{\tilde{f}} & \tilde{Z} & \xrightarrow{\tilde{g}} & Y \\
 & f'' \swarrow & \downarrow \tilde{v} & \searrow g'' & \\
 & & Z'' & &
 \end{array}$$

where \tilde{f} and \tilde{f} are \mathcal{D} -morphisms. By Lemma 2.1.2 \tilde{u} , \tilde{v} , \tilde{u} , \tilde{v} are all \mathcal{D} -morphisms.

Via the homotopy pullback \tilde{Y} we have the commutative square

$$\begin{array}{ccc}
 \tilde{Y} & \xrightarrow{w} & \tilde{Z} \\
 \downarrow w' & & \downarrow \tilde{v} \\
 \tilde{Z} & \xrightarrow{\tilde{u}} & Z'
 \end{array}$$

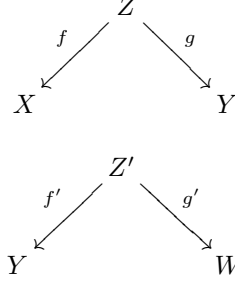
so by Lemma 2.1.2 w is a \mathcal{D} -morphism. $f^* = f' \circ \tilde{v} \circ w$ is a \mathcal{D} -morphism because it is a composition of \mathcal{D} -morphisms, so we have a diagram $[\tilde{Y}, f^*, g^*]$. The diagram

$$\begin{array}{ccccc}
 & & Z & & \\
 & f \swarrow & \uparrow \tilde{u} \circ w & \searrow g & \\
 X & \xleftarrow{f^*} & \tilde{Y} & \xrightarrow{g^*} & Y \\
 & f'' \swarrow & \downarrow \tilde{v} \circ w' & \searrow g'' & \\
 & & Z'' & &
 \end{array}$$

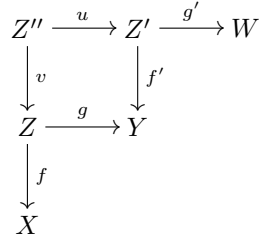
is commutative, so $[Z, f, g]$ is equivalent to $[Z'', f'', g'']$. □

We will call the equivalency classes of diagrams $[Z, f, g]$ with $f : Z \rightarrow X$ and $g : Z \rightarrow Y$ the *quotient morphisms* from X to Y .

If we have two diagrams $[Z, f, g]$ and $[Z', f', g']$ with



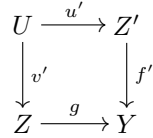
we can form the homotopy pullback Z'' of the morphisms g and f' in order to obtain a commutative diagram



v is a \mathcal{D} -morphism because it is obtained by homotopy pullback, so $f'' = f \circ v$ is a \mathcal{D} -morphisms.

We call the diagram $[Z'', f'', g'']$, where $g'' = g' \circ u$, the composition of the original two diagrams.

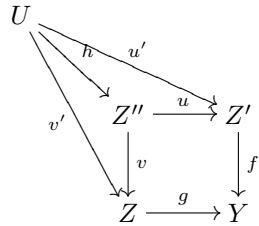
Lemma 2.1.5. *Let $[Z, f, g]$ and $[Z', f', g']$ be two diagrams as above, and let $[Z'', f'', g'']$ be their composition. If there exists a commutative square*



where v' is a \mathcal{D} -morphism, then the diagram $[U, f \circ v', g' \circ u']$ is equivalent to $[Z'', f'', g'']$.

Proof. $f \circ v'$ is a \mathcal{D} -morphism because it is a composition of \mathcal{D} -morphisms, so the diagram $[U, f \circ v', g' \circ u']$ is well defined.

Since Z'' is the homotopy pullback of g and f' , there is a morphism $h : U \rightarrow Z''$ such that the diagram



is commutative. So the diagram

$$\begin{array}{ccccc}
 & & U & & \\
 & f \circ v' \swarrow & \uparrow 1 & \searrow g' \circ u' & \\
 X & \xleftarrow{f \circ v'} & U & \xrightarrow{g' \circ u'} & Y \\
 & f \circ v \swarrow & \downarrow h & \searrow g' \circ u & \\
 & & Z'' & &
 \end{array}$$

commutes, hence $[Z'', f'', g'']$ is equivalent to $[U, f \circ v', g' \circ u']$. \square

Proposition 2.1.6. *The composition defined above is compatible with the equivalence relation on the diagrams.*

Proof. Let $[A, t, h]$ be a diagram equivalent to $[Z, f, g]$ via the commutative diagram

$$\begin{array}{ccccc}
 & & Z & & \\
 & f \swarrow & \uparrow u & \searrow g & \\
 X & \xleftarrow{s} & P & \xrightarrow{l} & Y \\
 & t \swarrow & \downarrow v & \searrow h & \\
 & & A & &
 \end{array}$$

and $[A', t', h']$ equivalent to $[Z', f', g']$ via the commutative diagram

$$\begin{array}{ccccc}
 & & Z' & & \\
 & f' \swarrow & \uparrow u' & \searrow g' & \\
 Y & \xleftarrow{s'} & P' & \xrightarrow{l'} & W \\
 & t' \swarrow & \downarrow v' & \searrow h' & \\
 & & A' & &
 \end{array}$$

The composition of $[Z, f, g]$ with $[Z', f', g']$ is the diagram $[Z'', f'', g'']$ given by

$$\begin{array}{ccccc}
 Z'' & \longrightarrow & Z' & \xrightarrow{g'} & W \\
 \downarrow & & \downarrow f' & & \\
 Z & \xrightarrow{f} & Y & & \\
 \downarrow g & & & & \\
 X & & & &
 \end{array}$$

while the composition of $[A, t, h]$ with $[A', t', h']$ is the diagram $[A'', t'', h'']$ given

by

$$\begin{array}{ccccc}
A'' & \longrightarrow & A' & \xrightarrow{h'} & W \\
\downarrow & & \downarrow t' & & \\
A & \xrightarrow{h} & Y & & \\
\downarrow t & & & & \\
X & & & &
\end{array}$$

We can also take the composition $[P'', s'', l'']$ of $[P, s, l]$ with $[P', s', l']$

$$\begin{array}{ccccc}
P'' & \longrightarrow & P' & \xrightarrow{l'} & W \\
\downarrow & & \downarrow s' & & \\
P & \xrightarrow{l} & Y & & \\
\downarrow s & & & & \\
X & & & &
\end{array}$$

Since $l = g \circ u = h \circ v$ and $s' = f' \circ u' = t' \circ v'$, we have the commutative squares

$$\begin{array}{ccc}
P'' & \longrightarrow & Z' \\
\downarrow & & \downarrow f' \\
Z & \xrightarrow{g} & Y
\end{array}$$

$$\begin{array}{ccc}
P'' & \longrightarrow & A' \\
\downarrow & & \downarrow t' \\
A & \xrightarrow{h} & Y
\end{array}$$

Then by Lemma 2.1.5 $[X'', f'', g'']$ and $[A'', t'', h'']$ are both equivalent to $[P'', s'', l'']$. □

Hence we have defined a composition law between quotient morphisms.

Proposition 2.1.7. *The composition between quotient morphisms defined above is associative.*

Proof. Let $[X, f, g]$ a quotient morphism between Y and Z , $[X', f', g']$ a quotient morphism between Z and W , $[X'', f'', g'']$ a quotient morphism between W and T . Let U be the homotopy pullback of g and f' , and U' the homotopy

pullback of g' and f'' , we have the homotopy cartesian squares

$$\begin{array}{ccc} U & \xrightarrow{u} & X' \\ \downarrow v & & \downarrow f' \\ X & \xrightarrow{g} & Z \end{array}$$

$$\begin{array}{ccc} U' & \xrightarrow{u'} & X'' \\ \downarrow v' & & \downarrow f'' \\ X' & \xrightarrow{g'} & W \end{array}$$

Then we can form the homotopy pullback V of u and v' , we obtain the homotopy cartesian square

$$\begin{array}{ccc} V & \xrightarrow{u''} & U' \\ \downarrow v'' & & \downarrow v' \\ U & \xrightarrow{u} & X' \end{array}$$

By Lemma 2.1.3 the morphisms v , v' and v'' are all \mathcal{D} -morphisms. So the diagram

$$\begin{array}{ccccccc} V & \xrightarrow{u''} & U' & \xrightarrow{u'} & X' & \xrightarrow{g''} & T \\ \downarrow v'' & & \downarrow v' & & \downarrow f'' & & \\ U & \xrightarrow{u} & X' & \xrightarrow{g'} & W & & \\ \downarrow v & & \downarrow f' & & & & \\ X & \xrightarrow{g} & Z & & & & \\ \downarrow f & & & & & & \\ Y & & & & & & \end{array}$$

commutes and every composition of the vertical morphisms is a \mathcal{D} -morphism. By Lemma 2.1.5 this means that composing in any of the two orders the three quotient morphisms we obtain the quotient morphism $[V, f \circ v \circ v'', g'' \circ u' \circ u'']$. \square

Note that the quotient morphism $[X, 1, 1]$ is the identity on X with respect to the composition.

Definition 2.1.8. We call Verdier quotient \mathcal{T}/\mathcal{D} the category whose objects are the objects of \mathcal{T} and the morphisms between two objects are the quotient morphisms between them.

We call F_{univ} the natural functor from \mathcal{T} to \mathcal{T}/\mathcal{D} which is the identity on the objects and sends a morphism $f : X \rightarrow Y$ to the quotient morphism $[X, 1, f]$.

2.2 Properties

Proposition 2.2.1. Let $f : X \rightarrow Y$ be a \mathcal{D} -morphism, then in \mathcal{T}/\mathcal{D} the quotient morphisms $[X, 1, f]$ and $[X, f, 1]$ are one the inverse of the other.

Proof. We have the commutative diagram

$$\begin{array}{ccccc}
 X & \xrightarrow{1} & X & \xrightarrow{1} & X \\
 \downarrow 1 & & \downarrow f & & \\
 X & \xrightarrow{f} & Y & & \\
 \downarrow 1 & & & & \\
 X & & & &
 \end{array}$$

so the composition of $[X, 1, f]$ and $[X, f, 1]$ is the equivalency class of $[X, 1, 1]$, which is the identity on X .

Since the diagram

$$\begin{array}{ccccc}
 X & \xrightarrow{1} & X & \xrightarrow{f} & Y \\
 \downarrow 1 & & \downarrow 1 & & \\
 X & \xrightarrow{1} & X & & \\
 \downarrow f & & & & \\
 Y & & & &
 \end{array}$$

is commutative, the composition of $[X, f, 1]$ and $[X, 1, f]$ is the equivalency class of $[X, f, f]$. Since the diagram

$$\begin{array}{ccccc}
 & & Y & & \\
 & \swarrow 1 & \uparrow f & \searrow 1 & \\
 Y & \xleftarrow{f} & X & \xrightarrow{f} & Y \\
 & \swarrow f & \downarrow 1 & \searrow f & \\
 & & X & &
 \end{array}$$

commutes, $[X, f, f]$ is equivalent to $[Y, 1, 1]$, which is the identity on Y . □

The proposition above means that the natural functor F_{univ} sends a \mathcal{D} -morphism in an invertible quotient morphism.

Note that any quotient morphism $[Z, f, g]$ from X to Y can be factorized as the composition $[Z, f, 1]$ with $[Z, 1, g]$. In fact, the diagram

$$\begin{array}{ccccc}
 Z & \xrightarrow{1} & Z & \xrightarrow{g} & Y \\
 \downarrow 1 & & \downarrow 1 & & \\
 Z & \xrightarrow{1} & Z & & \\
 \downarrow f & & & & \\
 X & & & &
 \end{array}$$

commutes. Hence we can write that $[Z, f, g] = F_{univ}(g) \circ F_{univ}(f)^{-1}$.

For any object X in \mathcal{D} we have the triangle

$$X \longrightarrow 0 \longrightarrow \Sigma X \xrightarrow{-1_{\Sigma X}} \Sigma X$$

Since \mathcal{D} is a triangulated subcategory, ΣX is in \mathcal{D} . Then $X \rightarrow 0$ is a \mathcal{D} -morphism, and consequently applying F_{univ} we obtain an isomorphism in \mathcal{T}/\mathcal{D} . Hence \mathcal{D} is contained in the kernel of F_{univ} .

Lemma 2.2.2. *Let f and g be two morphisms from X to Y in \mathcal{T} . Then $F_{univ}(f) = F_{univ}(g)$ if and only if there exists a \mathcal{D} -morphism $h : Z \rightarrow X$ such that $f \circ h = g \circ h$, and if and only if $f - g : X \rightarrow Y$ factors through an object in \mathcal{D} .*

Proof. $F_{univ}(f) = F_{univ}(g)$ if and only if there is a commutative diagram

$$\begin{array}{ccccc} & & X & & \\ & \swarrow 1 & \uparrow h & \searrow f & \\ X & \xleftarrow{u} & Z & \xrightarrow{v} & Y \\ & \swarrow 1 & \downarrow t & \searrow g & \\ & & X & & \end{array}$$

with u a \mathcal{D} -morphism. Then $h = t$ and by Lemma 2.1.2 it is a \mathcal{D} -morphism. So $F_{univ}(f) = F_{univ}(g)$ if and only if there exists a \mathcal{D} -morphism $h : Z \rightarrow X$ such that $f \circ h = g \circ h$, so that $(f - g) \circ h = 0$.

Let T be the mapping cone on a morphism $h' : Z \rightarrow X$, then we have a triangle

$$Z \xrightarrow{h'} X \longrightarrow T \longrightarrow \Sigma Z$$

Applying the cohomological functor $\text{Hom}(-, Y)$ to this triangle we obtain an exact sequence, so $(f - g) \circ h' = 0$ if and only if $f - g$ factors through T , and h' is a \mathcal{D} -morphism if and only if T is in \mathcal{D} . □

Proposition 2.2.3. *The Verdier's quotient \mathcal{T}/\mathcal{D} is an additive category and F_{univ} is an additive functor.*

Proof. Let 0 be the zero object for \mathcal{T} .

Let $[Y, f, 0]$ be any quotient morphism from an object X to 0 , then the diagram

$$\begin{array}{ccccc} & & Y & & \\ & \swarrow f & \uparrow 1 & \searrow & \\ X & \xleftarrow{f} & Y & \xrightarrow{} & 0 \\ & \swarrow f & \downarrow f & \searrow & \\ & & X & & \end{array}$$

commutes, so $[Y, f, 0]$ is equivalent to $[X, 1, 0]$. We conclude that there is only one quotient morphism from X to 0 , so it is a terminal object for \mathcal{T}/\mathcal{D} .

Let $[Y, 0, g]$ be any quotient morphism from 0 to an object X , then the diagram

$$\begin{array}{ccccc}
 & & Y & & \\
 & \swarrow & \uparrow 1 & \searrow g & \\
 0 & \xleftarrow{\quad} & Y & \xrightarrow{g} & X \\
 & \swarrow & \downarrow g & \searrow 1 & \\
 & & X & &
 \end{array}$$

commutes, so $[Y, 0, g]$ is equivalent to $[Y, 0, 1]$. We conclude that there is only one quotient morphism from 0 to X , so it is an initial object for \mathcal{T}/\mathcal{D} .

Then 0 is an initial and terminal object, hence it is the zero object for \mathcal{T}/\mathcal{D} .

Let X and Y be two objects in \mathcal{T} , we will prove that $X \oplus Y$ is a biproduct in \mathcal{T}/\mathcal{D} .

In \mathcal{T} there are morphisms p_1, p_2, i_1, i_2 ,

$$\begin{array}{ccc}
 & X \oplus Y & \\
 p_1 \swarrow & & \searrow p_2 \\
 X & & Y \\
 i_1 \searrow & & \swarrow i_2 \\
 & X \oplus Y &
 \end{array}$$

which give to $X \oplus Y$ the structure of product and coproduct in \mathcal{T} . We claim that the images of these morphisms via F_{univ} give to $X \oplus Y$ the structure of product and coproduct in \mathcal{T}/\mathcal{D} .

By duality, it suffices to show that $X \oplus Y$ is a coproduct in \mathcal{T}/\mathcal{D} .

Let $[Z, f, g]$ be a quotient morphism from X to an object T and $[Z', f', g']$ a quotient morphism from Y to T . f and f' are \mathcal{D} -morphisms, so in the triangles

$$Z \xrightarrow{f} X \longrightarrow P \longrightarrow \Sigma Z$$

$$Z' \xrightarrow{f'} Y \longrightarrow P' \longrightarrow \Sigma Z'$$

the mapping cones P and P' are in \mathcal{D} . Then by Proposition 1.2.9 the diagram

$$Z \oplus Z' \xrightarrow{f \oplus f'} X \oplus Y \longrightarrow P \oplus P' \longrightarrow \Sigma Z \oplus Z'$$

is a triangle in \mathcal{T} , so the mapping cone on $f \oplus f'$ is in \mathcal{D} , thus $f \oplus f'$ is a \mathcal{D} -morphism and $[Z \oplus Z', f \oplus f', (g \ g')]$ is a quotient morphism. If we compose

it with $[X, 1, i_1]$, since the diagram

$$\begin{array}{ccccc}
Z & \longrightarrow & Z \oplus Z' & \xrightarrow{(g \ g')} & T \\
\downarrow f & & \downarrow f \oplus f' & & \\
X & \xrightarrow{i_1} & X \oplus Y & & \\
\downarrow 1 & & & & \\
X & & & &
\end{array}$$

commutes, we obtain $[Z, f, g]$. If we instead compose it with $[X, 1, i_2]$ we obtain $[Z', f', g']$

So every pair of quotient morphisms from X and Y to an object T factors through $[X, 1, i_1]$ and $[X, 1, i_2]$.

Now we have to prove that the factorization is unique, so let $[T, u, v]$ a quotient morphism from $X \oplus Y$ to T that composed with $[X, 1, i_1]$ gives $[Z, f, g]$ and composed with $[X, 1, i_2]$ gives $[Z', f', g']$.

Let V be the homotopy pullback of $f \oplus f'$ and u , we have the homotopy cartesian square

$$\begin{array}{ccc}
V & \xrightarrow{t} & T \\
\downarrow h & & \downarrow u \\
Z \oplus Z' & \xrightarrow{f \oplus f'} & X \oplus Y
\end{array}$$

Suppose $f'' = (f \oplus f') \circ h = u \circ t$, $g'' = (g \ g') \circ h$ and $v'' = v \circ t$, then $[V, f'', g'']$ is equivalent to $[Z \oplus Z', f \oplus f', (g \ g')]$ and $[V, f'', v'']$ is equivalent to $[T, u, v]$.

By our assumption on the compositions of these quotient morphisms with $[X, 1, i_1]$ and $[X, 1, i_2]$ we have

$$F_{univ}(g'' \circ i_1) \circ F_{univ}(f'')^{-1} = F_{univ}(v'' \circ i_1) \circ F_{univ}(f'')^{-1}$$

$$F_{univ}(g'' \circ i_2) \circ F_{univ}(f'')^{-1} = F_{univ}(v'' \circ i_2) \circ F_{univ}(f'')^{-1}$$

so it is $F_{univ}(g'' \circ i_1) = F_{univ}(v'' \circ i_1)$ and $F_{univ}(g'' \circ i_2) = F_{univ}(v'' \circ i_2)$. By Lemma 2.2.2 then $g'' \circ i_1 - v'' \circ i_1 = (g'' - v'') \circ i_1$ factors through an object W of \mathcal{D} and $(g'' - v'') \circ i_2$ factors through an object W' of \mathcal{D} . It follows that $g'' - v''$ factors through $W \oplus W'$ and by Lemma 2.2.2 it is $F_{univ}(g'') = F_{univ}(v'')$.

So $F_{univ}(g'') \circ F_{univ}(f'')^{-1} = F_{univ}(v'') \circ F_{univ}(f'')^{-1}$ and this prove the uniqueness.

Since F_{univ} respects biproducts and the zero object, we can define the addition of two quotient morphisms $\alpha, \beta : X \rightarrow Y$ as one of the two identical compositions of quotient morphisms

$$\begin{array}{ccc}
X & \xrightarrow{\Delta} & X \oplus X \xrightarrow{(f \ g)} Y \\
X & \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} & Y \oplus Y \xrightarrow{(1_Y \ 1_Y)} Y
\end{array}$$

With this definition, the natural functor F_{univ} respects sums of morphisms.

Let α be a quotient morphism, then $\alpha = F_{univ}(f)^{-1} \circ F_{univ}(g)$ for some morphisms f and g in \mathcal{T} , where f is a \mathcal{D} -morphism. Then

$$\begin{aligned} F_{univ}(g) \circ F_{univ}(f)^{-1} + F_{univ}(g) \circ F_{univ}(f)^{-1} &= (F_{univ}(g) + F_{univ}(-g)) \circ F_{univ}(f)^{-1} = \\ &= F_{univ}(g - g) \circ F_{univ}(f)^{-1} = F_{univ}(0) \circ F_{univ}(f)^{-1} = 0 \end{aligned}$$

so $-\alpha = F_{univ}(-g) \circ F_{univ}(f)^{-1}$ is an additive inverse to α . \square

Lemma 2.2.4. *Suppose we are given a commutative square in \mathcal{T}/\mathcal{D}*

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ Z' & \longrightarrow & W' \end{array}$$

Then there is a commutative square in \mathcal{T}

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & W \end{array}$$

whose image in \mathcal{T}/\mathcal{D} is isomorphic to the first diagram.

Proof. In the diagram used to define the composition of quotient morphisms all the vertical morphisms are \mathcal{D} -morphisms, so their images via F_{univ} are isomorphisms. This means that F_{univ} sends the first row of the diagram in a composite of quotient morphisms equivalent to the full diagram.

The compositions $X' \rightarrow Y' \rightarrow W'$ and $X' \rightarrow Z' \rightarrow W'$ in \mathcal{T}/\mathcal{D} are diagrams in \mathcal{T} . Let $T_1 \rightarrow Y \rightarrow W'$ and $T_2 \rightarrow Z \rightarrow W'$ be the first row of these diagrams. Let T be the homotopy pullback of $T_1 \rightarrow X$ and $T_2 \rightarrow X$, then we can replace T_1 and T_2 with T . The diagram in \mathcal{T}/\mathcal{D}

$$\begin{array}{ccc} T & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & W' \end{array}$$

commutes, so by Lemma 2.2.2 there exists a \mathcal{D} -morphism $X \rightarrow T$ that composed with $T \rightarrow Y \rightarrow W'$ and $T \rightarrow Z \rightarrow W'$ gives the equal composites $X \rightarrow Y \rightarrow W'$ and $X \rightarrow Z \rightarrow W'$. If we take $W = W'$ we have the required commutative diagram in \mathcal{T} . \square

Lemma 2.2.5. *If the quotient morphism $[Z, f, g]$ from X to X is equivalent to the identity quotient morphism $[X, 1_X, 1_X]$ then g is a \mathcal{D} -morphism.*

Proof. $[Z, f, g]$ and $[X, 1_X, 1_X]$ are equivalent if and only if there is a commutative diagram

$$\begin{array}{ccccc}
 & & Z & & \\
 & \swarrow & \uparrow u & \searrow g & \\
 X & \xleftarrow{f} & T & \xrightarrow{h'} & X \\
 & \swarrow h & \downarrow v & \searrow 1_X & \\
 & & X & &
 \end{array}$$

where h is a \mathcal{D} -morphism. u and v are \mathcal{D} -morphisms by Lemma 2.1.2 and $h' = h = v$ because the diagram is commutative. Since $h' = v = g \circ u$, by Lemma 2.1.2 we deduce that g is a \mathcal{D} -morphism. \square

Lemma 2.2.6. *Let $F_{univ}(g) \circ F_{univ}(f)^{-1}$ be a quotient morphism in \mathcal{T}/\mathcal{D} , with $f : X' \rightarrow X$ a \mathcal{D} -morphism and $g : X' \rightarrow Y$. It is an isomorphism if and only if there exist morphisms $h : Z \rightarrow X'$ and $h' : Y \rightarrow Z'$ in \mathcal{T} such that $h' \circ g$ and $g \circ h$ are \mathcal{D} -morphisms.*

Proof. Suppose there are morphisms $h : Z \rightarrow X'$ and $h' : Y \rightarrow Z'$ in \mathcal{T} such that $h' \circ g$ and $g \circ h$ are \mathcal{D} -morphisms. Then $F_{univ}(h' \circ g)$ and $F_{univ}(g \circ h)$ are invertible in \mathcal{T}/\mathcal{D} , so $F_{univ}(g)$ has left and right inverse in \mathcal{T}/\mathcal{D} and is therefore invertible. Since $F_{univ}(f)^{-1}$ is invertible, $F_{univ}(g) \circ F_{univ}(f)^{-1}$ is an isomorphism.

Suppose $F_{univ}(g) \circ F_{univ}(f)^{-1}$ is an isomorphism, then $F_{univ}(g)$ is also an isomorphism.

Let the quotient morphism

$$\begin{array}{ccc}
 Z & \xrightarrow{h} & X \\
 \downarrow t & & \\
 Y & &
 \end{array}$$

be a right inverse to $F_{univ}(g)$, then the composition of the two quotient morphisms $[Z, t, g \circ h]$ is equivalent to $[Y, 1_Y, 1_Y]$. By Lemma 2.2.5 $g \circ h$ is a \mathcal{D} -morphism.

Let the quotient morphism

$$\begin{array}{ccc}
 Z' & \xrightarrow{h'} & X' \\
 \downarrow t' & & \\
 Y & &
 \end{array}$$

be a right inverse to $F_{univ}(g)$, then the composition of the two quotient morphisms $[X, 1_X, h' \circ g]$ is equivalent to $[X, 1_X, 1_X]$. By Lemma 2.2.5 $h' \circ g$ is a \mathcal{D} -morphism. \square

Lemma 2.2.7. *Let f be the morphism $X \rightarrow 0$ in \mathcal{T} , then $F_{univ}(f)$ is an isomorphism in \mathcal{T}/\mathcal{D} if and only if X is a direct summand of an object in \mathcal{D} .*

Proof. Suppose $F_{univ}(f)$ is an isomorphism. By Lemma 2.2.6 there exists a morphism $g : 0 \rightarrow \Sigma Y$ such that $g \circ f = 0 : X \rightarrow \Sigma Y$ is a \mathcal{D} -morphism. Then from the triangle

$$X \xrightarrow{0} \Sigma Y \longrightarrow \Sigma X \oplus \Sigma Y \longrightarrow \Sigma X$$

we deduce that $\Sigma(X \oplus Y)$ is an object in \mathcal{D} , since \mathcal{D} is a triangulated subcategory $X \oplus Y$ is an object in \mathcal{D} .

Let X be a direct summand of an object in \mathcal{D} , then there is an object Y such that $X \oplus Y$ is in \mathcal{D} . Let $g : 0 \rightarrow X$ and $h : 0 \rightarrow \Sigma Y$, then $f \circ g : 0 \rightarrow 0$ is an isomorphism and $h \circ f : X \rightarrow \Sigma Y$ is the zero morphism. $h \circ f$ is a \mathcal{D} -morphism because the diagram

$$X \xrightarrow{0} \Sigma Y \longrightarrow \Sigma X \oplus \Sigma Y \longrightarrow \Sigma X$$

is a triangle and $\Sigma(X \oplus Y)$ is in \mathcal{D} since \mathcal{D} is a triangulated subcategory. By Lemma 2.2.6 $F_{univ}(f)$ is an isomorphism in \mathcal{T}/\mathcal{D} . □

Lemma 2.2.8. *The image $F_{univ}(f)$ of a morphism $f : X \rightarrow Y$ in \mathcal{T} is invertible if and only if the mapping cone on it is a direct summand of some object in \mathcal{D} .*

Proof. Suppose $F_{univ}(f)$ is invertible, then by Lemma 2.2.6 there is a morphism $g : Y \rightarrow Y'$ such that $g \circ f$ is a \mathcal{D} -morphism. We have the following morphism of triangles in \mathcal{T} :

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{h} & Z & \longrightarrow & \Sigma X \\ \downarrow g \circ f & & \downarrow \begin{pmatrix} g \\ h \end{pmatrix} & & \downarrow 1 & & \downarrow \Sigma(g \circ f) \\ Y' & \longrightarrow & Y' \oplus Z & \longrightarrow & Z & \xrightarrow{0} & \Sigma Y' \end{array}$$

The second row is a contractible triangle, so by Lemma 1.3.8 the mapping cone on this morphism of triangles is a triangle. Then the commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow g \circ f & & \downarrow \begin{pmatrix} g \\ h \end{pmatrix} \\ Y' & \longrightarrow & Y' \oplus Z \end{array}$$

is homotopy cartesian by Lemma 1.3.1. The morphism

$$\alpha = \begin{pmatrix} g \\ h \end{pmatrix} : Y \rightarrow Y' \oplus Z$$

is a \mathcal{D} -morphism because it is a pushout of the \mathcal{D} -morphism $g \circ f$. Then $F_{univ}(\alpha)$ is invertible in \mathcal{T}/\mathcal{D} , so $F_{univ}(\alpha) \circ F_{univ}(f) = F_{univ}(\alpha \circ f)$ is invertible.

Since $h \circ f = 0$ it is

$$\alpha \circ f = \begin{pmatrix} g \\ h \end{pmatrix} \circ f = \begin{pmatrix} g \circ f \\ 0 \end{pmatrix} = (g \circ f) \circ \begin{pmatrix} 1_{Y'} \\ 0 \end{pmatrix}$$

$F_{univ}(g \circ f)$ is invertible, so the inclusion $Y' \rightarrow Y' \oplus Z$ is an isomorphism in \mathcal{T}/\mathcal{D} and its inverse is the projection $Y' \oplus Z \rightarrow Y'$. Then applying F_{univ} to the composition

$$Y' \oplus Z \xrightarrow{\begin{pmatrix} 1_{Y'} & 0 \end{pmatrix}} Y' \xrightarrow{\begin{pmatrix} 1_Y \\ 0 \end{pmatrix}} Y' \oplus Z$$

we obtain the identity on $Y' \oplus Z$ in \mathcal{T}/\mathcal{D} . This implies that the images in \mathcal{T}/\mathcal{D} of the maps

$$Y' \oplus Z \xrightarrow{\begin{pmatrix} 1_{Y'} & 0 \\ 0 & 0 \end{pmatrix}} Y' \oplus Z$$

$$Y' \oplus Z \xrightarrow{\begin{pmatrix} 1_{Y'} & 0 \\ 0 & 1_Z \end{pmatrix}} Y' \oplus Z$$

coincide, which in turn means that $F_{univ}(1_Z)$ coincide with the zero morphism from Z to Z in \mathcal{T}/\mathcal{D} , so the morphisms $0 \rightarrow Z$ and $Z \rightarrow 0$ are inverse to each other in \mathcal{T}/\mathcal{D} . We have proved that Z is isomorphic to 0 in \mathcal{T}/\mathcal{D} , so by Lemma 2.2.7 it is a direct summand of an object in \mathcal{D} .

Suppose that the mapping cone Z on f is a direct summand of an object in \mathcal{D} , that is there exists an object Z' such that $Z \oplus Z'$ lies in \mathcal{D} . We have two triangles

$$X \xrightarrow{f} Y \xrightarrow{h} Z \longrightarrow \Sigma X$$

$$0 \longrightarrow X' \xrightarrow{1_{X'}} X' \longrightarrow 0$$

and their direct sum

$$X \xrightarrow{\begin{pmatrix} f \\ 0 \end{pmatrix}} Y \oplus X' \longrightarrow Z \oplus Z' \longrightarrow \Sigma X$$

is a triangle by Proposition 1.2.9. Since $Z \oplus Z'$ is in \mathcal{D} the morphism

$$\begin{pmatrix} f \\ 0 \end{pmatrix} : X \rightarrow Y \oplus X'$$

is a \mathcal{D} -morphism and it factors through Y via the morphism $f : X \rightarrow Y$ and the inclusion $g : Y \rightarrow Y \oplus X'$. Then $g \circ f$ is a \mathcal{D} -morphism.

Dually, we can find a morphism u such that $f \circ u$ is a \mathcal{D} -morphism, so by Lemma 2.2.6 $F_{univ}(f)$ is an isomorphism in \mathcal{T}/\mathcal{D} . □

Lemma 2.2.9. *If we have a commutative diagram in \mathcal{T}*

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ \downarrow 1 & & \downarrow g & & & & \downarrow 1 \\ X & \xrightarrow{g \circ f} & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X \end{array}$$

whose rows are triangles and with $F_{univ}(g)$ invertible in \mathcal{T}/\mathcal{D} , then there exists a morphism $h : Z \rightarrow Z'$ that extends the diagram to a morphism of triangles in \mathcal{T} and such that $F_{univ}(h)$ is invertible.

Dually, if there is a morphism $h : Z \rightarrow Z'$ whose image in the Verdier localisation is invertible and such that the diagram in \mathcal{T}

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ \downarrow 1 & & & & \downarrow h & & \downarrow 1 \\ X & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X \end{array}$$

commutes, then there is a morphism $g : Y \rightarrow Y'$ that extends the diagram to a morphism of triangles and such that $F_{univ}(g)$ is invertible.

Proof. By Lemma 1.3.3 there is a morphism $h : Z \rightarrow Z'$ such that the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ \downarrow 1 & & \downarrow g & & \downarrow h & & \downarrow 1 \\ X & \xrightarrow{g \circ f} & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X \end{array}$$

is a morphism of triangles in \mathcal{T} and the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ \downarrow 1 & & \downarrow g & & \downarrow h & & \downarrow 1 \\ X' & \xrightarrow{g \circ f} & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X' \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & W & \xrightarrow{1} & W & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Sigma X & \xrightarrow{\Sigma f} & \Sigma Y & \longrightarrow & \Sigma Z & \longrightarrow & \Sigma^2 X \end{array}$$

commutes in \mathcal{T} . By Lemma 2.2.8 $F_{univ}(g)$ is invertible if and only if the mapping cone W on it is a direct summand of an object in \mathcal{D} , since W is the mapping cone on h this means that $F_{univ}(g)$ is invertible if and only if $F_{univ}(h)$ is invertible. \square

Lemma 2.2.10. *If we are given two triangles in \mathcal{T}*

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow \Sigma X$$

$$X' \xrightarrow{f'} Y' \longrightarrow Z' \longrightarrow \Sigma X'$$

and there are two isomorphisms $X \rightarrow X'$, $Y \rightarrow Y'$ in \mathcal{T}/\mathcal{D} such that the square

$$\begin{array}{ccc} X & \xrightarrow{F_{univ}(f)} & Y \\ \downarrow & & \downarrow \\ X' & \xrightarrow{F_{univ}(f')} & Y' \end{array}$$

commutes, we can extend the square to an isomorphism of triangles in \mathcal{T}/\mathcal{D} .

Proof. We can write $Y \rightarrow Y'$ as a composition $F_{univ}(f) \circ F_{univ}(f')^{-1}$ with $f : B \rightarrow Y'$ invertible and $f' : B \rightarrow Y$ a \mathcal{D} -morphism, so $F_{univ}(f)$ and $F_{univ}(f')$ are both invertible. The diagrams

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & Z & \longrightarrow & \Sigma A \\ & & \downarrow f & & \downarrow 1 & & \\ X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \end{array}$$

$$\begin{array}{ccccccc} A' & \longrightarrow & B & \longrightarrow & Z & \longrightarrow & \Sigma A' \\ & & \downarrow f' & & \downarrow 1 & & \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X' \end{array}$$

whose rows are triangles in \mathcal{T} can be completed to isomorphisms of diagrams in \mathcal{T}/\mathcal{D} by Lemma 2.2.9. Then we can suppose $Y = Y'$ and that the isomorphism $Y \rightarrow Y'$ is the identity.

We can write $X \rightarrow X'$ as a composition $F_{univ}(g) \circ F_{univ}(g')^{-1}$ with $g : A \rightarrow X'$ invertible and $g' : A \rightarrow X$ a \mathcal{D} -morphism. Then there is a commutative square

$$\begin{array}{ccc} A & \xrightarrow{g} & X' \\ \downarrow g' & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

By Lemma 2.2.4 there is an element T and a \mathcal{D} -morphism $T \rightarrow A$ in \mathcal{T} such that the square

$$\begin{array}{ccc} T & \longrightarrow & X' \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

commutes in \mathcal{T} , so we can replace A with T . Then the diagrams

$$\begin{array}{ccccccc} A & \longrightarrow & Y & \longrightarrow & C & \longrightarrow & \Sigma A \\ \downarrow g' & & \downarrow 1 & & & & \downarrow \Sigma g' \\ X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \end{array}$$

$$\begin{array}{ccccccc} A & \longrightarrow & Y & \longrightarrow & C & \longrightarrow & \Sigma A \\ \downarrow g & & \downarrow 1 & & & & \downarrow \Sigma g \\ X' & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \end{array}$$

commute in \mathcal{T} and their rows are triangles, so they can be completed to isomorphisms in \mathcal{T}/\mathcal{D} by Lemma 2.2.9. □

Proposition 2.2.11. *\mathcal{T}/\mathcal{D} is a triangulated category and F_{univ} is a triangulated functor.*

Proof. Applying the suspension functor Σ of \mathcal{T} on the objects of \mathcal{T}/\mathcal{D} and on the diagrams that define quotient morphisms we obtain a suspension functor on \mathcal{T}/\mathcal{D} . This functor respects the equivalence relation on diagrams $[X, f, g]$, so it is well defined. We will call this functor $\tilde{\Sigma}$. Note that $F_{univ}(\Sigma X) = \tilde{\Sigma}F_{univ}(X)$ for any object X in \mathcal{T} .

We define the triangles in \mathcal{T}/\mathcal{D} as the candidate triangles that are isomorphic to a diagram

$$F_{univ}(X) \xrightarrow{F_{univ}(f)} F_{univ}(Y) \xrightarrow{F_{univ}(g)} F_{univ}(Z) \xrightarrow{F_{univ}(h)} F_{univ}(\Sigma X)$$

where the diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

is a triangle in \mathcal{T} .

Axioms [T1], [T2] and [T4] of Definition 1.2.1 obviously hold.

Let $F_{univ}(f) \circ F_{univ}(f')^{-1}$ be a quotient morphism from X to Y in \mathcal{T}/\mathcal{D} , where f' is a \mathcal{D} -morphism from an object X' to X and $f : X' \rightarrow Y$. Then there is a triangle

$$X' \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X'$$

in \mathcal{D} , so the diagram

$$F_{univ}(X') \xrightarrow{F_{univ}(f)} F_{univ}(Y) \xrightarrow{F_{univ}(g)} F_{univ}(Z) \xrightarrow{F_{univ}(h)} F_{univ}(\Sigma X')$$

is a triangle in \mathcal{T}/\mathcal{D} . The quotient morphisms $F_{univ}(f')$ and $F_{univ}(\Sigma f')$ are isomorphism, so the triangle obtained above is isomorphic to the candidate triangle

$$F_{univ}(X') \xrightarrow{F(f) \circ F(f')^{-1}} F_{univ}(Y) \xrightarrow{F(g)} F_{univ}(Z) \xrightarrow{F(\Sigma f') \circ F(h)} F_{univ}(\Sigma X')$$

where F stands for F_{univ} , which consequently is a triangle. This proves that axiom [T3] of Definition 1.2.1 holds in \mathcal{T}/\mathcal{D} .

Suppose we are given a commutative diagram in \mathcal{T}/\mathcal{D}

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ \downarrow & & \downarrow & & & & \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X' \end{array} \quad (2.1)$$

whose rows are triangles. By Lemma 2.2.4 there is a commutative square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B' \end{array}$$

in \mathcal{T} whose image in \mathcal{T}/\mathcal{D} is isomorphic to the first square of 2.1.

Taking the mapping cone C and C' on the horizontal morphisms in \mathcal{T} , by axiom [T5] of Definition 1.2.1 we obtain a morphism of triangles in \mathcal{T}

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & \Sigma A \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & \Sigma A' \end{array} \quad (2.2)$$

whose image in \mathcal{T}/\mathcal{D} is still a morphism of triangles. Furthermore, the mapping cone on this morphism of triangles is a triangle.

By Lemma 2.2.10 the commutative square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

in \mathcal{T}/\mathcal{D} where the vertical morphisms are isomorphisms can be extended to an isomorphism of triangles in \mathcal{T}/\mathcal{D}

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & \Sigma A \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \end{array}$$

Analogously, we can obtain an isomorphism of triangles in \mathcal{T}/\mathcal{D}

$$\begin{array}{ccccccc} A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & \Sigma A' \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X' \end{array}$$

Composing these isomorphisms of triangles with the morphism 2.2 we complete the initial diagram in \mathcal{T}/\mathcal{D} to a morphism of triangles whose mapping cone is a triangle. Then [T5] holds and \mathcal{T}/\mathcal{D} is a triangulated category.

We have already proved that F_{univ} is an additive functor, by our definition of triangles in \mathcal{T}/\mathcal{D} it is also a triangulated functor. \square

Proposition 2.2.12. *The functor F_{univ} is universal for all functors F such that $F(f)$ is an isomorphism for any \mathcal{D} -morphisms f in \mathcal{T} .*

Proof. Let $F : \mathcal{T} \rightarrow \mathcal{C}$ be a functor such that $F(f)$ is an isomorphism for any \mathcal{D} -morphisms f in \mathcal{T} . We can apply F to the diagrams $[Z, f, g]$. Since in the diagram that defines the equivalence relation the vertical morphisms are \mathcal{D} -morphisms, by Lemma 2.1.2 F takes equivalent diagrams to isomorphic diagrams.

Then the functor $F' : \mathcal{T}/\mathcal{D} \rightarrow \mathcal{C}$ with $F'(X) = F(X)$ for any object X in \mathcal{T}/\mathcal{D} and $F'([Z, f, g]) = F(g) \cdot F(f)^{-1}$ is well defined. We have that $F = F' \circ F_{univ}$, this shows the universality of F_{univ} . \square

An object Z is in the kernel of a functor F if and only if $F(h)$ is an isomorphism for $h : Z \rightarrow 0$. If \mathcal{D} is contained in the kernel of a triangulated functor F then if f is a \mathcal{D} -morphism with mapping cone Z we have a triangle

$$F(X) \xrightarrow{F(f)} F(Y) \longrightarrow F(Z) \longrightarrow F(\Sigma X)$$

We have the commutative diagram

$$\begin{array}{ccccc} F(X) & \xrightarrow{F(f)} & F(Y) & \longrightarrow & F(Z) \longrightarrow F(\Sigma X) \\ \downarrow 1_{F(X)} & & & & \downarrow 1_{F(\Sigma X)} \\ F(X) & \xrightarrow{1_{F(X)}} & F(X) & \longrightarrow & 0 \longrightarrow F(\Sigma X) \end{array}$$

By Definition 1.2.1[T5] and Lemma 1.2.8 we can complete it to an isomorphism of triangles, so $F(f)$ is an isomorphism. Requiring that \mathcal{D} is contained in the kernel of F then is equivalent to require that $F(f)$ is an isomorphism for any \mathcal{D} -morphism f in \mathcal{T} .

The following theorem summarize the results of this section.

Theorem 2.2.13. *Let \mathcal{T} be a triangulated category and $\mathcal{D} \subset \mathcal{T}$ a triangulated subcategory. Then there exists a triangulated category \mathcal{T}/\mathcal{D} , called the Verdier quotient of \mathcal{T} by \mathcal{D} , and a triangulated functor $F_{univ} : \mathcal{T} \rightarrow \mathcal{T}/\mathcal{D}$ such that \mathcal{D} is in the kernel of F_{univ} , and F_{univ} is universal with this property. If $F : \mathcal{T} \rightarrow \mathcal{C}$ is a triangulated functor whose kernel contains \mathcal{D} , then it factors uniquely as*

$$\mathcal{T} \xrightarrow{F_{univ}} \mathcal{T}/\mathcal{D} \longrightarrow \mathcal{C}.$$

Chapter 3

Brown representability

In this chapter we define compactly generated triangulated categories, studying their behaviour with regard to localisation. Finally, we state Brown representability's theorem and obtain as a corollary a theorem of existence of a right adjoint functor for triangulated functors that respect coproducts. This results are in [8], though some of the proofs have been presented in [7].

3.1 Compact objects and Thomason's localisation theorem

Throughout this section, let \mathcal{T} be a triangulated category that contain countable coproducts.

Definition 3.1.1. *An object T of \mathcal{T} is called compact if, for any coproduct of objects in \mathcal{T} , we have*

$$\mathrm{Hom}\left(T, \coprod_{\lambda \in \Lambda} X_{\lambda}\right) = \coprod_{\lambda \in \Lambda} \mathrm{Hom}(T, X_{\lambda}).$$

We will call \mathcal{T}^c the full subcategory of compact objects in \mathcal{T} .

It is trivial that the suspension of a compact object is compact.

Lemma 3.1.2. *Let T be a compact object of \mathcal{T} and*

$$X_0 \xrightarrow{j_1} X_1 \xrightarrow{j_2} X_2 \xrightarrow{j_3} \dots$$

be a sequence of objects and morphisms in \mathcal{T} .

Then there is a natural isomorphism

$$\mathrm{colim} \mathrm{Hom}(T, X_n) \rightarrow \mathrm{Hom}(T, \mathrm{Hocolim}(X_n)).$$

Proof. Applying the homological functor $\mathrm{Hom}(T, -)$ to the triangle defining $\mathrm{Hocolim}(X_n)$ we obtain in particular the exact sequence

$$\mathrm{Hom}(T, \mathrm{Hocolim}(X_n)) \longrightarrow \mathrm{Hom}\left(T, \Sigma \coprod_{n=0}^{\infty} X_n\right) \xrightarrow{\circ 1-shift} \mathrm{Hom}\left(T, \Sigma \coprod_{n=0}^{\infty} X_n\right)$$

Since Σ commutes with the coproduct and T is a compact object, we have the commutative diagram

$$\begin{array}{ccccc} \mathrm{Hom}(T, \mathrm{Hocolim}(X_n)) & \longrightarrow & \mathrm{Hom}(T, \Sigma \coprod_{n=0}^{\infty} X_n) & \longrightarrow & \mathrm{Hom}(T, \Sigma \coprod_{n=0}^{\infty} X_n) \\ \downarrow & & \downarrow 1\text{-shift} & & \downarrow \\ \mathrm{Hom}(T, \mathrm{Hocolim}(X_n)) & \longrightarrow & \bigoplus_{n=0}^{\infty} \mathrm{Hom}(T, \Sigma X_n) & \xrightarrow{1\text{-shift}} & \bigoplus_{n=0}^{\infty} \mathrm{Hom}(T, \Sigma X_n) \end{array}$$

where the vertical maps are isomorphisms. The map $1 - \text{shift}$ on the second row is obviously injective, so the morphism

$$\mathrm{Hom}(T, \mathrm{Hocolim}(X_n)) \rightarrow \mathrm{Hom}\left(T, \Sigma \coprod_{n=0}^{\infty} X_n\right)$$

is the zero morphism.

Then, since T is compact, we have the commutative diagram

$$\begin{array}{ccccc} \mathrm{Hom}(T, \coprod_{n=0}^{\infty} X_n) & \xrightarrow{1\text{-shift}} & \mathrm{Hom}(T, \coprod_{n=0}^{\infty} X_n) & \longrightarrow & \\ \downarrow & & \downarrow & & \\ \bigoplus_{n=0}^{\infty} \mathrm{Hom}(T, X_n) & \xrightarrow{1\text{-shift}} & \bigoplus_{n=0}^{\infty} \mathrm{Hom}(T, X_n) & \longrightarrow & \\ & & & & \\ & & \longrightarrow \mathrm{Hom}(T, \mathrm{Hocolim}(X_n)) \longrightarrow 0 & & \\ & & \downarrow & & \\ & & \longrightarrow \mathrm{Hom}(T, \mathrm{Hocolim}(X_n)) \longrightarrow 0 & & \end{array}$$

where the columns are isomorphisms and the first row is an exact sequence. Hence the second row is exact, so it identifies the direct limit $\mathrm{colim} \mathrm{Hom}(T, X_n)$ with $\mathrm{Hom}(T, \mathrm{Hocolim}(X_n))$ via a natural isomorphism. \square

Definition 3.1.3. Let R be a collection of objects in \mathcal{T} . We say that an object X is R -local if for any object Y in R we have $\mathrm{Hom}(Y, X) = 0$.

\mathcal{T} is said compactly generated if there exists a set Ω of compact objects of \mathcal{T} such that the zero object is the only Ω -local object (that is, $\mathrm{Hom}(\Omega, T) = 0 \Rightarrow T = 0$).

A set Ω of compact object of \mathcal{T} is called a generating set if

1. $\mathrm{Hom}(\Omega, T) = 0 \Rightarrow T = 0$;
2. Ω is closed under suspension ($\Omega = \Sigma\Omega$).

From now on, we will assume that \mathcal{T} is compactly generated, that R is a set of compact objects closed under suspension and \mathcal{R} the smallest full, triangulated subcategory of \mathcal{T} containing R and closed with respect to coproducts.

Lemma 3.1.4. Let $X \in \mathcal{T}$, then there exists an R -local element Y in \mathcal{T} and a \mathcal{R} -morphism $f : X \rightarrow Y$.

Proof. Let $X \in \mathcal{T}$, we define by induction a sequence of objects and morphisms in \mathcal{T} .

Let $X = X_0$, then let I be the set of all morphisms $\alpha_i : T_i \rightarrow X_n$ with $T_i \in R$ and define X_{n+1} as the mapping cone on

$$\coprod_{i \in I} T_i \rightarrow X_n,$$

so we have a \mathcal{R} -morphism $X_n \rightarrow X_{n+1}$.

We define $Y = \text{Hocolim}(X_n)$. Since every $X_n \rightarrow X_{n+1}$ is an isomorphism in \mathcal{T}/\mathcal{R} , the canonical map $X = X_0 \rightarrow Y$ is also an isomorphism in \mathcal{T}/\mathcal{R} , so it is a \mathcal{R} -morphism.

Let $T \in R$, by Lemma 3.1.2 there is an isomorphism

$$\text{colim Hom}(T, X_n) \rightarrow \text{Hom}(T, Y).$$

From our construction it follows that

$$\text{Hom}(T, X_n) \rightarrow \text{Hom}(T, X_{n+1})$$

is the zero map, so $\text{colim Hom}(T, X_n) = 0$. We conclude that Y is R -local. \square

Proposition 3.1.5. *The Verdier localisation functor $j^* : \mathcal{T} \rightarrow \mathcal{T}/\mathcal{R}$ has a right adjoint $j_* : \mathcal{T}/\mathcal{R} \rightarrow \mathcal{T}$.*

The functor j_ preserves coproducts.*

Proof. Let X be an object of \mathcal{T}/\mathcal{R} , we can see it as an object in \mathcal{T} and take an R -local object Y of \mathcal{T} as in Lemma 3.1.4, following the construction in the proof. We define $j_*X = Y$. If $f : T_1 \rightarrow T_2$ is a morphism in \mathcal{T}/\mathcal{R} we take as j_*f any morphism f' from T_1 to T_2 in \mathcal{S} such that $j^*f' = f$.

We need to prove that for any object X_1 in \mathcal{T} and X_2 in \mathcal{T}/\mathcal{R} it is $\text{Hom}_{\mathcal{T}}(X_1, X_2) \simeq \text{Hom}_{\mathcal{S}}(X_1, j_*X_2)$. Let $Y_2 = j_*X_2$, by Lemma 3.1.4 Y_2 is isomorphic to X_2 in \mathcal{T}/\mathcal{R} , so we must prove $\text{Hom}_{\mathcal{T}}(X_1, Y_2) \simeq \text{Hom}_{\mathcal{S}}(X_1, Y_2)$.

Let $f : X \rightarrow X'$ be an \mathcal{R} -morphism in \mathcal{T} , then the mapping cone on it is an object Z in \mathcal{R} . Let Y be a R -local object in \mathcal{T} , then $\text{Hom}_{\mathcal{T}}(Z, Y) = 0$, so by the definition of morphisms in \mathcal{T}/\mathcal{R} we have

$$\text{Hom}_{\mathcal{T}}(X, Y) \simeq \text{Hom}_{\mathcal{T}/\mathcal{R}}(X', Y).$$

We have the claim taking $X = X' = X_1$ and $Y = Y_2$.

Now we prove that j_* preserves coproducts. It suffices to prove that the full subcategory of R -local objects in \mathcal{T} is closed under the formation of coproducts. Let $\{X_i, i \in I\}$ a small set of R -local objects in \mathcal{T} , then for any object T in R it is

$$\text{Hom}(T, \coprod_{i \in I} X_i) = \bigoplus_{i \in I} \text{Hom}(T, X_i)$$

because T is compact, since every X_i is R -local it is $\text{Hom}(T, X_i) = 0$ for every $i \in I$, so $\coprod_{i \in I} X_i$ is R -local. \square

Lemma 3.1.6. *Let \mathcal{U} be the smallest full, triangulated subcategory of \mathcal{T} containing R and closed under the formation of direct sums. Let $X \rightarrow Y$ be a morphism in \mathcal{T} with X in \mathcal{R}^c . Suppose we are given a morphism $Y' \rightarrow Y$ in \mathcal{T} such that the mapping cone on it is a finite extension of direct sums of objects of R . Then there exists a morphism $X' \rightarrow X$ such that the mapping cone on it is an object in \mathcal{U} and the composite $X' \rightarrow X \rightarrow Y$ factors through Y' .*

Proof. Let E be the mapping cone on $Y' \rightarrow Y$, let n be the length of E , that is the number of extension needed to express E as an extension of coproducts of objects in R . We will make induction on n .

If $n = 1$ then E is a coproduct of objects of R . Consider the composite map $X \rightarrow Y \rightarrow E$. X is a compact object in \mathcal{R} and E is a coproduct of objects in R , so this map factor through a finite direct sum F of objects of R that is a direct summand of E . Then we have the commutative diagram

$$\begin{array}{ccccccc} X' & \longrightarrow & X & \longrightarrow & F & \longrightarrow & \Sigma X' \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Y' & \longrightarrow & Y & \longrightarrow & E & \longrightarrow & \Sigma Y' \end{array}$$

that is a morphism of triangles. Then F , an object in \mathcal{U} , is the mapping cone on $X' \rightarrow X$ and the composite map $X' \rightarrow X \rightarrow Y$ factors through Y' .

Suppose $n > 1$, then we can factor $Y' \rightarrow Y$ through an object Y'' such that the mapping cones on $Y' \rightarrow Y''$ and $Y'' \rightarrow Y$ are both extensions of coproducts of objects of R whose length is strictly less than n . By induction there exist morphisms $X'' \rightarrow X$ and $X' \rightarrow X''$ such that the mapping cones on them are in \mathcal{U} , so $X'' \rightarrow X \rightarrow Y$ factors through Y'' and $X' \rightarrow X'' \rightarrow Y''$ factors through Y' . We have the commutative diagram

$$\begin{array}{ccccc} X' & \longrightarrow & X'' & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ Y' & \longrightarrow & Y'' & \longrightarrow & Y \end{array}$$

so we can factor $X' \rightarrow X \rightarrow Y$ through Y' . Finally, by the octahedral lemma the mapping cone on the composite $X' \rightarrow X'' \rightarrow X$ is in \mathcal{U} . □

Lemma 3.1.7. *The Verdier localisation functor $j^* : \mathcal{T} \rightarrow \mathcal{T}/\mathcal{R}$ sends compact objects in compact objects.*

Proof. Let X be a compact object in \mathcal{T} . Let $\coprod_{i \in I} X_i$ be a coproduct in \mathcal{T}/\mathcal{R} . By adjunction, since j_* respect coproducts it is

$$\mathrm{Hom}(j^* X, \coprod_{i \in I} X_i) = \mathrm{Hom}(X, j_* \coprod_{i \in I} j_* X_i)$$

Then by compactness of X it is

$$\mathrm{Hom}(X, j_* \coprod_{i \in I} j_* X_i) = \coprod_{i \in I} \mathrm{Hom}(X, j_* X_i),$$

applying the adjunction relation yields

$$\mathrm{Hom}(j^*X, \coprod_{i \in I} X_i) = \prod_{i \in I} \mathrm{Hom}(j^*X, X_i)$$

so j^*X is a compact object. \square

Theorem 3.1.8 (Thomason's localisation theorem). *Let \mathcal{S} be the Verdier quotient \mathcal{T}/\mathcal{R} . Then*

1. *the category \mathcal{R} is compactly generated, with R as a generating set;*
2. *if R is a generating set for \mathcal{T} , then $\mathcal{R} = \mathcal{T}$;*
3. *if R is closed under formation of triangles and direct sums, then $R = \mathcal{R}^c$. In any case, $\mathcal{R}^c = \mathcal{R} \cap \mathcal{T}^c$ and it is the smallest triangulated full subcategory of \mathcal{T} closed under direct sums;*
4. *if T is a compact object in \mathcal{S} then there are objects T' in \mathcal{S}^c and S in \mathcal{T}^c and an isomorphism in \mathcal{S} such that*

$$S \simeq T \oplus T'$$

Then, even if T is not isomorphic in \mathcal{S} to a compact object in \mathcal{T} , it is always a direct summand of an object isomorphic in \mathcal{S} to a compact object in \mathcal{T} . Furthermore, T' can be chosen to be ΣT or any other object whose sum with T is zero in K_0 ;

5. *given two objects S and S' in \mathcal{T} with S' compact in \mathcal{T} and a morphism $S \rightarrow S'$ in \mathcal{S} , there is an object S'' in \mathcal{T}^c and morphisms $S'' \rightarrow S$ and $S'' \rightarrow S'$ in \mathcal{T} such that the mapping cone on $S'' \rightarrow S$ is in \mathcal{R}^c and in \mathcal{S} the map $S \rightarrow S'$ is the composition of $S'' \rightarrow S'$ with the inverse of $S'' \rightarrow S$.*

Proof. 1. \mathcal{R} is a triangulated category that contains countable coproducts by hypothesis, and the triangles in \mathcal{R} are the triangles in \mathcal{T} with objects in \mathcal{R} .

Let X be an object of \mathcal{R} such that for any $Y \in R$ it is $\mathrm{Hom}(Y, X) = 0$. Consider the following full subcategory of \mathcal{R}

$${}^\perp X = \{Y \text{ object of } \mathcal{R} \mid \mathrm{Hom}(\Sigma^n Y, X) = 0 \text{ for every } n \in \mathbb{Z}\}$$

It is closed with respect to coproducts and contains R . We claim that ${}^\perp X$ is a triangulated category. Since it is contained in \mathcal{R} , it suffices to prove that, given a morphism in ${}^\perp X$, it can be completed to a triangle in ${}^\perp X$. So let $f : Y_1 \rightarrow Y_2$ be a morphism in ${}^\perp X$, then in \mathcal{R} we have a triangle

$$Y_1 \xrightarrow{f} Y_2 \xrightarrow{v} Z \xrightarrow{w} \Sigma Y_1$$

$\mathrm{Hom}(-, X)$ is a cohomological functor, so the long sequence

$$\begin{aligned} \cdots \longrightarrow \mathrm{Hom}(\Sigma Y_1, X) &\longrightarrow \mathrm{Hom}(Z, X) \longrightarrow \mathrm{Hom}(Y_2, X) \longrightarrow \\ &\longrightarrow \mathrm{Hom}(Y_1, X) \longrightarrow \mathrm{Hom}(\Sigma^{-1} Z, X) \longrightarrow \cdots \end{aligned}$$

is exact, from Y_1 and Y_2 in ${}^\perp X$ it follows $\text{Hom}(\Sigma^n Z, X) = 0$ for every $n \in \mathbb{Z}$, so Z is in ${}^\perp X$ and the triangle lies in ${}^\perp X$.

Since \mathcal{R} is the smallest triangulated full subcategory of \mathcal{T} containing R and closed with respect to coproducts, it is ${}^\perp X = \mathcal{R}$, then $X \in {}^\perp X$ and $\text{Hom}(X, X) = 0$, hence X is the zero object.

We conclude that \mathcal{R} is compactly generated with R as a generating set.

2. Let $j^* : \mathcal{T} \rightarrow \mathcal{S}$ be the Verdier localisation functor and j_* its right adjoint as in Proposition 3.1.5. Then for any object S in \mathcal{T} it is $\text{Hom}(R, j_* j^* S) = 0$, then $j_* j^* S = 0$ since R generates \mathcal{T} .

Since the identity $1 : j^* S \rightarrow j^* S$ factors through $j^* j_* j^* S = 0$, it is $j^* S = 0$, it follows that 0 is the only object in \mathcal{S} , so it is $\mathcal{R} = \mathcal{S}$.

3. Obviously any compact object in \mathcal{T} which is also an object of \mathcal{R} is compact in \mathcal{R} .

Let X be a compact object in \mathcal{R} , since X is in \mathcal{R} it is $j_* j^* X = 0$. Following the construction in the proof of Lemma 3.1.4 we have a sequence of objects X_n and morphisms $X_n \rightarrow X_{n+1}$ in \mathcal{R} and $\text{Hocolim}(X_n) = 0$. By Lemma 3.1.2 it is

$$\text{colim Hom}_{\mathcal{R}}(X, X_n) \simeq \text{Hom}_{\mathcal{R}}(X, \text{Hocolim}(X_n)) = 0.$$

It follows that for some $n > 0$ the natural map $X_0 \rightarrow X_n$ is the zero map.

By construction the mapping cone Z on the natural map $X_0 \rightarrow X_n$ is a finite extension of coproducts of objects of R , since Z is also the mapping cone on the zero map we have $Z = X_n \oplus \Sigma X_0$. Then $X_0 = X$ is a direct summand of a finite extension of elements of R .

Let $Y' = 0$ and apply Lemma 3.1.6. Then exists a morphism $X' \rightarrow X$ such that the mapping cone on it lies in the smallest full triangulated subcategory \mathcal{U} of \mathcal{T} containing R and closed under formation of direct sums and such that the composite $X' \rightarrow X \rightarrow Y$ factors through $Y' = 0$. Note that the morphism $X \rightarrow Y$ is split, composing $X' \rightarrow X \rightarrow Y$ with the splitting $Y \rightarrow X$ yields the zero map $X' \rightarrow X$. Then X is a direct summand of the mapping cone, that by hypothesis is in \mathcal{U} .

Then $\mathcal{R}^c = \mathcal{U}$ which is contained in \mathcal{S} , so $\mathcal{R}^c = \mathcal{R} \cap \mathcal{T}^c$.

4. To prove the first part it suffices to prove that \mathcal{S}^c is the smallest triangulated full subcategory of \mathcal{S} containing the image of \mathcal{T} under j^* and closed under formation of direct sums.

\mathcal{T} is compactly generated, let S be a generating set for \mathcal{T} . Then by the second point of this Proposition the smallest triangulated full subcategory of \mathcal{R} containing \mathcal{T} and closed with respect to coproducts is all \mathcal{T} . It follows that $j^*(S)$ is a generating set for \mathcal{S} . Then for the second and third point of this proposition \mathcal{S}^c is the smallest triangulated full subcategory of \mathcal{S} closed under direct sums. The fact that we can choose ΣT for T' is in [7].

5. Let S be an object in \mathcal{T}^c , S' an object in \mathcal{T} . By adjunction $\text{Hom}_{\mathcal{S}}(j^* S, j^* S') \simeq \text{Hom}_{\mathcal{T}}(S, j_* j^* S')$, so given a morphism $S \rightarrow S'$ in \mathcal{S} we have also a unique morphism $S \rightarrow j_* j^* S'$ whose image in \mathcal{S} is the given morphism. By

Lemma 3.1.5 and 3.1.4 j_*j^*S' is the homotopy colimit of a sequence S'_n with $S'_0 = S'$ and such that the mapping cones on the morphisms $S'_n \rightarrow S'_{n+1}$ are direct sums of objects in R . Then, since S is compact, the morphism $S \rightarrow j_*j^*S'$ factors through some S'_n .

Now apply Lemma 3.1.6 to the map $S \rightarrow S'_n$. Then there exists an object S'' and a morphism $S'' \rightarrow S$, the mapping cone on which is in the smallest full triangulated subcategory of \mathcal{T} containing R and closed under direct sums, that is \mathcal{R}^c by the third point of this proposition, and such that the composition $S'' \rightarrow S \rightarrow S'_n$ factors through S' . The conclusion follows from the fact that $S'' \rightarrow S$ is an isomorphism in \mathcal{S} . \square

3.2 Brown representability theorem

Theorem 3.2.1. *Let \mathcal{T} be a compactly generated triangulated category. Let $H : \mathcal{T} \rightarrow \mathcal{A}b$ be a cohomological functor. Suppose that the natural map*

$$H\left(\coprod_{\lambda \in \Lambda} T_\lambda\right) \rightarrow \prod_{\lambda \in \Lambda} H(T_\lambda)$$

is an isomorphism for all small coproducts in \mathcal{T} . Then H is representable.

Proof. Let Ω be a generating set of \mathcal{T} and $U_0 = \bigcup_{T \in \Omega} H(T)$. We can identify the elements of U_0 with pairs (α, T) with $\alpha \in H(T)$. Let $X_0 = \coprod_{(\alpha, T) \in U_0} T$, then by hypothesis

$$H(X_0) = \prod_{(\alpha, T) \in U_0} H(T)$$

Let α_0 be the element of $H(X_0)$ given by $\alpha \in H(T)$ for $(\alpha, T) \in U_0$ in the product. Observe that if $T \rightarrow X_0$ is the inclusion corresponding to $(\alpha, T) \in U_0$ the induced map $H(X_0) \rightarrow H(T)$ takes α_0 in $\alpha \in H(T)$.

By Yoneda's Lemma the object X_0 and $\alpha_0 \in H(X_0)$ identify a natural transformation

$$\Phi_0 : \text{Hom}(-, X_0) \rightarrow H$$

and by our previous observation

$$\Phi_0(T) : \text{Hom}(T, X_0) \rightarrow H(T)$$

is surjective for any $T \in \Omega$.

Now we define by induction a sequence of objects X_i , $i \in \mathbb{N}$, and morphisms $X_i \rightarrow X_{i+1}$.

Suppose we have a natural transformation

$$\text{Hom}(-, X_i) \rightarrow H$$

and let

$$U_{i+1} = \bigcup_{T \in \Omega} \ker\{\text{Hom}(T, X_i) \rightarrow H(T)\},$$

we can identify an element of U_{i+1} with a pair (f, T) with $f : T \rightarrow X_i$. Let

$$K_{i+1} = \prod_{(f, T) \in U_{i+1}} T$$

and $K_{i+1} \rightarrow X_i$ be the map which is f on the factor T corresponding to the pair (f, T) . We define X_{i+1} as the mapping cone on this morphism.

The natural transformation

$$\mathrm{Hom}(-, X_i) \rightarrow H$$

identify by Yoneda's Lemma an element $\alpha_i \in H(X_i)$. Consider the map

$$j : H(X_i) \rightarrow H(K_{i+1}) = H \left(\coprod_{(f,T) \in U_{i+1}} T \right) = \prod_{(f,T) \in U_{i+1}} H(T),$$

since the morphisms $f : T \rightarrow X_i$ were chosen so that they induce map $\mathrm{Hom}(T, X_i) \rightarrow H(T)$ is zero, $j(\alpha_i)$ is zero. Since H is cohomological we have an exact sequence

$$H(X_{i+1}) \xrightarrow{k} H(X_i) \xrightarrow{j} H(K_{i+1})$$

so there is an element $\alpha_{i+1} \in H(X_{i+1})$ such that $k(\alpha_{i+1}) = \alpha_i$. By Yoneda's Lemma α_{i+1} identifies a natural transformation $\mathrm{Hom}(-, X_{i+1}) \rightarrow H$ such that the triangle

$$\begin{array}{ccc} & \mathrm{Hom}(-, X_i) & \\ \swarrow & & \searrow \\ \mathrm{Hom}(-, X_{i+1}) & \xrightarrow{\quad} & H \end{array}$$

is commutative.

Let $X = \mathrm{Hocolim}(X_i)$.

Consider the triangle

$$\coprod_i X_i \xrightarrow{1-shift} \coprod_i X_i \longrightarrow \mathrm{Hocolim} X_i = X$$

and apply H , we obtain the exact sequence

$$H(X) \longrightarrow H(\coprod_i X_i) = \prod_i H(X_i) \xrightarrow{1-shift} H(\coprod_i X_i) = \prod_i H(X_i)$$

$\prod_i \alpha_i \in \prod_i H(X_i)$ is in the kernel of the $1 - shift$, so there exists an element $\alpha \in H(X)$ whose image is $\prod_i \alpha_i$. By Yoneda's Lemma α identifies a natural transformation

$$\Phi : \mathrm{Hom}(-, X) \rightarrow H$$

such that the diagram

$$\begin{array}{ccc} & \mathrm{Hom}(-, X_i) & \\ \swarrow & & \searrow \\ \mathrm{Hom}(-, X) & \xrightarrow{\quad \Phi \quad} & H \end{array}$$

commutes for every i .

Let $T \in \Omega$, we have the commutative diagram

$$\begin{array}{ccc} & \text{Hom}(T, X_0) & \\ \swarrow & & \searrow \\ \text{Hom}(T, X) & \xrightarrow{\Phi(T)} & H(T) \end{array}$$

Since $\text{Hom}(T, X_0) \rightarrow H(T)$ is surjective, $\text{Hom}(T, X) \rightarrow H(T)$ is also surjective.

Let $f \in \text{Hom}(T, X)$ such that $\Phi(T)(f) = 0$. Since T is compact by Lemma 3.1.2 it is

$$\text{Hom}(T, X) = \text{colim } \text{Hom}(T, X_i)$$

so f factors through a X_i

$$T \xrightarrow{f_i} X_i \longrightarrow X$$

Then in the commutative diagram

$$\begin{array}{ccc} & \text{Hom}(T, X_i) & \\ \swarrow j & & \searrow \\ \text{Hom}(T, X) & \xrightarrow{\Phi(T)} & H(T) \end{array}$$

we have $j(f_i) = f$ and $\Phi(T)(f) = 0$, so $f_i \in \ker\{\text{Hom}(T, X_i) \rightarrow H(T)\}$ and this means $(f_i, T) \in U_{i+1}$. Then f_i factors through X_i via the morphism h in the triangle

$$K_{i+1} \xrightarrow{h} X_i \xrightarrow{g} X_{i+1}$$

hence $g \circ f_i = 0$. Consider the map $X_i \xrightarrow{g} X_{i+1} \xrightarrow{\bar{g}} X$, then

$$f = (\bar{g} \circ g) \circ f_i = \bar{g} \circ (g \circ f_i) = 0$$

and so $\Phi(T) : \text{Hom}(-, X) \rightarrow H(T)$ is injective and therefore an isomorphism for any $T \in \Omega$.

Let \mathcal{S} be the full subcategory of the objects Y in \mathcal{T} such that for all $n \in \mathbb{Z}$ the map $\Phi(\Sigma^n Y) : \text{Hom}(\Sigma^n Y, X) \rightarrow H(\Sigma^n Y)$ is an isomorphism. \mathcal{S} contains Ω , it is triangulated and closed under formation of coproducts. By Proposition 3.1.8 it is $\mathcal{S} = \mathcal{T}$, so H is representable. □

Theorem 3.2.2. *Let \mathcal{S} be a compactly generated triangulated category and \mathcal{T} any triangulated category. Let $F : \mathcal{S} \rightarrow \mathcal{T}$ be a triangulated functor that respects coproducts. Then F has a right adjoint $G : \mathcal{T} \rightarrow \mathcal{S}$.*

Proof. Let T be an object in \mathcal{T} , consider the functor $\text{Hom}_{\mathcal{T}}(F(-), T)$ on \mathcal{S} . This is a cohomological functor that takes coproducts to coproducts since

$$\text{Hom}_{\mathcal{T}} \left(F \left(\coprod_{\lambda \in \Lambda} S_{\lambda} \right), T \right) = \text{Hom}_{\mathcal{T}} \left(\coprod_{\lambda \in \Lambda} F(S_{\lambda}), T \right) = \prod_{\lambda \in \Lambda} \text{Hom}_{\mathcal{T}}(F(S_{\lambda}), T).$$

Then by the previous theorem this functor is representable, so there is an object $G(T)$ in \mathcal{S} such that

$$\mathrm{Hom}_{\mathcal{T}}(F(S), T) = \mathrm{Hom}_{\mathcal{S}}(S, G(T))$$

and we can extend $G(-)$ to a functor that is the right adjoint to F . \square

Proposition 3.2.3. *Let $F : \mathcal{S} \rightarrow \mathcal{T}$ be a triangulated functor as in the theorem above, let R be a generating class of compact objects for \mathcal{S} . Then G , the right adjoint of F , respects coproducts if and only if $F(S)$ is a compact object in \mathcal{T} for any S object of R .*

Proof. Suppose G respect coproducts and let T be an object of R . Then, by adjunction and because G respect coproducts, for any collection $\{T_\lambda\}_{\lambda \in \Lambda}$ of objects in \mathcal{T} we have

$$\mathrm{Hom}_{\mathcal{T}}\left(F(S), \coprod_{\lambda \in \Lambda} T_\lambda\right) = \mathrm{Hom}_{\mathcal{S}}\left(S, G\left(\coprod_{\lambda \in \Lambda} T_\lambda\right)\right) = \mathrm{Hom}_{\mathcal{S}}\left(S, \coprod_{\lambda \in \Lambda} G(T_\lambda)\right)$$

Since S is a compact object and using adjunction we have also

$$\mathrm{Hom}_{\mathcal{S}}\left(S, \coprod_{\lambda \in \Lambda} G(T_\lambda)\right) = \coprod_{\lambda \in \Lambda} \mathrm{Hom}_{\mathcal{S}}(S, G(T_\lambda)) = \coprod_{\lambda \in \Lambda} \mathrm{Hom}_{\mathcal{T}}(F(S), T_\lambda)$$

so we conclude

$$\mathrm{Hom}_{\mathcal{T}}\left(F(S), \coprod_{\lambda \in \Lambda} T_\lambda\right) = \coprod_{\lambda \in \Lambda} \mathrm{Hom}_{\mathcal{T}}(F(S), T_\lambda)$$

Hence $F(S)$ is compact in \mathcal{T} .

Suppose $F(S)$ is compact in \mathcal{T} for any object S in R . Let $\{T_\lambda\}_{\lambda \in \Lambda}$ be a collection of objects in \mathcal{T} .

The natural morphism $\tau : \coprod_{\lambda \in \Lambda} G(T_\lambda) \rightarrow G(\coprod_{\lambda \in \Lambda} T_\lambda)$ induces a natural transformation

$$\phi : \mathrm{Hom}_{\mathcal{S}}\left(-, \coprod_{\lambda \in \Lambda} G(T_\lambda)\right) \rightarrow \mathrm{Hom}_{\mathcal{S}}\left(-, G\left(\coprod_{\lambda \in \Lambda} T_\lambda\right)\right)$$

For any S in R we have, by adjunction and because $F(S)$ and S are compact

$$\begin{aligned} \mathrm{Hom}\left(S, G\left(\coprod_{\lambda \in \Lambda} T_\lambda\right)\right) &= \mathrm{Hom}\left(F(S), \coprod_{\lambda \in \Lambda} T_\lambda\right) = \\ &= \coprod_{\lambda \in \Lambda} \mathrm{Hom}(F(S), T_\lambda) = \coprod_{\lambda \in \Lambda} \mathrm{Hom}(S, G(T_\lambda)) = \mathrm{Hom}\left(S, \coprod_{\lambda \in \Lambda} G(T_\lambda)\right) \end{aligned}$$

so $\phi(S)$ is an isomorphism.

Let Z be the mapping cone on τ , applying the homological functor $\mathrm{Hom}(S, -)$ to the triangle

$$\coprod_{\lambda \in \Lambda} G(T_\lambda) \xrightarrow{\tau} G(\coprod_{\lambda \in \Lambda} T_\lambda) \longrightarrow Z \longrightarrow \Sigma(\coprod_{\lambda \in \Lambda} G(T_\lambda))$$

we obtain that $\text{Hom}(S, Z) = 0$. Since S is an arbitrary element in R and R is a generating class for \mathcal{S} , it is $Z = 0$. Then τ is an isomorphism. \square

Chapter 4

Grothendieck duality

4.1 Derived categories

Let \mathcal{A} be an abelian category, let $\text{Coch}(\mathcal{A})$ be the category of cochain complexes in \mathcal{A} . In this section we define derived categories and prove that they are triangulated categories.

Definition 4.1.1. *The homotopy category $\mathcal{K}(\mathcal{A})$ of complexes in \mathcal{A} is a category whose objects are complexes in \mathcal{A} and whose morphisms are homotopy equivalence classes of complexes.*

Proposition 4.1.2. *$\mathcal{K}(\mathcal{A})$ is an additive category and the natural functor $\text{Coch}(\mathcal{A}) \rightarrow \mathcal{K}(\mathcal{A})$ is additive.*

Proof. $\mathcal{K}(\mathcal{A})$ admits a zero object, the complex that is identically zero.

Let f and g be two morphisms between two complexes X and Y , both homotopic to the zero morphism. Then $f + g$ and $-f$ are homotopic to zero, so the set of morphisms homotopic to zero is a subgroup of the group of morphisms between X and Y . So the morphisms in $\mathcal{K}(\mathcal{A})$ inherits a group structure as a quotient of the morphisms between complexes. This also prove that the natural functor $\text{Coch}(\mathcal{A}) \rightarrow \mathcal{K}(\mathcal{A})$ is additive.

Finite products and coproducts exist and coincide because they are inherited from $\text{Coch}(\mathcal{A})$. □

Consider the functor $\Sigma : \mathcal{K}(\mathcal{A}) \rightarrow \mathcal{K}(\mathcal{A})$ with $\Sigma X^n = X^{n+1}$ and $\delta_{\Sigma X}^n = -\delta_X^{n+1}$. This endofunctor is clearly invertible and additive. Then we can define candidate triangles with respect to the suspension Σ .

Given a morphism of complexes $f : X \rightarrow Y$, we define the mapping cone on f as the complex Z_f given by $Z_f^n = X^{n+1} \oplus Y^n$ and $\delta_{Z_f}^n = \begin{pmatrix} -\delta_X^{n+1} & 0 \\ f^{n+1} & \delta_Y^n \end{pmatrix}$.

We have two canonical morphisms $g : Y \rightarrow Z_f$ and $h : Z_f \rightarrow \Sigma X$ defined by the inclusion and the projection.

Lemma 4.1.3.

$$X \xrightarrow{f} Y \xrightarrow{g} Z_f \xrightarrow{h} \Sigma X$$

is a candidate triangle in $\mathcal{K}(\mathcal{A})$.

Proof. The composite $g \circ f$ is homotopic to zero via the morphisms $\begin{pmatrix} 0 \\ 1_{X^n} \end{pmatrix}$.

The composite $h \circ g$ is zero.

The composite $\Sigma f \circ h$ is homotopic to zero via the morphisms $\begin{pmatrix} 0 & 1_{Y^n} \end{pmatrix}$.

We conclude that the diagram above is a candidate triangle in $\mathcal{K}(\mathcal{A})$. \square

We define the class of triangles in $\mathcal{K}(\mathcal{A})$ as the candidate triangles that are isomorphic to a candidate triangle as in the lemma above.

Consider the canonical morphism $k = \Sigma^{-1}h : \Sigma^{-1}Z_f \rightarrow X$, we define *mapping cylinder* \tilde{C}_f of f the mapping cone on k , that is $\tilde{C}_f = Z_k$. We write $\tilde{f} : X \rightarrow \tilde{C}_f$ and $\tilde{g} : \tilde{C}_f \rightarrow Z_f$ for the canonical morphisms defined respectively by inclusion and projection.

Lemma 4.1.4. *The diagram*

$$X \xrightarrow{\tilde{f}} \tilde{C}_f \xrightarrow{-\tilde{g}} Z_f \xrightarrow{h} \Sigma X.$$

is a triangle in $\mathcal{K}(\mathcal{A})$ and every triangle in $\mathcal{K}(\mathcal{A})$ is isomorphic to a triangle in this form.

Proof. Note that $\tilde{C}_f^n = X^{n+1} \oplus Y^n \oplus X^n$ and the differentials are given by

$$\delta_{\tilde{C}_f}^n = \begin{pmatrix} -\delta_X^{n+1} & 0 & 0 \\ f^{n+1} & \delta_Y^n & 0 \\ 1_{X^{n+1}} & 0 & \delta_X^n \end{pmatrix}$$

We have that $-\tilde{g} \circ \tilde{f} = 0$, $h \circ -\tilde{g}$ is homotopic to zero via the morphisms

$$\begin{pmatrix} 0 & 0 & 1_{X^n} \end{pmatrix}$$

and that $\Sigma \tilde{f} \circ h$ is homotopic to zero via the morphisms

$$\begin{pmatrix} 1_{X^{n+1}} & 0 \\ 0 & 1_{Y^n} \\ 0 & 0 \end{pmatrix}$$

So the diagram is a candidate triangle.

Let $\phi : \tilde{C}_f \rightarrow Y$ be the morphism of complexes given by

$$\phi^n = \begin{pmatrix} 0 & -1_{Y^n} & f^n \end{pmatrix}$$

and consider the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{\tilde{f}} & \tilde{C}_f & \xrightarrow{-\tilde{g}} & Z_f & \xrightarrow{h} & \Sigma X \\ \downarrow 1 & & \downarrow \phi & & \downarrow 1 & & \downarrow 1 \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z_f & \xrightarrow{h} & \Sigma X \end{array} \quad (4.1)$$

We have that $\phi \circ \tilde{f} = f$ and that $g \circ \Phi$ is homotopic to $-\tilde{g}$ via the morphisms

$$\begin{pmatrix} 0 & 0 & 1_{X^n} \\ 0 & 0 & 0 \end{pmatrix}$$

so 4.1 is a morphism of candidate triangles.

Let $\theta : Y \rightarrow \tilde{C}_f$ be the morphism given by

$$\theta^n = \begin{pmatrix} 0 \\ -1_{Y^n} \\ 0 \end{pmatrix}$$

Then $\phi \circ \theta = 1_Y$ and $\theta \circ \phi$ is homotopic to $1_{\tilde{C}_f}$ via the morphisms

$$\begin{pmatrix} 0 & 0 & 1_{X^n} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

so ϕ is an isomorphism in $\mathcal{K}(\mathcal{A})$ and 4.1 is an isomorphism of candidate triangles, so it is a triangle. Also, by our definition of triangles in $\mathcal{K}(\mathcal{A})$, every triangle is isomorphic to a triangle of this form. \square

Lemma 4.1.5. *Let $f : X \rightarrow Y$ be a morphism of complexes.*

1. *A morphism of complexes $m : Y \rightarrow Q$ factors through $g : Y \rightarrow Z_f$ if and only if $m \circ f$ is homotopic to 0.*
2. *A morphism of complexes $s : Q \rightarrow X$ factors through $\Sigma^{-1}h : \Sigma^{-1}Z_f \rightarrow X$ if and only if $f \circ s$ is homotopic to 0.*

Proof. 1. A morphism $v : Z_f \rightarrow Q$ is defined by morphisms $v^n : X^{n+1} \oplus Y^n \rightarrow Q^n$ with $v^n = \begin{pmatrix} u^n \\ w^n \end{pmatrix}$ such that

$$(w \circ f)^n = \delta_Q^{n-1} \circ u^n + u^{n+1} \circ \delta_X^n$$

$$w^{n+1} \circ \delta_Y^n = \delta_Q^n \circ w^n$$

Then to give v is dequivalent to give a morphism of complexes $w : Y \rightarrow Q$ and an homotopy between 0 and $w \circ f$. Furthermore, $w = v \circ f$.

A morphism of complexes $m : Y \rightarrow Q$ then factors through g if and only if $m \circ f$ is homotopic to 0, moreover, there is a bijection between the factorizations $Z_f \rightarrow Q$ and the homotopies between $m \circ f$ and 0.

2. To give a morphism of complexes $v : Q \rightarrow Z_f$ is equivalent to give a morphism of complexes $w : Q \rightarrow X$ and an homotopy between $f \circ w$ and 0, and $w = \Sigma^{-1}h \circ v$.

A morphism of complexes $s : Q \rightarrow X$ then factors through $\Sigma^{-1}h$ if and only if $f \circ s$ is homotopic to 0, and there is a bijection between the factorisations and the homotopies between $f \circ s$ and 0. \square

Lemma 4.1.6. *Suppose we are given an exact sequence of complexes*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{t} Z \longrightarrow 0$$

Then there is a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & X & \xrightarrow{\tilde{f}} & \tilde{C}_f & \xrightarrow{-\tilde{g}} & Z_f \longrightarrow 0 \\
& & \downarrow 1_X & & \downarrow \phi & & \downarrow v \\
0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{t} & Z \longrightarrow 0
\end{array}$$

with exact rows.

Proof. By exactness, $t \circ f = 0$, so 0 provides an homotopy between $t \circ f$ and 0, so by Lemma 4.1.5 a factorization $v : Z_f \rightarrow Z$ of t through $g : Y \rightarrow Z_f$, with $v^n = \begin{pmatrix} 0 & t^n \end{pmatrix}$. The morphism ϕ is defined in Lemma 4.1.4.

The commutativity of the diagram and the exactness of the first row is a simple check. \square

Proposition 4.1.7. $\mathcal{K}(\mathcal{A})$ with the class of triangles previously defined is a triangulated category.

Proof. We will prove that the conditions in Definition 1.2.1 are satisfied.

T1 Obvious by our definition of triangles in $\mathcal{K}(\mathcal{A})$.

T2 Let X be a complex, consider the candidate triangle

$$X \xrightarrow{1_X} X \longrightarrow 0 \longrightarrow \Sigma X \quad (4.2)$$

The canonical morphism $g : X \rightarrow Z_{1_X}$ is homotopic to zero via the morphisms $\begin{pmatrix} 1_X^n \\ 0 \end{pmatrix}$, so the diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{1_X} & X & \longrightarrow & 0 & \longrightarrow & \Sigma X \\
\downarrow 1_X & & \downarrow 1_X & & \downarrow & & \downarrow 1_{\Sigma X} \\
X & \xrightarrow{1_X} & X & \xrightarrow{g} & Z_{1_X} & \xrightarrow{h} & \Sigma X
\end{array}$$

is a morphism of candidate triangles. Since the identity on Z_{1_X} is homotopic to zero via the morphisms $\begin{pmatrix} 0 & 1_X^n \\ 0 & 0 \end{pmatrix}$ the diagram is also an isomorphism of candidate triangles and by the previous point 4.2 is a triangle.

T3 Given a morphism $f : X \rightarrow Y$, we have the mapping cone Z_f and canonical morphisms $g : Y \rightarrow Z_f$, $h : Z_f \rightarrow \Sigma X$, by definition

$$X \xrightarrow{f} Y \xrightarrow{g} Z_f \xrightarrow{h} \Sigma X$$

is a triangle.

T4 It will suffice to prove that rotating back or forward a triangle we obtain a triangle. Furthermore, by Lemma 4.1.4 it is enough to prove that this is true for a triangle of the form

$$X \xrightarrow{\tilde{f}} \tilde{C}_f \xrightarrow{-\tilde{g}} Z_f \xrightarrow{h} \Sigma X \quad (4.3)$$

Rotating back 4.3 we obtain the candidate triangle

$$\Sigma^{-1}Z_f \xrightarrow{-\Sigma^{-1}h} X \xrightarrow{-\tilde{f}} \tilde{C}_f \xrightarrow{\tilde{g}} Z_f$$

\tilde{C}_f is the mapping cone of $\Sigma^{-1}h$, so we have the isomorphism of candidate triangles

$$\begin{array}{ccccccc} \Sigma^{-1}Z_f & \xrightarrow{-\Sigma^{-1}h} & X & \xrightarrow{-\tilde{f}} & \tilde{C}_f & \xrightarrow{\tilde{g}} & Z_f \\ \downarrow 1_{\Sigma^{-1}Z_f} & & \downarrow -1_x & & \downarrow 1_{\tilde{C}_f} & & \downarrow 1_{Z_f} \\ \Sigma^{-1}Z_f & \xrightarrow{\Sigma^{-1}h} & X & \xrightarrow{\tilde{f}} & \tilde{C}_f & \xrightarrow{\tilde{g}} & Z_f \end{array}$$

Since the second is a triangle, the first is also a triangle.

Rotating forward 4.3 we obtain the candidate triangle

$$\tilde{C}_f \xrightarrow{\tilde{g}} Z_f \xrightarrow{-h} \Sigma X \xrightarrow{-\Sigma\tilde{f}} \Sigma\tilde{C}_f$$

We can form the mapping cone on \tilde{g} and obtain a triangle via the canonical morphisms r and s

$$\tilde{C}_f \xrightarrow{\tilde{g}} Z_f \xrightarrow{r} Z_{\tilde{g}} \xrightarrow{s} \Sigma\tilde{C}_f$$

Let $t : \Sigma X \rightarrow Z_{\tilde{g}}$ be the morphism of complexes given by the inclusions of X^{n+1} in the middle summand of $Z_{\tilde{g}}^n = X^{n+2} \oplus Y^{n+1} \oplus X^{n+1} \oplus X^{n+1} \oplus Y^n$, then

$$\begin{array}{ccccccc} \tilde{C}_f & \xrightarrow{\tilde{g}} & Z_f & \xrightarrow{-h} & \Sigma X & \xrightarrow{-\Sigma\tilde{f}} & \Sigma\tilde{C}_f \\ \downarrow 1_{\tilde{C}_f} & & \downarrow 1_{Z_f} & & \downarrow -t & & \downarrow 1_{\Sigma\tilde{C}_f} \\ \tilde{C}_f & \xrightarrow{\tilde{g}} & Z_f & \xrightarrow{r} & Z_{\tilde{g}} & \xrightarrow{s} & \Sigma\tilde{C}_f \end{array} \quad (4.4)$$

is a morphism of triangles. Consider the morphism of complexes $u : Z_{\tilde{g}} \rightarrow \Sigma X$ given by the morphisms $u^n = \begin{pmatrix} 0 & 0 & 1_{X^{n+1}} & 1_{X^{n+1}} & 0 \end{pmatrix}$, then $u \circ t = 1_X$ and $t \circ u$ is homotopic to $1_{Z_{\tilde{g}}}$ via the morphisms

$$\begin{pmatrix} 0 & 0 & 0 & 1_{X^{n+1}} & 0 \\ 0 & 0 & 0 & 0 & 1_{Y^n} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

So t is an isomorphism in $\mathcal{K}(\mathcal{A})$ and 4.4 is an isomorphism of candidate triangles. Since the second row is a triangle, the first row is also a triangle.

T5 Consider the commutative diagram in $\mathcal{K}(\mathcal{A})$

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z_f & \xrightarrow{h} & \Sigma X \\ \downarrow u & & \downarrow v & & & & \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z_{f'} & \xrightarrow{h'} & \Sigma X' \end{array}$$

whose rows are triangles. This means that $v \circ f$ is homotopic to $f' \circ u$ via some morphisms Φ^n . Then the morphism $w : Z_f \rightarrow Z_{f'}$ defined by

$$w^n = \begin{pmatrix} f^{n+1} & 0 \\ -\Phi^{n+1} & v^n \end{pmatrix}$$

gives the commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z_f & \xrightarrow{h} & \Sigma X \\ \downarrow u & & \downarrow v & & \downarrow w & & \downarrow \Sigma u \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z_{f'} & \xrightarrow{h'} & \Sigma X' \end{array}$$

so this is a morphism of triangles. Now consider the mapping cone

$$Y \oplus X' \xrightarrow{\begin{pmatrix} -g & 0 \\ v & f' \end{pmatrix}} Z_f \oplus Y' \xrightarrow{\begin{pmatrix} -h & 0 \\ w & g' \end{pmatrix}} \Sigma X \oplus Z_{f'} \xrightarrow{\begin{pmatrix} -\Sigma f & 0 \\ \Sigma u & h' \end{pmatrix}} \Sigma Y \oplus \Sigma X'$$

Let $f'' = \begin{pmatrix} -g & 0 \\ v & f' \end{pmatrix}$, then we have the mapping cone $Z_{f''}$. Let $q : Z_{f''} \rightarrow \Sigma X \oplus Z_{f'}$ be the morphism defined by

$$q^n = \begin{pmatrix} 0 & 0 & -1_{X^{n+1}} & 0 & 0 \\ 0 & 1_{X'^{n+1}} & u^{n+1} & 0 & 0 \\ 0 & 0 & -\Phi^{n+1} & g^n & 1 \end{pmatrix}$$

Then the diagram

$$\begin{array}{ccccccc} Y \oplus X' & \xrightarrow{f''} & Z_f \oplus Y' & \xrightarrow{g''} & Z_{f''} & \xrightarrow{h''} & \Sigma Y \oplus \Sigma X' \\ \downarrow 1 & & \downarrow 1 & & \downarrow t & & \downarrow 1 \\ Y \oplus X' & \xrightarrow{f''} & Z_f \oplus Y' & \xrightarrow{\begin{pmatrix} -h & 0 \\ w & g' \end{pmatrix}} & \Sigma X \oplus Z_{f'} & \xrightarrow{\begin{pmatrix} -\Sigma f & 0 \\ \Sigma u & h' \end{pmatrix}} & \Sigma Y \oplus \Sigma X' \end{array}$$

is commutative in $\mathcal{K}(\mathcal{A})$.

Since t has an homotopy inverse, τ , with

$$\tau^n = \begin{pmatrix} -f^{n+1} & 0 & 0 \\ u^{n+1} & 1_{X'^{n+1}} & 0 \\ -1_{\Sigma X} & 0 & 0 \\ 0 & 0 & 0 \\ -\Phi^{n+1} & 0 & 1_{Y'} \end{pmatrix}$$

the diagram is an isomorphism of candidate triangles. The first row is a triangle, so the second row is also a triangle.

□

The n -th cohomology functor on complexes, H^n , is an additive functor and obviously $H^n = H^0 \circ \Sigma^n$. The triangle

$$X \xrightarrow{f} Y \longrightarrow Z_f \longrightarrow \Sigma X$$

is isomorphic to the triangle

$$X \xrightarrow{\tilde{f}} \tilde{C}_f \longrightarrow Z_f \longrightarrow \Sigma X$$

and we have an exact sequence

$$0 \longrightarrow X \xrightarrow{\tilde{f}} \tilde{C}_f \longrightarrow Z_f \longrightarrow 0$$

Applying H^n to this exact sequence shows that it is a cohomological functor from the triangulated category $\mathcal{K}(\mathcal{A})$ to \mathcal{A} .

Definition 4.1.8. A morphism of complexes $f : X \rightarrow Y$ in $\mathcal{K}(\mathcal{A})$ is a quasi-isomorphism if the morphism $H^n(f) : H^n(X) \rightarrow H^n(Y)$ is an isomorphism in \mathcal{A} for any $n \in \mathbb{Z}$. This property respects homotopy equivalences, so the definition is well given.

Lemma 4.1.9. A morphism $f : X \rightarrow Y$ in $\mathcal{K}(\mathcal{A})$ is a quasi-isomorphism if and only if the mapping cone Z_f on it is exact.

Proof. Given a triangle

$$0 \longrightarrow X \xrightarrow{\tilde{f}} \tilde{C}_f \longrightarrow Z_f \longrightarrow 0$$

we have a long exact sequen

$$\dots \rightarrow H^{n-1}(Z_f) \rightarrow H^n(X) \rightarrow H^n(Y) \rightarrow H^n(Z_f) \rightarrow H^{n+1}(X) \rightarrow \dots$$

Then $H^n(f)$ is an isomorphism for any $n \in \mathbb{Z}$ if and only if $H^n(Z_f) = 0$ for any $n \in \mathbb{Z}$. This means that f is a quasi-isomorphism if and only if Z_f is exact. \square

Proposition 4.1.10. The full subcategory \mathcal{D} of exact complexes is a triangulated subcategory of $\mathcal{K}(\mathcal{A})$. The \mathcal{D} -morphisms are the quasi-isomorphisms.

Proof. Obviously \mathcal{D} is closed under the action of Σ and Σ^{-1} , finite coproducts of exact complexes are exact and a complex isomorphic to an exact complex is exact.

If we have a triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

with X and Y exact, by the long exact sequence

$$\dots \rightarrow H^{n-1}(Z) \rightarrow H^n(X) \rightarrow H^n(Y) \rightarrow H^n(Z) \rightarrow H^{n+1}(X) \rightarrow \dots$$

it follows that $H^n(Z) = 0$ for any $n \in \mathbb{Z}$, so Z is exact. This prove that \mathcal{D} is triangulated.

The characterization of \mathcal{D} -morphisms follows from Lemma 4.1.9. \square

Definition 4.1.11. The derived category $\mathcal{D}(\mathcal{A})$ of \mathcal{A} is the Verdier quotient $\mathcal{K}(\mathcal{A})/\mathcal{D}$.

Proposition 4.1.12. *If there is an exact sequence of complexes*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{t} Z \longrightarrow 0$$

then there exists a morphism $w : Z \rightarrow \Sigma X$ in $\mathcal{D}(\mathcal{A})$ such that the following diagram

$$X \xrightarrow{f} Y \xrightarrow{t} Z \xrightarrow{-w} \Sigma X$$

is a triangle.

Proof. By Lemma 4.1.4 and Lemma 4.1.6 there is a commutative diagram in $\mathcal{D}(\mathcal{A})$

$$\begin{array}{ccccc} X & \xrightarrow{\tilde{f}} & \tilde{C}_f & \xrightarrow{-\tilde{g}} & Z_f & \xrightarrow{h} & \Sigma X \\ \downarrow 1_X & & \downarrow \phi & & \downarrow v & & \\ X & \xrightarrow{f} & Y & \xrightarrow{t} & Z & & \end{array}$$

Let $F : K(\mathcal{A}) \rightarrow \mathcal{D}(\mathcal{A})$ be the localisation functor, we define $w = -F(h) \circ F(v)^{-1} : Z \rightarrow \Sigma X$. Then

$$X \xrightarrow{f} Y \xrightarrow{t} Z \xrightarrow{-w} \Sigma X$$

is a triangle in $\mathcal{D}(\mathcal{A})$ because it is isomorphic to the triangle in the first row. \square

4.2 Derived categories of quasi-coherent sheaves on a quasi-compact, separated scheme

Let X be a quasi-compact, separated scheme, then we can form the triangulated category $D(\mathcal{Qcoh}(X))$, as seen in the previous section. In this section we will prove that it is compactly generated.

The category $D(\mathcal{Qcoh}(X))$ contains arbitrary coproducts: a coproduct of chain complexes is a chain complex in which the object at any degree is the coproduct of the objects at that degree.

Lemma 4.2.1. *Let Y be a scheme and $f : X \rightarrow Y$ be a separated morphism of schemes. Then the direct image functor Rf_* from $D(\mathcal{Qcoh}(X))$ to $D(\mathcal{Qcoh}(Y))$ respects coproducts.*

Proof. We have to prove that, given a set $\{T_\lambda | \lambda \in \Lambda\}$ of objects in $D(\mathcal{Qcoh}(X))$, the natural map

$$\coprod_{\lambda \in \Lambda} Rf_* T_\lambda \rightarrow Rf_* \left(\coprod_{\lambda \in \Lambda} T_\lambda \right)$$

is an isomorphism. Since this happens if and only if for any affine subscheme U of X we have

$$\coprod_{\lambda \in \Lambda} Rf_* T_{\lambda|U} \rightarrow Rf_* \left(\coprod_{\lambda \in \Lambda} T_\lambda \right)_{|U}$$

we can assume Y affine.

Since X is quasi-compact, it can be covered by finitely many open affines U_1, \dots, U_n , with n the least integer for which such a covering exists. We will make induction on n .

If $n = 1$, X is affine, let $X = \operatorname{Spec}(S)$ and $Y = \operatorname{Spec}(R)$ with R and S rings. Then the morphism $f : X \rightarrow Y$ is induced by an homomorphism of rings $\phi : R \rightarrow S$. Since the category of quasi-coherent modules on $\operatorname{Spec}(S)$ is equivalent to the category of S -modules, we have that $D(\mathcal{Qcoh}(X))$ is equivalent to $D(\mathcal{Mod}(S))$, the derived category of chain complexes of S -modules. Then we can see Rf_* as a functor from $D(\mathcal{Mod}(S))$ to $D(\mathcal{Mod}(R))$ that sends chain complexes of S -modules to chain complexes of R -modules through the homomorphism ϕ . The coproduct of a collection $\{M_\lambda | \lambda \in \Lambda\}$ of S -modules seen as an R -module through ϕ is trivially isomorphic to the coproduct of the collection $\{M_\lambda | \lambda \in \Lambda\}$ seen as a collection of R -modules through ϕ , so the functor Rf_* respects coproducts.

If $n > 1$, let $U = U_n$ and $V = \bigcup_{i=1}^{n-1} U_i$. Note that $U \cap V = \bigcup_{i=1}^{n-1} (U_n \cap U_i)$ and the $U_n \cap U_i$ are affine since X is separated, so both V and $U \cap V$ are unions of $n-1$ affines. By induction, the restrictions of Rf_* on U , V and $U \cap V$ respect coproducts.

For any $T \in \mathcal{D}(\mathcal{Qcoh}(X))$ we have an exact sequence

$$0 \rightarrow T \rightarrow (i_U)_* i_U^* T \oplus (i_V)_* i_V^* T \rightarrow (i_{U \cap V})_* i_{U \cap V}^* T \rightarrow 0$$

so by Proposition 4.1.12 we have a triangle

$$\begin{aligned} T &\longrightarrow (i_U)_* i_U^* T \oplus (i_V)_* i_V^* T \longrightarrow \\ &\longrightarrow (i_{U \cap V})_* i_{U \cap V}^* T \longrightarrow \Sigma T \end{aligned}$$

Every complex in $\mathcal{K}(\mathcal{Qcoh}(X))$ is quasi-isomorphic to a K -injective complex (see [10], Theorem 4.5), so we can replace T with a K -injective complex. Then by [6], Corollary 2.3.2.3 and Lemma 2.4.5.2, we have an isomorphism between $(i_U)_* i_U^* T$ and $R(i_U)_* i_U^* T$ for any open U . Then, applying Rf^* , we obtain a triangle

$$\begin{aligned} Rf_* T &\longrightarrow Rf_* R(i_U)_* i_U^* T \oplus Rf_* R(i_V)_* i_V^* T \longrightarrow \\ &\longrightarrow Rf_* R(i_{U \cap V})_* i_{U \cap V}^* T \longrightarrow \Sigma Rf_* T \end{aligned}$$

$Rf_* R(i_U)_*$ is the restriction $R(f_U)_*$ of Rf_* on U , so it commutes with coproducts, and the same holds for $Rf_* R(i_V)_* = R(f_V)_*$ and $Rf_* R(i_{U \cap V})_* = R(f_{U \cap V})_*$, while Ri_U^* , Ri_V^* and $Ri_{U \cap V}^*$ commute with coproducts because they have right adjoints.

The coproduct of triangles is a triangle, so we have a morphism of triangles

$$\begin{array}{ccccc}
\coprod_{\lambda \in \Lambda} Rf_*(T_\lambda) & \longrightarrow & \coprod_{\lambda \in \Lambda} (R(f_U)_* i_u^*(T_\lambda) \oplus R(f_V)_* i_v^*(T_\lambda)) & \longrightarrow & \\
\downarrow \alpha & & \downarrow \beta & & \\
Rf_* (\coprod_{\lambda \in \Lambda} T_\lambda) & \longrightarrow & R(f_V)_* i_U^* (\coprod_{\lambda \in \Lambda} T_\lambda) \oplus R(f_V)_* i_V^* (\coprod_{\lambda \in \Lambda} T_\lambda) & \longrightarrow & \\
& & \downarrow \gamma & & \downarrow \Sigma \alpha \\
& \longrightarrow & \coprod_{\lambda \in \Lambda} R(f_{U \cap V})_* i_{U \cap V}^*(T_\lambda) & \longrightarrow & \Sigma \coprod_{\lambda \in \Lambda} Rf_*(T_\lambda) \\
& & \downarrow \gamma & & \downarrow \Sigma \alpha \\
& \longrightarrow & R(f_{U \cap V})_* i_{U \cap V}^* (\coprod_{\lambda \in \Lambda} T_\lambda) & \longrightarrow & \Sigma Rf_* (\coprod_{\lambda \in \Lambda} T_\lambda)
\end{array}$$

where β and γ are isomorphism. By Proposition 1.2.8, α is also an isomorphism. \square

Corollary 4.2.2. *The functor $H^i(X, -)$ respect coproducts.*

Proof. Take $Y = \text{Spec}(\mathbb{Z})$, then Rf_* is the derived functor of the global section functor. So it suffice to apply the previous lemma to conclude that the functor $H^i(X, -)$ respect coproducts. \square

Lemma 4.2.3. *If there is an ample line bundle \mathcal{L} on X then $D(\mathcal{Q}coh(X))$ is compactly generated.*

Proof. We can see \mathcal{L} as an object in $D(\mathcal{Q}coh(X))$ by taking the complex which is \mathcal{L} in degree 0 and 0 at every other degree.

Let $\{T_\lambda | \lambda \in \Lambda\}$ be a set of objects of $D(\mathcal{Q}coh(X))$. Then

$$\text{Hom}(\mathcal{L}, \coprod_{\lambda \in \Lambda} T_\lambda) = H^0(\mathcal{L}^{-1} \otimes \coprod_{\lambda \in \Lambda} T_\lambda)$$

The tensor product respect coproducts because it has a right adjoint, and H^0 respect coproducts by Lemma 4.2.1, so

$$\text{Hom}(\mathcal{L}, \coprod_{\lambda \in \Lambda} T_\lambda) = \coprod_{\lambda \in \Lambda} H^0(\mathcal{L}^{-1} \otimes T_\lambda) = \coprod_{\lambda \in \Lambda} \text{Hom}(\mathcal{L}, T_\lambda)$$

and \mathcal{L} is a compact object.

Since \mathcal{L} is ample, it is trivial that for any $n, m \in \mathbb{Z}$ also $\Sigma^n \mathcal{L}^m$ are compact objects. Let $\Omega = \{\Sigma^n \mathcal{L}^m | m, n \in \mathbb{Z}\}$ and V an object in $D(\mathcal{Q}coh(X))$ not isomorphic to 0.

There is a $k \in \mathbb{Z}$ such that the sheaf cohomology $\mathcal{H}^{-k}(V) \neq 0$, but $\mathcal{H}^{-k}(V)$ is a quasi-coherent sheaf on X , so $\mathcal{L}^t \otimes \mathcal{H}^{-k}(V)$ has non trivial global sections for some $t \gg 0$ because \mathcal{L} is ample.

This means that we have a surjective morphism from the kernel of $X^{-k} \rightarrow X^{-k+1}$, given by the complex X , to $\mathcal{H}^{-k}(X)$ and that for $t \gg 0$ there is an element $f \in H^0(\mathcal{L} \otimes X^{-k})$ that is sent to 0 in $H^0(\mathcal{L}^t \otimes X^{-k+1})$ and to a non-zero element in $H^0(\mathcal{H}^{-k}(\mathcal{L}^t \otimes X))$. This identify a non-zero map from $\Sigma^n \mathcal{L}^{-k}$ to X .

So if $\text{Hom}(\Omega, X) = 0$ it is $X = 0$. \square

Definition 4.2.4. A complex L in $\mathcal{D}(\mathcal{Qcoh}(X))$ is called perfect if it is isomorphic, locally on X , to a bounded complex of finitely generated, projective \mathcal{O}_X -modules.

For instance, the complex associated to an ample line bundle \mathcal{L} is perfect.

Lemma 4.2.5. A perfect complex on X is compact.

Proof. Let $\{T_\lambda\}_{\lambda \in \Lambda}$ be a collection of objects in $\mathcal{D}(\mathcal{Qcoh}(X))$ and L a perfect complex. Take the sheaf $RHom(L, \coprod_{\lambda \in \Lambda} T_\lambda)$ and consider it as an object in $\mathcal{D}(\mathcal{Qcoh}(X))$. Then there is a natural map

$$\phi_L : \coprod_{\lambda \in \Lambda} RHom(L, T_\lambda) \rightarrow RHom\left(L, \coprod_{\lambda \in \Lambda} T_\lambda\right)$$

We want to prove that ϕ_L is an isomorphism, this is a local problem, so we can assume X affine. We can replace L with a bounded complex of finitely generated projective \mathcal{O}_X -modules.

If $L = \Sigma^m \mathcal{O}_X^n$, that is L is a direct sum of n -times $\Sigma^m \mathcal{O}_X$, then ϕ_L is obviously an isomorphism (because $Hom(\mathcal{O}_X, T) = T$ for any sheaf T on X).

If $\tilde{L} = L' \oplus L''$ and $\phi_{\tilde{L}}$ is an isomorphism, then $\phi_{L'}$ and $\phi_{L''}$ are also isomorphism. So if L is a suspension of a finitely generated projective module, and hence a direct summand of a suspension of a direct sum of \mathcal{O}_X , then ϕ_L is an isomorphism.

Suppose we are given a triangle in $\mathcal{D}(\mathcal{Qcoh}(X))$

$$L \longrightarrow L' \longrightarrow L'' \longrightarrow \Sigma L$$

with L' and L'' such that $\phi_{\Sigma^m L'}$ and $\phi_{\Sigma^m L''}$ are isomorphism for every $m \in \mathbb{Z}$. Then by the 5-lemma it follows that $\phi_{\Sigma^m L}$ is an isomorphism for every $m \in \mathbb{Z}$.

It follows that the full subcategory of the objects J such that $\phi_{\Sigma^m J}$ is an isomorphism for every $m \in \mathbb{Z}$ is triangulated. Since it contains the finitely generated projective \mathcal{O}_X -modules as proven above, it contains also every bounded complex of finitely generated projectives.

If L is a perfect complex, we have proved that

$$\phi_L : \coprod_{\lambda \in \Lambda} RHom(L, T_\lambda) \rightarrow RHom\left(L, \coprod_{\lambda \in \Lambda} T_\lambda\right)$$

is an isomorphism.

Then we have

$$Hom\left(L, \coprod_{\lambda \in \Lambda} T_\lambda\right) = H^0\left(RHom\left(L, \coprod_{\lambda \in \Lambda} T_\lambda\right)\right) = H^0\left(\coprod_{\lambda \in \Lambda} RHom(L, T_\lambda)\right)$$

By Corollary 4.2.2 H^0 commutes with coproducts, so

$$H^0\left(\coprod_{\lambda \in \Lambda} RHom(L, T_\lambda)\right) = \coprod_{\lambda \in \Lambda} H^0(RHom(L, T_\lambda)) = \coprod_{\lambda \in \Lambda} Hom(L, T_\lambda)$$

Thus L is compact. □

Let $X = \operatorname{Spec} A$ and U a quasi-compact open affine, with $U = \bigcup_{i=1}^n X_{f_i}$. Let $j : U \rightarrow X$ be the inclusion of U in X and $i : X - U \rightarrow X$ the inclusion of $X - U$ in X . Let \mathcal{M} be the full subcategory of $\mathcal{D}(\mathcal{Qcoh}(X))$ composed by all the sheaves of \mathcal{O}_X -modules on X with support in $X \setminus U$. Then $i_* : \mathcal{D}(\mathcal{Qcoh}(X - U)) \rightarrow \mathcal{D}(\mathcal{Qcoh}(X))$ induces an isomorphism between $\mathcal{D}(\mathcal{Qcoh}(X - U))$ and \mathcal{M} .

Lemma 4.2.6. *Let $X = \operatorname{Spec} A$ and U a quasi-compact open affine, with $U = \bigcup_{i=1}^n X_{f_i}$. Let T be in $\mathcal{D}(\mathcal{Qcoh}(X))$ with $j^*T = 0$ and F be the complex*

$$F = \otimes_{i=1}^n \left(A \xrightarrow{f_i} A \right).$$

Then $RHom(F, T) = 0$ if and only if $T = 0$.

Proof. Since $j^*T = 0$, the cohomology of the complex T is (f_1, \dots, f_n) -torsion. F is self-dual, so $RHom(F, T) = F \otimes T$.

By induction, it suffice to prove the case with $n = 1$. Then $F = (A \xrightarrow{f} A)$, T has an f -torsion cohomology and $(A \xrightarrow{g} A) \times T = 0$. This means that the multiplication by f induces an isomorphism on the cohomology of T , but this cohomology is f -torsion, so it must be identically 0. We conclude that $T = 0$ in $\mathcal{D}(\mathcal{Qcoh}(X))$. \square

Lemma 4.2.7. *Let $X = \operatorname{Spec} A$ and U a quasi-compact open affine, with $U = \bigcup_{i=1}^n X_{f_i}$. An object T in $\mathcal{D}(\mathcal{Qcoh}(X))$ is \mathcal{M} -local if and only if for any $n \in \mathbb{Z}$*

$$\operatorname{Hom}(\Sigma^n F, T) = 0$$

where F is the complex

$$F = \otimes_{i=1}^n \left(A \xrightarrow{f_i} A \right)$$

Proof. Since $\operatorname{Hom}(\Sigma^n F, T) = H^n(RHom(F, T))$, it will suffice to prove that T is \mathcal{M} -local if and only if $RHom(F, T) = 0$.

Suppose that T is \mathcal{M} -local. T is \mathcal{M} -local if and only if $T = j_*G$ for some G in $\mathcal{D}(\mathcal{Qcoh}(X - U))$, so $RHom(F, T) = RHom(F, j_*G) = RHom(j^*F, G) = 0$ since $j^*F = 0$.

There is an exact sequence

$$0 \rightarrow i_*i^!T \rightarrow T \rightarrow j_*j^*T \rightarrow 0$$

so by Lemma 4.1.12 there is a triangle

$$i_*i^!T \longrightarrow T \longrightarrow j_*j^*T \longrightarrow \Sigma i_*i^!T$$

If we apply $RHom(F, -)$ to this triangle, since $RHom(F, j_*j^*T) = 0$, we obtain an isomorphism $\phi : RHom(F, i_*i^!T) \rightarrow RHom(F, T)$. By Lemma 4.2.6 we have that $RHom(F, i_*i^!T) = 0$ if and only if $i_*i^!T = 0$, and this happens if and only if T is \mathcal{M} -local. \square

So if T is a complex in $\mathcal{D}(\mathcal{Qcoh}(X))$ supported on $X - U$, and $\operatorname{Hom}(\Sigma^n F, T) = 0$ for any n , then it is \mathcal{M} -local, but then it must be zero. Since F is a compact object, we have that $\{\Sigma^n F\}$ is a generating set of compact objects for the full subcategory of complexes in $\mathcal{D}(\mathcal{Qcoh}(X))$ supported on $X - U$, so it is compactly generated.

Lemma 4.2.8. *Let $U \subseteq X$ be a quasi-compact open subscheme of X . Let H be an object in $\mathcal{D}(\mathcal{Qcoh}(X))$ and J a perfect complex in $\mathcal{D}(\mathcal{Qcoh}(U))$, suppose there is a morphism $f : J \rightarrow H$ in $\mathcal{D}(\mathcal{Qcoh}(U))$. Then there is a perfect complex J' in $\mathcal{D}(\mathcal{Qcoh}(U))$ such that $f \circ \pi$ (where π is the projection $J \oplus J' \rightarrow J$), can be lifted to a morphism in $\mathcal{D}(\mathcal{Qcoh}(X))$, that is, there exist a perfect complex J'' in $\mathcal{D}(\mathcal{Qcoh}(X))$ and a morphism $f' : J'' \rightarrow H$ which restricted on U gives $f \circ \pi$.*

Proof. Suppose X is affine, then the trivial line bundle is ample. By Lemma 4.2.3 we have that $\mathcal{D}(\mathcal{Qcoh}(X))$ is compactly generated, and by Lemma 4.2.5 we can take the perfect complexes as the generating class of compact objects.

Let $\mathcal{D}_{X-U}(\mathcal{Qcoh}(X))$ the full subcategory of $\mathcal{D}(\mathcal{Qcoh}(X))$ whose objects are complexes supported on $X - U$. By Lemma 4.2.7 then $\mathcal{D}_{X-U}(\mathcal{Qcoh}(X))$ is compactly generated.

Now we apply Theorem 3.1.8 with $\mathcal{T} = \mathcal{D}(\mathcal{Qcoh}(X))$, $\mathcal{R} = \mathcal{D}_{X-U}(\mathcal{Qcoh}(X))$ and R a generating set for \mathcal{R} . Then we can identify the quotient $\mathcal{S} = \mathcal{T}/\mathcal{R}$ with $\mathcal{D}(\mathcal{Qcoh}(U))$.

J is a perfect complex \mathcal{S} , so it is a compact object. By Theorem 3.1.8.4 there is a perfect complex \tilde{J} in \mathcal{T} isomorphic in \mathcal{S} to the complex $J \oplus \Sigma J$. Then we can apply Theorem 3.1.8.5, lifting the map $J \oplus \Sigma J \rightarrow H$ in \mathcal{S} to a map $\tilde{J} \rightarrow H$ in \mathcal{T} , up to a different choice of \tilde{J} .

X is quasi-compact, so it can be covered by finitely many open affines. By induction, it will suffice to prove the lemma in the case where X is the union of two open affines, $X = U \cup W$. We have just proved that the restriction of the map $f : J \rightarrow H$ to $U \cap W$ can be extended to all of W , that is, there exists a perfect complex \tilde{J} on W and a map $\tilde{J} \rightarrow H$ such that its restriction to $U \cap W$ is isomorphic to $J \oplus \Sigma J \rightarrow H$.

Let $j_U, j_W, j_{U \cap W}$ be the open immersions. We have that on $U \cap W$ there is an isomorphism

$$\phi : (j_{U \cap W})^*(J \oplus \Sigma J) \rightarrow (j_{U \cap W})^* \tilde{J}$$

From the exact sequence

$$0 \rightarrow H \rightarrow (j_U)_*(j_U)^* H \oplus (j_W)_*(j_W)^* H \rightarrow (j_{U \cap W})_*(j_{U \cap W})^* H \rightarrow 0$$

we deduce by Lemma 4.1.12 a triangle

$$H \longrightarrow (j_U)_*(j_U)^* H \oplus (j_W)_*(j_W)^* H \longrightarrow (j_{U \cap W})_*(j_{U \cap W})^* H \longrightarrow \Sigma H$$

We can complete the morphism $(j_U)_*(J \oplus \Sigma J) \oplus (j_W)_* \tilde{J} \rightarrow (j_{U \cap W})_*(j_{U \cap W})^*(J \oplus \Sigma J)$ to the triangle (up to a rotation)

$$J'' \longrightarrow (j_U)_*(J \oplus \Sigma J) \oplus (j_W)_* \tilde{J} \longrightarrow (j_{U \cap W})_*(j_{U \cap W})^*(J \oplus \Sigma J) \longrightarrow \Sigma J''$$

where J'' is a perfect complex because it is a mapping cone on a morphism of perfect complexes.

There are two obvious morphisms

$$\alpha : (j_U)_*(J \oplus \Sigma J) \oplus (j_W)_* \tilde{J} \rightarrow (j_U)_*(j_U)^* H \oplus (j_W)_*(j_W)^* H$$

and

$$\beta : (j_{U \cap W})_*(j_{U \cap W})^*(J \oplus \Sigma J) \rightarrow (j_{U \cap W})_*(j_{U \cap W})^*H$$

so by axiom [T5] of Definition 1.2.1 there is a morphism

$$f' : J'' \rightarrow H$$

such that the diagram

$$\begin{array}{ccccccc} J'' & \longrightarrow & (j_U)_*(J \oplus \Sigma J) \oplus (j_W)_*\tilde{J} & \longrightarrow & (j_{U \cap W})_*(j_{U \cap W})^*(J \oplus \Sigma J) & \longrightarrow & \Sigma J'' \\ \downarrow f' & & \downarrow \alpha & & \downarrow \beta & & \downarrow \Sigma f' \\ H & \longrightarrow & (j_U)_*(j_U)^*H \oplus (j_W)_*(j_W)^*H & \longrightarrow & (j_{U \cap W})_*(j_{U \cap W})^*H & \longrightarrow & \Sigma H \end{array}$$

is a morphism of triangles. f' is a lifting of $J \oplus \Sigma J \rightarrow H$ to $\mathcal{D}(\mathcal{Qcoh}(X))$. \square

Proposition 4.2.9. *Let X be a quasi-compact, separated scheme, then the category $\mathcal{D}(\mathcal{Qcoh}(X))$ is compactly generated.*

Proof. Let U be an open affine of X , then the trivial line bundle on U is ample, so by Lemma 4.2.3 $\mathcal{D}(\mathcal{Qcoh}(U))$ is compactly generated. By Lemma 4.2.5 we can take the perfect complexes on U as the generating class of compact objects. Let $i : U \rightarrow X$ be the inclusion.

Let H be an object in $\mathcal{D}(\mathcal{Qcoh}(X))$ such that its restriction i^*H to U is non-zero. Since an object L in $\mathcal{D}(\mathcal{Qcoh}(U))$ is zero if and only if $\text{Hom}(T, L) = 0$ for any T perfect complex on U , there is a perfect complex J on U and a non-zero map $f : J \rightarrow H$ on U . By Lemma 4.2.8 f can be extended to a map $f' : J'' \rightarrow H$ on X , with J'' a perfect complex on X .

Then, if K is an object in $\mathcal{D}(\mathcal{Qcoh}(X))$, there is a non-zero morphism from a perfect complex to K if and only if the restriction of K to some open affine is non-zero, that is if and only if $K \neq 0$. So $\mathcal{D}(\mathcal{Qcoh}(X))$ is compactly generated, and we can choose the perfect complexes as a generating class of compact objects. \square

Lemma 4.2.10. *Let A be a ring, $\mathcal{D}^b(A)$ the derived category of finite complexes of finitely generated projective A -modules. Then every idempotent in $\mathcal{D}^b(A)$ is split.*

Proof. Let $e : X \rightarrow X$ be an idempotent in $\mathcal{D}^b(A)$. By Proposition 1.2.14 we have that, in $\mathcal{D}(A)$, e have a split given by $X = Y \oplus Z$, where Y is the totalization of the complex

$$\cdots \longrightarrow X \xrightarrow{e} X \xrightarrow{1-e} X \xrightarrow{e} X$$

and Z is the totalization of the complex

$$\cdots \longrightarrow X \xrightarrow{1-e} X \xrightarrow{e} X \xrightarrow{1-e} X$$

Y and Z are homotopy colimits of sequences $\{Y_i\}$ and $\{Z_i\}$ of objects in $\mathcal{D}^b(A)$, and there is a quasi-isomorphism between $\text{hocolim}(Y_i)$ and $\text{colim}(Y_i)$. The Y_i and Z_i are obtained through mapping cones on morphisms of the form

$X \rightarrow Y_i$ or $X \rightarrow Z_i$, that are morphisms between objects in $\mathcal{D}^b(A)$. By the construction of mapping cones in derived categories, it follows that the sequences are stable in each degree. Then their colimits are bounded above complexes, and they are complexes of finitely generated projective A -modules because at each degree there is a finite direct sum of finitely generated projective A -modules. Since $X = Y \oplus Z$, X is a bounded above complex of finitely generated projective A -modules.

Since X is a finite complex, also Y and Z are finite complexes, so e is split in $\mathcal{D}^b(A)$. □

Proposition 4.2.11. *Let A be a ring. If X is an object in $\mathcal{D}^b(A)$ and $X = Y \oplus Z$ in $\mathcal{D}(A)$, then Y and Z are in $\mathcal{D}^b(A)$.*

Proof. $e = \begin{pmatrix} 1_Y & 0 \\ 0 & 0 \end{pmatrix}$ is an idempotent in $\mathcal{D}^b(A)$, so it is split in $\mathcal{D}^b(A)$ by Lemma 4.2.10. Since the split is given by $X = Y \oplus Z$ by construction of e , it follows that X and Y are in $\mathcal{D}^b(A)$. □

Proposition 4.2.12. *Let X be a quasi-compact, separated scheme. Then the category of all perfect complexes on X is $\mathcal{D}(\mathcal{Qcoh}(X))^c$.*

Proof. From Proposition 4.2.9 we know that $\mathcal{D}(\mathcal{Qcoh}(X))$ is compactly generated and that the perfect complexes are a generating class of compact objects. We apply Theorem 3.1.8 with $\mathcal{T} = \mathcal{D}(\mathcal{Qcoh}(X))$ and R the generating class formed by the perfect complexes. Since R generates \mathcal{T} , it is $\mathcal{R} = \mathcal{T}$, where \mathcal{R} is the smallest full, triangulated subcategory of \mathcal{T} containing R and closed with respect to coproducts.

R is obviously closed with respect to triangles. R is closed with respect to direct sums if and only if a direct summand of a perfect complex is perfect. Since this is a local problem, we can suppose $X = \text{Spec } A$, with A a ring. Then the category of perfect complexes is equivalent to the derived category of finite complexes of finitely generated projective A -modules, and this category is closed with respect to direct sums by Proposition 4.2.11.

By Theorem 3.1.8.3 it is $R = \mathcal{R}^c$, so the compact objects in $\mathcal{D}(\mathcal{Qcoh}(X))$ are exactly the perfect complexes. □

4.3 Grothendieck duality

In this final section, we will prove a version of Grothendieck duality similar to that given in Theorem 11.1 of [4], that is an isomorphism

$$Rf_*(R\text{Hom}(S, f^!T)) \simeq R\text{Hom}(Rf_*S, T).$$

Previously, we have established that if X is a quasi-compact, separated scheme then $\mathcal{D}(\mathcal{Qcoh}(X))$ is a compactly generated triangulated category. We have also determined that the subcategory of perfect complexes is in fact the subcategory of compact objects.

Let $f : X \rightarrow Y$ be a morphism of noetherian, separated schemes. Since X is noetherian it is quasi-compact, so $\mathcal{D}(\mathcal{Qcoh}(X))$ is compactly generated. Then

$Rf_* : \mathcal{D}(\mathcal{Qcoh}(X)) \rightarrow \mathcal{D}(\mathcal{Qcoh}(Y))$ is a triangulated functor, and by Proposition 3.2.2 it has a right adjoint $f^!$.

Let S be an object in $\mathcal{D}(\mathcal{Qcoh}(X))$ and T an object in $\mathcal{D}(\mathcal{Qcoh}(Y))$. We have a natural map $\alpha : Rf_*(RHom(S, f^!T)) \rightarrow Rf_*(RHom(Rf_*S, Rf_*f^!T))$. Then the counit of the adjunction $u : Rf_*f^!T \rightarrow T$ defines through composition a natural map

$$\phi : Rf_*(RHom(S, f^!T)) \rightarrow RHom(Rf_*S, T).$$

We want to show that ϕ is an isomorphism if f is proper. First of all, we need to find a more explicit way to express $f^!$.

Proposition 4.3.1 (Projection formula). *Let $f : X \rightarrow Y$ be a morphism of separated, quasi-compact schemes. Let $\mathcal{D}(X)$ be the derived category of \mathcal{O}_X -modules and $\mathcal{D}(Y)$ the derived category of \mathcal{O}_Y -modules. If T is an object in $\mathcal{D}(Y)$ and S an object in $\mathcal{D}(X)$, there is a natural map in $\mathcal{D}(Y)$*

$$T^L \otimes Rf_*S \rightarrow Rf_*(Lf^*T^L \otimes S)$$

If T is in $\mathcal{D}(\mathcal{Qcoh}(Y))$ and S is in $\mathcal{D}(\mathcal{Qcoh}(X))$ then this natural map is an isomorphism.

The proof can be found in [8]. For a particular case see Proposition 5.6, [4], where the proof is for bounded below complexes.

Theorem 4.3.2. *Let $f : X \rightarrow Y$ be a morphism of schemes with Y quasi-compact and separated, and f such that Rf_* has a right adjoint $f^!$.*

$f^!$ respects coproducts if and only if there is a natural isomorphism of functors between $f^!$ and $(Lf^-)\otimes_{\mathcal{O}_X}(f^!\mathcal{O}_Y)$.*

Proof. Lf^* commutes with coproducts because it has a right adjoint, and tensor products respect coproducts, so $(Lf^*-)\otimes_{\mathcal{O}_X}(f^!\mathcal{O}_Y)$. Then, if there is the required isomorphism of functors, $f^!$ also respect coproducts.

Suppose now that $f^!$ respect coproducts.

Let T be an object in $\mathcal{D}(\mathcal{Qcoh}(Y))$. The counit of the adjunction is a natural morphism $\mu : Rf_*f^!T \rightarrow T$. By Proposition 4.3.1 for any object S in $\mathcal{D}(\mathcal{Qcoh}(X))$ it is

$$Rf_*(Lf^*(S) \otimes_{\mathcal{O}_X} f^!T) = S \otimes_{\mathcal{O}_Y} Rf_*f^!T$$

so we can see $1_S \otimes \mu$ as a natural morphism from $Rf_*(Lf^*(S) \otimes_{\mathcal{O}_X} f^!T)$ to $S \otimes_{\mathcal{O}_Y} T$. Then by adjunction we obtain a natural morphism

$$\beta : Lf^*(S) \otimes_{\mathcal{O}_X} f^!T \rightarrow f^!(S \otimes_{\mathcal{O}_Y} T)$$

We want to show that the morphism above is an isomorphism. Suppose S is a compact object in $\mathcal{D}(\mathcal{Qcoh}(Y))$ and Z an object in $\mathcal{D}(\mathcal{Qcoh}(X))$, applying $\text{Hom}(Z, -)$ to β we obtain a morphism α , we need to show that α is in fact an isomorphism.

By definition, α sends a morphism γ from Z to $Lf^*(S) \otimes_{\mathcal{O}_X} f^!T$ to the morphism $\gamma' = \beta \circ \gamma$ from Z to $f^!(S \otimes_{\mathcal{O}_Y} T)$. By adjunction, γ' in turn yields a

map γ'' from Rf_*Z to $S \otimes_{\mathcal{O}_Y} T$. But we have that γ'' is the composite of $Rf_*\gamma$ with the natural counit

$$Rf_*(Lf^*(S) \otimes_{\mathcal{O}_X} f^!T) = S \otimes_{\mathcal{O}_Y} Rf_*f^!T \xrightarrow{1_S \otimes \mu} S \otimes_{\mathcal{O}_Y} T$$

since β has been determined by adjunction from this morphism.

Let $\tilde{S} = RHom_{\mathcal{O}_Y}(S, \mathcal{O}_Y)$, then $Lf^*\tilde{S} = RHom_{\mathcal{O}_X}(Lf^*S, \mathcal{O}_X)$. S is a perfect complex and hence a compact object. Since $f^!$ is an adjoint of Lf^* and $f^!$ respects coproducts, by Proposition 3.2.3 Lf^*S is a compact object, so it is a perfect complex by Proposition 4.2.12. It follows that $Hom_X(- \otimes Lf^*\tilde{S}, -)$ and $Hom_Y(- \otimes \tilde{S}, -)$ are naturally isomorphic respectively to $Hom_X(-, Lf^*S \otimes -)$ and $Hom_X(-, S \otimes -)$.

So we have that $Hom_{\mathcal{O}_X}(Z, Lf^*S \otimes_{\mathcal{O}_X} f^!T) = Hom_{\mathcal{O}_X}(Z \otimes Lf^*\tilde{S}, f^!T)$, and by adjunction this is equal to $Hom_{\mathcal{O}_Y}(Rf_*(Z \otimes Lf^*\tilde{S}), T)$. By the projection formula (see Proposition 4.3.1) we have

$$Hom_{\mathcal{O}_Y}(Rf_*(Z \otimes Lf^*\tilde{S}), T) = Hom_{\mathcal{O}_Y}(Rf_*(Z) \otimes \tilde{S}, T)$$

and the right half of the equality is in turn naturally isomorphic to $Hom_{\mathcal{O}_Y}(Rf_*(Z), S \otimes_{\mathcal{O}_Y} T)$, and again by adjunction this is equal to $Hom_{\mathcal{O}_X}(Z, f^!(S \otimes T))$. Thus we have obtained an isomorphism from $Hom_{\mathcal{O}_X}(Z, Lf^*S \otimes_{\mathcal{O}_X} f^!T)$ to $Hom_{\mathcal{O}_Y}(Rf_*(Z), S \otimes_{\mathcal{O}_Y} T)$, and we have found it through adjunctions and the projection formula. An easy check shows that this isomorphism is equal to α . Then β is an isomorphism for any S compact in $\mathcal{D}(\mathcal{Qcoh}(Y))$.

The tensor product and Lf^* are both triangulated functors, so $(Lf^*-) \otimes_{\mathcal{O}_X} f^!T$ is triangulated. By Lemma 2.1.1 $f^!$ is triangulated because it is the adjoint of a triangulated functor, so $f^!(- \otimes_{\mathcal{O}_Y} T)$ is also triangulated. It follows that β is a natural transformation of triangulated functors that respect coproducts. Let \mathcal{S} be the full subcategory of the objects L in $\mathcal{D}(\mathcal{Qcoh}(Y))$ such that $\beta(\Sigma^n L)$ is an isomorphism for all $n \in \mathbb{Z}$. \mathcal{S} is triangulated and closed with respect to coproducts and it contains a generating class of compact objects for $\mathcal{D}(\mathcal{Qcoh}(Y))$, so $\mathcal{S} = \mathcal{D}(\mathcal{Qcoh}(Y))$. It follows that β is a natural isomorphism.

If we take $T = \mathcal{O}_Y$, we have the proof of the theorem. □

Note that if f is a proper morphism between noetherian, separated schemes, by [11] we have that Rf_* sends perfect complexes on X to perfect complexes on Y , so it sends compact objects in compact objects. It follows that $f^!$ commute with coproducts, and we can use the previous isomorphism between $f^!$ and $(Lf^*-) \otimes_{\mathcal{O}_X} (f^!\mathcal{O}_Y)$.

Theorem 4.3.3 (Grothendieck's duality). *Let $f : X \rightarrow Y$ be a proper morphism of noetherian, separated schemes. Then the map ϕ defined at the beginning of the section is an isomorphism.*

Proof. Since $f^!$ is the right adjoint of Rf_* and ϕ is obtained through the counit of the adjunction, if we apply $H^0(Y, -)$ to ϕ we obtain the natural isomorphism between $Hom(S, f^!T)$ and $Hom(Rf_*S, T)$.

ϕ is an isomorphism of sheaves if and only if the right derived functor $R\Gamma(U, -)$ of $\Gamma(U, -)$ provides an isomorphism $R\Gamma(U, \phi)$ for any U open subscheme of Y . So let U be an open subscheme of Y , $V = f^{-1}U$, then there is a

commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{i'} & X \\ \downarrow f|_V & & \downarrow f \\ U & \xrightarrow{i} & Y \end{array}$$

where i and i' are the inclusions. We need to show that $((f|_V)^! \circ i^*)(T) = ((i')^* \circ f^!)(T)$.

We have that for any object \tilde{S} in $\mathcal{D}(\mathcal{Qcoh}(X))$ it is $(i^* \circ Rf_*)(\tilde{S}) = (R(f|_V)_* \circ (i')^*)(\tilde{S})$, so by adjunction for any object W in $\mathcal{D}(\mathcal{Qcoh}(V))$ we have $(f^! \circ Ri_*)(W) = (Ri'_* \circ (f|_V)^!)(W)$.

Since $(i')^* \circ Ri'_*$ is the identity functor, it is $((f|_V)^! \circ i^*)(T) = ((i')^* \circ Ri'_* \circ (f|_V)^! \circ i^*)(T) = ((i')^* \circ f^! \circ Ri_* \circ j^*)(T)$.

Then we need only to show that the counit of the adjunction $1 \rightarrow Ri_* \circ i^*$ induces an isomorphism between $((i')^* \circ f^!)(T)$ and $((i')^* \circ f^! \circ Ri_* \circ i^*)(T)$.

Let $Z = Y \setminus U$ with the induced reduced structure of closed subscheme, let j be the closed immersion $j : Z \rightarrow Y$. If $j^!$ is the closed restriction, we have an exact sequence

$$0 \rightarrow (j_* \circ j^!)(T) \rightarrow T \rightarrow (i_* \circ i^*)(T) \rightarrow 0$$

and by Proposition 4.1.12 we deduce a triangle

$$(j_* \circ j^!)(T) \longrightarrow T \longrightarrow (i_* \circ i^*)(T) \longrightarrow \Sigma(j_* \circ j^!)(T)$$

Replacing T with a K -injective complex, we have that $(i_* \circ i^*)(T)$ is isomorphic to $(Ri_* \circ i^*)(T)$, so we obtain the triangle

$$(j_* \circ j^!)(T) \longrightarrow T \longrightarrow (Ri_* \circ i^*)(T) \longrightarrow \Sigma(i_* \circ i^*)(T)$$

Let K be an object in $\mathcal{D}(\mathcal{Qcoh}(Y))$ of the form $j_* j^! N$ for some N , then K is a complex with support in Z . By Proposition 4.3.2 we have $f^!(K) = Lf^*(K) \otimes f^!(\mathcal{O}_Y)$, but the support of $Lf^*(K)$ is in $f^{-1}(Z)$, so also $f^!(K)$ is supported on $f^{-1}(Z)$. This means that $f^!(Z)$ is acyclic on $U = X \setminus Z$, so $((i')^* \circ f^!)(Z) = 0$.

It follows that applying $(i')^* f^!$ to the triangle

$$(j_* \circ j^!)(T) \longrightarrow T \longrightarrow (Ri_* \circ i^*)(T) \longrightarrow \Sigma(i_* \circ i^*)(T)$$

we deduce an isomorphism

$$((i')^* \circ f^!)(T) \rightarrow ((i')^* \circ f^! \circ Ri_* \circ i^*)(T)$$

□

If we take $X = \mathbb{P}_k^n$, the n -dimensional projective space over an algebraically closed field k , and $Y = \text{Spec}(k)$, with $S = \mathcal{F}$ a quasi-coherent sheaf over X and $T = \mathcal{O}_Y$, then we obtain the Serre duality for a projective space.

First of all, we must identify $f^! \mathcal{O}_Y$. By duality we have

$$\begin{aligned} Rf_* \mathcal{O}_X &= R\text{Hom}_Y(Rf_* \mathcal{O}_X, \mathcal{O}_Y) = Rf_* R\text{Hom}_X(\mathcal{O}_X, f^! \mathcal{O}_Y) = \\ &= R\text{Hom}_Y(Rf_* R\text{Hom}_X(\mathcal{O}_X, f^! \mathcal{O}_Y), \mathcal{O}_Y) = Rf_* R\text{Hom}_X(R\text{Hom}_X(\mathcal{O}_X, \mathcal{O}_Y), f^! \mathcal{O}_Y) \end{aligned}$$

Since \mathcal{O}_X and \mathcal{O}_Y are quasi-coherent and f_* is faithful, we have that

$$\mathcal{O}_X = RHom_X(RHom_X(\mathcal{O}_X, \mathcal{O}_Y), f^! \mathcal{O}_Y)$$

so $f^! \mathcal{O}_Y$ is a dualizing complex for X . By unicity of the dualizing complex (see [4], Theorem 3.1, pag. 266), then we have $f^! \mathcal{O}_Y = \omega[n] = \Sigma^n \omega$, where ω is the sheaf of n -differentials.

Then, taking global sections and then the cohomology group H^i , we deduce from

$$Rf_*(RHom(S, \omega[n])) \simeq RHom(Rf_* S, T)$$

an isomorphism

$$\text{Ext}_X^i(S, \omega[n]) \simeq \text{Ext}_Y^i(R^i f_* S, \mathcal{O}_Y) = \text{Hom}_Y(R^i f_* S, \mathcal{O}_Y[i])$$

Then by [3], Proposition 8.5 pag. 251, and since the global section of \mathcal{O}_Y is isomorphic to k , we have

$$\text{Ext}^i(S, \omega[n]) \simeq \text{Hom}(H^{-i}(X, S), k)$$

and taking into account the shift on ω we get

$$\text{Ext}^{i+n}(S, \omega) \simeq \text{Hom}(H^{-i}(X, S), k)$$

so we have obtained Serre duality.

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