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TESI DI DOTTORATO DI RICERCA

ON 1-MOTIVES WITH TORSION AND THEIR l-ADIC REALISATIONS

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ABSTRACT

This thesis deals with 1-motives with torsion and their l-adic realisations. The category $\mathcal{M}$ of 1-motives with torsion is abelian, and it is $\mathbb{Z}[1/p]$-linear when the base field is of positive characteristic p. We relate the homological dimension $\mathrm{d}(\mathcal{M})$ of the abelian category $\mathcal{M}$ of 1-motives with torsion over a perfect field k, to the cohomological dimension $\mathrm{cd}(\Gamma)$ of the absolute Galois group Γ of k, and prove $\mathrm{d}(\mathcal{M}) = \mathrm{cd}(\Gamma)+1$.

We compare the Hom-group and 1-Ext group between two 1-motives with torsion, with the corresponding Hom-group and 1-Ext group of their l-adic realisations. In particular, we generalise Falting's theorem on homomorphisms of abelian varieties over finite fields (Tate Theorem in this case) and number fields to 1-motives with torsion. We show the 1-Ext group between 1-motives with torsion injects to the 1-Ext group of l-adic realisations. Over finite fields, we give a very explicit description to the maps T_l for Ext^i groups for all $i \geq 0$.

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Conventions

A field is always perfect, and p denotes its characteristic. l is a prime number. An algebraic variety over a field k is a reduced separated scheme which is of finite type over k. An algebraic curve is an algebraic variety of dimension 1.

$\mathcal{M}_1$, ${}^t\mathcal{M}_1^{\mathrm{eff}}$, and ${}^t\mathcal{M}_1$ denote the category of Deligne's 1-motives over k, the category of effective 1-motives with torsion over k, and the category of 1-motives with torsion over k respectively. $\mathcal{M}$ denotes the abelian category of 1-motives with torsion over k, which is just ${}^t\mathcal{M}_1$ in character zero case and ${}^t\mathcal{M}_1 \otimes \mathbb{Z}[1/p]$ in positive characteristic case. Let G be a profinite group, we denote the abelian category of discrete G-modules and the abelian category of finitely generated discrete G-modules by $\mathcal{C}_G$ and $\mathcal{C}_G^f$ respectively. In the case G being the absolute Galois group of the field k, we use the notation $\mathcal{C}_k$ instead of $\mathcal{C}_G^f$. We denote by $\mathcal{R}$ the abelian category of finitely generated $\mathbb{Z}_l$-modules with continuous Galois action.

Throughout this thesis, by k-group schemes we always mean commutative ones. Given any two k-group schemes X and Y, we use the notation $\mathrm{Ext}^i_{k-\mathrm{fppf}}(X,Y)$ to denote the i-th Yoneda extension group of X by Y in the abelian category of fppf-sheaves over k, and we write $\mathrm{Hom}_k(X,Y)$ as the group of homomorphisms in the category of k-group schemes, which is the same as $\mathrm{Hom}_{k-\mathrm{fppf}}(X,Y)$. If both X and Y are algebraic, we write the i-th Yoneda extension group of X by Y in the abelian category of commutative algebraic group schemes over k as $\mathrm{Ext}^i_k(X,Y)$. If both X and Y are étale locally constant defined by finitely generated abelian groups, we denote the i-th Yoneda extension group of X by Y in $\mathcal{C}_k$ by $\mathrm{Ext}^i_{\mathcal{C}_k}(X,Y)$.

Throughout this thesis, we fix a universe, and suppose the objects and the morphisms of the categories we consider form sets which belong to this universe.

Introduction

The Tao produces One;
One produces Two;
Two produces Three;
Three produces Everything.
—Tao Te Ching, by Laozi

It's well-known that cohomology theories are linearisations of geometric objects. In algebraic geometry, motives were introduced by Grothendieck as a universal cohomology theory which lies below the "good" cohomology theories. In other words, they are the best linearisations of algebraic varieties in the sense that all good linearisations (Weil cohomology theories) can be obtained from them, like "one comes from Tao" as said in Tao Te Ching.[1] More intuitively, they are the bricks of which algebraic varieties are made in some sense, like topological spaces are made up of cells up to weakly homotopy equivalence.

Grothendieck had constructed a category of pure motives, but a good category of mixed motives is still missing today. The first step towards mixed motives was made by Deligne, in the paper [6, sec. 10] he defined 1-motives (which are called Deligne's 1-motives in this thesis in order to distinguish these from the 1-motives with torsion). The 1-motives should be the motives of level ≤ 1 in the missing category of mixed motives. A Deligne's 1-motive $M = [L \xrightarrow{u} G]$ over a field k consists of an étale locally constant sheaf L defined by a finitely generated free abelian group, a semi-abelian variety G, and a morphism $u : L \to G$ of groups schemes over k. Given any algebraic curve C over k satisfying certain properties as in I.5, we can define the 1-motive associated to C, which is an analogue of the Jacobian variety associated to a smooth

[1] In our case, it should be "The Tao produces One, One produces Everything". The Two are usually interpreted as Yin and Yang, it would be great if we could relate something in mathematics to Yin-Yang.

proper algebraic curve over k.

A morphism between two Deligne's 1-motives is defined to be a morphism of the complexes underlying the 1-motives. Then Deligne's 1-motives form a category $\mathcal{M}_1$. Abelian categories are very handy for doing homology theory. But the category $\mathcal{M}_1$ is far from being abelian. In order to make an abelian category out of $\mathcal{M}_1$, one has to add the "torsion 1-motives" first. In their proof of Deligne's conjecture on 1-motives in characteristic zero case (a modified version) in [4], L. Barbieri-Viale, A. Rosenschon and M. Saito defined 1-motives with torsion, and constructed an abelian category of 1-motives with torsion. We denote this category by $\mathcal{M}$ which is the base on which are going to work in this thesis. Given any semi-abelian variety G over a field k, let n be a positive integer which is coprime to the characteristic of k, then the multiplication map $G \xrightarrow{[n]_G} G$ gives a short exact sequence

$$0 \to {}_nG \to G \xrightarrow{[n]_G} G \to 0$$

of commutative group schemes over k. Let n be l^r, passing to the l-adic Tate modules for some prime number l different from char(k), we then get a short exact sequence

$$0 \to T_lG \to T_lG \to {}_{l^r}G \to 0$$

of $\mathbb{Z}_l$-modules with continuous Galois action. Now, if we regard G as an object of the abelian category $\mathcal{M}$, the multiplication map gives a short exact sequence

$$0 \to G \xrightarrow{[l^r]_G} G \to {}_{l^r}G[1] \to 0,$$

which is more coherent with the short exact sequence of l-adic realisations. This justifies why 1-motives are *motives* in some sense. And also such coherence is very useful in practise, and has been used very often in this thesis.

In [35], Serre described the properties of the category of commutative quasi-algebraic groups by introducing pro-algebraic groups. Later, Oort determined that the homological dimension of the abelian category $\mathcal{G}$ of commutative algebraic group schemes over an algebraically closed field of positive characteristic is two in his book [27]. When the field is not algebraically closed, Milne related the homological dimension of $\mathcal{G}$ over a perfect field k to the cohomological dimension of the Galois group of k. Following the ideas from the above work, in the third chapter we are going to prove that the homological dimension $\mathrm{d}(\mathcal{M})$ of the abelian category $\mathcal{M}$ over a perfect field k equals $d+1$, where d is the cohomological dimension of the absolute Galois group of k. In particular, $\mathrm{d}(\mathcal{M}) = 2$ over a finite field, $\mathrm{d}(\mathcal{M}) = 3$ over a totally imaginary number field, and $\mathrm{d}(\mathcal{M}) = d+1 = \infty$ over a number field which is not totally imaginary. For number fields which are not totally imaginary, although $\mathrm{d}(\mathcal{M}) = \infty$ in general, we have $\mathrm{d}(\mathcal{M} \otimes \mathbb{Z}[1/2]) = 2+1 = 3$.

Let A and B be two abelian varieties over a finitely generated field k, a theorem of Faltings (in the finite field case Tate theorem) gives an isomorphism

$$T_l : \mathrm{Hom}_k(A, B) \otimes \mathbb{Z}_l \to \mathrm{Hom}_{\mathcal{R}}(T_l A, T_l B),$$

where $\mathcal{R}$ denotes the category of finitely generated $\mathbb{Z}_l$-modules with continuous Galois action. Since 1-motives are generalisations of abelian varieties, it is natural to ask if we can generalise Faltings' theorem to the case of 1-motives for even higher Ext^i ($i \geq 0$), i.e. to understand the maps

$$T_l : \mathrm{Ext}^i_?(M, M') \otimes \mathbb{Z}_l \to \mathrm{Hom}_{\mathcal{R}}(T_l M, T_l M')$$

for two 1-motives M, M' in a suitable category "?" and $i \geq 0$. The Hom case has been dealt with by Fengsheng in [9] for semi-abelian varieties over number fields, by Jannsen in [15] for Deligne's 1-motives over number fields, and by Jossen in [16] for his 1-motives with torsion. Jossen has also dealt with Ext^1 for 1-motives over number fields, and shown that T_l is injective. In the last chapter of this thesis, we will investigate the homomorphisms T_l with respect to the category $\mathcal{M}$, and show that T_l is an isomorphism for $i = 0$ and injective for $i = 1$ over finite fields and number fields. In particular, over finite fields, we will give full description for the homomorphism T_l for all $i \geq 0$. In particular, we will describe the kernels and cokernels of T_l.

CHAPTER I

1-motives

In this chapter, we will give an introduction to the theory of 1-motives with torsion, which serves as the base on which this thesis is built. Proofs will usually be sketched, or even omitted. The main reference for this chapter is [2, App. C]. In each section we will also give specific further references.

1.1 Deligne's 1-motive

In this section, we will introduce 1-motive, which was defined by Deligne in [6, 10.1]. Historically, this was the first step towards the theory of mixed motives. And the construction is very concrete.

Let S be a scheme.

Definition I.1. An **abelian scheme** over S is a smooth proper group scheme $\pi : A \to S$, such that all its geometric fibres are connected. A **torus** T over S is a commutative S-group scheme such that locally on S_{fppf} it is isomorphic to a product of finitely many copies of the multiplicative group $\mathbb{G}_{\text{m},S}$. A **semi-abelian scheme** over S is a smooth separated commutative group scheme $\pi : G \to S$ with connected fibres, such that each fibre is an extension of an abelian variety by a torus.

And like abelian varieties being commutative group varieties, abelian schemes are commutative group schemes (see [25, chap. 6]).

Remark I.2. Let S be Spec$k$ for some field k. Then the above definition of abelian schemes coincides with the usual definition of abelian varieties. In this case, being a semi-abelian scheme is the same as being an extension of an abelian scheme by a torus in the category of group schemes over S, since S consists of only one point in this case.

Definition I.3. A 1-**motive** M over S is a two-term complex, concentrated in degree -1 and 0, of S-commutative group schemes

$$M = [X \overset{u}{\to} G],$$

where X is an étale locally constant sheaf defined by some finitely generated free $\mathbb{Z}$-module on S, G is an extension of an abelian S-scheme A by an S-torus T, and u is a morphism of S-group schemes. We also write $M = (X, A, T, G, u)$ in order to emphasize the roles of A and T.

Remark I.4. The above definition implies that G is a semi-abelian scheme over S. However, not every semi-abelian scheme is an extension of an abelian scheme by a torus. For example, let E_q be the Tate curve over $\mathbb{Q}_p$ with invariant q for some $q \in \mathbb{Q}_p$ with $v_p(q) = 1$, where v_p denotes the canonical p-adic valuation, then the Néron model of E_q is a semi-abelian scheme over Spec$\mathbb{Z}_p$, and it has its generic fibre the elliptic curve E_q and closed fibre the multiplicative group over $\mathbb{F}_p$, hence cannot be an extension of an abelian scheme by a torus. However, if the base is a field, it's enough to require G to be a semi-abelian scheme as we have seen before.

Given 1-motives $M_1 = [X_1 \overset{u_1}{\to} G_1]$ and $M_2 = [X_2 \overset{u_2}{\to} G_2]$, a morphism from M_1 to M_2 is defined to be a morphism of complexes of commutative groups schemes, i.e. a commutative diagram of the form:

$$\begin{tikzcd} X_1 \arrow[r, "f"] \arrow[d, "u_1"'] & X_2 \arrow[d, "u_2"] \\ G_1 \arrow[r, "g"] & G_2 \, . \end{tikzcd}$$

Then 1-motives over S form a category, and we denote it by $\mathcal{M}_1$. Given $M = [X \overset{u}{\to} G] = (X, A, T, G, u) \in \mathcal{M}$, we have two canonical extensions of 1-motives, namely:

$$0 \to G \to [X \overset{u}{\to} G] \to X[1] \to 0$$

$$0 \to T \to G \to A \to 0$$

where T (resp. G, resp. A, resp. $X[1]$) is regarded as the 1-motive $[0 \to T]$ (resp. $[0 \to G]$, resp. $[0 \to A]$, resp. $[X \to 0]$).

These two sequences give a natural increasing filtration $W_\bullet$ on M, which is called the **weight filtration**, defined as follows:

$$W_i(M) = \begin{cases} 0, & \text{if } i \le -3 \\ T, & \text{if } i = -2 \\ G, & \text{if } i = -1 \\ M, & \text{if } i \ge 0. \end{cases}$$

The associated graded pieces are:

$$gr_i^W(M) = \begin{cases} T, & \text{if } i = -2 \\ A, & \text{if } i = -1 \\ X[1], & \text{if } i = 0 \\ 0, & \text{otherwise.} \end{cases}$$

The definition of 1-motives looks somehow artificial. However, the following example will show how close 1-motives over a field k are related to certain algebraic curves over k, as abelian varieties over k are related to smooth projective algebraic curves over k.

Example I.5 (cf. [29] 2.1)**.** Suppose S is the spectrum of a field k. Let C be a geometrically reduced algebraic curve over k. Let $\bar{C}$ be a compactification of C such that the complement of C in $\bar{C}$ consists of regular points. Assume that every singular point as a prime divisor is a normal crossing divisor, i.e. it's the zero locus of coordinate function étale locally, and also assume that the residue fields at $\bar{C} - C$ are separable over k. Then the generalised Jacobian $J = J(\bar{C})$ of $\bar{C}$ is a semi-abelian variety. Let Y be the étale sheaf generated by the divisors D with support in $\bar{C} - C$, whose restrictions to each geometric component of $\bar{C}$ have degree zero. Then the canonical map $D \to Cl(\mathscr{O}_{\bar{C}}(D))$ gives a morphism $u : Y \to J$. This is called the 1-motive associated to the curve C.

Example I.6. In the definition of 1-motive, if taking X to be constant, i.e. $X = \mathbb{Z}^r$ for some non-negative integer r, and $S = \mathrm{Spec}k$ for some field k, then giving such a 1-motive M is equivalent to specifying r k-rational points on G.

Example I.7. Let $S = \mathrm{Spec}(k)$, where k is a finite extension of $\mathbb{Q}_p$ with a discrete valuation ν, let M be the 1-motive $[\mathbb{Z} \xrightarrow{u} \mathbb{G}_m]$ with $u(1) = q$, where q is in k^* with $\nu(q) \geq 1$. Then regarded as a complex in $\mathcal{D}^b_{\mathrm{rig}}(\mathrm{fppf})$, M is isomorphic to the sheaf represented by the Tate curve E_q over k with q-invariant $q(E) = q$. Here $\mathcal{D}^b_{\mathrm{rig}}(\mathrm{fppf})$ is the derived category of bounded complexes of fppf-sheaves over the small rigid site $\mathrm{Spec}(k)$, see [29] for details.

1.2 1-motives with torsion over field k

The torsion group schemes play a very important role in the study of group schemes. For example, the integral l-adic realisation of an abelian variety is given by the inverse limit of its l-power torsion subgroup schemes. When we embed abelian varieties into the category of 1-motives, unfortunately the n-torsion subgroup doesn't fit into the later resulting from the definition of 1-motives. In order to fix this problem, there are several constructions of 1-motives with torsion. In their proof of Deligne's conjecture on 1-motives in characteristic zero case (a modified version) in [4], L. Barbieri-Viale, A. Rosenschon and M. Saito define 1-motives with torsion, and construct an abelian category of 1-motives with torsion. Later, L. Barbieri-Viale and B. Kahn extend the construction to any characteristic in [2]. In [16], Jossen's category of 1-motives with torsion admits Cartier duality as the category of Deligne's 1-motives does, however it is not abelian. In [30], H. Russell constructs a category of 1-motives with torsion, which extends Laumon's 1-motives with unipotent parts and admits Cartier duality. In [3], L. Barbieri-Viale and A. Bertapelle construct an abelian category of Laumon's 1-motives with torsion. For our purpose here (we are going to deal with Yoneda extensions), we use the first one, and follow the construction in [2] closely. Note one should not confuse the category of effective 1-motives with torsion with the category of 1-motives with torsion, the first will be defined first and is not abelian.

From now on, our base scheme S will always be $\mathrm{Spec}k$, where k is a perfect field. We will omit the base provided no ambiguity arises.

Definition I.8. An **effective 1-motive with torsion** over k is a complex of group schemes $M = [L \to G]$, where L is finitely generated and locally constant for the étale topology, and G is a semi-abelian scheme over k. From now on, we will call effective

1-motives with torsion simply 1-motives, the motives defined before as Deligne's 1-motives.

The L appeared in the above definition can be written as an extension

$$0 \to L_{\text{tor}} \to L \to L_{\text{fr}} \to 0 \tag{1.1}$$

where L_{tor} is a finite étale group scheme and L_{fr} is free. And the semi-abelian scheme G can be written as an extension

$$0 \to T \to G \to A \to 0 \tag{1.2}$$

where A is an abelian scheme and T is a torus.

Definition I.9. Given two 1-motives with torsion $M = [L \stackrel{u}{\to} G]$ and $M' = [L' \stackrel{u'}{\to} G']$, an **effective map** from M to M' is a commutative square

$$\begin{array}{ccc} L & \xrightarrow{f} & L' \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u'} \\ G & \xrightarrow{g} & G' \end{array}$$

in the category of group schemes. We denote such a map by

$$(f, g) : M \to M'.$$

The natural composition of squares makes 1-motives with torsion into a category, the category of 1-motives with torsion, denoted by ${}^{\mathrm{t}}\mathcal{M}_1^{\mathrm{eff}}$. We will denote by $\mathrm{Hom}_{\mathrm{eff}}(M, M')$ the abelian group of effective morphisms.

Since 1-motives with torsion are supposed to be generalizations of Deligne's 1-motives in order to have torsion, we would like to cut out the torsion part of a 1-motive with torsion and to see how Deligne's 1-motives fit into the category of 1-motives with torsion. Given a 1-motive $M = [L \to G]$, we have (in the category of commutative group schemes) a commutative diagram

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \operatorname{Ker}(u)\cap L_{\mathrm{tor}} \arrow[r] \arrow[d] & L_{\mathrm{tor}} \arrow[r, "u"] \arrow[d] & u(L_{\mathrm{tor}}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{Ker}(u) \arrow[r] & L \arrow[r, "u"] \arrow[d] & G \arrow[d] & \\
 & & L_{\mathrm{fr}} \arrow[r, "\bar{u}"] \arrow[d] & G/u(L_{\mathrm{tor}}) \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}
\tag{1.3}
$$

with exact rows and columns. We then get three effective 1-motives

$$
\begin{aligned}
M_{\mathrm{fr}} &:= [L_{\mathrm{fr}} \to G/u(L_{\mathrm{tor}})] \\
M_{\mathrm{tor}} &:= [L_{\mathrm{tor}} \cap \operatorname{Ker}(u) \to 0] \\
M_{\mathrm{tf}} &:= [L/(L_{\mathrm{tor}} \cap \operatorname{Ker}(u)) \to G].
\end{aligned}
$$

From the above diagram, there are also three canonical effective maps $M \to M_{\mathrm{tf}}$, $M_{\mathrm{tor}} \to M$ and $M_{\mathrm{tf}} \to M_{\mathrm{fr}}$.

Definition I.10. A 1-motive $M = [L \to G]$ is free if L is free, i.e. if $M = M_{\mathrm{fr}}$. It is torsion, if L is torsion and $G = 0$, i.e. if $M = M_{\mathrm{tor}}$. It is torsion-free, if $\operatorname{Ker}(u) \cap L_{\mathrm{tor}} = 0$, i.e. if $M = M_{\mathrm{tf}}$.

Denote by ${}^t\mathcal{M}_1^{\mathrm{eff,fr}}$, ${}^t\mathcal{M}_1^{\mathrm{eff,tor}}$ and ${}^t\mathcal{M}_1^{\mathrm{eff,tf}}$, the full subcategories of ${}^t\mathcal{M}_1^{\mathrm{eff}}$ given by free, torsion and torsion-free 1-motives respectively. Then the category ${}^t\mathcal{M}_1^{\mathrm{eff,fr}}$ is nothing else but the category $\mathcal{M}_1$ of Deligne's 1-motives and we will henceforth use this notation for simplicity. It's obviously that the category ${}^t\mathcal{M}_1^{\mathrm{eff,tor}}$ is equivalent to the category of finite étale group schemes. From diagram 1.3 we can see that, if M is torsion-free, the morphism $L_{\mathrm{tor}} \to u(L_{\mathrm{tor}})$ is an isomorphism, hence L is the pull-back of L_{fr} along the isogeny $G \to G/u(L_{\mathrm{tor}})$.

Proposition I.11. *The categories ${}^t\mathcal{M}_1^{\mathrm{eff}}$ and $\mathcal{M}_1$ have all finite limits and colimits. In particular, they admit kernel and cokernel. And given two 1-motives $M = [L \xrightarrow{u} G]$, $M' = [L' \xrightarrow{u'} G']$ and an effective morphism $\varphi = (f, g) : M \to M'$ in ${}^t\mathcal{M}_1^{\mathrm{eff}}$ (resp.*

$\mathcal{M}_1$), the kernel of φ is given by $\mathrm{Ker}(\varphi) = [\mathrm{Ker}^0(f) \xrightarrow{u} \mathrm{Ker}^0(g)]$ *(resp.* $\mathrm{Ker}(\varphi) = [\mathrm{Ker}^0(f) \xrightarrow{u} \mathrm{Ker}^0(g)]$*), and the cokernel of φ is given by* $\mathrm{Coker}(\varphi) = [\mathrm{Coker}(f) \xrightarrow{\bar{u}'} \mathrm{Coker}(g)]$ *(resp.* $\mathrm{Coker}(\varphi) = [\mathrm{Coker}(f) \xrightarrow{\bar{u}'} \mathrm{Coker}(g)]_{\mathrm{fr}}$*), where* $\mathrm{Ker}^0(g)$ *is the reduced connected component of the kernel of g in the category of commutative group schemes,* $\mathrm{Ker}^0(f)$ *is the pullback of* $\mathrm{Ker}^0(g)$ *along* $u : \mathrm{Ker}(f) \xrightarrow{u} G$*, and $\bar{u}'$ is the map induced by u'.*

Proof. See [2, prop. C.1.3]. □

Although ${}^t\mathcal{M}_1^{\mathrm{eff}}$ and $\mathcal{M}_1$ have all finite limits and colimits, they turn out to be not abelian. In order to get an abelian category out of ${}^t\mathcal{M}_1^{\mathrm{eff}}$, we are going to define quasi-isomorphism in ${}^t\mathcal{M}_1^{\mathrm{eff}}$, and then take localization with respect to quasi-isomorphisms.

Definition I.12. Given $M = [L \xrightarrow{u} G]$ and $M' = [L' \xrightarrow{u'} G']$ in ${}^t\mathcal{M}_1^{\mathrm{eff}}$, an effective morphism of 1-motives $M \to M'$ is a quasi-isomorphism (q.i. for short) of 1-motives if it yields a pullback diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \longrightarrow & L & \longrightarrow & L' & \longrightarrow & 0 \\ & & \| & & {\scriptstyle u}\downarrow & & {\scriptstyle u'}\downarrow & & \\ 0 & \longrightarrow & F & \longrightarrow & G & \longrightarrow & G' & \longrightarrow & 0, \end{array} \tag{1.4}$$

where F is a finite étale group.

Remark I.13. In general, quasi-isomorphisms are not isomorphisms. For example, for G a non-trivial semi-abelian variety over a field k of positive characteristic and n a positive integer coprime to the characteristic of k, we have that

$$\begin{array}{ccc} {}_nG & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ G & \xrightarrow{[n]_G} & G \end{array}$$

is a quasi-isomorphism, but not an isomorphism.

Remark I.14. In fact quasi-isomorphisms between 1-motives M and M' are the same as quasi-isomorphisms of the corresponding complexes of group schemes. This can be verified by using snake lemma and the fact that the cokernel of any morphism between two semi-abelian varieties is connected.

Proposition I.15. *Quasi-isomorphisms are simplifiable on the left and on the right. And the class of quasi-isomorphisms admits a calculus of right fractions in the sense of (the dual of) [11].*

Proof. See [2, prop. C.2.3., C.2.4.]. □

Thank to I.15, we can formally inverting quasi-isomorphisms.

Definition I.16. The category ${}^t\mathcal{M}_1$ of 1-motives with torsion is the localization of ${}^t\mathcal{M}_1^{\mathrm{eff}}$ with respect to the multiplicative class {q.i.} of quasi-isomorphisms.

The category ${}^t\mathcal{M}_1$ is (almost in positive characteristic case) what we want. It has the same objects as ${}^t\mathcal{M}_1^{\mathrm{eff}}$ does, and the morphisms in ${}^t\mathcal{M}_1$ are given by the formula

$$\mathrm{Hom}(M, M') = \varinjlim_{q.i.} \mathrm{Hom}_{\mathrm{eff}}(\widetilde{M}, M'),$$

where the limit is taken over the filtering set of all quasi-isomorphisms $\widetilde{M} \to M'$. And any morphism of 1-motives $M \to M'$ can be represented by a diagram

$$\begin{tikzcd} M & & M' \\ & \widetilde{M}. \arrow[ul, "\text{q.i.}"] \arrow[ur, "\text{eff}"'] & \end{tikzcd}$$

The composition is given by the following commutative diagram

$$\begin{tikzcd} M & & M' & & M'' \\ & \widetilde{M} \arrow[ul, "\text{q.i.}"] \arrow[ur, "\text{eff}"'] & & \widetilde{M'} \arrow[ul, "\text{q.i.}"] \arrow[ur, "\text{eff}"'] & \\ & & \widehat{M}, \arrow[ul, "\text{q.i.}"] \arrow[ur, "\text{eff}"'] & & \end{tikzcd}$$

where the existence of $\widehat{M}$ is guaranteed by the condition of calculus of right fractions in I.15.

Now we introduce the notion of strict morphism, which is useful for investigating the morphisms between 1-motives explicitly.

Definition I.17. Let $(f, g) : M \to M'$ be an effective morphism of 1-motives. It is strict, if we have

$$\mathrm{Ker}(f, g) = [\mathrm{Ker}(f) \to \mathrm{Ker}(g)],$$

i.e. if $\mathrm{Ker}(g)$ is a semi-abelian variety.

Proposition I.18. *Any effective morphism $\varphi \in \mathrm{Hom}_{\mathrm{eff}}(M, M')$ can be factored as*

$$\begin{tikzcd} M \arrow[rr, "\varphi"] \arrow[dr, "\bar{\varphi}"'] & & M' \\ & \widetilde{M} \arrow[ur] & \end{tikzcd}$$

where $\bar{\varphi}$ is a strict morphism and $\widetilde{M} \to M'$ is a composition of a quasi-isomorphism with a p-power isogeny. Note the p-power isogeny can only happen when the base field is of postive characteristic.

Proof. See [2, C.4.3.] . □

This proposition reveals that strict morphisms give the essential part of morphisms between 1-motives in characteristic 0 case. But in characteristic p case, the same thing only happens if we kill the p-power isogenies, i.e. tensoring with $\mathbb{Z}[1/p]$. In [2, C.5.], with the help of strict morphisms, Barbieri-Viale and Kahn give explicit description of the morphisms in ${}^t\mathcal{M}_1$ explicitly, and show the following proposition.

Proposition I.19. *The category ${}^t\mathcal{M}_1$ is abelian, if $\mathrm{char}(k) = 0$. If $\mathrm{char}(k) = p > 0$, the category ${}^t\mathcal{M}_1[1/p] = {}^t\mathcal{M}_1 \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$ is abelian. Given an effective morphism $\varphi : M \to M'$, the kernel of φ in ${}^t\mathcal{M}_1$ is just its kernel in ${}^t\mathcal{M}_1^{\mathrm{eff}}$, and the cokernel of φ is given by the cokernel of the strict morphism appearing in the factorization of φ as in the last proposition, up to quasi-isomorphism.*

Proof. See [2, C.5.2, C.5.3.]. □

From now on, we will simply denote by $\mathcal{M}$ the category ${}^t\mathcal{M}_1$ if $\mathrm{char}(k) = 0$; the category ${}^t\mathcal{M}_1[1/p]$ if $\mathrm{char}(k) = p > 0$. Then we can formulate the Yoneda extensions in the abelian category $\mathcal{M}$.

Proposition I.20. *A short exact sequence of 1-motives in $\mathcal{M}$*

$$0 \to M' \to M \to M'' \to 0$$

can be represented up to isomorphisms by a strict effective epimorphism $(f, g) : M \to M''$ with kernel M', i.e. by an exact sequence of complexes.

Example I.21. Let M be a 1-motive, there are two canonical short exact sequences in $\mathcal{M}$ fitting into the following diagram

$$\begin{tikzcd}
0 \arrow[r] & M_{\mathrm{tor}} \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d, equal] & M_{\mathrm{tf}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_{\mathrm{tor}} \arrow[r] & M \arrow[r] & M_{\mathrm{fr}} \arrow[r] & 0
\end{tikzcd}$$

where the effective morphism $M \to M_{\mathrm{tf}}$ is a strict epimorphism with kernel M_{tor} and $M_{\mathrm{tf}} \to M_{\mathrm{fr}}$ is a quasi-isomorphism.

1.3 l-adic realization

The Tate modules of tori and abelian varieties carries a lot of information in both geometric and arithmetic aspects. In this section, we are going to define the analogue of Tate modules for 1-motives, namely the l-adic realizations of 1-motives.

Similar to the definition of Tate modules of abelian varieties and tori, our definition of l-adic realizations of 1-motives involves multiplication morphisms. First let us understand the kernel and cokernel of the multiplication morphism $n : M \to M$, where $M = [L \xrightarrow{u} G]$ is a 1-motive and n is invertible in k. By I.11, we can define

$$ {}_nM := \mathrm{Ker}(M \xrightarrow{n} M) = [\mathrm{Ker}(u) \cap {}_nL \to 0]. $$

Thus ${}_nM = 0$ for all n with $(n, \mathrm{char}(k)) = 1$ if and only if $M_{\mathrm{tor}} = 0$. By I.19,

$$ M/n := \mathrm{Coker}(M \to M) $$

is always a torsion 1-motive. And if $L = 0$, then we have an extension in $\mathcal{M}$

$$ 0 \to G \xrightarrow{n} G \to {}_nG[1] \to 0, $$

with ${}_nG := \ker(G \xrightarrow{n} G)$. In the general case, we apply snake lemma to the canonical exact sequence

$$ 0 \to G \to M \to L[1] \to 0 $$

of effective 1-motives, which is also exact in $\mathcal{M}$. Then we get the following long exact sequence in ${}^t\mathcal{M}_1^{\mathrm{tor}}[1/p]$

$$ 0 \to {}_nM \to {}_nL[1] \to {}_nG[1] \to M/n \to L/n[1] \to 0. $$

Now let $n = l^v$, where $v \in \mathbb{N}$ and $l \neq \mathrm{char}(k)$.

Definition I.22. The l-adic realisation of a 1-motive M is

$$T_l(M) := \text{“}\varprojlim_v\text{''} L_v$$

in the category of l-adic sheaves, where L_v is a finite étale group scheme such that $M/l^v = [L_v \to 0]$ for each v.

Since every term in the above long exact sequence is finite, the inverse limit functor is exact. And ${}_{l^v}L$ is stable for v big enough, so “$\varprojlim_v$'' ${}_{l^v}L = 0$. Hence we get a short exact sequence

$$0 \to T_l(G) \to T_l(M) \to L \otimes \mathbb{Z}_l \to 0$$

where $T_l(G)$ is the Tate module of the semi-abelian variety G.

Given a short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

in $\mathcal{M}$, apply snake lemma to it, we get a long exact sequence

$$0 \to {}_{l^v}M' \to {}_{l^v}M \to {}_{l^v}M'' \to M'/l^v \to M/l^v \to M''/l^v \to 0.$$

These terms are finite for each v. By the similar argument as in the above case, we have a short exact sequence

$$0 \to T_l(M') \to T_l(M) \to T_l(M'') \to 0.$$

So we have:

Proposition I.23. *The functor T_l is exact on $\mathcal{M}$, and extends canonically to $\mathcal{M} \otimes \mathbb{Z}_l$.*

1.4 Extensions

In this section, we are going to give some easy descriptions to the homomorphisms and the Yoneda 1-extensions in the abelian category $\mathcal{M}$, in terms of the homomorphisms and the Yoneda extensions in other categories which are easier to understand, namely the abelian category of commutative algebraic k-group schemes, the category of k-group schemes, and the category of finitely generated Galois modules. We will denote by $\mathrm{Hom}_k(-,-)$ the group of homomorphisms in the category of commutative

k-group schemes, by $\mathrm{Ext}^i_k(-,-)$ the i-th Yoneda extension group in the abelian category of commutative algebraic k-group schemes (note for $i=0$, this doesn't conflict with the previous notation), and by $\mathrm{Ext}^i_{\mathcal{C}_k}(-,-)$ the i-th Yoneda extension group in the category of finitely generated Galois modules. Further investigations will be given in Chapter III and Chapter IV.

Note that in the following context, we are going to use the fact that there are no nontrivial quasi-isomorphisms to a 1-motive without semi-abelian part.

Proposition I.24. *Let $M=[L\to G], M=[L'\to G']\in\mathcal{M}$, and let T (resp. T') and A (resp. A') be the torus and abelian variety corresponding to G (resp. G') given by Chevalley decomposition. Then the following holds:*
(a)

$$\mathrm{Hom}_{\mathcal{M}}(L[1],L'[1])=\begin{cases}\mathrm{Hom}_k(L,L')=\mathrm{Hom}_{\mathcal{C}_k}(L,L') & \text{, } \textit{if } \mathrm{char}(k)=0\\ \mathrm{Hom}_k(L,L')\otimes\mathbb{Z}[1/p]=\mathrm{Hom}_{\mathcal{C}_k}(L,L')\otimes\mathbb{Z}[1/p], & \textit{otherwise}\end{cases};$$

(b) $\mathrm{Hom}_{\mathcal{M}}(L[1],G')=0$*;*
(c)

$$\mathrm{Hom}_{\mathcal{M}}(G,G')\subseteq\begin{cases}\mathrm{Hom}_k(A,A')\times\mathrm{Hom}_k(T,T'), & \textit{if } \mathrm{char}(k)=0\\ \mathrm{Hom}_k(A,A')\otimes\mathbb{Z}[1/p]\times\mathrm{Hom}_k(T,T')\otimes\mathbb{Z}[1/p], & \textit{otherwise}\end{cases};$$

(d)

$$\mathrm{Hom}_{\mathcal{M}}(G,L'[1])=\begin{cases}\mathrm{Hom}_k({}_nG,L'_{\mathrm{tor}}), & \textit{if } \mathrm{char}(k)=0\\ \mathrm{Hom}_k({}_nG,L'_{\mathrm{tor}})\otimes\mathbb{Z}[1/p], & \textit{otherwise}\end{cases}$$

where n is a positive integer such that $nL'_{\mathrm{tor}}=0$. In particular, the group $\mathrm{Hom}_{\mathcal{M}}(G,L'[1])$ is a finite group.
Hence the group $\mathrm{Hom}_{\mathcal{M}}(M,M')$ is finitely generated as a module over $\mathbb{Z}$ (resp. $\mathbb{Z}[1/p]$), if $\mathrm{char}(k)=0$ (resp. $\mathrm{char}(k)=p>0$).

Proof. It is enough to show the characteristic zero case. Since there are no nontrivial quasi-isomorphisms to $L[1]$, we have

$$\mathrm{Hom}_{\mathcal{M}}(L[1],L'[1])=\mathrm{Hom}_{\mathrm{eff}}(L[1],L'[1])=\mathrm{Hom}_k(L,L')$$

and

$$\mathrm{Hom}_{\mathcal{M}}(L[1], G') = \mathrm{Hom}_{\mathrm{eff}}(L[1], G') = 0,$$

this proves (a) and (b). Any quasi-isomorphism to G has the form $(0, f) : [F \to \tilde{G}] \to G$, where F is an étale subgroup scheme of $\tilde{G}$ and $f : \tilde{G} \to G$ is an isogeny with kernel F. Any morphism $(f, g) : [F \to \tilde{G}] \to G'$ must have g mapping the subgroup F of $\tilde{G}$ into 0. Hence (f, g) actually factor through G, and it follows that

$$\mathrm{Hom}_{\mathcal{M}}(G, G') = \mathrm{Hom}_{\mathrm{eff}}(G, G') = \mathrm{Hom}_k(G, G').$$

Given $\alpha \in \mathrm{Hom}_k(G, G')$ let $\bar{\alpha}$ be the composition $T \hookrightarrow G \xrightarrow{f} G' \to A'$. Since $\mathrm{Hom}_k(T, A') = 0$, we have $\bar{\alpha} = 0$. Hence $T \to G'$ factors through T' and $G \to A'$ factors through A. This gives a map

$$\mathrm{Hom}_{\mathcal{M}}(G, G') \to \mathrm{Hom}_k(A, A') \times \mathrm{Hom}_k(T, T'),$$

which is obviously injective by snake lemma, hence (c) follows. For (d), by the above description of quasi-isomorphism to G, we have

$$\mathrm{Hom}_{\mathcal{M}}(G, L'[1]) = \varinjlim_{\text{q.i.}} \mathrm{Hom}_{\mathrm{eff}}([F \to G], L'[1]) = \varinjlim_{\text{q.i.}} \mathrm{Hom}_k(F, L'_{\mathrm{tor}}).$$

If $nL'_{\mathrm{tor}} = 0$, then the limit is bounded by the quasi-isomorphism $(0, n_G) : [{}_nG \to G] \to G$, thus we have (d).

The last statement is an easy consequence of (a), (b), (c), (d) and a devissage for Hom with respect to $0 \to G \to M \to L[1] \to 0$ and $0 \to G' \to M' \to L'[1] \to 0$. $\square$

The above proposition is about groups of homomorphisms in both characteritic 0 case and positive characteristic case. The next two propositions concern 1-extensions, but only for characteristic zero case (the corresponding positive characteristic statement will be given in charpter III).

Proposition I.25. *Notations as before, and suppose the characteristic of the base field is zero. Then we have the following canonical isomorphisms.*

(a) $\mathrm{Ext}^1_{\mathcal{C}_k}(L, L') \xrightarrow{\cong} \mathrm{Ext}^1_{\mathcal{M}}(L[1], L'[1])$;

(b) $\mathrm{Hom}_k(L, G') \xrightarrow{\cong} \mathrm{Ext}^1_{\mathcal{M}}(L[1], G')$;

(c) $\mathrm{Ext}^1_k(G, G') \xrightarrow{\cong} \mathrm{Ext}^1_{\mathcal{M}}(G, G')$.

Proof. First of all, any element of $\mathrm{Ext}^1_{\mathcal{C}_k}(L, L')$ gives an element of $\mathrm{Ext}^1_{\mathcal{M}}(L[1], L'[1])$, this gives the map in (a). The existence of the map in (b) is given by associating a map $\alpha : L \to G'$ to the diagram:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & L \arrow[r, equal] \arrow[d, "\alpha"'] & L \arrow[d] \\ G \arrow[r, equal] & G \arrow[r] & 0. \end{tikzcd}$$

By Chevalley's structure theorem of commutative algebraic groups, any extension of a semi-abelian variety by another semi-abelian variety in the category of commutative algebraic k-group is still a semi-abelian variety, so we also have the map in (c).

By Proposition I.20, any short exact sequence of 1-motives can be represented up to isomorphism by a short exact sequence of complexes in which each term is a 1-motive.

Then (a) is just an immediate consequence of the fact that there are no nontrivial quasi-isomorphisms from or to a 1-motive with zero semi-abelian part.

For (b), an extension of $L[1]$ by G' in $\mathcal{M}$ is given by an exact sequence of of complexes of the form

$$\begin{tikzcd} 0 \arrow[r] & F' \arrow[r] \arrow[d] & L'' \arrow[r] \arrow[d, "v"'] & L \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \tilde{G}' \arrow[r, equal] & \tilde{G}' \arrow[r] & 0 \arrow[r] & 0, \end{tikzcd}$$

where $\tilde{M}' = [F' \to \tilde{G}']$ is quasi-isomorphic to G'. We can mod out F', i.e. take push-out along $\tilde{M}' \to G'$, and get a quasi-isomorphic sequence which is an element of $\mathrm{Hom}_k(L, G')$, hence the map in (b) is surjective. It's also injective, since the existence of a section of the sequence

$$0 \to G' \to [L \xrightarrow{v} G'] \to L[1] \to 0$$

means exactly that v is zero.

For (c), we see that an extension of G by G' in $\mathcal{M}$ can be represented by a diagram

$$\begin{tikzcd} 0 \arrow[r] & F' \arrow[r] \arrow[d] & L'' \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \tilde{G}' \arrow[r] & G'' \arrow[r] & \tilde{G} \arrow[r] & 0 \end{tikzcd}$$

with $\tilde{M'} = [F' \to \tilde{G'}]$ is quasi-isomorphic to G', and $\tilde{M} = [F \to \tilde{G}]$ is quasi-isomorphic to G. We can mod out F', and get a quasi-isomorphic exact sequence

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & L''/F' & \xrightarrow{\cong} & F & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G' & \longrightarrow & G''/F' & \longrightarrow & \tilde{G} & \longrightarrow & 0.
\end{array}$$

We can further mod out $L''/F' \cong F$, and get

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G' & \longrightarrow & G''/L'' & \longrightarrow & G & \longrightarrow & 0.
\end{array}$$

Hence the map in (c) is surjective. Its injectivity is obtained by similar argument as in the proof to (b). $\square$

Proposition I.26. *Notations as before, and suppose that the characteristic of the base field is zero. Then we have a canonical isomorphism*

$$\Phi : \varinjlim_n \mathrm{Ext}^1_{\mathcal{C}_k}({}_nG, L') \longrightarrow \mathrm{Ext}^1_{\mathcal{M}}(G, L'[1]).$$

In particular, when $NL' = 0$ for some positive integer N, i.e. L' is torsion, the map Φ becomes

$$H^1(k, \mathrm{Hom}_{\mathcal{C}_{\bar{k}}}({}_NG, L')) \cong \varinjlim_n \mathrm{Ext}^1_{\mathcal{C}_k}({}_nG, L') \xrightarrow{\Phi} \mathrm{Ext}^1_{\mathcal{M}}(G, L'[1]),$$

and these groups are zero when k is algebraically closed.

Proof. First, we construct a map $\Phi_n : \mathrm{Ext}^1_{\mathcal{C}_k}({}_nG, L') \to \mathrm{Ext}^1_{\mathcal{M}}(G, L'[1])$ for all positive integers n. Let $L'' \in \mathrm{Ext}^1_{\mathcal{C}_k}({}_nG, L')$ and consider the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L' & \longrightarrow & L'' & \longrightarrow & {}_nG & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & G & = & G & \longrightarrow & 0
\end{array} \tag{1.5}$$

where the map $L'' \to G$ is given by the composition of $L'' \to {}_nG \to G$. Since $[{}_nG \to G]$ is canonically quasi-isomorphic to G, this provides an extension of G by $L'[1]$ in $\mathcal{M}$. For n variable $\{\mathrm{Ext}^1_{\mathcal{C}_k}({}_nG, L')\}_n$ is a direct system. The maps Φ_n's are compatible with respect to pull-back, and hence give a well-defined map

$$\Phi : \varinjlim_n \mathrm{Ext}^1_{\mathcal{C}_k}({}_nG, L') \to \mathrm{Ext}^1_{\mathcal{M}}(G, L'[1]).$$

This map is surjective since any extension of G by $L'[1]$ can be represented by a diagram 1.5 for some n (as multiplications by positive integers are cofinal in the direct system of isogenies). So we are left to show the injectivity.

Before going to the injectivity, let us first give another description to the map Φ. From the short exact sequence $0 \to G \xrightarrow{n} G \to {}_nG[1] \to 0$, we get a long exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{M}}(G, L'[1]) \otimes \mathbb{Z}/n\mathbb{Z} \to \mathrm{Ext}^1_{\mathcal{M}}({}_nG[1], L'[1]) \to {}_n\mathrm{Ext}^1_{\mathcal{M}}(G, L'[1]) \to 0.$$

Taking dierect limit, we get a short exact sequence

$$0 \to \varinjlim_n \mathrm{Hom}_{\mathcal{M}}(G, L'[1]) \otimes \mathbb{Z}/n\mathbb{Z} \to \varinjlim_n \mathrm{Ext}^1_{\mathcal{M}}({}_nG[1], L'[1]) \to \varinjlim_n {}_n\mathrm{Ext}^1_{\mathcal{M}}(G, L'[1]) \to 0.$$

The group $\mathrm{Hom}_{\mathcal{M}}(G, L'[1])$ is finite by I.24 (d), so we get

$$\varinjlim_n \mathrm{Hom}_{\mathcal{M}}(G, L'[1]) \otimes \mathbb{Z}/n\mathbb{Z} = 0.$$

The map Φ being surjective implies that the group $\mathrm{Ext}^1_{\mathcal{M}}(G, L'[1])$ is torsion, so we have

$$\mathrm{Ext}^1_{\mathcal{M}}(G, L'[1]) = \varinjlim_n {}_n\mathrm{Ext}^1_{\mathcal{M}}(G, L'[1]).$$

By I.25 (a), we have

$$\mathrm{Ext}^1_{\mathcal{M}}({}_nG[1], L'[1]) = \mathrm{Ext}^1_{\mathcal{C}_k}({}_nG, L').$$

Combining all the above, we get an isomorphism

$$\varinjlim_n \mathrm{Ext}^1_{\mathcal{C}_k}({}_nG, L') \xrightarrow{\cong} \mathrm{Ext}^1_{\mathcal{M}}(G, L'[1]).$$

This isomorphism is nothing but the map Φ.

Now we consider the case that L' is killed by N. According to II.28, there is a spectral sequence

$$H^i(\Gamma, \mathrm{Ext}^j_{\mathcal{C}_{\bar{k}}}({}_nG, L')) \Longrightarrow \mathrm{Ext}^{i+j}_{\mathcal{C}_k}({}_nG, L')$$

for each positive integers n. This gives an exact sequence of low degree terms

$$(1.6) \qquad 0 \to H^1(k, \mathrm{Hom}_{\mathcal{C}_{\bar{k}}}({}_nG, L')) \to \mathrm{Ext}^1_{\mathcal{C}_k}({}_nG, L') \to \mathrm{Ext}^1_{\mathcal{C}_{\bar{k}}}({}_nG, L')^{\Gamma}.$$

Taking the direct limit over n, we get an exact sequence

$$0 \to \varinjlim_n H^1(k, \mathrm{Hom}_{\mathcal{C}_{\bar{k}}}({}_nG, L')) \to \varinjlim_n \mathrm{Ext}^1_{\mathcal{C}_k}({}_nG, L') \to \varinjlim_n \mathrm{Ext}^1_{\mathcal{C}_{\bar{k}}}({}_nG, L')^{\Gamma}. \tag{1.7}$$

Note that as an abelian group $\mathrm{Ext}^1_{\mathcal{C}_{\bar{k}}}({}_nG, L')$ is just an extension group of abelian groups, and it's a standard homological computation of $\mathbb{Z}$-modules to shows that

$$\varinjlim_n \mathrm{Ext}^1_{\bar{k}}({}_nG, L') = 0. \tag{1.8}$$

By the isomorphism $\varinjlim_n H^1(k, \mathrm{Hom}_{\mathcal{C}_{\bar{k}}}({}_nG, L')) = \varinjlim_r H^1(k, \mathrm{Hom}_{\mathcal{C}_{\bar{k}}}({}_{rN}G, L'))$ and the fact that the maps $\mathrm{Hom}_{\mathcal{C}_{\bar{k}}}({}_NG, L') \to \mathrm{Hom}_{\mathcal{C}_{\bar{k}}}({}_{rN}G, L')$ are isomorphisms, we have

$$H^1(k, \mathrm{Hom}_{\mathcal{C}_{\bar{k}}}({}_NG, L')) \xrightarrow{\cong} \varinjlim_n H^1(k, \mathrm{Hom}_{\mathcal{C}_{\bar{k}}}({}_nG, L')). \tag{1.9}$$

Combining 1.7, 1.8 and 1.9, we get the isomorphism

$$H^1(k, \mathrm{Hom}_{\mathcal{C}_{\bar{k}}}({}_NG, L')) \cong \varinjlim_n \mathrm{Ext}^1_{\mathcal{C}_k}({}_nG, L').$$

When k is algebraic closed, we have

$$\varinjlim_n \mathrm{Ext}^1_{\mathcal{C}_k}({}_nG, L') \cong H^1(k, \mathrm{Hom}_{\mathcal{C}_{\bar{k}}}({}_NG, L')) = 0,$$

hence so is $\mathrm{Ext}^1_{\mathcal{M}}(G, L'[1])$. $\square$

CHAPTER II

Some cohomology theories

In this chapter, we are going to formulate some homological results related to 1-motives. A large part of this chapter is taking from the literature, and the main references are [26], [33] and [38].

2.1 Comparison between extension groups of 1-motives and extension groups of their l-adic realisations

Let $\mathcal{R}$ denote the abelian category of finitely generated $\mathbb{Z}_l$-modules with continuous Γ-action, where Γ denotes the absolute Galois group of the base field k. Recall that in I.23, the l-adic realisation functor T_l is exact on $\mathcal{M}$, and hence sends any Yoneda n-extension

$$0 \to M' \to M_1 \cdots M_n \to M \to 0$$

in $\mathcal{M}$ to a Yoneda n-extension

$$0 \to T_l M' \to T_l M_1 \cdots T_l M_n \to T_l M \to 0$$

in $\mathcal{R}$.

Suppose given two commutative diagrams with exact rows

$$\begin{tikzcd} \mathcal{E} & 0 \arrow[r] & M' \arrow[r] \arrow[d, equal] & M_1 \arrow[r] \arrow[d] & N \arrow[r] \arrow[d, "\beta"'] & 0 \\ \mathcal{E}' & 0 \arrow[r] & M' \arrow[r] & \tilde{M}_1 \arrow[r] & \tilde{N} \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{F} & 0 \arrow[r] & N \arrow[r] \arrow[d, "\beta"'] & M_2 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\ \mathcal{F}' & 0 \arrow[r] & \tilde{N} \arrow[r] & \tilde{M}_2 \arrow[r] & M \arrow[r] & 0, \end{tikzcd}$$

i.e. given

$$\mathcal{E} \cdot \mathcal{F} = (\mathcal{E}'\beta) \cdot \mathcal{F} \sim \mathcal{E}' \cdot (\beta\mathcal{F}) = \mathcal{E}' \cdot \mathcal{F}',$$

applying the functor T_l, we get two commutative diagrams with exact rows

$$\begin{array}{lc}
T_l\mathcal{E} & 0 \longrightarrow T_lM' \longrightarrow T_lM_1 \longrightarrow T_lN \longrightarrow 0 \\
 & \qquad\quad \Vert \qquad\qquad\quad \downarrow \qquad\quad\ \ {\scriptstyle\beta}\downarrow \\
T_l\mathcal{E}' & 0 \longrightarrow T_lM' \longrightarrow T_l\tilde{M}_1 \longrightarrow T_l\tilde{N} \longrightarrow 0
\end{array}$$

$$\begin{array}{lc}
T_l\mathcal{F} & 0 \longrightarrow T_lN \longrightarrow T_lM_2 \longrightarrow T_lM \longrightarrow 0 \\
 & \qquad {\scriptstyle\beta}\downarrow \qquad\qquad\quad \downarrow \qquad\qquad\quad \Vert \\
T_l\mathcal{F}' & 0 \longrightarrow T_l\tilde{N} \longrightarrow T_l\tilde{M}_2 \longrightarrow T_lM \longrightarrow 0.
\end{array}$$

Hence we have

$$T_l\mathcal{E} \cdot T_l\mathcal{F} = (T_l\mathcal{E}'\beta) \cdot T_l\mathcal{F} \sim T_l\mathcal{E}' \cdot (\beta T_l\mathcal{F}) = T_l\mathcal{E}' \cdot T_l\mathcal{F}',$$

i.e. the functor T_l keeps the relation $\mathcal{E} \cdot \mathcal{F} \sim \mathcal{E}' \cdot \mathcal{F}'$. Since the equivalent relation used to define the Yoneda extension groups is generated by the relations of the form $\mathcal{E} \cdot \mathcal{F} \sim \mathcal{E}' \cdot \mathcal{F}'$, we get a map $\mathrm{Ext}^n_{\mathcal{M}}(M, M') \to \mathrm{Ext}^n_{\mathcal{R}}(T_lM, T_lM')$. Actually, this is not only a map, but also a homomorphism of abelian groups. Here we only check that it keeps the group operation, and the rest can be shown in the same way. Given any two n-extensions $\mathcal{E}$ and $\mathcal{E}'$ of M by M' in $\mathcal{M}$, recall that the addition $[\mathcal{E}] + [\mathcal{E}']$ in the Yoneda n-extension group $\mathrm{Ext}^n_{\mathcal{M}}(M, M')$ is defined by the rule

$$[\mathcal{E}] + [\mathcal{E}'] := [\nabla_{M'}(\mathcal{E} \oplus \mathcal{E}')\triangle_M].$$

It is obvious that we have

$$T_l(\nabla_{M'}(\mathcal{E} \oplus \mathcal{E}')\triangle_M) = \nabla_{T_lM'}(T_l\mathcal{E} \oplus T_l\mathcal{E}')\triangle_{T_lM},$$

hence the functor T_l indeed gives a group homomorphism from $\mathrm{Ext}^n_{\mathcal{M}}(M, M')$ to $\mathrm{Ext}^n_{\mathcal{R}}(T_lM, T_lM')$. Moreover, the group $\mathrm{Ext}^n_{\mathcal{R}}(T_lM, T_lM')$ has a natural $\mathbb{Z}_l$-module structure, so we get a homomorphism

$$T_l : \mathrm{Ext}^n_{\mathcal{M}}(M, M') \otimes \mathbb{Z}_l \to \mathrm{Ext}^n_{\mathcal{R}}(T_lM, T_lM') \tag{2.1}$$

of $\mathbb{Z}_l$-modules. Here our notation T_l should be $(T_l)_n$, however we will abuse the notation T_l for any n, whenever the index n is clear from the context.

Let M be a 1-motive in $\mathcal{M}$, given a short exact sequence

$$0 \to N' \xrightarrow{u} N \xrightarrow{v} N'' \to 0$$

in $\mathcal{M}$, we then get a canonical long exact sequence

$$\mathrm{Ext}^i_{\mathcal{M}}(M, N') \to \mathrm{Ext}^i_{\mathcal{M}}(M, N) \to \mathrm{Ext}^i_{\mathcal{M}}(M, N'') \xrightarrow{\delta} \mathrm{Ext}^{i+1}_{\mathcal{M}}(M, N') \to,$$

where δ is the connection morphism. Since the ring $\mathbb{Z}_l$ is flat over $\mathbb{Z}$, we get another long exact sequence

$$\mathrm{Ext}^i_{\mathcal{M}}(M, N')\otimes\mathbb{Z}_l \to \mathrm{Ext}^i_{\mathcal{M}}(M, N)\otimes\mathbb{Z}_l \to \mathrm{Ext}^i_{\mathcal{M}}(M, N'')\otimes\mathbb{Z}_l \xrightarrow{\delta} \mathrm{Ext}^{i+1}_{\mathcal{M}}(M, N')\otimes\mathbb{Z}_l \to .$$

We also have a short exact sequence for the l-adic realisations

$$0 \to T_lN' \to T_lN \to T_lN'' \to 0,$$

hence get another canonical long exact sequence

$$\mathrm{Ext}^i_{\mathcal{R}}(T_lM, T_lN') \to \mathrm{Ext}^i_{\mathcal{R}}(T_lM, T_lN) \to \mathrm{Ext}^i_{\mathcal{R}}(T_lM, T_lN'') \to \mathrm{Ext}^{i+1}_{\mathcal{R}}(T_lM, T_lN') \to .$$

These two sequences fit into the following diagram
(2.2)

$$\begin{tikzcd}
{} \arrow[r] & \mathrm{Ext}^i_{\mathcal{M}}(M,N)\otimes\mathbb{Z}_l \arrow[r] \arrow[d] & \mathrm{Ext}^i_{\mathcal{M}}(M,N'')\otimes\mathbb{Z}_l \arrow[r, "\delta"] \arrow[d] & \mathrm{Ext}^{i+1}_{\mathcal{M}}(M,N')\otimes\mathbb{Z}_l \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \mathrm{Ext}^i_{\mathcal{R}}(T_lM,T_lN) \arrow[r] & \mathrm{Ext}^i_{\mathcal{R}}(T_lM,T_lN'') \arrow[r, "\delta"] & \mathrm{Ext}^{i+1}_{\mathcal{R}}(T_lM,T_lN') \arrow[r] & {} .
\end{tikzcd}$$

This diagram is actually commutative. To prove the commutativity, it suffices to check the commutativity of the two squares. Given any i-extension

$$\mathcal{E} : 0 \to N \to P_1 \cdots P_i \to M \to 0,$$

we consider the push-out diagram

$$\begin{tikzcd}
\mathcal{E} & 0 \arrow[r] & N \arrow[r] \arrow[d, "v"'] & P_1 \arrow[r] \arrow[d] & P_2 \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & M \arrow[r] \arrow[d, equal] & 0 \\
v\mathcal{E} & 0 \arrow[r] & N'' \arrow[r] & \tilde{P}_1 \arrow[r] & P_2 \arrow[r] & \cdots \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

By applying the functor T_l, we get a commutative diagram with exact rows

$$\begin{tikzcd}
T_l\mathcal{E} & 0 \arrow[r] & T_lN \arrow[r] \arrow[d, "T_lv"'] & T_lP_1 \arrow[r] \arrow[d] & T_lP_2 \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & T_lM \arrow[r] \arrow[d, equal] & 0 \\
T_l(v\mathcal{E}) & 0 \arrow[r] & T_lN'' \arrow[r] & T_l\tilde{P}_1 \arrow[r] & T_lP_2 \arrow[r] & \cdots \arrow[r] & T_lM \arrow[r] & 0.
\end{tikzcd}$$

This diagram has to be a push-out diagram, hence we have $T_l(v\mathcal{E}) = (T_l v)(T_l\mathcal{E})$. This shows the commutativity of the first square. For the second square, just notice that the connection map is given by splicing a given i-extension of M by N'' with the short exact sequence $0 \to N' \xrightarrow{u} N \xrightarrow{v} N'' \to 0$, then the commutativity follows from applying the functor T_l in a similar way.

Later in the fourth chapter, we are going to use the commutativity of the diagram 2.2 repeatedly.

2.2 Galois cohomology

The definition of 1-motives with torsion involves discrete sheaves over the base field k with respect to the étale topology. In the characteristic zero case, such sheaves are just finitely generated Galois modules. In the positive characteristic case, since we invert the multiplication-by-p map, the finite étale p-group schemes regarded as 1-motives become isomorphic to zero, hence we only need to consider the discrete sheaves without p-torsion. Such sheaves are again just finitely generated Galois modules. Hence Galois cohomology is quite useful in the study of 1-motives.

In this section, we are going to give a quick introduction to the cohomology of profinite groups, and list some results which are needed for our investigation of 1-motives, but proofs will be omitted mostly. The main reference for this section is [33].

Definition II.1. Let G be a profinite group, a G-module M is said to be discrete if one has $M = \cup M^U$, where U runs over all open subgroups of G.

The discrete G-modules can also be defined as abelian groups with discrete topology, on which G acts continuously. The discrete G-modules form an abelian category $\mathcal{C}_G$, in which there are enough injective objects.

Definition II.2. Let M be a discrete G-module, the q-th cohomology group $H^q(G, M)$ of G with coefficient in M is defined to be $R^qF(M)$. Here R^qF denotes the q-th right derived functor of the functor $M \mapsto F(M) = M^G$, with M^G being the maximal subgroup of M fixed by G.

From this definition, we get the usual formal results for cohomology groups, see [13, chap. II thm. 1.1A.]. In particular, we have the very useful long exact sequences associated to short exact sequences of discrete G-modules. This definition is not very helpful for computation. There is another definition for the cohomology groups of G via cochain.

Let $C^0(G,M)$ be M, and $C^n(G,M)$ be the abelian group of all continuous maps from G^n to M (the topology on M is the discrete one) for $n>0$. We define the differential map $d: C^n(G,M) \to C^{n+1}(G,M)$ by the formula

$$\begin{aligned}(df)(g_1,\cdots,g_{n+1}) = & g_1 \cdot f(g_2,\cdots,g_{n+1}) \\ & + \sum_{i=1}^{n} (-1)^i f(g_1,\cdots,g_i g_{i+1},\cdots,g_{n+1}) \\ & + (-1)^{n+1} f(g_1,\cdots,g_n).\end{aligned}$$

It's a formal check to see that $d \circ d = 0$, hence we get a complex $C^\bullet(G,M)$. Then the cohomology groups $H^q(G,M)$ can be computed as the cohomology groups of the following complex

$$C^\bullet(G,M):\ 0 \to C^0(G,M) \to C^1(G,M) \to \cdots \to C^n(G,M) \to \cdots. \tag{2.3}$$

By using the cochain complex, we are able to do some useful computation. The following results are taken from Serre's book [33].

Proposition II.3. *Let (G_i) be a projective system of profinite groups, and let (M_i) be an inductive system of discrete G_i-modules (the homomorphisms $M_i \to M_j$ have to be compatible with the morphisms $G_j \to G_i$). Let G be $\varprojlim_i G_i$, M be $\varinjlim_i M_i$. Then we have $H^q(G,M) = \varinjlim_i H^q(G_i,M_i)$ for each $q \geq 0$.*

Proof. The canonical homomorphism $\varinjlim C^\bullet(G_i,M_i) \longrightarrow C^\bullet(G,M)$ is an isomorphism, whence the result follows by passing to homology. □

Corollary II.4. *Let M be a discrete G-module, then we have*

$$H^q(G,M) = \varinjlim H^q(G/U, M^U)$$

for each $q \geq 0$, where U runs over all open normal subgroups of G.

Proof. Since we have $G = \varprojlim G/U$ and $M = \varinjlim M^U$, then the result follows from II.3. □

Corollary II.5. *Let M be a discrete G-module, then we have*

$$H^q(G, M) = \varinjlim H^q(G, N)$$

for each $q \geq 0$, where N runs over the set of finitely generated sub-G-modules of M.

Proof. The result follows from II.3 with the help of the expression $M = \varinjlim N$. $\square$

Corollary II.6. *The groups $H^q(G, M)$ are torsion for $q > 0$.*

Proof. The case G being a finite group is a classical result, see [34, chap. VII prop. 6]. The general case follows from this and II.4. $\square$

Example II.7. The cohomology groups in degree zero, one and two can be described very explicitly via the cochain complex $C^\bullet(G, M)$ as follows:
(1) $H^0(G, M) = M^G$;
(2) $H^1(G, M)$ is the group of classes of continuous crossed-homomorphisms from G to M, and in particular it is the group $\mathrm{Hom}(G, M)$ in the category of topological groups when M is a constant module;
(3) $H^2(G, M)$ is the group of classes of continuous factor systems from G to M.

Let G and G' be two profinite groups, and let $f : G \to G'$ be a morphism. Take $M \in \mathcal{C}_G$ and $M' \in \mathcal{C}_{G'}$, suppose that we have a morphism $h : M' \to M$ which is compatible with f, i.e. h is a G-morphism with M' regarded as a G-module via f. Such a pair (f, h) defines a homomorphism $H^q(G', M') \to H^q(G, M)$ for each $q \geq 0$. In particular, when G' is a closed subgroup H of G, and $M' = M$ is a discrete G-module, we obtain the restriction homomorphisms

$$\mathrm{Res} : H^q(G, M) \longrightarrow H^q(H, M), \quad q \geq 0.$$

When H is a closed normal subgroup, we obtain the inflation homomorphisms

$$\mathrm{Inf} : H^q(G/H, M^H) \longrightarrow H^q(G, M), \quad q \geq 0.$$

When H is an open subgroup of G with finite index n, we have the corestriction homomorphisms

$$\mathrm{Cor} : H^q(H, M) \longrightarrow H^q(G, M), \quad q \geq 0.$$

Similar as in the case that G is finite, we have $\mathrm{Cor} \circ \mathrm{Res} = n$.

Proposition II.8. *Let H be a closed normal subgroup of G and let M be a discrete G-module. Then we have an exact sequence*

$$0 \to H^1(G/H, M^H) \xrightarrow{\text{Inf}} H^1(G, M) \xrightarrow{\text{Res}} H^1(H, M).$$

Proof. This is actually just part of the five term exact sequence of the Hochschild-Serre spectral sequence, which will be given in 2.5. □

Definition II.9. Let l be a prime number, and G a profinite group. The l-cohomological dimension (resp. strict l-cohomological dimension) of G, denoted by $\mathrm{cd}_l(G)$ (resp. $\mathrm{scd}_l(G)$), is the smallest integer n such that the l-primary component of $H^q(G, M)$ is null for every discrete torsion (resp. not necessary torsion) G-module M and every $q > n$. If there is no such integer, then we define $\mathrm{cd}_l(G)$ (resp. $\mathrm{scd}_l(G)$) to be $+\infty$. The cohomological dimension (resp. strict cohomological dimension) of G is defined to be $\mathrm{cd}(G) := \sup_l \mathrm{cd}_l(G)$ (resp. $\mathrm{scd}(G) := \sup_l \mathrm{scd}_l(G)$).

Proposition II.10. *Let G be a profinite group, n be an integer, and l be a prime number. The following are equivalent:*
(a) $\mathrm{cd}_l(G) \leq n$.
(b) $H^q(G, M) = 0$ *for all $q > n$ and every discrete G-module which is an l-primary torsion group.*
(c) $H^{n+1}(G, M) = 0$ *when M is a simple discrete G-module killed by l.*

Proof. See [33, chap. I prop. 11]. □

Proposition II.11. $\mathrm{scd}_l(G)$ *is equal to either* $\mathrm{cd}_l(G)$ *or* $\mathrm{cd}_l(G) + 1$.

Proof. See [33, chap I prop 13]. □

Let k be a field, and Γ be its absolute Galois group which is a profinite group. Let M be a discrete Γ-module. We will write $H^q(k, M)$ instead of $H^q(\Gamma, M)$ as in most text books.

Example II.12. Let k be a finite field, then the absolute Galois group of k is isomorphic to $\hat{\mathbb{Z}}$. For any prime l, we have $\mathrm{cd}_l(\hat{\mathbb{Z}}) = 1$, for reference see [34, XIII prop 2]. It follows that we also have $\mathrm{cd}(\hat{\mathbb{Z}}) = 1$. Consider the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

of constant $\hat{\mathbb{Z}}$-modules, we get the cohomological long exact sequence

$$H^1(k,\mathbb{Q}) \to H^1(k,\mathbb{Q}/\mathbb{Z}) \to H^2(k,\mathbb{Z}) \to H^2(k,\mathbb{Q}).$$

Since we have $H^1(k,\mathbb{Q}) = \mathrm{Hom}_{\mathrm{cts}}(\hat{\mathbb{Z}},\mathbb{Q}) = 0$, and the fact $H^2(k,\mathbb{Q}) = 0$ can be deduced by [34, XIII prop 2] with the help of II.4. Hence we get

$$H^2(k,\mathbb{Z}) \cong H^1(k,\mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}_{\mathrm{cts}}(\hat{\mathbb{Z}},\mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z},$$

whence $\mathrm{scd}(\hat{\mathbb{Z}}) = \mathrm{scd}_l(\hat{\mathbb{Z}}) = 2$ by II.11.

Example II.13. Let p be a prime number, k be a p-adic field (i.e. a finite field extension of $\mathbb{Q}_p$), and Γ be the absolute Galois group of k. Then we have $\mathrm{scd}_l(\Gamma) = \mathrm{cd}_l(\Gamma) = 2$ for all prime number l, see [33, chap. II, 5.3].

Example II.14. Let k be an algebraic number field, and Γ be its absolute Galois group. If $l \neq 2$, or k is totally imaginary, we have $\mathrm{cd}_l(\Gamma) \leq 2$. Otherwise, we have $\mathrm{cd}_l(\Gamma) = \infty$. Although $\mathrm{cd}_2(\Gamma)$ could be ∞, we can still control the group $H^q(k,M)$ very well for any discrete torsion Γ-module M and $q > 2$. Actually, we have

$$H^q(k,M) = \prod_v H^q(k_v,M)$$

for $q > 2$, where v varies over all the real archimedean places of k. This group is in fact a 2-torsion group, since the absolute Galois group of k_v is just the cyclic group of order two for any real archimedean place v.

2.3 Continuous cochain cohomology

The l-adic realisations of 1-motives lie in the category $\mathcal{R}$ of finitely generated $\mathbb{Z}_l$-modules with continuous Galois action. In order to study the l-adic realisations of 1-motives, it is necessary to study the category $\mathcal{R}$. The continuous cochain cohomology is a useful tool. In this section, we are going to formulate the continuous cochain cohomology, and collect some propositions on continuous cochain cohomology groups. The final aim is to give some applications in our context.

Let G be a profinite group, and let M be a topological G-module (i.e. a topological abelian group with continuous G-action). We construct the continuous cochain complex $C^{\bullet}_{\mathrm{cts}}(G,M)$ of G with coefficients in M in the same way as in 2.3, except that we write $C^n_{\mathrm{cts}}(G,M)$ instead of $C^n(G,M)$ to indicate that we are using the topology of M itself.

Definition II.15. The q-th continuous cochain cohomology group of G is defined to be the q-th cohomology group of the complex $C^{\bullet}_{\text{cts}}(G, M)$, which we denote by $H^q_{\text{cts}}(G, M)$.

Remark II.16. If the topology on M is the discrete one, then the continuous cochain cohomology groups coincide with the ones defined in II.2. However, for arbitrary topological G-module M, the groups $H^q(G, M)$ may not be defined.

Since the continuous cochain cohomology is not defined via the standard derived functor method, given any short exact sequence of topological G-modules, we don't get the cohomological long exact sequence automatically. However, we have the following proposition from [26].

Proposition II.17. *Let*

$$0 \to M' \to M \xrightarrow{\beta} M'' \to 0$$

be a short exact sequence of topological G-modules such that the topology of M' is induced from that of M and such that β has a continuous section (not necessary a homomorphism). Then there exist canonical boundary homomorphisms

$$\delta : H^q_{\text{cts}}(G, M'') \to H^{q+1}_{\text{cts}}(G, M')$$

and we obtain a long exact sequence

$$\cdots \to H^q_{\text{cts}}(G, M') \to H^q_{\text{cts}}(G, M) \to H^q_{\text{cts}}(G, M'') \xrightarrow{\delta} H^{q+1}_{\text{cts}}(G, M') \to \cdots .$$

Proof. See [26, chap. II, 2.7.2]. □

Remark II.18. We can apply this proposition in the particular case when M' is an open submodule of M and $M'' = M/M'$ is the quotient module with the quotient topology, which is discrete. For our purpose, we are going to consider the short exact sequence

$$0 \to T_l M \xrightarrow{l^n} T_l M \to T_l M/l^n \to 0$$

coming from applying the l-adic realisation functor to the short exact sequence

$$0 \to M \xrightarrow{l^n} M \to M/l^n \to 0$$

associated to a torsion-free 1-motive M.

We are particularly interested in the l-adic realisations of 1-motives, which are the inverse limits of finite étale group schemes. The following proposition (taken from [26]) relates the continuous cochain cohomology groups of compact topological G-modules, whose underlying topology is profinite, to the Galois cohomology groups of finite G-modules.

Proposition II.19. *Let M be a compact topological G-module which has a presentation*

$$M = \varprojlim_{n\in\mathbb{N}} M_n$$

as a countable inverse limit of finite G-modules. Then there exists a natural exact sequence

$$0 \to \varprojlim_n{}^1 H^{i-1}(G, M_n) \to H^i_{\text{cts}}(G, M) \to \varprojlim_n H^i(G, M_n) \to 0$$

for each $i > 0$, where $\varprojlim_n{}^1$ denotes the first right derived functor of $\varprojlim_n$.

Proof. See [26, chap. II, 2.7.2]. □

Corollary II.20. *If $H^i(G, M_n)$ is finite for each $i \leq N$ and each n, then we have*

$$H^i_{\text{cts}}(G, M) = \varprojlim_n H^i(G, M_n)$$

for all $i \leq N + 1$.

Proof. For $i \leq N+1$, since the groups $H^{i-1}(G, M_n)$ are all finite for all n, the inverse system satisfies the Mittag-Leffler condition. So we get $\varprojlim_n{}^1 H^{i-1}(G, M_n) = 0$, whence the result follows from II.19. □

At last, we give an application of continuous cochain cohomology to some extension groups in the category $\mathcal{R}$.

Proposition II.21. *Let $M, N \in \mathcal{R}$, and suppose that M is free as a $\mathbb{Z}_l$-module. Then we have*

$$\mathrm{Ext}^1_{\mathcal{R}}(M, N) \cong H^1_{\text{cts}}(\Gamma, \mathrm{Hom}_{\mathbb{Z}_l}(M, N)).$$

Proof. First suppose that the continuous Γ-actions on M and N are given by the continuous homomorphisms

$$\rho_M : \Gamma \to \mathrm{Aut}_{\mathbb{Z}_l}(M)$$

and

$$\rho_N : \Gamma \to \mathrm{Aut}_{\mathbb{Z}_l}(N)$$

respectively. Recall that the continuous Γ-module structure on $\mathrm{Hom}_{\mathbb{Z}_l}(M,N)$ is given by

$$f^\sigma(m) = (\rho_N(\sigma) f \rho_M(\sigma^{-1}))(m)$$

for any $\sigma \in \Gamma$, $f \in \mathrm{Hom}_{\mathbb{Z}_l}(M,N)$, and $m \in M$.

Given any element in $\mathrm{Ext}^1_{\mathcal{R}}(M,N)$, which is represented by an short exact sequence

$$0 \to N \to E \to M \to 0 \tag{2.4}$$

in $\mathcal{R}$, let $\rho_E : \Gamma \to Aut_{\mathbb{Z}_l}(E)$ be the continuous homomorphism giving the continuous Γ-module structure. Since M is free as a $\mathbb{Z}_l$-module, the short exact sequence 2.4 splits as a sequence of $\mathbb{Z}_l$-modules, i.e. $E \cong N \oplus M$ as a $\mathbb{Z}_l$-module via some section $s : M \to E$. Then under this expression, for any element $\sigma \in \Gamma$, $\rho_E(\sigma)$ can be written as a matrix

$$\begin{pmatrix} \rho_N(\sigma) & f_\sigma \\ 0 & \rho_M(\sigma) \end{pmatrix},$$

where $f_\sigma : M \to N$ is a homomorphism of $\mathbb{Z}_l$-modules. Given any two elements $\sigma, \tau \in \Gamma$, we have $\rho_E(\sigma\tau) = \rho_E(\sigma)\rho_E(\tau)$, i.e.

$$\begin{pmatrix} \rho_N(\sigma) & f_\sigma \\ 0 & \rho_M(\sigma) \end{pmatrix} \cdot \begin{pmatrix} \rho_N(\tau) & f_\tau \\ 0 & \rho_M(\tau) \end{pmatrix} = \begin{pmatrix} \rho_N(\sigma\tau) & f_{\sigma\tau} \\ 0 & \rho_M(\sigma\tau) \end{pmatrix}.$$

So we get

$$f_{\sigma\tau} = \rho_N(\sigma) f_\tau + f_\sigma \rho_M(\tau),$$

which is equivalent to the equality

$$\begin{aligned} f_{\sigma\tau}\rho_M((\sigma\tau)^{-1}) &= (\rho_N(\sigma) f_\tau + f_\sigma \rho_M(\tau))\rho_M((\sigma\tau)^{-1}) \\ &= \rho_N(\sigma) f_\tau \rho_M((\sigma\tau)^{-1}) + f_\sigma \rho_M(\tau)\rho_M((\sigma\tau)^{-1}) \\ &= \rho_N(\sigma) f_\tau \rho_M(\tau^{-1})\rho_M(\sigma^{-1}) + f_\sigma \rho_M(\sigma^{-1}) \\ &= (f_\tau \rho_M(\tau^{-1}))^\sigma + f_\sigma \rho_M(\sigma^{-1}) \end{aligned}$$

for any σ and τ in Γ. This equality says nothing but that the collection

$$S_s := \{f_\sigma \rho_M(\sigma^{-1}) |\, \sigma \in \Gamma\}$$

gives a 1-cocycle, i.e. it represents an element of $H^1_{\text{cts}}(\Gamma, \text{Hom}_{\mathbb{Z}_l}(M, N))$.

Now we prove the cohomological class associated to this extension is independent of the choice of the splitting. Given another section s', the difference $t := s - s'$ gives an element of $\text{Hom}_{\mathbb{Z}_l}(M, N)$. Suppose that $\rho_E(\sigma)$ can be written as a matrix

$$\begin{pmatrix} \rho_N(\sigma) & g_\sigma \\ 0 & \rho_M(\sigma) \end{pmatrix}$$

under the splitting given by the section s', where $g_\sigma : M \to N$ is a homomorphism of $\mathbb{Z}_l$-modules, then the collection $S_{s'} = \{g_\sigma \rho_M(\sigma^{-1}) |\, \sigma \in \Gamma\}$ gives another 1-cocycle. We have two equalities

$$\begin{aligned} f_\sigma(m) &= \rho_E(\sigma)(s(m)) - s(\rho_M(\sigma)(m)) \\ g_\sigma(m) &= \rho_E(\sigma)(s'(m)) - s'(\rho_M(\sigma)(m)) \end{aligned}$$

for any $m \in M$. The difference between them gives

$$\begin{aligned} f_\sigma(m) - g_\sigma(m) &= \rho_E(\sigma)(t(m)) - t(\rho_M(\sigma)(m)) \\ &= \rho_N(\sigma)(t(m)) - t(\rho_M(\sigma)(m)), \end{aligned}$$

i.e. $f_\sigma - g_\sigma = \rho_N(\sigma)t - t\rho_M(\sigma)$. This is equivalent to

$$\begin{aligned} f_\sigma \rho_M(\sigma^{-1}) - g_\sigma \rho_M(\sigma^{-1}) &= \rho_N(\sigma) t \rho_M(\sigma^{-1}) - t \\ &= t^\sigma - t. \end{aligned}$$

This shows the difference between the two 1-cocycles is a coboundary, hence we get a well-defined map

$$\varphi : \text{Ext}^1_{\mathcal{R}}(M, N) \to H^1_{\text{cts}}(\Gamma, \text{Hom}_{\mathbb{Z}_l}(M, N)).$$

On the other hand, given any 1-cocycle, we can construct an extension of M by N easily, from what we have seen above. And it's easy to see the map φ is bijective.

So we are left to show that φ is a homomorphism of abelian groups. It's easy to see that φ map the trivial extension to zero. Then it suffices to show that φ keeps the group operations. Before continuing the proof, we digress to describe the functorial behavior of the map φ.

Claim. Given morphisms $a : N \to N'$ and $b : M' \to M$ with M' free as a $\mathbb{Z}_l$-module, let $0 \to N \xrightarrow{u} E \xrightarrow{v} M \to 0$ represents an element of $\mathrm{Ext}^1_{\mathcal{R}}(M, N)$. Suppose that this extension corresponds to a 1-cocycle $(f_\sigma \rho_M(\sigma^{-1}))_{\sigma\in\Gamma}$ for some section s of v, then:
(1). The canonical push-out morphism

$$u_* : \mathrm{Ext}^1_{\mathcal{R}}(M, N) \to \mathrm{Ext}^1_{\mathcal{R}}(M, N')$$

maps the extension class $[E]$ to the extension class corresponding to the 1-cocycle $(af_\sigma \rho_M(\sigma^{-1}))_{\sigma\in\Gamma}$.
(2). The canonical pullback morphism

$$v^* : \mathrm{Ext}^1_{\mathcal{R}}(M, N) \to \mathrm{Ext}^1_{\mathcal{R}}(M', N)$$

maps the extension class $[E]$ to the extension class corresponding to the 1-cocycle $(f_\sigma \rho_M(\sigma^{-1})b)_{\sigma\in\Gamma}$.

We only prove (1), whilst the proof to (2) goes similarly. Consider the push-out diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "u"] \arrow[d, "a"'] & E \arrow[r, "v"] \arrow[d, "c"'] & M \arrow[r] \arrow[l, bend left, "s"] \arrow[d, equal] & 0 \\
0 \arrow[r] & N' \arrow[r, "u'"] & E' \arrow[r, "v'"] & M \arrow[r] \arrow[l, bend left, "s'"] & 0,
\end{tikzcd}$$

where $s' := cs$ is a section of v'. Denote the 1-cocycle associated to the section s' for the extension class $[E']$ by $(f'_\sigma \rho_M(\sigma^{-1}))_{\sigma\in\Gamma}$, where by definition f'_σ is such that $u'f'_\sigma = \rho_{E'}(\sigma)s' - s'\rho_M(\sigma)$ for each $\sigma \in \Gamma$. Since $uf_\sigma = \rho_E(\sigma)s - s\rho_M(\sigma)$, we get

$$\begin{aligned}
u'af_\sigma = cuf_\sigma &= c(\rho_E(\sigma)s - s\rho_M(\sigma)) \\
&= \rho_{E'}(\sigma)cs - cs\rho_M(\sigma) \\
&= \rho_E(\sigma)s' - s'\rho_M(\sigma) \\
&= u'f'_\sigma.
\end{aligned}$$

The injectivity of u' implies $f'_\sigma = af_\sigma$, which shows the (1) of the Claim.

Now we go back to the proof of the proposition. Given two extensions of M by N

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] & E \arrow[r, "v"] & M \arrow[r] \arrow[l, bend left, "s"] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r] & N \arrow[r] & E' \arrow[r, "v'"] & M \arrow[r] \arrow[l, bend left, "s'"] & 0
\end{tikzcd},$$

by definition we have $E + E' = \nabla_N(E \oplus E')\triangle_M$. Let s and s' be sections of v and v' respectively, and denote the corresponding 1-cocycles by $(f_\sigma \rho_M(\sigma^{-1}))_{\sigma\in\Gamma}$ and $(f'_\sigma \rho_M(\sigma^{-1}))_{\sigma\in\Gamma}$ respectively. It's easy to see that the 1-cocycle corresponding to the section $s \oplus s'$ of $v \oplus v'$ for the extension $E \oplus E'$ is just

$$((f_\sigma \oplus f'_\sigma)(\rho_M \oplus \rho_M)(\sigma^{-1}))_{\sigma\in\Gamma}.$$

Then by the claim, we have the extension $E + E'$ can be represented by the 1-cocycles $((f_\sigma + f'_\sigma)\rho_M(\sigma^{-1}))_{\sigma\in\Gamma}$. And this just shows that the map φ keeps the group operations. $\square$

2.4 The five lemma

We are going to use the five lemma repeatedly, hence it's worth stating it.

Proposition II.22 (The five lemma)**.** *Let $\mathcal{C}$ be a small abelian category. Given a commutative diagram with exact rows*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f_1"'] & B \arrow[r] \arrow[d, "f_2"'] & C \arrow[r] \arrow[d, "f_3"'] & D \arrow[r, "u"] \arrow[d, "f_4"'] & E \arrow[d, "f_5"'] \\
A' \arrow[r, "v"] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

in $\mathcal{C}$. Then we have the following:

(1) *If both f_2 and f_4 are monomorphisms, and f_1 is an epimorphism, then f_3 is a monomorphism.*

(2) *If both f_2 and f_4 are epimorphisms, and f_5 is a monomorphism, then f_3 is a epimorphism.*

(3) *If both f_2 and f_4 are isomorphisms, f_1 is an epimorphism, and f_5 is a monomorphism, then f_3 is an isomorphism.*

Proof. See [23, chap. I, thm. 21.1] for the case $\mathcal{C}$ being the category of abelian groups. The general case follows from [23, chap. IV, metathm 1.1 and thm. 2.6] $\square$

Remark II.23. Let $\mathcal{C}$ be an abelian category in which we can do diagram chasing (eg. the category of modules over a fixed ring), we can make the five lemma slightly stronger. For (1), the condition concerning f_1 can be weaken to that for any $b' \in \operatorname{im}(v)$ there exists an $a \in A$ such that $vf_1(a) = b'$. For (2), the condition concerning f_5 can be weakened to f_5 being injective when restricted to the image of u.

2.5 Some spectral sequences

Spectral sequences are very useful in computing (co)homology groups. In this section, we just give the definition of (cohomological) spectral sequences, then list several (cohomological) spectral sequences needed for our purpose.

Definition II.24. Let $\mathcal{A}$ be an abelian category, m be a positive integer. An E_m-spectral sequence in $\mathcal{A}$ consists of
(1) objects $E_r^{p,q} \in \mathcal{A}$ for all $p, q \in \mathbb{Z}$ and all integer $r \geq m$.
(2) morphisms $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ such that: (a) $d_r^{p,q} d_r^{p-r,q+r-1} = 0$, i.e. we have a complex $\cdots \to E_r^{p-r,q+r-1} \to E_r^{p,q} \to E_r^{p+r,q-r+1} \to \cdots$ passing through $E_r^{p,q}$; (b) $E_{r+1}^{p,q} \cong \ker(d_r^{p,q})/\mathrm{im}(d_r^{p-r,q+r-1})$, i.e. $E_{r+1}^{p,q}$ comes from the cohomology at (p,q) place of the complex in (a); (c) for each fixed pair $(p,q) \in \mathbb{Z}^2$ the morphisms $d_r^{p,q}$ and $d_r^{p-r,q+r-1}$ vanish for sufficient large r.
(3) finite decreasingly filtrated objects $(E^n \in \mathcal{A}, F^\bullet)$ for all $n \in \mathbb{Z}$, such that $E_\infty^{p,q} \cong gr_p E^n$, where $E_\infty = E_r^{p,q}$ for some r large enough such that $d_r^{p,q} = d_r^{p-r,q+r-1} = 0$.
We denote the spectral sequence by $E_m^{p,q} \Longrightarrow E^{p+q}$. If $E_r^{p,q} = 0$ for $p < 0$ or $q < 0$, then we call such a spectral sequence a first quadrant spectral sequence.

For a first quadrant spectral sequence $E_m^{p,q} \Longrightarrow E^{p+q}$, we have that $F^{n+1}E^n = 0$ and $F^0 E^n = E^n$ for all $n \geq 0$. Hence we get an injection $E_\infty^{n,0} = F^n E^n \hookrightarrow E^n$ and a quotient map $E^n \twoheadrightarrow E_\infty^{0,n}$. For any $r \geq 2$, the morphisms $E_r^{n,0} \xrightarrow{d_r^{n,0}} E_r^{n+r,-r+1} = 0$ and $0 = E_r^{-r,n+r-1} \xrightarrow{d_r^{-r,n+r-1}} E_r^{0,n}$ are forced to be zero for all $n \geq 0$, so we have inclusions $E_{r+1}^{n,0} \hookrightarrow E_r^{n,0}$ and quotient maps $E_r^{0,n} \twoheadrightarrow E_{r+1}^{0,n}$. Then we get two chains of morphisms

$$E_2^{n,0} \twoheadrightarrow E_3^{n,0} \twoheadrightarrow \cdots \twoheadrightarrow E_\infty^{n,0} \hookrightarrow E^n$$

and

$$E^n \twoheadrightarrow E_\infty^{0,n} \hookrightarrow \cdots \hookrightarrow E_3^{0,n} \hookrightarrow E_2^{0,n}.$$

The compositions of these two chains give two morphisms $E_2^{n,0} \to E^n$ and $E^n \to E_2^{0,n}$ for each n, and these morphism are called the edge morphisms.

Proposition II.25 (The five term exact sequence)**.** *Given any first quadrant spectral sequence $E_2^{p,q} \Longrightarrow E^{p+q}$, we have an exact sequence*

$$0 \to E_2^{1,0} \xrightarrow{\text{edge}} E^1 \xrightarrow{\text{edge}} E_2^{0,1} \to E_2^{2,0} \xrightarrow{\text{edge}} E^2. \tag{2.5}$$

Proof. Since the spectral sequence given is a first quadrant one, we have

$$E_\infty^{1,0} = E_2^{1,0} \tag{2.6}$$

$$E_\infty^{0,1} = E_3^{0,1} = \ker(d_2^{0,1}) \tag{2.7}$$

$$E_\infty^{2,0} = E_3^{2,0} = \operatorname{coker}(d_2^{0,1}). \tag{2.8}$$

We then have a short exact sequence

$$0 \to E_\infty^{1,0} \to E^1 \to E_\infty^{0,1} \to 0$$

from the filtration on E^1, an exact sequence

$$0 \to \ker(d_2^{0,1}) \to E_2^{0,1} \to E_2^{2,0} \to \operatorname{coker}(d_2^{0,1}) \to 0$$

from the morphism $d_2^{0,1}$, and an injection

$$0 \to E_\infty^{2,0} \to E^2$$

from the filtration on E^2. Under the identifications from 2.6, 2.7 and 2.8, splicing the above three exact sequences gives the required one. $\square$

The five term exact sequence is very useful for dealing with lower degree cohomology groups. And it can be extended further in various cases. The following is taken from [26, chap. 2.13].

Proposition II.26. *Suppose that we have a first quadrant spectral sequence*

$$E_2^{p,q} \Longrightarrow E^{p+q}.$$

(1) *If $E_2^{p,q} = 0$ for all $q > 1$ and all p, then we have a long exact sequence*

$$\begin{aligned}
0 \longrightarrow E_2^{1,0} &\xrightarrow{\text{edge}} E^1 \xrightarrow{\text{edge}} E_2^{0,1} \\
\longrightarrow E_2^{2,0} &\xrightarrow{\text{edge}} E^2 \longrightarrow E_2^{1,1} \\
\longrightarrow E_2^{3,0} &\xrightarrow{\text{edge}} E^3 \longrightarrow E_2^{2,1} \longrightarrow \cdots .
\end{aligned}$$

(2) *If $E_2^{p,q} = 0$ for all $p > 1$ and all q, then we have short exact sequences*

$$0 \to E_2^{1,n-1} \to E^n \xrightarrow{\text{edge}} E_2^{0,n} \to 0$$

for all $n \geq 1$.

Proof. Under the condition of (1) (resp. (2)), the differentials $d_r^{p,q}$ vanish for all $r > 2$ (resp. $r \geq 2$). The proofs are similar to the proof of the five term exact sequence. □

Theorem II.27 (The Hochschild-Serre spectral sequence)**.** *Let G be a profinite group, H be a closed normal subgroup of G, and M be a discrete G-module. Then there is a first quadrant spectral sequence*

$$E_2^{p,q} = H^p(G/H, H^q(H, M)) \Longrightarrow H^{p+q}(G, M).$$

Proof. See [26, chap II, 2.4.1]. □

Theorem II.28. *Let G be a profinite group, and M, N be two discrete G-modules. As in [22, chap. I, sec. 0] set*

$$\begin{aligned}\mathcal{H}om(M,N) &:= \bigcup_{U\,open} \mathrm{Hom}(M,N)^U \\ &= \{f \in \mathrm{Hom}(M,N) \,|\, \sigma f = f\sigma \textit{ for all } \sigma \textit{ in some open subgroup } U\},\end{aligned}$$

let $\mathcal{E}xt^r(M,N)$ denote the r-th derived functor of the left exact functor

$$\mathcal{C}_G \to \mathcal{C}_G, N \mapsto \mathcal{H}om(M,N).$$

Then there is a first quadrant spectral sequence

$$E_2^{p,q} = H^p(G, \mathcal{E}xt^q(M,N)) \Longrightarrow \mathrm{Ext}^{p+q}_{\mathcal{C}_G}(M,N).$$

In particular, when M is finitely generated as an abelian group, then for any $q \geq 0$ the group underlying the discrete G-module $\mathcal{E}xt^q(M,N)$ is just the abelian group $\mathrm{Ext}^q_{\mathbb{Z}}(M,N)$*, and we follow Milne's notation in [22] to write* $\mathrm{Ext}^q(M,N)$ *instead of* $\mathcal{E}xt^q(M,N)$.

Proof. See [22, chap. I 0.8]. □

Theorem II.29. *Let k be a perfect field, and let A and B be commutative algebraic group schemes over k. Then there is a first quadrant spectral sequence*

$$E_2^{p,q} = H^p(k, \mathrm{Ext}^q_{k^s}(A,B)) \Longrightarrow \mathrm{Ext}^{p+q}_k(A,B),$$

where k^s denotes the separable closure of k.

Proof. See [21]. □

2.6 Yoneda extensions in abelian subcategories

In this section, we are going to compare the Yoneda extension groups in a given abelian category and its abelian subcategories.

First let us give a description of Yoneda extension groups from the point of view of derived category. Given any abelian category $\mathcal{A}$, let $C(\mathcal{A})$ be the category of complexes in $\mathcal{A}$ and $C^*(\mathcal{A})$ $(* = \mathrm{b}, +, -)$ be the full subcategory of $C(\mathcal{A})$ consisting of bounded (resp. bounded below, bounded above) complexes. The homotopy category $K(\mathcal{A})$ is a triangulated category which has the same objects as $C(\mathcal{A})$, but has the morphisms being the morphisms in $C(\mathcal{A})$ modulo chain homotopy equivalence. Similarly, we can also define the category $K^*(\mathcal{A})$ $(* = \mathrm{b}, +, -)$ in the same manner. Then the derived category $\mathcal{D}(\mathcal{A})$ is defined to be the localisation of $K(\mathcal{A})$ with respect to quasi-isomorphism, and is a triangulated category naturally. Similarly, the derived category $\mathcal{D}^*(\mathcal{A})$ $(* = \mathrm{b}, +, -)$ can also be defined. We have four natural functors fitting into the following commutative diagram

$$\begin{tikzcd}
 & \mathcal{D}^+(\mathcal{A}) \arrow[dr] & \\
\mathcal{D}^{\mathrm{b}}(\mathcal{A}) \arrow[ur] \arrow[dr] & & \mathcal{D}(\mathcal{A}) \\
 & \mathcal{D}^-(\mathcal{A}) \arrow[ur] &
\end{tikzcd}$$

and all of these functors are all fully faithful, see [37, chap. III, them. 1.2.3.]. For details about derived category, see [17], [37] or [38].

The categories $K^*(\mathcal{A})$ and $\mathcal{D}^*(\mathcal{A})$ are additive. Given two objects $X, Y \in \mathcal{A}$, regarded as objects of $\mathcal{D}(\mathcal{A})$ with X and Y in degree zero and 0 in all other degrees, the group of homomorphism between X and Y can be described as follows:

$$\begin{aligned}
&\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X, Y) \\
&= \varinjlim_{(\mathrm{Qis}/X)^\circ} \mathrm{Hom}_{K(\mathcal{A})}(\cdot, Y) = \varinjlim_{(\mathrm{Qis}^-/X)^\circ} \mathrm{Hom}_{K(\mathcal{A})}(\cdot, Y) = \varinjlim_{(\mathrm{Qis}^{\mathrm{b}}/X)^\circ} \mathrm{Hom}_{K(\mathcal{A})}(\cdot, Y) \\
&= \varinjlim_{Y\backslash \mathrm{Qis}} \mathrm{Hom}_{K(\mathcal{A})}(X, \cdot) = \varinjlim_{Y\backslash \mathrm{Qis}^+} \mathrm{Hom}_{K(\mathcal{A})}(X, \cdot) = \varinjlim_{Y\backslash \mathrm{Qis}^{\mathrm{b}}} \mathrm{Hom}_{K(\mathcal{A})}(X, \cdot),
\end{aligned}$$

where Qis/X denotes the category of quasi-isomorphisms into X in $K(\mathcal{A})$ and its opposite category $(\mathrm{Qis}^{\mathrm{b}}/X)^\circ$ is filtrant, $Y\backslash\mathrm{Qis}$ denotes the category of quasi-isomorphisms with domain Y in $K(\mathcal{A})$ which is filtrant, Qis^-/X, $\mathrm{Qis}^{\mathrm{b}}/X$, $Y\backslash\mathrm{Qis}^+$

and $Y\backslash\mathrm{Qis}^{\mathrm{b}}$ are defined in a similar way. See [37, chap. III, prop. 3.1.3.] for details of these descriptions.

Now given a Yoneda n-extension

$$0 \to Y \to Z_{n-1} \to Z_{n-2} \to \cdots \to Z_0 \to X \to 0$$

in $\mathcal{A}$, let $Z^\bullet$ be the complex

$$\cdots \to 0 \to Y \to Z_{n-1} \to \cdots Z_0 \to 0 \to \cdots$$

where Y lies in degree $-n$ and Z_i lies in degree $-i$ for each i. Then the extension gives a canonical quasi-isomorphism $Z^\bullet \to X$, the complex $Z^\bullet$ itself gives a canonical element $Z^\bullet \to Y[n]$ in $\mathrm{Hom}_{K(\mathcal{A})}(Z^\bullet, Y[n])$, so we get an element of $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X, Y[n])$. This gives a map δ^n from the set of Yoneda extensions to $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X, Y[n])$. Actually, we can say more.

Proposition II.30. *The map δ^n induces an isomorphism*

$$\bar{\delta}^n : \mathrm{Ext}^n_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X, Y[n])$$

of groups.

Proof. See [37, chap III, prop. 3.2.2.]. □

The proposition II.30 reveals that the Yoneda extension group $\mathrm{Ext}^n_{\mathcal{A}}(X, Y)$ can be defined alternatively as the group $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X, Y[n])$ of morphisms in the derived category, so in the rest of this thesis we will use these two definitions freely.

Now we go back to the category $\mathcal{C}_G$ of discrete G-modules for a profinite group G. Let $\mathcal{C}^f_G$ be the full subcategory of $\mathcal{C}_G$ consisting of all finitely generated discrete G-modules, this category is obviously a full abelian subcategory of $\mathcal{C}_G$. When G is the absolute Galois group of a field k, we will use the notation $\mathcal{C}_k$ instead of $\mathcal{C}^f_G$. Let $\mathcal{D}^{\mathrm{b}}_{\mathcal{C}^f_G}(\mathcal{C}_G)$ be the full additive subcategory of $\mathcal{D}^{\mathrm{b}}(\mathcal{C}_G)$ consisting of all objects X such that $H^i(X) \in \mathcal{C}^f_G$ for all i. Then we have natural functors

$$\mathcal{D}^{\mathrm{b}}(\mathcal{C}^f_G) \xrightarrow{\delta^{\mathrm{b}}} \mathcal{D}^{\mathrm{b}}_{\mathcal{C}^f_G}(\mathcal{C}_G) \hookrightarrow \mathcal{D}^{\mathrm{b}}(\mathcal{C}_G).$$

Proposition II.31. *The above functor δ^{b} is an equivalence of categories, and hence the canonical morphism*

$$\mathrm{Ext}^i_{\mathcal{C}^f_G}(X, Y) \to \mathrm{Ext}^i_{\mathcal{C}_G}(X, Y)$$

is actually an isomorphism for any $X, Y \in \mathcal{C}^f_G$.

Proof. Recall that a thick subcategory of an abelian category is a full subcategory which is closed by kernels, cokernels, and extensions, see [17, chap. 8, def. 8.3.21]. It's easy to see that $\mathcal{C}_G^f$ is a thick subcategory of $\mathcal{C}_G$. Given any epimorphism $f : X \to Y$ in $\mathcal{C}_G$ with $Y \in \mathcal{C}_G^f$, let $y_1, \cdots, y_n$ be a set of generators for Y, take preimage $x_1, \cdots, x_n \in X$ such that $f(x_i) = y_i$. Since X is a discrete G-module, there exists an open subgroup U of G such that X^U contains all the x_i's, let X' be the sub-G/U-module of X^U generated by the x_i's. Then Y' is finitely generated and is naturally a discrete sub-G-module of X, hence an object of the category $\mathcal{C}_G^f$. The composition $Y' \hookrightarrow X \xrightarrow{f} Y$ is obviously an epimorphism, hence by the dual version of [17, chap. 13, them. 13.2.8], the functor δ^{b} is an equivalence of categories.

The category $\mathcal{D}^{\mathrm{b}}_{\mathcal{C}_G^f}(\mathcal{C}_G)$ is a full subcategory of $\mathcal{D}^{\mathrm{b}}(\mathcal{C}_G)$, hence we have

$$\mathrm{Hom}_{\mathcal{D}^{\mathrm{b}}_{\mathcal{C}_G^f}(\mathcal{C}_G)}(X, Y[i]) = \mathrm{Hom}_{\mathcal{D}^{\mathrm{b}}(\mathcal{C}_G)}(X, Y[i])$$

for any $X, Y \in \mathcal{C}_G^f$. On the other hand, the functor δ^{b} gives a canonical isomorphism

$$\mathrm{Hom}_{\mathcal{D}^{\mathrm{b}}(\mathcal{C}_G^f)}(X, Y[i]) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{D}^{\mathrm{b}}_{\mathcal{C}_G^f}(\mathcal{C}_G)}(X, Y[i]).$$

So we get a canonical isomorphism $\mathrm{Hom}_{\mathcal{D}^{\mathrm{b}}(\mathcal{C}_G^f)}(X, Y[i]) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{D}^{\mathrm{b}}(\mathcal{C}_G)}(X, Y[i])$ which is just the morphism appearing in the statement under the identification in II.30. □

Corollary II.32. *Given $X, Y \in \mathcal{C}_G^f$, we have a first quadrant spectral sequence*

$$E_2^{p,q} = H^p(G, \mathrm{Ext}^q(X, Y)) \Longrightarrow \mathrm{Ext}^{p+q}_{\mathcal{C}_G^f}(X, Y).$$

Proof. Easy consequence of II.28 and II.31. □

Theorem II.33. *Let $\mathcal{A}$ be an abelian category, $\mathcal{B}$ a full abelian subcategory which is thick. Suppose that $\mathcal{B}$ is also closed by subobjects and quotients, see [17, chap. 8, def. 8.3.21] for definitions. Suppose given any two objects $X, Y \in \mathcal{B}$ and a positive integer $i > 1$, the canonical map $\varphi_{i-1} : \mathrm{Ext}^{i-1}_{\mathcal{B}}(X, Y) \to \mathrm{Ext}^{i-1}_{\mathcal{A}}(X, Y)$ for Yoneda extension groups is an isomorphism, then the canonical map $\varphi_i : \mathrm{Ext}^i_{\mathcal{B}}(X, Y) \to \mathrm{Ext}^i_{\mathcal{A}}(X, Y)$ is injective. In particular, the map φ_2 is always injective.*

Proof. Before going to the proof, we first make a claim.

Claim. For any short exact sequence $0 \to A' \to A \to A'' \to 0$ in $\mathcal{A}$, A lies in $\mathcal{B}$ if and only if both A' and A'' lie in $\mathcal{B}$.

The claim is just an easy consequence of the fact that $\mathcal{B}$ is thick and closed by subobjects and quotients.

Now suppose that the map φ_{i-1} is an isomorphism. Given any i-extension in $\mathcal{B}$, which represents the trivial element of the group $\mathrm{Ext}^i_{\mathcal{A}}(X, Y)$, write it as $\mathcal{E} \cdot \mathcal{F}$ for some $(i-1)$-extension $\mathcal{E}$ and some 1-extension $\mathcal{F}$.

$$\mathcal{E} \quad 0 \to Y \to Y_1 \to \cdots \to Y_{i-1} \to P \to 0 \qquad 0 \to P \xrightarrow{\alpha} Y_i \to X \to 0 \quad \mathcal{F}.$$

By the claim, it is easy to see that the extension $\mathcal{F}$ lies in the category $\mathcal{B}$. We have the long exact sequence associated to $\mathcal{F}$

$$\to \mathrm{Ext}^{i-1}_{\mathcal{A}}(Y_i, Y) \xrightarrow{\alpha^*} \mathrm{Ext}^{i-1}_{\mathcal{A}}(P, Y) \xrightarrow{\delta} \mathrm{Ext}^i_{\mathcal{A}}(P, X) \to .$$

Note that the map δ is just the map splicing with $\mathcal{F}$. Then $[\mathcal{E} \cdot \mathcal{F}] = 0$ implies that $\mathcal{E} \sim \alpha^*(\mathcal{E}')$ and $\mathcal{E} \cdot \mathcal{F} \sim \alpha^*(\mathcal{E}') \cdot \mathcal{F} \sim \mathcal{E}' \cdot (\alpha_* \mathcal{F})$ for some $(i-1)$-extension $\mathcal{E}'$ of Y_i by Y. From the isomorphism

$$\mathrm{Ext}^{i-1}_{\mathcal{B}}(Y_i, Y) \to \mathrm{Ext}^{i-1}_{\mathcal{A}}(Y_i, Y),$$

there exists an extension $\mathcal{E}''$ of Y_i by Y in $\mathcal{B}$ which is equivalent to $\mathcal{E}'$. It follows that $[\mathcal{E} \cdot \mathcal{F}] = [\mathcal{E}' \cdot (\alpha_* \mathcal{F})] = [\mathcal{E}'' \cdot (\alpha_* \mathcal{F})] = 0$ in $\mathcal{B}$. This shows the injectivity of φ_i.

Note the claim implies that the map φ_1 is an isomorphism, hence φ_2 is injective.

□

Lemma II.34. *Let G be the profinite group $\hat{\mathbb{Z}}$, R be the ring $\mathbb{Z}_l[T, T^{-1}]$ with T some indeterminate, $\mathcal{B}$ be the abelian category of finitely generated $\mathbb{Z}_l$-modules with continuous G-action. Then the abelian category $\mathcal{B}$ is equivalent to the abelian category $R - \mathrm{Mod}^f$ consisting of all R-modules which are finitely generated over $\mathbb{Z}_l$. In particular, given any $X, Y \in \mathcal{B}$, we have a canonical isomorphism*

$$\mathrm{Ext}^i_{\mathcal{B}}(X, Y) \xrightarrow{\cong} \mathrm{Ext}^i_{R-\mathrm{Mod}^f}(X, Y)$$

for each positive integer i.

Proof. For any finitely generated $\mathbb{Z}_l$-module X, the topology on $\mathrm{Aut}_{\mathbb{Z}_l}(X)$ is the l-adic one which is complete and compact. Hence a continuous homomorphism from $\hat{\mathbb{Z}}$ to $\mathrm{Aut}_{\mathbb{Z}_l}(X)$ is uniquely determined by the image of the topological generator of $\hat{\mathbb{Z}}$, or equivalently is uniquely determined by the induced homomorphism from the dense subgroup $\mathbb{Z}$ of $\hat{\mathbb{Z}}$ to $\mathrm{Aut}_{\mathbb{Z}_l}(X)$. Note that we have an isomorphism $\mathbb{Z}_l[\mathbb{Z}] \cong \mathbb{Z}_l[T, T^{-1}]$ sending $1 \in \mathbb{Z}$ to T. It follows that we get an equivalence from the category $\mathcal{B}$ to the category $R - \mathrm{Mod}^f$ consisting of all the R-modules which are finitely generated over $\mathbb{Z}_l$. □

Lemma II.35. *Let A be a noetherian ring, and M, N two finitely generated A-modules. Then any element of $\mathrm{Ext}^i_A(M, N)$ can be represented by an i-extension of M by N which consists of only finitely generated A-modules.*

Proof. First the category of A-modules admits enough projectives, so the group $Ext^i_A(M, N)$ can also be computed via resolution. Since M is finitely generated and A is notherian, so there exists a projective resolution of M

$$\cdots \to L_{i+1} \xrightarrow{d_{i+1}} L_i \xrightarrow{d_i} \cdots \to L_1 \xrightarrow{d_1} L_0 \to M,$$

in which all L_j's are finitely generated A-modules. Given any $\alpha \in \mathrm{Ext}^i_A(M, N)$, i.e. an element of the group $\ker(\mathrm{Hom}_A(d_{i+1}, N))/\mathrm{im}(\mathrm{Hom}_A(d_i, N))$, choose a representative $f \in \mathrm{Hom}_A(L_i, N)$, then we have $f \circ d_{i+1} = 0$. It follows the map f factors through $\tilde{L}_i := L_i/\mathrm{im}(d_{i+1})$ which is a finitely generated A-module. We also have the following diagram

$$\begin{tikzcd} \mathcal{E}: 0 \arrow[r] & \tilde{L}_i \arrow[r] \arrow[d, "\tilde{f}"] & L_{i-1} \arrow[r] & \cdots \arrow[r] & L_0 \arrow[r] & M \arrow[r] & 0 \\ & N. \end{tikzcd}$$

Then the Yoneda extension class corresponding to α can be represented by the extension $\tilde{f}_*\mathcal{E}$, which consists of finitely generated A-modules. □

Theorem II.36. *Let notations be as in II.34, then we have canonical isomorphisms*

$$\varphi_i : \mathrm{Ext}^i_{R-\mathrm{Mod}^f}(X, Y) \xrightarrow{\cong} \mathrm{Ext}^i_{R-\mathrm{Mod}}(X, Y)$$

for all positive integers i. In particular, the group $\mathrm{Ext}^i_{R-\mathrm{Mod}^f}(X, Y)$ vanishes for each $i > 2$.

Proof. Firstly it's easy to see that φ_1 is an isomorphism. By theorem II.33, the map φ_2 is injective. Secondly, the groups $\mathrm{Ext}^i_R(X, Y)$ vanish for all $i > 2$, since the global dimension of the ring R is two. So the surjectivity of the maps φ_i is obvious for all $i > 2$. If φ_2 is an isomorphism, then φ_i's are all injective by using theorem II.33 repeatedly. So we are left to show the surjectivity of the map φ_2.

Given any element $\alpha \in \operatorname{Ext}^i_R(X, Y)$ regarded as a class of Yoneda extensions, choose a representative

$$0 \to Y \to L_1 \to L_0 \to X \to 0$$

with L_1, L_0 finitely generated R-modules.

Since Y is finitely generated as a $\mathbb{Z}_l$-module, the annihilator ideal $\operatorname{ann}(Y)$ of Y must contain an element f which doesn't lie in $\mathbb{Z}_l$. Let L_1^f be the set

$$\{x \in L_1 | f^r x = 0 \,\text{for some}\, r \in \mathbb{N}\},$$

it's easy to see that L_1^f is a R-submodule of L_1, hence a finitely generated R-module. And also we have $Y \subset L_1^f$. Let $\{x_1, x_2, \cdots, x_t\}$ be a set of generators of L_1^f as a R-module, and suppose that $f^{n_i} x_i = 0$ for some positive integer n_i. Let n be $\max_i\{n_i\}$, then we have $f^n L_i^f = 0$.

We claim that $Y \cap f^n L_1 = \{0\}$. Take $y \in Y \cap f^n L_1$, then y can be written as $f^n x$. $y \in Y$ implies that $f^{n+1} x = f y = 0$, so x lies in L_1^f. It follows that $y = f^n x = 0$.

So we get the following commutative diagram with exact rows

(2.9)
$$\begin{tikzcd}
 & & f L_1 \arrow[d, hook] & & & \\
0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & L_1 \arrow[r, "u"] \arrow[d] & L_0 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Y \arrow[r] & L_1/f^n L_1 \arrow[r] & L_0/u(f^n L_1) \arrow[r] & X \arrow[r] & 0.
\end{tikzcd}$$

Since L_1 is a finitely generated R-module, there exists a surjective map $R^r \xrightarrow{v} L_1$ for some positive integer r. Consider the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker(v) \arrow[r] \arrow[d, "s_1"] & R^r \arrow[r, "v"] \arrow[d, "s_2"] & L_1 \arrow[r] \arrow[d, "s_3"] & 0 \\
0 \arrow[r] & \ker(v) \arrow[r] & R^r \arrow[r, "v"] & L_1 \arrow[r] & 0,
\end{tikzcd}$$

where the three vertical maps are the multiplication maps by f^n. By the snake lemma, we have an exact sequence

$$0 \to \ker(s_3) \to \operatorname{coker}(s_1) \to \operatorname{coker}(s_2) \to \operatorname{coker}(s_3) \to 0.$$

f doesn't lie in $\mathbb{Z}_l$, so R/f^nR is finitely generated over $\mathbb{Z}_l$, hence so is $L_1/f^nL_1 = \mathrm{coker}(s_3)$. Hence $L_0/u(f^nL_1)$ is also finite generated over $\mathbb{Z}_l$, and α lies in the image of φ_2. $\square$

2.7 The noetherianity of $\mathcal{M}$

First recall that in the theory of rings and modules, the chain condition gives two special kinds of objects, the artinian ones and the noetherian ones, which satisfy descending chain condition and ascending chain condition respectively. These kinds of rings and modules are very useful and relatively easy to understand. In the abelian categories, we can do the similar thing.

Definition II.37 (cf. [10] chap. II, sec. 4)**.** Let $\mathcal{C}$ be an abelian category. An object M in $\mathcal{C}$ is noetherian (resp. artinian) if every ascending (resp. descending) chains of subobjects of M is stationary. The category $\mathcal{C}$ is noetherian (resp. artinian) if all the objects of $\mathcal{C}$ are noetherian (resp. artinian).

Proposition II.38. *The category $\mathcal{M}$ is not artinian.*

Proof. Let L be a torsion-free finitely generated locally constant sheaf for the étale topology, then $L[1]$ is a 1-motive. Let r be a prime number which is not equal to $\mathrm{char}(k)$, the family

$$\{L[1] \xrightarrow{r^n} L[1] \,|\, n \in \mathbb{N}\}$$

of monomorphisms shows that $\mathcal{M}$ is not artinian. This can also be shown by another example, the family

$$\{G \xrightarrow{r^n} G \,|\, n \in \mathbb{N}\}$$

of monomorphisms, where G is a nonzero semi-abelian variety. $\square$

Theorem II.39. *The category $\mathcal{M}$ is noetherian.*

Proof. Given any 1-motive $M = [L \to G]$, we have a canonical short exact sequence

$$0 \to G \to M \to L[1] \to 0.$$

Then by [10, lemme 1], in order to show M is noetherian, it is enough to that both G and $L[1]$ are noetherian. Note that any subobject of $L[1]$ has to be of the form

$L'[1]$, where L' is a subsheaf of L, then the noetherianity of $L[1]$ follows from the noetherianity of L which is obvious.

Now we are left to show the noetherianity of G. Again by [10, lemme 1], we can assume that the semi-abelian variety G doesn't contain any proper subgroup variety. Any morphism to G can be represented by an effective map of the form

$$\begin{tikzcd} F \arrow[r, "0"] \arrow[d, "u"'] & 0 \arrow[d] \\ G' \arrow[r, "g"] & G \end{tikzcd}$$

such that $u(F)$ goes to zero under g. The morphism $(0, g)$ is a monomorphism if and only if the map u is injective. If this is the case, then $(0, g)$ factors through the quasi-isomorphism $[F \xrightarrow{u} G'] \to G'/F$, as in the following diagram

$$\begin{tikzcd} {[F \xrightarrow{u} G']} \arrow[rr, "{(0,g)}"] \arrow[dr, "\text{q.i.}"'] & & G \\ & G'/F. \arrow[ur, dotted, "\bar{g}"'] & \end{tikzcd}$$

Then we can go further, any monomorphism to G can be represented by a morphism of the form $G' \xrightarrow{g} G$ with g a morphism of k-group schemes such that $\ker(g)$ is a finite étale subgroup of G'. Since G doesn't contain any proper subgroup variety, g has to be an isogeny of semi-abelian varieties. It follows that any chain of subobjects of G can be represented by the diagram:

$$\begin{tikzcd} G_0 \arrow[r, "i_0"] \arrow[dr, "g_0"'] & G_1 \arrow[r, "i_1"] \arrow[d, "g_1"] & G_2 \arrow[r, "i_2"] \arrow[dl, "g_2"] & \cdots, \\ & G & & \end{tikzcd}$$

with g_i's isogenies to G. Note that we can choose i_j's to be effective maps which again have to be isogenies. Since $\ker(g_0)$ (note here it is the kernel of the morphism of group schemes) is finite, so the chain is stationary. $\square$

Remark II.40. In [27], the author shows that the category $\mathcal{G}$ of commutative group schemes over an algebraically closed field is artinian, then embeds $\mathcal{G}$ as a full subcategory into its pro-category $\mathrm{Pro}(\mathcal{G})$ in which there are enough projectives. Then the groups $\mathrm{Ext}^i_{\mathcal{G}}(A, B)$ can be computed as $\mathrm{Ext}^i_{\mathrm{Pro}(\mathcal{G})}(A, B)$ for any $A, B \in \mathcal{G}$ by [28, them. 3.5]. Here we have shown $\mathcal{M}$ is not artinian, but noetherian. We can embed $\mathcal{M}$ into its Ind-category (instead of Pro-category) in which there are enough injectives, and the groups $\mathrm{Ext}^i_{\mathcal{M}}(M, M')$ can be computed as $\mathrm{Ext}^i_{\mathrm{Ind}(\mathcal{M})}(M, M')$ for any $M, M' \in \mathcal{M}$ by the dual version of [28, them. 3.5].

CHAPTER III

Higher Yoneda extensions in the abelian category of 1-motives with torsion

Throughout this chapter, $M = [L \to G]$ and $M' = [L' \to G']$ will be two 1-motives over the base filed k, $\Gamma = \mathrm{Gal}(\bar{k}/k)$ will be the absolute Galois group of k, and p will be the characteristic of k.

In 1.4, we have discussed the groups $\mathrm{Hom}_{\mathcal{M}}(M, M')$ and $\mathrm{Ext}^1_{\mathcal{M}}(M, M')$ for simple M and M' in characteristic zero case. In this chapter we are going to investigate systematically the Yoneda extension groups in the abelian category $\mathcal{M}$. The main result we want to prove in this chapter is the following theorem.

Theorem III.1. *The homological dimension of the abelian category $\mathcal{M}$ is*

$$\mathrm{d}(\mathcal{M}) = \mathrm{cd}(\Gamma) + 1,$$

where $\mathrm{cd}(\Gamma)$ *denotes the cohomological dimension of the absolute Galois group* Γ *of the base field k.*

We are going to prove III.1 in the first section for the characteristic zero case with the help of I.25 and I.26. In the second section, we are going to give the analogues of I.25 and I.26 in the positive characteristic case, and then finish the proof of III.1. The last two sections are devoted to some applications of III.1.

3.1 Proof of theorem III.1 in characteristic zero case

Throughout this section, the characteristic of k will be zero. Recall that the homological dimension of an abelian category $\mathcal{C}$ is defined to be the non-negative integer n such that the functor $\mathrm{Ext}^i_{\mathcal{C}}(-,-)$ is zero for $i > n$; if such n doesn't exist, then the homological dimension is defined to be infinity, see [27, I.3-2].

Lemma III.2. *Let G, G' be two semi-abelian varieties, then the group* $\mathrm{Ext}^2_{\mathcal{M}}(G, G')$ *is torsion.*

Proof. Take any element in $\mathrm{Ext}^2_{\mathcal{M}}(G, G')$, then it can be expressed as the product of some $\mathcal{F} \in \mathrm{Ext}^1_{\mathcal{M}}(G, [Y \to J])$ and $\mathcal{E} \in \mathrm{Ext}^1_{\mathcal{M}}([Y \to J], G')$ for a suitable 1-motive $[Y \to J]$.

From the canonical short exact sequence $0 \to J \xrightarrow{\alpha} [Y \to J] \xrightarrow{\beta} Y[1] \to 0$, we have an exact sequence:

$$\to \mathrm{Ext}^1_{\mathcal{M}}(G, J) \xrightarrow{\alpha_*} \mathrm{Ext}^1_{\mathcal{M}}(G, [Y \to J]) \xrightarrow{\beta_*} \mathrm{Ext}^1_{\mathcal{M}}(G, Y[1]) \to .$$

By I.26, the group $\mathrm{Ext}^1_{\mathcal{M}}(G, Y[1])$ is torsion, so there exists a positive integer r such that $\beta_*(r \cdot \mathcal{F}) = r \cdot \beta_*(\mathcal{F}) = 0$. It follows that $r \cdot \mathcal{F} = \alpha_*(\mathcal{F}')$ for some $\mathcal{F}' \in \mathrm{Ext}^1_{\mathcal{M}}(G, J)$. Then we have

$$r \cdot (\mathcal{E} \cdot \mathcal{F}) = \mathcal{E} \cdot (\alpha_*(\mathcal{F}')) \sim (\alpha^*(\mathcal{E})) \cdot \mathcal{F}' = \mathcal{E}' \cdot \mathcal{F}',$$

where $\mathcal{E}' := \alpha^*(\mathcal{E}) \in \mathrm{Ext}^1_{\mathcal{M}}(J, G')$. By I.25 (c), both $\mathcal{E}'$ and $\mathcal{F}'$ can be represented by the short exact sequences in the category of commutative group schemes as follows:

$$\mathcal{E}' \quad 0 \longrightarrow G' \longrightarrow J_1 \longrightarrow J \longrightarrow 0 \qquad 0 \longrightarrow J \longrightarrow J_2 \longrightarrow G \longrightarrow 0 \quad \mathcal{F}'.$$

Now let us write G and J as $0 \to T \to G \to A \to 0$ and $0 \to T_1 \xrightarrow{\mu} J \xrightarrow{\lambda} B \to 0$ respectively. By Poincaré's complete reducibility theorem, the group $\mathrm{Ext}^1_k(A, B)$ is torsion, the torsioness of the group $\mathrm{Ext}^1_k(T, T_1)$ follows from the torsioness of the extension group of their character groups, and $\mathrm{Ext}^1_k(T, B)$ is torsion by Chevalley's theorem on the structure of algebraic groups over k, hence we have both $\mathrm{Ext}^1_k(G, B)$ and $\mathrm{Ext}^1_k(T, J)$ are torsion (note that these results still hold if we replace G and J by any semi-abelian varieties). Applying the functor $\mathrm{Hom}_k(G, -)$ to the short exact sequence $0 \to T_1 \xrightarrow{\mu} J \xrightarrow{\lambda} B \to 0$, we get an exact sequence

$$\mathrm{Ext}^1_k(G, T_1) \xrightarrow{\mu_*} \mathrm{Ext}^1_k(G, J) \xrightarrow{\lambda_*} \mathrm{Ext}^1_k(G, B).$$

Hence there exists a positive integer s such that $\lambda_*(s \cdot \mathcal{F}') = s \cdot \lambda_*(\mathcal{F}') = 0$, and it follows that $s \cdot \mathcal{F}' = \mu_*(\mathcal{F}'')$ for some $\mathcal{F}'' \in \mathrm{Ext}^1_k(G, T_1)$. So we have

$$(sr) \cdot (\mathcal{E} \cdot \mathcal{F}) \sim s \cdot (\mathcal{E}' \cdot \mathcal{F}') = \mathcal{E}' \cdot \mu_*(\mathcal{F}'')) \sim \mu^*(\mathcal{E}') \cdot \mathcal{F}'' = \mathcal{E}'' \cdot \mathcal{F}'',$$

where $\mathcal{E}'' = \mu^*(\mathcal{E}')$. So we get:

$$\mathcal{E}'' \quad 0 \longrightarrow G' \longrightarrow J_1' \longrightarrow T_1 \longrightarrow 0 \qquad 0 \longrightarrow T_1 \longrightarrow J_2' \longrightarrow G \longrightarrow 0 \quad \mathcal{F}''.$$

Now $\mathrm{Ext}_k^1(T_1, G')$ is torsion by the same reason as $\mathrm{Ext}_k^1(T, J)$ is torsion, hence $\mathcal{E}'' \cdot \mathcal{F}''$ represents a torsion element, so does $\mathcal{E} \cdot \mathcal{F}$. Therefore, the group $\mathrm{Ext}^2_{\mathcal{M}}(G, G')$ is torsion. □

Let $\Gamma = \mathrm{Gal}(\bar{k}/k)$ be the absolute Galois group of the base field k, and let $\mathcal{C}_k = \mathcal{C}_\Gamma^f$ be the abelian category of finitely generated abelian groups on which Γ acts discretely as in 2.6.

We have seen in I.24 (a) and I.25 (a) that $\mathrm{Hom}_{\mathcal{C}_k}(L, L') \cong \mathrm{Hom}_{\mathcal{M}}(L[1], L'[1])$ and $\mathrm{Ext}^1_{\mathcal{C}_k}(L, L') \cong \mathrm{Ext}^1_{\mathcal{M}}(L[1], L'[1])$. In fact, this is true for any Ext^i by the following lemma.

Lemma III.3. *For each positive integer i, there is a canonical morphism*

$$\Psi_i : \mathrm{Ext}^i_{\mathcal{C}_k}(L, L') \to \mathrm{Ext}^i_{\mathcal{M}}(L[1], L'[1])$$

sending the Yoneda i-extension

$$0 \to L' \to L_1 \to \cdots \to L_i \to L \to 0$$

in $\mathcal{C}_k$ to the Yoneda i-extension

$$0 \to L'[1] \to L_1[1] \to \cdots \to L_i[1] \to L[1] \to 0$$

in $\mathcal{M}$ with each L_j regarded as a group scheme over k. And they are all isomorphisms.

Proof. The $i = 1$ case is just I.25 (a), so we only need to prove the lemma for $i > 1$. Surjectivity:

We prove the surjectivity of Ψ_i by using induction on i. Suppose the morphism Ψ_i is surjective. Given any element of $\mathrm{Ext}^{i+1}_{\mathcal{M}}(L[1], L'[1])$, we can express it as the product of some $\mathcal{E} \in \mathrm{Ext}^1_{\mathcal{M}}([Y \to J], L'[1])$ and $\mathcal{F} \in \mathrm{Ext}^i_{\mathcal{M}}(L[1], [Y \to J])$ for some $[Y \to J] \in \mathcal{M}$, such that we can find a short exact sequence of complexes as a representative of $\mathcal{E}$ as follows:

$$\mathcal{E}: \quad \begin{tikzcd} 0 \arrow[r] & L' \arrow[r] \arrow[d] & Y_1 \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & 0 \arrow[r] & J \arrow[r, equal] & J \arrow[r] & 0. \end{tikzcd}$$

Let $\mathcal{E}'$ be the short exact sequence $0 \to L'[1] \to Y_1[1] \to Y[1] \to 0$ coming from $\mathcal{E}$ by forgetting the semiabelian part, and let α be the canonical map $[Y \to J] \to Y[1]$. Then it's easy to see that $\mathcal{E} = \alpha^*(\mathcal{E}')$, whence

$$\mathcal{E} \cdot \mathcal{F} = \alpha^*(\mathcal{E}') \cdot \mathcal{F} \sim \mathcal{E}' \cdot (\alpha_*(\mathcal{F})).$$

By induction, $\alpha_*(\mathcal{F}) \in \mathrm{Ext}^i_{\mathcal{M}}(L[1], Y[1])$ can be represented by some element $\mathcal{F}'$ of $\mathrm{Ext}^i_{\mathcal{C}_k}(L, Y)$. And $\mathcal{E}'$ comes from an element of $\mathrm{Ext}^1_{\mathcal{C}_k}(Y, L')$, hence $\mathcal{E} \cdot \mathcal{F} \sim \mathcal{E}' \cdot \mathcal{F}'$ can be represented by some element of $\mathrm{Ext}^{i+1}_{\mathcal{C}_k}(L, L')$.
Injectivity:

For any element in the kernel of Ψ_{i+1}, we express it as the product $\mathcal{E} \cdot \mathcal{F}$ of $\mathcal{E} \in \mathrm{Ext}^1_{\mathcal{C}_k}(Y, L')$ and $\mathcal{F} \in \mathrm{Ext}^i_{\mathcal{C}_k}(L, Y)$ for some $Y \in \mathcal{C}_k$. We pick representatives for $\mathcal{E}$ and $\mathcal{F}$ as follows:

$$\mathcal{E} \quad 0 \longrightarrow L' \longrightarrow L_1 \longrightarrow Y \longrightarrow 0 \qquad 0 \longrightarrow Y \longrightarrow L_2 \longrightarrow \cdots \longrightarrow L \longrightarrow 0 \quad \mathcal{F}.$$

By [23, chap. VII lemma 4.1.], there exists a morphism $\gamma : [X \to J] \to Y[1]$ in $\mathcal{M}$, and $\mathcal{F}' \in \mathrm{Ext}^i_{\mathcal{M}}(L, [X \to J])$, such that $\mathcal{F} = \gamma_*(\mathcal{F}')$ and $\gamma^*(\mathcal{E}) = 0$. We can assume γ to be an effective map, after replacing $[X \to J]$ by another 1-motive $[\tilde{X} \to \tilde{J}]$ from which there is an quasi-isomorphism s to $[X \to J]$, since s being an isomorphism in $\mathcal{M}$ induces an isomorphism between $\mathrm{Ext}^i_{\mathcal{M}}(L, [\tilde{X} \to \tilde{J}])$ and $\mathrm{Ext}^i_{\mathcal{M}}(L, [X \to J])$.

It is easy to see that $\mathcal{E}' := \gamma^*(\mathcal{E})$ is represented by the extension

$$\mathcal{E}' : \quad \begin{tikzcd} 0 \arrow[r] & L' \arrow[r] \arrow[d] & L_1 \times_Y X \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & 0 \arrow[r] & J \arrow[r, equal] & J \arrow[r] & 0. \end{tikzcd}$$

And $\mathcal{E}' = 0$ means the map $[L_1 \times_Y X \to J] \to [X \to J]$ admits a section s. After replacing $[X \to J]$ by some suitable (i.e. "big" enough) 1-motive which admits a quasi-isomorphism to $[X \to J]$, we can assume the section s is an effective map, i.e. the sequence $\mathcal{E}'' : 0 \to L' \to L_1 \times_Y X \to X \to 0$ splits. Let β be the canonical map $[X \to J] \to X[1]$, then we have $\mathcal{E}' = \beta^*(\mathcal{E}'')$. Hence we get $\mathcal{E} \cdot \mathcal{F} \sim \mathcal{E}' \cdot \mathcal{F}' = \beta^*(\mathcal{E}'') \cdot \mathcal{F}' \sim \mathcal{E}'' \cdot \beta_*(\mathcal{F}') = \mathcal{E}'' \cdot \mathcal{F}''$, where $\mathcal{F}''$ denotes the extension $\beta_*(\mathcal{F}') \in \mathrm{Ext}^i_{\mathcal{M}}(L[1], X[1])$. We have already proven the surjectivity of the morphisms Ψ_i's, so $\mathcal{F}''$ can be represented by an i-extension $\mathcal{F}'''$ in the category $\mathcal{C}_k$. Hence we have $\mathcal{E} \cdot \mathcal{F} \sim \mathcal{E}'' \cdot \mathcal{F}'''$ and $\mathcal{E}'' \cdot \mathcal{F}''' \in \mathrm{Ext}^{i+1}_{\mathcal{C}_k}(L, L')$. The fact that $\mathcal{E}''$ splits implies that $[\mathcal{E} \cdot \mathcal{F}] = [\mathcal{E}'' \cdot \mathcal{F}'''] = 0$ in $\mathcal{C}_k$, hence Φ_{i+1} is injective. $\square$

Corollary III.4. *The groups* $\mathrm{Ext}^i_{\mathcal{M}}(L[1], L'[1])$ *are all torsion for* $i \geq 1$*.*

Proof. By III.3, it is enough to prove that the groups $\mathrm{Ext}^i_{\mathcal{C}_k}(L, L')$ are torsion for all $i \geq 1$. We are going to use the spectral sequence II.32

$$E_2^{i,j} = H^i(k, \mathrm{Ext}^j(L, L')) \Longrightarrow \mathrm{Ext}^{i+j}_{\mathcal{C}_k}(L, L'). \tag{3.1}$$

Since the homological dimension of the category of abelian groups is equal to the global dimension of the ring $\mathbb{Z}$ which is one, we have that $E_2^{i,j} = H^i(k, \mathrm{Ext}^j_{\mathbb{Z}}(L, L')) = 0$ for all $j > 1$. So we must have the morphisms $d_r^{i,j} = 0$ for $r > 2$, and the following holds:

$$E_\infty^{i,1} = E_3^{i,1} = \mathrm{ker} d_2^{i,1}$$
$$E_\infty^{i,0} = E_3^{i,0} = E_2^{i,0}/\mathrm{im} d_2^{i-2,1}$$

for $i \geq 0$. In particular $E_\infty^{1,0} = E_2^{1,0}$ and $E_\infty^{0,0} = E_2^{0,0} = E^0$. We also have the following exact sequences:

$$0 \to E_\infty^{i,0} \to E^i \to E_\infty^{i-1,1} \to 0$$
$$0 \to E_3^{i-1,1} \to E_2^{i-1,1} \xrightarrow{d_2^{i-1,1}} E_2^{i+1,0} \to E_3^{i+1,0} \to 0$$

for all $i \geq 1$. Combining all the above, we get exact sequences

$$0 \to E_3^{i,0} \to E^i \to E_2^{i-1,1} \xrightarrow{d_2^{i-1,1}} E_2^{i+1,0} \to E_3^{i+1,0} \to 0$$

for all $i \geq 1$. Taking into account $E_2^{i,j} = H^i(k, \mathrm{Ext}^j_{\mathbb{Z}}(L, L'))$ and $E^{i+j} = \mathrm{Ext}^{i+j}_{\mathcal{C}_k}(L, L')$, we can rewrite the above exact sequences as

$$0 \to H^i(k, \mathrm{Hom}_{\mathbb{Z}}(L, L'))/\mathrm{im} d_2^{i-2,1} \to \mathrm{Ext}^i_{\mathcal{C}_k}(L, L') \to H^{i-1}(k, \mathrm{Ext}^1_{\mathbb{Z}}(L, L')) \to \tag{3.2}$$

for all $i \geq 1$. In particular for $i = 1$, the morphism $d_2^{i-2,1} = 0$, this is just the exact sequence of lower degree associated to our spectral sequence. According to II.6, the group $H^i(k, \mathrm{Hom}_{\mathbb{Z}}(L, L'))$ is torsion, so is the first term of the above short exact sqeuence. And $\mathrm{Ext}^1_{\mathbb{Z}}(L, L'))$ is torsion, hence so is the third term of the above exact sequence. It follows that the middle term is also torsion. $\square$

Lemma III.5. *There is a canonical epimorphism*

$$\Phi_i : \varinjlim_n \mathrm{Ext}^i_{\mathcal{C}_k}({}_nG, L') \longrightarrow \mathrm{Ext}^i_{\mathcal{M}}(G, L'[1]),$$

for each $i \geq 0$*. In particular, the groups* $\mathrm{Ext}^i_{\mathcal{M}}(G, L'[1])$ *are all torsion for* $i \geq 0$*.*

Proof. For the case $i = 1$, see I.26. The case $i = 0$ follows from I.24 (d).

In general, the reason for the existence of the morphism Φ_i is the same reason as for the existence of Φ_1 given in I.26, though the notation there is just Φ instead of Φ_1 here. We need to show that Φ_i is surjective for i. We prove this by using induction on i. Suppose that for any semiabelian variety G and discrete sheaf L which is defined by a finitely generated abelian group, the morphism Φ_i is surjective. Any element of $\mathrm{Ext}^{i+1}_{\mathcal{M}}(G, L'[1])$ can be represented by the product $\mathcal{E} \cdot \mathcal{F}$ for some 1-motive $[Y \to J]$, and $\mathcal{E} \in \mathrm{Ext}^1_{\mathcal{M}}([Y \to J], L'[1])$, $\mathcal{F} \in \mathrm{Ext}^i_{\mathcal{M}}(G, [Y \to J])$. Replacing $[Y \to J]$ by another 1-motive which is quasi-isomorphic to $[Y \to J]$, we can assume that $\mathcal{E}$ is represented by the short exact sequence of complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L' & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & 0 & \longrightarrow & J & = & J & \longrightarrow & 0. \end{array}$$

Let $\mathcal{E}'$ be the short exact sequence $0 \to L' \to X \to Y \to 0$ coming from $\mathcal{E}$ by forgetting the semi-abelian part. Let α be the canonical map $[Y \to J] \to Y[1]$, then we have $\mathcal{E} = \alpha^*(\mathcal{E}')$. It follows that

$$\mathcal{E} \cdot \mathcal{F} = \alpha^*(\mathcal{E}') \cdot \mathcal{F} \sim \mathcal{E}' \cdot \alpha_*(\mathcal{F}) = \mathcal{E}' \cdot \mathcal{F}',$$

where $\mathcal{F}'$ denotes $\alpha_*(\mathcal{F})$. By induction, the class represented by $\mathcal{F}'$ comes from the group $\mathrm{Ext}^i_{\mathcal{C}_k}({}_nG, Y)$ for some positive integer n, thus we can choose its representative $\mathcal{F}'' \in \mathrm{Ext}^i_{\mathcal{C}_k}({}_nG, Y)$. Hence the class represented by $\mathcal{E} \cdot \mathcal{F}$ can also be represented by $\mathcal{E}' \cdot \mathcal{F}''$, which represents an element in the group $\mathrm{Ext}^{i+1}_{\mathcal{C}^f_\Gamma}({}_nG, L')$. This shows the morphism Φ_{i+1} is surjective, hence Φ_i's are all surjective for all $i > 0$.

Since the groups $\mathrm{Ext}^i_{\mathcal{C}_k}({}_nG, L')$ are torsion for all $i \geq 0$ and $n \in \mathbb{N}$, so are the groups $\mathrm{Ext}^i_{\mathcal{M}}(G, L'[1])$ for all $i \geq 0$. $\square$

Lemma III.6. *The group* $\mathrm{Ext}^2_{\mathcal{M}}(L[1], G')$ *is torsion.*

Proof. Take any element in $\mathrm{Ext}^2_{\mathcal{M}}(L[1], G')$, and we can write it as the product $\mathcal{E} \cdot \mathcal{F}$ for $\mathcal{E} \in \mathrm{Ext}^1_{\mathcal{M}}([X \to J], G')$ and $\mathcal{F} \in \mathrm{Ext}^1_{\mathcal{M}}(L[1], [X \to J])$ for some 1-motive $[X \to J]$.

Applying the functor $\mathrm{Hom}_{\mathcal{M}}(L[1], -)$ to the canonical short exact sequence

$$0 \to J \xrightarrow{\alpha} [X \to J] \xrightarrow{\beta} X[1] \to 0,$$

we get a long exact sequence

$$\mathrm{Ext}^1_{\mathcal{M}}(L[1],J) \to \mathrm{Ext}^1_{\mathcal{M}}(L[1],[X\to J]) \to \mathrm{Ext}^1_{\mathcal{M}}(L[1],X[1]).$$

By III.4, $\mathrm{Ext}^1_{\mathcal{M}}(L[1],X[1])$ is a torsion group, hence there exists a positive integer n such that $\beta_*(n\mathcal{F}) = n\beta_*(\mathcal{F}) = 0$. Then $n\mathcal{F}$ lies in the image of α_*, i.e. $n\mathcal{F} = \alpha_*(\mathcal{F}')$ for some $\mathcal{F}' \in \mathrm{Ext}^1_{\mathcal{M}}(L[1],J)$. Now we get

$$n\mathcal{E}\cdot\mathcal{F} = \mathcal{E}\cdot(n\mathcal{F}) = \mathcal{E}\cdot\alpha_*(\mathcal{F}') \sim \alpha^*(\mathcal{E})\cdot\mathcal{F}' = \mathcal{E}'\cdot\mathcal{F}',$$

where $\mathcal{E}'$ denotes $\alpha^*(\mathcal{E})$. By I.25 (c), $\mathcal{E}'$ can be represented by a short exact sequence of semiabelian varieties

$$0 \to G' \to J_1 \to J \to 0.$$

By I.25 (b), $\mathcal{F}'$ can be represented by a short exact sequence of complexes as follows:

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] \arrow[d] & L \arrow[r, equal] \arrow[d] & L \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & J \arrow[r, equal] & J \arrow[r] & 0 \arrow[r] & 0. \end{tikzcd}$$

Consider the short exact sequence $0 \to T_1 \xrightarrow{u} J \xrightarrow{v} B \to 0$ associated to J, where T_1 is the torus part of J and B is the maximal abelian quotient of J. Then we have a long exact sequence

$$\mathrm{Ext}^1_k(B,G') \xrightarrow{v^*} \mathrm{Ext}^1_k(J,G') \xrightarrow{u^*} \mathrm{Ext}^1_k(T_1,G').$$

The group $\mathrm{Ext}^1_k(T_1,G')$ being torsion implies that there exists a positive integer n' such that $n'u^*(\mathcal{E}') = 0$. Hence $n'\mathcal{E}'$ equals $v^*(\mathcal{E}'')$ for some $\mathcal{E}'' \in \mathrm{Ext}^1_k(B,G')$. So we have

$$nn'(\mathcal{E}\cdot\mathcal{F}) \sim n'(\mathcal{E}'\cdot\mathcal{F}') = v^*(\mathcal{E}'')\cdot\mathcal{F}' \sim \mathcal{E}''\cdot v_*(\mathcal{F}') = \mathcal{E}''\cdot\mathcal{F}'',$$

where $\mathcal{F}''$ denotes $v_*(\mathcal{F}')$. So we can express $\mathcal{E}''$ and $\mathcal{E}''$ into the following forms

$$\mathcal{E}'' \quad \begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & G' \arrow[r] & J_2 \arrow[r] & B \arrow[r] & 0 \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] \arrow[d] & L \arrow[r, equal] \arrow[d] & L \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & B \arrow[r, equal] & B \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd} \quad \mathcal{F}''$$

for some semiabelian variety J_2.

Consider the short exact sequence $0 \to T_2 \xrightarrow{\lambda} G' \xrightarrow{\mu} A \to 0$ associated to G', where T_2 is the torus part of G' and A is the maximal abelian quotient of G'. Then we have a long exact sequence

$$\mathrm{Ext}_k^1(B, T_2) \xrightarrow{\lambda_*} \mathrm{Ext}_k^1(B, G') \xrightarrow{\mu_*} \mathrm{Ext}_k^1(B, A).$$

The group $\mathrm{Ext}_k^1(B, A)$ being torsion implies that there exists a positive integer n'' such that $n''\mu_*(\mathcal{E}'') = 0$. Hence $n''\mathcal{E}''$ lies in the image of λ_*, i.e. $n''\mathcal{E}'' = \lambda_*(\mathcal{E}''')$ for some $\mathcal{E}''' \in \mathrm{Ext}_k^1(B, T_2)$. So we get

$$n''(\mathcal{E}'' \cdot \mathcal{F}'') = \lambda_*(\mathcal{E}''') \cdot \mathcal{F}'' = \lambda_*(\mathcal{E}''' \cdot \mathcal{F}'').$$

It follows that to prove $\mathcal{E} \cdot \mathcal{F}$ is torsion, it's enough to prove $\mathcal{E}''' \cdot \mathcal{F}''$ is torsion. And we can express $\mathcal{E}'''$ and $\mathcal{F}''$ as follows:

$$\begin{tikzcd}
\mathcal{E}''' & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
& 0 \arrow[r] & T_2 \arrow[r] & J_3 \arrow[r] & B \arrow[r] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & L \arrow[r, equal] \arrow[d] & L \arrow[r] \arrow[d] & 0 & \mathcal{F}'' \\
0 \arrow[r] & B \arrow[r, equal] & B \arrow[r] & 0 \arrow[r] & 0 &
\end{tikzcd}$$

for some semiabelian variety J_3.

Applying the functor $\mathrm{Hom}_{k-\mathrm{fppf}}(L, -)$ to the short exact sequence

$$0 \to T_2 \to J_3 \to B \to 0,$$

we get a long exact sequence

$$0 \to \mathrm{Hom}_{k-\mathrm{fppf}}(L, T_2) \to \mathrm{Hom}_{k-\mathrm{fppf}}(L, J_3) \to \mathrm{Hom}_{k-\mathrm{fppf}}(L, B) \xrightarrow{\delta} \mathrm{Ext}^1_{k-\mathrm{fppf}}(L, T_2).$$

Claim. The group $\mathrm{Ext}^1_{k-\mathrm{fppf}}(L, T_2)$ is torsion.

Proof of the Claim: Decompose the group L as $0 \to L_{\mathrm{t}} \to L \to L_{\mathrm{tf}} \to 0$, where L_{t} is the torsion part of L and L_{tf} is the quotient. From this, we get a long exact sequence

$$\mathrm{Ext}^1_{k-\mathrm{fppf}}(L_{\mathrm{tf}}, T_2) \to \mathrm{Ext}^1_{k-\mathrm{fppf}}(L, T_2) \to \mathrm{Ext}^1_{k-\mathrm{fppf}}(L_{\mathrm{t}}, T_2).$$

The third term is obviously torsion, so we can assume L to be torsion-free. Take a finite Galois extension K/k such that L (resp. T_2) becomes isomorphic to $\mathbb{Z}^r$ (resp. $\mathbb{G}_m^s$) for some $r \in \mathbb{N}$ (resp. $s \in \mathbb{N}$) over K. By the local-global spectral sequence for Exts

$$E_2^{i,j} = H^i_{\mathrm{fppf}}(\mathrm{Spec} k, \mathcal{E}xt^j(L, T_2)) \Rightarrow \mathrm{Ext}^{i+j}_{k-\mathrm{fppf}}(L, T_2),$$

we have an exact sequence

$$0 \to H^1_{\mathrm{fppf}}(\mathrm{Spec}k, \mathcal{H}om(L, T_2)) \to \mathrm{Ext}^1_{k-\mathrm{fppf}}(L, T_2) \to H^0_{\mathrm{fppf}}(\mathrm{Spec}k, \mathcal{E}xt^1(L, T_2)).$$

The fppf-sheaf $\mathcal{E}xt^1(L, T_2))$ is zero by [5, lem. 1.1.6.], so we have

$$H^1_{\mathrm{fppf}}(\mathrm{Spec}k, \mathcal{H}om(L, T_2)) \cong \mathrm{Ext}^1_{k-\mathrm{fppf}}(L, T_2).$$

Let $X(T_2)$ denotes the groups of characters of the torus T_2, then we have

$$\begin{aligned} \mathcal{H}om(L, T_2)) &\cong \mathcal{H}om(L, \mathcal{H}om(X(T_2), \mathbb{G}_m)) \\ &\cong \mathcal{H}om(L \otimes_{\mathbb{Z}} X(T_2), \mathbb{G}_m), \end{aligned}$$

whence the fppf-sheaf $\mathcal{H}om(L, T_2))$ is represented by the torus $\mathcal{H}om(L \otimes_{\mathbb{Z}} X(T_2), \mathbb{G}_m)$. Fppf-torsors under smooth group schemes are the same as the étale-torsors, it follows that

$$\begin{aligned} H^1_{\mathrm{fppf}}(\mathrm{Spec}k, \mathcal{H}om(L, T_2)) &= H^1_{\text{ét}}(\mathrm{Spec}k, \mathcal{H}om(L, T_2)) \\ &= H^1(k, \mathcal{H}om(L, T_2)). \end{aligned}$$

The Hochschild-Serre spectral sequence gives an exact sequence

$$0 \to H^1(\mathrm{Gal}(K/k), \mathcal{H}om(L, T_2)) \to H^1(k, \mathcal{H}om(L, T_2)) \to H^1(K, \mathcal{H}om(L, T_2))^{\mathrm{Gal}(K/k)}.$$

The torus $\mathcal{H}om(L, T_2)$ becomes isomorphic to $\mathbb{G}_m^{rs}$, and $H^1(K, \mathbb{G}_m)$ equals zero by Hilbert's 90, then the torsioness of the group $\mathrm{Ext}^1_{k-\mathrm{fppf}}(L, T_2) \cong H^1(k, \mathcal{H}om(L, T_2))$ follows from the torsiness of the group $H^1(\mathrm{Gal}(K/k), \mathcal{H}om(L, T_2))$ which is a standard result of group cohomology of finite groups. $\square$

Now we go back to the proof of III.6. By I.25 (b), we have $\mathrm{Ext}^1_{\mathcal{M}}(L[1], B) = \mathrm{Hom}_k(L, B)$, and we assume that $\mathcal{F}''$ corresponds to $f \in \mathrm{Hom}_k(L, B)$. By the claim, there exists $m \in \mathbb{N}$ such that $\delta(mf) = m\delta(f) = 0$. Then the homomorphism mf can be lifted to $\tilde{f} \in \mathrm{Hom}_k(L, J_3)$.

Let ι be the canonical embedding $B \to [L \xrightarrow{mf} B]$, and $\mathcal{E}''''$ be the following short exact sequence of complexes

$$\begin{tikzcd} \mathcal{E}'''' & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & L \arrow[r, equal] \arrow[d, "\tilde{f}"'] & L \arrow[r] \arrow[d, "mf"'] & 0 \\ & 0 \arrow[r] & T_2 \arrow[r] & J_3 \arrow[r] & B \arrow[r] & 0. \end{tikzcd}$$

Then we have that $\mathcal{E}''' = \iota^*(\mathcal{E}'''')$. Since mf corresponds to $m\mathcal{F}''$, so we get that

$$m(\mathcal{E}''' \cdot \mathcal{F}'') = \mathcal{E}''' \cdot (m\mathcal{F}'') = (\iota^*\mathcal{E}'''') \cdot (m\mathcal{F}'') \sim \mathcal{E}'''' \cdot (\iota_*(m\mathcal{F}'')).$$

Note that ι is part of the extension

$$m\mathcal{F}'' : 0 \to B \xrightarrow{\iota} [L \xrightarrow{mf} B] \to L[1] \to 0,$$

whence the extension $\iota_*(m\mathcal{F}'')$ is trivial. It follows that $\mathcal{E}''' \cdot \mathcal{F}''$ is torsion, so is $\mathcal{E} \cdot \mathcal{F}$. □

Theorem III.7. *The groups* $\mathrm{Ext}^i_{\mathcal{M}}(M, M')$ *are all torsion for* $i \geq 2$.

Proof. Combining III.2, III.4, III.5 and III.6, it's easy to see that the group $\mathrm{Ext}^2_{\mathcal{M}}(M, M')$ is torsion by diagram chasing. Hence the groups $\mathrm{Ext}^i_{\mathcal{M}}(M, M')$ are torsion for all $i \geq 2$. □

Remark III.8. The result in theorem III.7 is not true for $i = 1$ in general. For example, if taking the base field k to be the field of rational numbers $\mathbb{Q}$, M to be an elliptic curve E with a non-torsion rational point, M' to be the multiplicative group $\mathbb{G}_m$, then we have $\mathrm{Ext}^1_{\mathcal{M}}(E, \mathbb{G}_m) \cong \mathrm{Ext}^1_k(E, \mathbb{G}_m) \cong \hat{E}(\mathbb{Q}) \cong E(\mathbb{Q})$ is not a torsion group, where $\hat{E}$ is the elliptic curve dual to E, which is canonical isomorphic to E.

Proof of III.1 in characteristic zero case: Let d be the cohomological dimension of the absolute Galois group of the base field k. To prove the theorem, we are going to prove the following first.

(1) $\mathrm{Ext}^{d+i}_{\mathcal{M}}(L[1], L'[1]) = 0$ for $i \geq 2$;

(1')$\mathrm{Ext}^{d+1}_{\mathcal{M}}(L[1], L'[1]) = 0$ provided that L is torsion free and L' is torsion;

(2) There exists a finite Galois module M which is killed by some prime number l, such that

$$\begin{aligned}\mathrm{Ext}^{d+1}_{\mathcal{M}}(\mathbb{Z}/l\mathbb{Z}[1], M[1]) &= \mathrm{Ext}^{d+1}_{\mathcal{C}_k}(\mathbb{Z}/l\mathbb{Z}, M)\\ &= H^d(k, \mathrm{Ext}^1_{\mathbb{Z}}((\mathbb{Z}/l\mathbb{Z}, M))\\ &\neq 0;\end{aligned}$$

(3) $\mathrm{Ext}^{d+2}_{\mathcal{M}}(G, L'[1]) = 0$;

(4) $\mathrm{Ext}^{d+i}_{\mathcal{M}}(L[1], G') = 0$ for $i \geq 2$;

(5) $\mathrm{Ext}^{d+2}_{\mathcal{M}}(G, G') = 0$.

For (1) and (2), we are going to use the spectral sequence 3.1 again. Recall that the exact sequence 3.2,

$$0 \to H^{d+i}(k, \mathrm{Hom}_{\mathbb{Z}}(L, L'))/\mathrm{im} d_2^{d+i-2,1} \to \mathrm{Ext}^{d+i}_{\mathcal{C}_k}(L, L') \to H^{d+i-1}(k, \mathrm{Ext}^1_{\mathbb{Z}}(L, L')) \to .$$

Since the Galois module $\mathrm{Ext}^1_{\mathbb{Z}}(L, L')$ is torsion and the cohomological dimension of the absolute Galois group of k is d, both $H^{d+i}(k, \mathrm{Hom}_{\mathbb{Z}}(L, L'))$ and $H^{d+i-1}(k, \mathrm{Ext}^1_{\mathbb{Z}}(L, L'))$ are zero for $i \geq 2$, hence the group $\mathrm{Ext}^{d+i}_{\mathcal{M}}(L[1], L'[1]) \cong \mathrm{Ext}^{d+i}_{\mathcal{C}_k}(L, L')$ is zero. This proves (1).

To prove (1'), just notice that both $H^{d+1}(k, \mathrm{Hom}_{\mathbb{Z}}(L, L'))$ and $H^d(k, \mathrm{Ext}^1_{\mathbb{Z}}(L, L')))$ are zero in this case, hence we have $\mathrm{Ext}^{d+1}_{\mathcal{M}}(L[1], L'[1]) = 0$ by considering the exact sequence above.

For (2), since the cohomological dimension of the absolute Galois group of k is d, by II.10 (c) we can find a simple Galois module M killed by some prime number l (which is finte) such that $H^d(k, M) \neq 0$. The short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{l} \mathbb{Z} \to \mathbb{Z}/l\mathbb{Z} \to 0$$

of discrete G-modules gives a short exact sequence

$$0 \to \mathrm{Hom}(\mathbb{Z}, M) \to \mathrm{Ext}^1(\mathbb{Z}/l\mathbb{Z}, M) \to \mathrm{Ext}^1(\mathbb{Z}, M) \to 0$$

of discrete G-modules. Here to avoid confusion, we remind that the notation $\mathrm{Ext}^i(-,-)$ denotes a discrete G-module, for its definition see the end of II.28. Since $\mathrm{Hom}(\mathbb{Z}, M)$ is canonically isomorphic to M as discrete G-modules, and $\mathrm{Ext}^1(\mathbb{Z}, M)$ is zero, so we have $\mathrm{Ext}^1(\mathbb{Z}/l\mathbb{Z}, M) \cong M$. By proposition II.26 (1) and the spectral sequence II.32, we have exact sequence

$$\begin{aligned}
&\to H^{d+1}(k, \mathrm{Hom}(\mathbb{Z}/l\mathbb{Z}, M)) \to \mathrm{Ext}^{d+1}_{\mathcal{C}_k}(\mathbb{Z}/l\mathbb{Z}, M) \to H^d(k, \mathrm{Ext}^1(\mathbb{Z}/l\mathbb{Z}, M)) \\
&\to H^{d+2}(k, \mathrm{Hom}(\mathbb{Z}/l\mathbb{Z}, M)) \to .
\end{aligned}$$

The group $\mathrm{Hom}(\mathbb{Z}/l\mathbb{Z}, M)$ being torsion implies

$$H^{d+1}(k, \mathrm{Hom}(\mathbb{Z}/l\mathbb{Z}, M)) = H^{d+2}(k, \mathrm{Hom}(\mathbb{Z}/l\mathbb{Z}, M)) = 0,$$

hence we have

$$\mathrm{Ext}^{d+1}_{\mathcal{C}_k}(\mathbb{Z}/l\mathbb{Z}, M) \cong H^d(k, \mathrm{Ext}^1(\mathbb{Z}/l\mathbb{Z}, M)) \cong H^d(k, M) \neq 0.$$

Applying the functor $\mathrm{Hom}_{\mathcal{M}}(-, L'[1])$ to the short exact sequence

$$0 \to G \xrightarrow{n} G \to {}_nG[1] \to 0$$

in $\mathcal{M}$, we get a long exact sequence

$$\to \mathrm{Ext}^{d+2}_{\mathcal{M}}({}_nG[1], L'[1]) \to \mathrm{Ext}^{d+2}_{\mathcal{M}}(G, L'[1]) \xrightarrow{n} \mathrm{Ext}^{d+2}_{\mathcal{M}}(G, L'[1]) \to \mathrm{Ext}^{d+3}_{\mathcal{M}}({}_nG[1], L'[1]).$$

We know from (1) that both $\mathrm{Ext}^{d+2}_{\mathcal{M}}({}_nG[1], L'[1])$ and $\mathrm{Ext}^{d+3}_{\mathcal{M}}({}_nG[1], L'[1])$ are zero, hence the multiplication-by-n map is an isomorphism on $\mathrm{Ext}^{d+2}_{\mathcal{M}}(G, L'[1])$. Let n vary in $\mathbb{N}$, then we know the group $\mathrm{Ext}^{d+2}_{\mathcal{M}}(G, L'[1])$ is actually a $\mathbb{Q}$-vector space. But we already know that $\mathrm{Ext}^{d+2}_{\mathcal{M}}(G, L'[1])$ is a torsion group by III.7, which cannot be a nontrivial $\mathbb{Q}$-vector space, so it has to be the zero group. This proves (3).

For the proof to (4), it suffices to prove it for the cases L being torsion and being torsion free separately. If L is torsion, i.e. there exists a positive integer m such that $m \cdot L = 0$. Applying the functor $\mathrm{Hom}_{\mathcal{M}}(L[1], -)$ to the short exact sequence

$$0 \to G' \xrightarrow{m} G' \to {}_mG'[1] \to 0$$

in $\mathcal{M}$, we get a long exact sequence

$$\to \mathrm{Ext}^{d+i}_{\mathcal{M}}(L[1], G') \xrightarrow{m} \mathrm{Ext}^{d+i}_{\mathcal{M}}(L[1], G') \to \mathrm{Ext}^{d+i}_{\mathcal{M}}(L[1], {}_mG'[1]) \to .$$

The fact $m \cdot L = 0$ implies that the multiplication-by-m map on $\mathrm{Ext}^{d+i}_{\mathcal{M}}(L[1], G')$ is just the zero map. And by (1), we have that $\mathrm{Ext}^{d+i}_{\mathcal{M}}(L[1], {}_mG'[1]) = 0$ for $i \geq 2$. Hence the group $\mathrm{Ext}^{d+i}_{\mathcal{M}}(L[1], G')$ is zero for $i \geq 2$. Now we come to the torsion free case. In this case, similarly as above, we have the long exact sequence

$$\mathrm{Ext}^{d+i-1}_{\mathcal{M}}(L[1], {}_nG'[1]) \to \mathrm{Ext}^{d+i}_{\mathcal{M}}(L[1], G') \xrightarrow{n} \mathrm{Ext}^{d+i}_{\mathcal{M}}(L[1], G') \to \mathrm{Ext}^{d+i}_{\mathcal{M}}(L[1], {}_nG'[1])$$

for each positive integer n. Both $\mathrm{Ext}^{d+i-1}_{\mathcal{M}}(L[1], {}_nG'[1])$ and $\mathrm{Ext}^{d+i}_{\mathcal{M}}(L[1], {}_nG'[1])$ are zero for $i \geq 2$ by (1) and (1'), hence the multiplication-by-n map on $\mathrm{Ext}^{d+i}_{\mathcal{M}}(L[1], G')$ must be an isomorphism. It follows that $\mathrm{Ext}^{d+i}_{\mathcal{M}}(L[1], G')$ is actually a $\mathbb{Q}$-vector space. Since it is also a torsion group, it is forced to be the trivial group. This proves (4).

Applying the functor $\mathrm{Hom}_{\mathcal{M}}(-, G')$ to the short exact sequence

$$0 \to G \xrightarrow{n} G \to {}_nG[1] \to 0$$

in $\mathcal{M}$, we get a long exact sequence

$$\mathrm{Ext}^{d+2}_{\mathcal{M}}({}_nG[1], G') \to \mathrm{Ext}^{d+2}_{\mathcal{M}}(G, G') \xrightarrow{n} \mathrm{Ext}^{d+2}_{\mathcal{M}}(G, G') \to \mathrm{Ext}^{d+3}_{\mathcal{M}}({}_nG[1], G').$$

By (4), both the leftmost side and the rightmost side are zero, hence the multiplication-by-n map is an isomorphism for all positive integers n. Then we know $\mathrm{Ext}^{d+2}_{\mathcal{M}}(G, G')$ is a $\mathbb{Q}$-vector space and also a torsion abelian group, hence it has to be zero. So (5) is proven.

At last, combining (1), (3), (4) and (5), it's a standard conclusion of diagram chasing that the group $\mathrm{Ext}^{d+2}_{\mathcal{M}}(M, M')$ is zero. Hence, the homological dimension of the category $\mathcal{M}$ is at most $d+1$. And (2) tells us that the homological dimension of $\mathcal{M}$ is at least $d+1$, so it is $d+1$ indeed. □

3.2 Proof of theorem III.1 in positive characteristic case

Throughout this section, we assume the characteristic of k is positive and denote it by p.

First let's describe homomorphisms in the category $\mathcal{M}$. Recall that the category $\mathcal{M}$ is defined to be the category ${}^t\mathcal{M}_1[1/p]$, i.e. the localisation of ${}^t\mathcal{M}_1$ with respect to the multiplicative system $\{M \xrightarrow{p^i} M | M \in {}^t\mathcal{M}_1, i \geq 0\}$. Note that this multiplicative system is both right and left, hence it admits calculus of both right and left fractions. Given any two 1-motives $M, M' \in \mathcal{M}$, any homomorphism between M and M' can be represented either by the diagram (corresponding to right multiplicative structure)

(3.3)
$$\begin{tikzcd}
M & & & \\
& M \arrow[ul, "p^i"'] & & M' \\
& & \tilde{M} \arrow[ul, "\text{q.i.}"] \arrow[ur] &
\end{tikzcd}$$

in ${}^t\mathcal{M}_1^{\mathrm{eff}}$, or by the diagram (corresponding to left multiplicative structure)

(3.4)

$$\begin{tikzcd}
 & & & M' \arrow[dl, "p^i"] \\
M & & M' & \\
 & \tilde{M} \arrow[ul, "\text{q.i.}"'] \arrow[ur] & &
\end{tikzcd}$$

in ${}^t\mathcal{M}_1^{\mathrm{eff}}$. And the diagram 3.3 can be also rewritten as

(3.5)

$$\begin{tikzcd}
M & & & \\
 & \tilde{M} \arrow[ul, "\text{q.i.}"'] & & M' \\
 & & \tilde{M} \arrow[ul, "p^i"'] \arrow[ur] &
\end{tikzcd}$$

since the multiplication-by-p^i map commutes with any map.

Now we turn to the analogues of I.25 and I.26 in positive characteristic case.

Proposition III.9. *Notations as in I.25, we have the followings canonical isomorphisms:*

(a) $\mathrm{Ext}^1_{\mathcal{C}_k}(L, L') \otimes \mathbb{Z}[1/p] \xrightarrow{\cong} \mathrm{Ext}^1_{\mathcal{M}}(L[1], L'[1])$;

(b) $\mathrm{Hom}_k(L, G') \otimes \mathbb{Z}[1/p] \xrightarrow{\cong} \mathrm{Ext}^1_{\mathcal{M}}(L[1], G')$;

(c) $\mathrm{Ext}^1_k(G, G') \otimes \mathbb{Z}[1/p] \xrightarrow{\cong} \mathrm{Ext}^1_{\mathcal{M}}(G, G')$.

Here the morphism in (a) is given by sending $[\mathcal{E}] \otimes p^i \in \mathrm{Ext}^1_{\mathcal{C}_k}(L, L') \otimes \mathbb{Z}[1/p]$ to the extension class represented by $p^i\mathcal{E}$, and $p^i\mathcal{E}$ can be taken as either the pushout of $\mathcal{E}$ along the multiplication map by p^i on L' in the category $\mathcal{M}$, or the pullback of $\mathcal{E}$ along the multiplication map by p^i on L in the same category (note that p^i is also a morphism in $\mathcal{M}$ for negative i according to 3.3 and 3.4). The morphisms in (b) and (c) are similar to the morphism in (a).

Proof. The injectivity in (a), (b) and(c) is just an immediate consequence of the following. Any short exact sequence $0 \to A \to B \to C \to 0$ in the left side become zero in the right side means that it admits an section in $\mathcal{M}$. By 3.3, this amouts to giving a commutative diagram of the form

$$\begin{tikzcd}
 & & & C \arrow[dl, "u"'] \arrow[d, "p^t"] & \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0.
\end{tikzcd}$$

Note that the map u can be chosen as an effective map in all the three cases. And such a diagram means exactly that the extension becomes zero after tensoring with $\mathbb{Z}_l$.

Now we turn to the proof of the surjectivity in (a), (b) and (c). Recall in I.20, any short exact sequence of 1-motives can be represented up to isomorphism by a short exact sequence of complexes in which each term is an effective 1-motive.

By the same argument as in the proof of (a) of I.25, given any element Θ in $\mathrm{Ext}^1_{\mathcal{M}}(L[1], L'[1])$, it can be represented up to isomorphism by an short exact sequence

$$\mathcal{E}: \quad 0 \to \tilde{L}' \to Y \to \tilde{L} \to 0$$

in $\mathcal{C}_k$ with $\tilde{L}'[1]$ (resp. $\tilde{L}[1]$) isomorphic to $L'[1]$ (resp. $L[1]$) in $\mathcal{M}$. By the diagram 3.4, the isomorphism from $\tilde{L}'[1]$ to $L'[1]$ can be expressed as

$$p^{-i}\alpha : \tilde{L}'[1] \xrightarrow{\alpha} L'[1] \xleftarrow{p^i} L'[1]$$

for some nonnegative integer i. By the diagram 3.3, the isomorphism from $L[1]$ to $\tilde{L}[1]$ can be expressed as

$$\beta p^{-j} : L[1] \xleftarrow{p^j} L[1] \xrightarrow{\beta} \tilde{L}[1]$$

for some nonnegative integer j. Note that both α and β are forced to be isomorphisms in $\mathcal{M}$. So we get the following diagram:

$$\begin{tikzcd}
 & & & L & \\
 & & & L \arrow[u, "p^j"'] \arrow[d, "\beta"'] & \\
0 \arrow[r] & \tilde{L}' \arrow[r] \arrow[d, "\alpha"'] & Y \arrow[r] & \tilde{L} \arrow[r] & 0 \\
 & L' & & & \\
 & L' \arrow[u, "p^i"'] & & &
\end{tikzcd}$$

Then Θ can also be represented by $\alpha_*\beta^*(\mathcal{E}) \in \mathrm{Ext}^1_{\mathcal{C}_k}(L, L')$ up to isomorphism, and it actually lies in $\mathrm{Ext}^1_{\mathcal{C}_k}(L, L') \otimes p^{-i-j}$. This shows the surjectivity in (a).

By the same argument as in the proof of (b) of I.25, any element Θ of $\mathrm{Ext}^1_{\mathcal{M}}(L[1],G')$ can be represented up to isomorphism by a short exact sequence of complexes

$$\begin{tikzcd} \mathcal{E}: 0 \arrow[r] & 0 \arrow[r] \arrow[d] & \tilde{L} \arrow[r, equal] \arrow[d, "f"'] & \tilde{L} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \tilde{G}' \arrow[r, equal] & \tilde{G}' \arrow[r] & 0 \arrow[r] & 0. \end{tikzcd}$$

with $\tilde{L}[1]$ (resp. $\tilde{G}'$) isomorphic to $L[1]$ (resp. G') in $\mathcal{M}$. To have such a short exact sequence amounts to have the homomorphism f. By 3.4 and 3.3, the isomorphisms from $\tilde{G}'$ to G' and from $L[1]$ to $\tilde{L}[1]$ can be expressed as

$$p^{-i}\alpha : \tilde{G}' \xrightarrow{\alpha} G' \xleftarrow{p^i} G'$$

and

$$\beta p^{-j} : L[1] \xleftarrow{p^j} L[1] \xrightarrow{\beta} \tilde{L}[1]$$

respectively, for some i and j nonnegative integers, $\alpha \in \mathrm{Hom}_k(\tilde{G}', G')$ and $\beta \in \mathrm{Hom}_k(L, \tilde{L})$. Then Θ can also be represented by the short exact sequence

$$\begin{tikzcd} \alpha_*\beta^*(\mathcal{E}): 0 \arrow[r] & 0 \arrow[r] \arrow[d] & L \arrow[r, equal] \arrow[d, "\alpha f\beta"'] & L \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & G' \arrow[r, equal] & G' \arrow[r] & 0 \arrow[r] & 0. \end{tikzcd}$$

up to isomorphisms in $\mathcal{M}$. Hence Θ corresponds to $\alpha f\beta \otimes p^{-i-j}$, and this shows the surjectivity of (b).

The surjectivity follows the same strategy as in the proof of the surjectivity in (a) and (b), with help of the proof of (c) in I.25. $\square$

Proposition III.10. *Let notations be as in I.26. Then we have a canonical isomorphism*

$$\Phi : \varinjlim_n \mathrm{Ext}^1_{\mathcal{C}_k}({}_nG, L') \otimes \mathbb{Z}[1/p] \longrightarrow \mathrm{Ext}^1_{\mathcal{M}}(G, L'[1]),$$

hence $\mathrm{Ext}^1_{\mathcal{M}}(G, L'[1])$ *is a torsion group.*

Proof. The injectivity is obvious, so we are left to show the surjectivity. Given an element Θ in $\mathrm{Ext}^1_{\mathcal{M}}(G, L'[1])$, it can be represented up to isomorphism by a short exact sequence of complexes

$$\begin{tikzcd} 0 \arrow[r] & \tilde{L}' \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & 0 \arrow[r] & \tilde{G} \arrow[r, equal] & \tilde{G} \arrow[r] & 0 \end{tikzcd}$$

with $\tilde{L}'[1]$ (resp. $[F \to \tilde{G}]$) isomorphic to $L'[1]$ (resp. G) in $\mathcal{M}$. By 3.4, the isomorphism from $\tilde{L}'[1]$ to $L'[1]$ can be expressed as $p^{-i}\alpha : \tilde{L}' \xrightarrow{\alpha} L' \xleftarrow{p^i} L'$ for some nonnegative integer i, and $\alpha \in \mathrm{Hom}_k(\tilde{L}', L')$. By 3.3, the isomorphism from G to $[F \to \tilde{G}]$ can be expressed as $\beta^{-j}p^{-j} : G \xleftarrow{p^j} G \xleftarrow{\beta} [F \to \tilde{G}]$ for some nonnegative integer j, and β the canonical quasi-isomorphism from $[F \to \tilde{G}]$ to G. Since the multiplication-by-n isogenies are cofinal, we can assume $[F \to \tilde{G}]$ to be $[{}_nG \to G]$. So we get the following commutative diagram:

$$\begin{tikzcd}
 & & & G \\
 & & & G \arrow[u, "p^j"'] \\
0 \arrow[r] & \tilde{L}'[1] \arrow[r] \arrow[d, "\alpha"'] & {[X \to G]} \arrow[r] & {[{}_nG \to G]} \arrow[r] \arrow[u, "\beta"'] & 0 \\
 & L'[1] & & & \\
 & L'[1] \arrow[u, "p^i"'] & & &
\end{tikzcd}$$

We denote the extension $0 \to \tilde{L}' \to X \to {}_nG \to 0$ by $\mathcal{E}$, then Θ can also be represented by $\alpha_*\mathcal{E} \in \mathrm{Ext}^1_{\mathcal{C}_k}({}_nG, L')$ up to isomorphisms. It follows Θ lies in $\mathrm{Ext}^1_{\mathcal{C}_k}({}_nG, L') \otimes p^{j-i}$. This shows the surjectivity of Φ. $\square$

Lemma III.11. *Notations as in III.3, then for each positive integer i, there is a canonical isomorphism $\Psi_i : \mathrm{Ext}^i_{\mathcal{C}_k}(L, L') \otimes \mathbb{Z}[1/p] \to \mathrm{Ext}^i_{\mathcal{M}}(L[1], L'[1])$, sending $[\mathcal{E}] \otimes p^i$ to the extension class $[p^i\mathcal{E}]$, with $\mathcal{E}$ an i-extension*

$$0 \to L' \to L_1 \to \cdots \to L_i \to L \to 0$$

in $\mathcal{C}_k$. Note for negative integer i, $p^i\mathcal{E}$ makes sense as before, since p^i is a homomorphism in $\mathcal{M}$.

Proof. The $i = 1$ case is just III.9 (a). So we are left to prove it for the case $i > 1$.

For the surjectivity, we only need to make a small modification to the proof of the surjectivity in III.3. The upper left term of $\mathcal{E}$ is not necessary L' anymore, but some $\tilde{L}'$ which is isomorphic to L' in $\mathcal{M}$. Hence by III.9 (a), $\mathcal{E}'$ comes from an element of $\mathrm{Ext}^1_{\mathcal{C}_k}(Y, L') \otimes \mathbb{Z}[1/p]$. At the same time, by induction $\mathcal{F}'$ can be represented by an element of $\mathrm{Ext}^i_{\mathcal{C}_k}(L, Y) \otimes \mathbb{Z}[1/p]$. Hence the extension class represented by $\mathcal{E} \cdot \mathcal{F}$ comes from an element of $\mathrm{Ext}^i_{\mathcal{C}_k}(L, L') \otimes \mathbb{Z}[1/p]$.

And the proof for injectivity is also a slight modification to the proof of the injectivity in III.3. The homomorphism α in III.3 should be changed to

$$[X \to J] \xleftarrow{p^i} [X \to J] \xleftarrow{s} [\tilde{X} \to \tilde{J}] \xrightarrow{\alpha} Y[1]$$

by 3.3, with s an quasi-isomorphism and i some nonnegative integer. Then the extension class represented by $(\alpha p^{-i} s^{-1})^*(\mathcal{E}) = (s^{-1})^*(p^{-i}\alpha^*\mathcal{E})$ being zero in $\mathcal{M}$ implies that the class represented by $\mathcal{E}' := \alpha^*\mathcal{E}$ is zero in $\mathcal{M}$. The extension $\mathcal{E}'$ can be expressed as a short exact sequence of complexes

$$\mathcal{E}': \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & L' & \longrightarrow & L_1 \times_Y \tilde{X} & \longrightarrow & \tilde{X} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & 0 & \longrightarrow & \tilde{J} & = & \tilde{J} & \longrightarrow & 0. \end{array}$$

And the extension $\mathcal{E}'$ being trivial implies that the extension $\mathcal{E}'$ admits a section on the right side, and such a section must be of the form

$$[\tilde{X} \to \tilde{J}] \xleftarrow{p^j} [\tilde{X} \to \tilde{J}] \xleftarrow{\text{q.i.}} ? \to [L_1 \times_Y \tilde{X} \to \tilde{J}]$$

for some nonnegative integer j, and some quasi-isomorphism to $[\tilde{X} \to \tilde{J}]$. We can replace $[\tilde{X} \to \tilde{J}]$ by "?" , i.e. making this quasi-isomorphism to be identity. Then we have that $\tilde{X}[1] \xleftarrow{p^j} \tilde{X}[1] \to L_1 \times_Y \tilde{X}[1]$ is an section to the extension

$$\mathcal{E}'': \qquad 0 \to L'[1] \to L_1 \times_Y \tilde{X}[1] \to \tilde{X}[1] \to 0$$

in $\mathcal{M}$. It follows that p^j kills the extension $0 \to L' \to L_1 \times_Y \tilde{X} \to \tilde{X} \to 0$ in $\mathcal{C}_k$. Then the injectivity follows as in the proof of III.3. $\square$

Corollary III.12. *The groups* $\mathrm{Ext}^i_{\mathcal{M}}(L[1], L'[1])$ *are all torsion for* $i > 0$.

Proof. This is an consequence of III.4 and III.11. $\square$

Lemma III.13. *Both the group* $\mathrm{Ext}^2_{\mathcal{M}}(G, G')$ *and the group* $\mathrm{Ext}^2_{\mathcal{M}}(L[1], G')$ *are torsion.*

Proof. These follow from the proof of III.2 and III.6 with the help of III.9 and III.10. $\square$

Lemma III.14. *There is a canonical epimorphism*

$$\Phi^i : \varinjlim_n \mathrm{Ext}^i_{\mathcal{C}_k}({}_nG, L') \otimes \mathbb{Z}[1/p] \longrightarrow \mathrm{Ext}^i_{\mathcal{M}}(G, L'[1]),$$

for each $i > 0$. In particular, the groups $\mathrm{Ext}^i_{\mathcal{M}}(G, L'[1])$ are all torsion for $i \geq 0$.

Proof. This can be proven by in exactly the same strategy as in III.5 with the help of III.9 (a) and III.10. □

Now we are ready to prove the positive characteristic version of III.7 and the positive characteristic case of our main theorem III.1.

Theorem III.15. *The groups $\mathrm{Ext}^i_{\mathcal{M}}(M, M')$ are all torsion for $i \geq 2$.*

Proof. The proof is the same as in III.7. □

Proof of theorem III.1 in positive characteristic case: This follows from the proof 3.1 with the help of III.15. □

3.3 Torsioness of Yoneda extension groups

By III.6 and III.15, the group $\mathrm{Ext}^2_{\mathcal{M}}(M, M')$ is torsion, hence so are the groups $\mathrm{Ext}^i_{\mathcal{M}}(M, M')$ for $i > 1$.

The group of homomorphisms between two lattices is not necessarily torsion, for example if both of them have constant torsion-free part. The group of homomorphisms between two abelian varieties is a finitely generated free abelian group unless it's zero. The group of homomorphism between tori $\mathbb{G}_m$ and $\mathbb{G}_m$ is isomorphic to $\mathbb{Z}$. Hence by I.24, the group $\mathrm{Hom}_{\mathcal{M}}(M, M')$ could be far from being torsion.

Now we are going to discuss the torsioness of the group $\mathrm{Ext}^1_{\mathcal{M}}(M, M')$. We first study its torsioness for both M and M' being concentrated only in one degree. By I.26 and III.10, we know that $\mathrm{Ext}^1_{\mathcal{M}}(G, L'[1])$ is a torsion group. By I.25 (a), III.9 (a) and III.4, we know the group $\mathrm{Ext}^1_{\mathcal{M}}(L[1], L'[1])$ is torsion. We need to investigate the two cases left.

Proposition III.16. *Let T (resp. T') and A (resp. A') be the torus and abelian variety corresponding to G (resp. G') given by Chevalley decomposition as algebraic groups over k. Then the following hold.*

(1) *The groups $\mathrm{Ext}^1_k(T, T')$, $\mathrm{Ext}^1_k(A, A')$ and $\mathrm{Ext}^1_k(T, A')$ are torsion.*

(2) *Let $\hat{A}$ be the dual abelian variety of A, and K be a finite Galois field extension of k such that T' becomes isomorphic to $\mathbb{G}_m^r$ over K for some positive integer r. Then the group* $\mathrm{Ext}_k^1(A, T')$ *is isomorphic to* $H^0(\mathrm{Gal}(K/k), M)$, *where the* $\mathrm{Gal}(K/k)$*-module M is the abelian group $\hat{A}(K)^r$ with action not only on the group $\hat{A}(K)$ but also on the components induced from the structure of the torus T'.*

Proof.

(1) By the spectral sequence II.29, we have the assocaited exact sequence of lower degree

$$0 \to H^1(k, \mathrm{Hom}_{\bar{k}}(T_{\bar{k}}, T'_{\bar{k}})) \to \mathrm{Ext}_k^1(T, T') \to H^0(k, \mathrm{Ext}_{\bar{k}}^1(T_{\bar{k}}, T'_{\bar{k}})).$$

The first term being a Galois cohomology group is obviously torsion. The third term is zero, since any torus over $\bar{k}$ is isomorphic to a direct sum of a finite number of copies of $\mathbb{G}_m$ and the group $\mathrm{Ext}_{\bar{k}}^1(\mathbb{G}_m, \mathbb{G}_m) \cong \mathrm{Ext}_{\mathcal{C}_k}^1(\mathbb{Z}, \mathbb{Z})$ equals zero, where the $\mathbb{Z}$ inside the bracket stands for the character group of $\mathbb{G}_m$. Hence the group $\mathrm{Ext}_k(T, T')$ is torsion.

The group $\mathrm{Ext}_k(A, A')$ being torsion is a simple conclusion of the Poincaré complete reducibility theorem of abelian varieties. More explicitly, any extension of A by A' is isogenous to the product $A \times A'$.

Given any extension $0 \to A' \to J \to T \to 0$, consider the decomposition of J given by Chavelley theorem as shown in the column of the diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & \tilde{T} \arrow[d, "u"] & & \\
0 \arrow[r] & A' \arrow[r, "s"] & J \arrow[r, "t"] \arrow[d, "v"] & T \arrow[r] & 0 \\
 & & \tilde{A} \arrow[d] & & \\
 & & 0. & &
\end{tikzcd}$$

Then the morphism vs has to be an isogeny, otherwise there will be an nontrivial abelian subvariety of A' which lies in the torus $\tilde{T}$ which is impossible. Then it is an easy exercise involving the universal property of pushout to conclude that the

extension $(vs)_*(J)$ splits. It follows that $n \cdot J$ also splits for $n = \deg(vu)$. Hence the group $\mathrm{Ext}_k(T, A')$ is torsion.
(2) By the spectral sequence II.29, we have the associated exact sequence of lower degree

$$0 \to H^1(k, \mathrm{Hom}_{\bar{k}}(A_{\bar{k}}, T'_{\bar{k}})) \to \mathrm{Ext}^1_k(A, T') \to H^0(k, \mathrm{Ext}^1_{\bar{k}}(A_{\bar{k}}, T'_{\bar{k}})) \to H^2(k, \mathrm{Hom}_{\bar{k}}(A_{\bar{k}}, T'_{\bar{k}})).$$

Since there is no nontrivial morphism from a complete variety to an affine variety, the Galois module $\mathrm{Hom}_{\bar{k}}(A_{\bar{k}}, T'_{\bar{k}})$ is zero. So we get $\mathrm{Ext}^1_k(A, T') \cong H^0(k, \mathrm{Ext}^1_{\bar{k}}(A_{\bar{k}}, T'_{\bar{k}}))$. By [32, Chap. VII, 16], the Galois module $\mathrm{Ext}^1_{\bar{k}}(A_{\bar{k}}, T'_{\bar{k}})$ is isomorphic to $\hat{A}(\bar{k})^r$ as an abelian group. Since $T'_K \cong \mathbb{G}^r_m$, we also have that $H^0(k, \mathrm{Ext}^1_{\bar{k}}(A_{\bar{k}}, T'_{\bar{k}})) \cong H^0(\mathrm{Gal}(K/k), M)$ where $M := H^0(K, \mathrm{Ext}^1_{\bar{k}}(A_{\bar{k}}, T'_{\bar{k}}))$ is isomorphic to $\hat{A}(K)^r$ as an abelian group. □

The above proposition tells us that the non-torsion elements of $\mathrm{Ext}^1_{\mathcal{M}}(G, G') \cong \mathrm{Ext}^1_k(G, G')$ (or $\mathrm{Ext}^1_k(G, G') \otimes \mathbb{Z}[1/p]$ in positive characteristic case) can only come from $\mathrm{Ext}^1_k(A, T')$, which is essentially related to the rational points of the dual abelian variety of A. So we have the following theorem.

Theorem III.17. *Notation as in III.16, then the group $\mathrm{Ext}^1_{\mathcal{M}}(G, G')$ is torsion if the group $H^0(\mathrm{Gal}(K/k), M)$ is torsion, where the Galois module M is isomorphic to $\hat{A}(K)^r$ as an abelian group.*

Proof. This is an easy corollary of the proposition III.16. □

Let L_{tor} be the torsion part of L and $L_{\mathrm{tf}} = L/L_{\mathrm{tor}}$. It is obvious that the group $\mathrm{Ext}^1_{\mathcal{M}}(L_{\mathrm{tor}}[1], G') \cong \mathrm{Hom}_k(L_{\mathrm{tor}}, G')$ is torsion. We have an exact sequence

$$0 \to \mathrm{Hom}_k(L_{\mathrm{tf}}, G') \to \mathrm{Hom}_k(L, G') \to \mathrm{Hom}_k(L_{\mathrm{tor}}, G'),$$

hence to understand the torsioness of the group $\mathrm{Ext}^1_{\mathcal{M}}(L[1], G') \cong \mathrm{Hom}_k(L, G')$, it suffices to study the torsioness of the group $\mathrm{Hom}_k(L_{\mathrm{tf}}, G')$.

Let K' be a finite Galois extension of the base field k such that $(L_{\mathrm{tf}})_{K'}$ is isomorphic to $\mathbb{Z}^r$ with $r \in \mathbb{N}$. Then we have

$$\begin{aligned}
\mathrm{Hom}_k(L_{\mathrm{tf}}, G') &= H^0(\mathrm{Gal}(K'/k), \mathrm{Hom}_{K'}(\mathbb{Z}^r, G'_{K'})) \\
&= H^0(\mathrm{Gal}(K'/k), \mathrm{Hom}_{K'}(\mathbb{Z}, G'_{K'})^r) \\
&\cong H^0(\mathrm{Gal}(K'/k), G'(K')^r).
\end{aligned}$$

Here the $\mathrm{Gal}(K'/k)$-module structure on $G'(K')^r$ is induced from L_{tf} and G', i.e. for $\sigma \in \mathrm{Gal}(K'/k)$, $(P_1, \cdots, P_r) \in G'(K')^r$, the action is given by

$$\sigma \cdot (P_1, \cdots, P_r) = (\sigma^{-1} \cdot P_1, \cdots, \sigma^{-1} \cdot P_r)\rho(\sigma),$$

where $\rho : \mathrm{Gal}(K'/k) \to \mathrm{GL}_r(\mathbb{Z})$ is the group homomorphism corresponding to the $\mathrm{Gal}(K'/k)$-module structure of L_{tf}. So we get the following theorem.

Theorem III.18. *The group* $\mathrm{Ext}^1_{\mathcal{M}}(L[1], G')$ *is torsion if the group* $H^0(\mathrm{Gal}(K'/k), G'(K')^r)$ *is torsion. In particular, if* L_{tf} *is constant and the group* $G'(k)$ *is torsion, then* $\mathrm{Ext}^1_{\mathcal{M}}(L[1], G')$ *is torsion.*

We can assume $K \supset K'$. If not, we can always enlarge the field K up to a finite extension. Now combining all of the above, we have the following theorem.

Theorem III.19. *If both* $H^0(\mathrm{Gal}(K/k), M)$ *(as in III.16) and* $H^0(\mathrm{Gal}(K/k), G'(K)^r)$ *(as in III.18) are torsion, then the group* $\mathrm{Ext}^1_{\mathcal{M}}(M, M')$ *is torsion.*

Proof. This is just an easy conclusion of diagram chasing. □

Remark III.20. We can see from the discussion that the torsioness of the first Yoneda extension group is closely related to the arithmetic structure of the group varieties of $\hat{A}$ and G'. It could be nontorsion, for example if letting $k = \mathbb{Q}$, $M = \mathbb{Z}[1]$, $M' = E$ for some elliptic curve E over $\mathbb{Q}$ such that $E(\mathbb{Q})$ is not a torsion group, then the group $\mathrm{Ext}^1_{\mathcal{M}}(M, M') \cong E(\mathbb{Q})$ is not torsion. Another example to this is $M = E$ and $M' = \mathbb{G}_m$.

3.4 Over special fields

In this section, we will make further study on the Yoneda extension groups for finite fields and number fields.

First let's consider the finite fields case. Finite fields are simple in two aspects. Firstly, the varieties defined over finite fields have only finitely many rational points. According to III.20, this give us torsioness property for the bifunctor $\mathrm{Ext}^1_{\mathcal{M}}(-, -)$. Secondly, the cohomological dimension of the absolute Galois group of a finite field is one.

Theorem III.21. *Let k be $\mathbb{F}_q$, the finite field with $q = p^r$ elements, where p is some prime number. Then we have that the homological dimension of the category $\mathcal{M}$ is 2, and the groups $\mathrm{Ext}^i_{\mathcal{M}}(M, M')$ are all torsion for any two 1-motives M, M' and $i > 0$.*

Proof. The absolute Galois group is isomorphic to the profinite group $\hat{\mathbb{Z}}$ with a topological generator the Frobenius automorphism of k. According to [33, Chap. I, 3.2], the cohomological dimension of $\hat{\mathbb{Z}}$ is one, hence by III.1 the homological dimension of the abelian category $\mathcal{M}$ is just 2. Since our base field is finite, both the varieties $\hat{A}$ and G', as in III.19, have only finitely many K-rational points. It follows that $\mathrm{Ext}^1_{\mathcal{M}}(M, M')$ is torsion by III.19. And we know that $\mathrm{Ext}^2_{\mathcal{M}}(M, M')$ is torsion by III.15. $\square$

In fact, we can go further than III.21 for Yoneda 1-extension. In [20], Milne proved that the group $\mathrm{Ext}^1_k(A, A')$ is finite for abelian varieties A, A' over finite field k. We are going to mimic his strategy to prove that the group $\mathrm{Ext}^1_{\mathcal{M}}(M, M')$ is finite for any two 1-motives over a finite field. The key point is the the duality pairing in [20, lem. 2, lem. 3]. And the key ingredient is the finiteness of the groups $\mathrm{Ext}^1_k(\mathbb{G}_m, A')$ and $\varinjlim_n \mathrm{Ext}^1_{\mathcal{C}_k}({}_nG, \mathbb{Z})$ for abelian variety A and semiabelian variety G.

We are going to use some of Milne's notations in [20]. To avoid confusion, let's first list those notations. Let k be a finite field of characteristic p, and l be a prime number (l is not necessarily different from p), write $\mathrm{Ext}^r_k(Z_1, Z_2)$ (resp. $\mathrm{Ext}^r_{k,v}(Z_1, Z_2)$) for the group of equivalence classes of r-fold Yoneda extensions of Z_1 by Z_2 in the category of algebraic group schemes over k (resp. of finite group schemes over k killed by l^v). If G is a semiabelian variety over k, we write $A(l^\infty)$ the l-divisible group $\varinjlim_v {}_{l^v}G$ of A , and T_lG the pro-l-group $\varprojlim_v l^vG$, which is essentially the Tate module of A when l is not equal to p. Also, if Z_1 or Z_2 is an ind-algebraic (resp. pro-algebraic) group schemes, then $\mathrm{Ext}^r_k(Z_1, Z_2)$ denotes the group formed in the category of ind-algebraic (resp. pro-algebraic) group schemes over k. Given any abelian group Z, we use the following notations:

$${}_nZ := \ker(Z \xrightarrow{n} Z), \qquad Z(l) := \varinjlim_v {}_{l^v}Z, \qquad T_lZ := \varprojlim_v {}_{l^v}Z.$$

Finally, if G and H are l-divisible groups over k, we write

$$\mathrm{Ext}^r_k(T_lG, H) := \varinjlim_v \mathrm{Ext}^r_{k,v}(G_v, H_v).$$

Note here $\mathrm{Ext}_k^r(T_lG, H)$ is really just a notation, since there is no suitable category in which one formulates the extensions group of a pro-algebraic group scheme by a ind-algebraic group scheme. Also note that r could be zero in the above notations, and we mean Hom by Ext^0.

Now we begin with the following theorem.

Theorem III.22. *Let k be as in III.21, and A be an abelian variety over k. Then the group*

$$\mathrm{Ext}_k^1(\mathbb{G}_m, A) \otimes \mathbb{Z}[1/p]$$

is dual to the group

$$\mathrm{Ext}_k^1(A, \mathbb{G}_m) \otimes \mathbb{Z}[1/p] \cong \hat{A}(k) \otimes \mathbb{Z}[1/p],$$

where $\hat{A}$ is the dual abelian variety of A. In particular $\mathrm{Ext}_k^1(\mathbb{G}_m, A)$ is a finite group.

Proof. We already know that the group $\mathrm{Ext}_k^1(\mathbb{G}_m, A)$ is torsion by III.16 (1), so we have that

$$\mathrm{Ext}_k^1(\mathbb{G}_m, A) \otimes \mathbb{Z}[1/p] = \bigcup_l \mathrm{Ext}_k^1(\mathbb{G}_m, A)(l),$$

where l varies over all prime numbers except p. First we prove that the group $\mathrm{Ext}_k^1(\mathbb{G}_m, A)(l)$ is finite. The torsion l-group $\mathrm{Ext}_k^1(\mathbb{G}_m, A)(l)$ can be written as $T \oplus (\mathbb{Q}_l/\mathbb{Z}_l)^t$, where $(\mathbb{Q}_l/\mathbb{Z}_l)^t$ is the l-divisible subgroup of $\mathrm{Ext}_k^1(\mathbb{G}_m, A)(l)$ (note that t is not necessary a finite number). Applying the functor $\mathrm{Hom}_k(-, A)$ to the short exact sequence $0 \to \mu_{l^n} \to \mathbb{G}_m \to \mathbb{G}_m \to 0$ gives us another short exact sequence

$$0 \to \mathrm{Hom}_k(\mathbb{G}_m, A) \otimes \mathbb{Z}/l^n\mathbb{Z} \to \mathrm{Hom}_k(\mu_{l^n}, A) \to {}_{l^n}\mathrm{Ext}_k(\mathbb{G}_m, A) \to 0.$$

From the fact that $\mathrm{Hom}_k(\mathbb{G}_m, A) = 0$, we get

$${}_{l^v}\mathrm{Ext}_k(\mathbb{G}_m, A) \cong \mathrm{Hom}_k(\mu_{l^v}, A) = \mathrm{Hom}_k(\mu_{l^v}, {}_{l^v}A)$$

is finite for each positive integer v, hence the group T is finite. We also have the expression

$$\mathrm{Ext}_k^1(\mathbb{G}_m, A)(l) \cong \varinjlim_v \mathrm{Hom}_k(\mu_{l^v}, {}_{l^v}A) = \mathrm{Hom}_k(\mathbb{Z}_l(1), A(l^\infty)). \tag{3.6}$$

Then the group $\mathrm{Ext}_k^1(\mathbb{G}_m, A)(l)$ is finite if and only if its l-divisible subgroup is zero. Applying the functor $\mathrm{Hom}_k(-, {}_{l^v}A)$ to the short exact sequence

$$0 \to \mathbb{Z}_l(1) \xrightarrow{l^v} \mathbb{Z}_l(1) \to \mu_{l^v} \to 0$$

gives us $\mathrm{Hom}_k(\mathbb{Z}_l(1), {}_{l^v}A) \cong \mathrm{Hom}_k(\mu_{l^v}, {}_{l^v}A)$. It follows that

$$\begin{aligned} T_l \operatorname{Ext}^1_k(\mathbb{G}_m, A) &= \varprojlim_v {}_{l^v}\operatorname{Ext}_k(\mathbb{G}_m, A) \\ &= \varprojlim_v \mathrm{Hom}_k(\mu_{l^v}, {}_{l^v}A) \\ &= \varprojlim_v \mathrm{Hom}_k(\mathbb{Z}_l(1), {}_{l^v}A) \\ &= \mathrm{Hom}_k(\mathbb{Z}_l(1), T_l(A)) \\ &= 0. \end{aligned}$$

This shows that $\operatorname{Ext}^1_k(\mathbb{G}_m, A)(l)$ is finite.

Applying the functor $\mathrm{Hom}_k(T_lA, -)$ to the short exact sequence

$$0 \to \mathbb{Z}_l(1) \xrightarrow{l^v} \mathbb{Z}_l(1) \to \mu_{l^v} \to 0,$$

we get the following short exact sequence

$$0 \to \operatorname{Ext}^1_k(T_lA, \mathbb{Z}_l(1)) \otimes \mathbb{Z}/l^v\mathbb{Z} \to \operatorname{Ext}^1_k(T_lA, \mu_{l^v}) \to {}_{l^v}\operatorname{Ext}^2_k(T_lA, \mathbb{Z}_l(1)) \to 0.$$

Taking inverse limit gives an exact sequence

$$0 \to \operatorname{Ext}^1_k(T_lA, \mathbb{Z}_l(1)) \to \varprojlim_v \operatorname{Ext}^1_k(T_lA, \mu_{l^v}) \to T_l\operatorname{Ext}^2_k(T_lA, \mathbb{Z}_l(1)) \to 0. \tag{3.7}$$

The non-degenerate pairing in [20, lemma 3.] gives a non-degenerate pairing

$$\varprojlim_v \operatorname{Ext}^1_k(T_lA, \mu_{l^v}) \times \mathrm{Hom}_k(\mathbb{Z}_l(1), A(l^\infty)) \longrightarrow \mathbb{Q}_l/\mathbb{Z}_l.$$

By 3.6, the finiteness of the group $\operatorname{Ext}^1_k(\mathbb{G}_m, A)(l)$ implies that $\mathrm{Hom}_k(\mathbb{Z}_l(1), A(l^\infty))$ is finite. Then the non-degeneracy of the above pairing implies that the group $\varprojlim_v \operatorname{Ext}^1_k(T_lA, \mu_{l^v})$ is finite. From the short exact sequence 3.7, it follows that $T_l\operatorname{Ext}^2_k(T_lA, \mathbb{Z}_l(1))$ is zero and

$$\operatorname{Ext}^1_k(T_lA, \mathbb{Z}_l(1)) \cong \varprojlim_v \operatorname{Ext}^1_k(T_lA, \mu_{l^v})$$

is dual to

$$\operatorname{Ext}^1_k(\mathbb{G}_m, A)(l) \cong \mathrm{Hom}_k(\mathbb{Z}_l(1), A(l^\infty)).$$

Also note that for $l \neq \mathrm{char}(k)$, we have

$$\operatorname{Ext}^1_k(T_lA, \mathbb{Z}_l(1)) = \operatorname{Ext}^1_{\mathbb{Z}_l[\Gamma]}(T_lA, \mathbb{Z}_l(1)),$$

and

$$\mathrm{Ext}^1_{\mathbb{Z}_l[\Gamma]}(T_l A, \mathbb{Z}_l(1)) \cong \mathrm{Ext}^1_k(A, \mathbb{G}_m) \otimes \mathbb{Z}_l \cong \hat{A}(k) \otimes \mathbb{Z}_l,$$

the later isomorphism comes from the Weil-Barsotti formula, see [27, chap. III, sec. 18]. Hence we can conclude that $\mathrm{Ext}^1_k(\mathbb{G}_m, A)(l)$ is dual to the group $\hat{A}(k) \otimes \mathbb{Z}_l$ for $l \neq \mathrm{char}(k)$. Let l varies over all prime numbers which are different from $p = \mathrm{char}(k)$, we get that the group $\mathrm{Ext}^1_k(\mathbb{G}_m, A) \otimes \mathbb{Z}[1/p]$ is dual to the group $\hat{A}(k) \otimes \mathbb{Z}[1/p]$, which is obviously a finite group. Since we already know $\mathrm{Ext}^1_k(\mathbb{G}_m, A)(l)$ is finite for all prime number l, in particular for p, then the finiteness of the group $\mathrm{Ext}^1_k(\mathbb{G}_m, A)$ follows. □

Theorem III.23. *Let k be as in III.21, and G be a semiabelian variety over k. Then the group*

$$\varinjlim_n \mathrm{Ext}^1_{\mathcal{C}_k}({}_nG, \mathbb{Z}) \otimes \mathbb{Z}[1/p]$$

is dual to the group

$$\mathrm{Hom}_k(\mathbb{Z}, G) \otimes \mathbb{Z}[1/p] \cong G(k) \otimes \mathbb{Z}[1/p],$$

where p denotes the characteristic of k as usual. In particular

$$\varinjlim_n \mathrm{Ext}^1_{\mathcal{C}_k}({}_nG, \mathbb{Z}) \otimes \mathbb{Z}[1/p]$$

is a finite group.

Proof. Let T denote the torsion group $\varinjlim_n \mathrm{Ext}^1_{\mathcal{C}_k}({}_nG, \mathbb{Z}) \otimes \mathbb{Z}[1/p]$, then we have $T = \cup_l T(l)$ with $T(l) = \varinjlim_v \mathrm{Ext}^1_{\mathcal{C}_k}({}_{l^v}G, \mathbb{Z})$ and l varying over all prime number except p. In order to have a duality between T and $G(k)$, it suffices to give a duality between $T(l)$ and $G(k)(l) = G(k) \otimes \mathbb{Z}_l$ for each prime number $l \neq p$.

Applying the functor $\mathrm{Hom}_{\mathcal{C}_k}({}_{l^v}G, -)$ to the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{l^v} \mathbb{Z} \to \mathbb{Z}/l^v\mathbb{Z} \to 0$$

gives

$$\mathrm{Ext}^1_{\mathcal{C}_k}({}_{l^v}G, \mathbb{Z}) \cong \mathrm{Hom}_{\mathcal{C}_k}({}_{l^v}G, \mathbb{Z}/l^v\mathbb{Z}) = \mathrm{Hom}_k({}_{l^v}G, \mathbb{Z}/l^v\mathbb{Z}).$$

Taking the direct limit, we get that

$$T(l) = \varinjlim_v \mathrm{Hom}_k({}_{l^v}G, \mathbb{Z}/l^v\mathbb{Z}) = \mathrm{Hom}_k(T_l G, \mathbb{Q}_l/\mathbb{Z}_l). \tag{3.8}$$

Similar as in III.22, we have a non-degenerate pairing

$$\varprojlim_{v} \operatorname{Ext}^1_k(\mathbb{Z}_l, {}_{l^v}G) \times \operatorname{Hom}_k(T_l G, \mathbb{Q}_l/\mathbb{Z}_l) \longrightarrow \mathbb{Q}_l/\mathbb{Z}_l. \tag{3.9}$$

Applying the functor $\operatorname{Hom}_{\mathcal{C}_k}(-, {}_{l^v}G)$ to the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{l^v} \mathbb{Z} \to \mathbb{Z}/l^v\mathbb{Z} \to 0$$

gives an exact sequence

$$0 \to \operatorname{Hom}_{\mathcal{C}_k}(\mathbb{Z}, {}_{l^v}G) \to \operatorname{Ext}^1_{\mathcal{C}_k}(\mathbb{Z}/l^v\mathbb{Z}, {}_{l^v}G) \to \operatorname{Ext}^1_{\mathcal{C}_k}(\mathbb{Z}, {}_{l^v}G) \to 0. \tag{3.10}$$

Applying the functor $\operatorname{Hom}_k(-, {}_{l^v}G)$ to the exact sequence

$$0 \to \mathbb{Z}_l \to \mathbb{Z}_l \to \mathbb{Z}/l^v\mathbb{Z} \to 0$$

gives an exact sequence

$$0 \to \operatorname{Hom}_k(\mathbb{Z}_l, {}_{l^v}G) \to \operatorname{Ext}^1_k(\mathbb{Z}/l^v\mathbb{Z}, {}_{l^v}G) \to \operatorname{Ext}^1_k(\mathbb{Z}_l, {}_{l^v}G) \to 0. \tag{3.11}$$

Since we have

$$\operatorname{Hom}_{\mathcal{C}_k}(\mathbb{Z}, {}_{l^v}G) = \operatorname{Hom}_k(\mathbb{Z}_l, {}_{l^v}G)$$

and

$$\operatorname{Ext}^1_{\mathcal{C}_k}(\mathbb{Z}/l^v\mathbb{Z}, {}_{l^v}G) \cong \operatorname{Ext}^1_k(\mathbb{Z}/l^v\mathbb{Z}, {}_{l^v}G),$$

the exact sequences 3.10 and 3.11 give that

$$\operatorname{Ext}^1_{\mathcal{C}_k}(\mathbb{Z}, {}_{l^v}G) \cong \operatorname{Ext}^1_k(\mathbb{Z}_l, {}_{l^v}G).$$

Applying the functor $\operatorname{Hom}_{k-\mathrm{fppf}}(\mathbb{Z}, -)$ to the short exact sequence

$$0 \to {}_{l^v}G \to G \xrightarrow{l^v} G \to 0$$

gives an exact sequence

(3.12)

$$\operatorname{Hom}_{k-\mathrm{fppf}}(\mathbb{Z}, G) \xrightarrow{l^v} \operatorname{Hom}_{k-\mathrm{fppf}}(\mathbb{Z}, G) \to \operatorname{Ext}^1_{k-\mathrm{fppf}}(\mathbb{Z}, {}_{l^v}G) \to \operatorname{Ext}^1_{k-\mathrm{fppf}}(\mathbb{Z}, G).$$

We have

$$\begin{aligned}
\operatorname{Ext}^1_{k-\mathrm{fppf}}(\mathbb{Z}, G) &\cong H^1_{\mathrm{fppf}}(\operatorname{Spec} k, G) \\
&\cong H^1_{\text{ét}}(\operatorname{Spec} k, G) \\
&\cong H^1(k, G) \\
&= 0.
\end{aligned}$$

Here the second isomorphim comes from the fact that the fppf-torsors over a smooth group scheme are the same as the étale-torsors over that group scheme, and the triviality of $H^1(k, G)$ is given by Lang's theorem (see [31, them. 20.3]). We also have that

$$\mathrm{Hom}_k(\mathbb{Z}, G) \cong G(k)$$

and

$$\mathrm{Ext}^1_{k-\mathrm{fppf}}(\mathbb{Z}, {}_{l^v}G) \cong \mathrm{Ext}^1_{\mathcal{C}_k}(\mathbb{Z}, {}_{l^v}G),$$

then the exact sequence 3.12 implies that

$$\mathrm{Ext}^1_k(\mathbb{Z}_l, {}_{l^v}G) \cong \mathrm{Ext}^1_{\mathcal{C}_k}(\mathbb{Z}, {}_{l^v}G) \cong \mathrm{Ext}^1_{k-\mathrm{fppf}}(\mathbb{Z}, {}_{l^v}G) \cong G(k) \otimes \mathbb{Z}/l^v\mathbb{Z}.$$

Taking inverse limit, we get

$$\varprojlim_v \mathrm{Ext}^1_{\mathcal{C}_k}(\mathbb{Z}_l, {}_{l^v}G) \cong G(k) \otimes \mathbb{Z}_l. \tag{3.13}$$

Combining 3.8 and 3.13, the required duality follows from the non-degenerate pairing 3.9. □

Theorem III.24. *Let k, M, M' be as in III.21, then the group $\mathrm{Ext}^1_{\mathcal{M}}(M, M')$ is finite.*

Proof. By diagram chasing, it's enough to check that the groups $\mathrm{Ext}^1_{\mathcal{M}}(L[1], L'[1])$, $\mathrm{Ext}^1_{\mathcal{M}}(L[1], G')$, $\mathrm{Ext}^1_{\mathcal{M}}(G, L'[1])$ and $\mathrm{Ext}^1_{\mathcal{M}}(G, G')$ are all finite.

Before going to the proof of the finiteness of the four groups, we first make a claim.

Claim. Let M be a Galois module over the field k, and k'/k be a finite Galois extension. Suppose that $H^0(k', M)$ is finite generated and $H^1(k', M)$ is finite, then we claim $H^1(k, M)$ is finite.

Consider the exact sequence of lower degree terms of Hochschild-Serre spectral sequence

$$0 \to H^1(\mathrm{Gal}(k'/k), M^{\mathrm{Gal}(\bar{\mathrm{k}}/\mathrm{k}')}) \to H^1(k, M) \to H^1(k', M)^{\mathrm{Gal}(k'/k)}.$$

The finiteness of the first term is a classical result of group cohomology of finite groups, then the finiteness of the middle term follows. This shows the claim.

By the spectral sequence II.32, we get an exact sequence

$$0 \to H^1(k, \mathrm{Hom}(L, L')) \to \mathrm{Ext}^1_k(L, L') \to H^0(k, \mathrm{Ext}^1(L, L')).$$

The term on the right side is obviously finite. The finiteness of the term on the left side is a consequence of the claim. Hence the middle term is finite. Then the finiteness of $\mathrm{Ext}^1_{\mathcal{M}}(L[1], L'[1])$ follows from III.9 (a).

By III.9 (b), to show that $\mathrm{Ext}^1_{\mathcal{M}}(L[1], G')$ is finite, it suffices to show $\mathrm{Hom}_k(L, G')$ is finite. This is an easy consequence of the finiteness of the group of rational points of a group scheme defined over a finite field.

By III.10, to show $\mathrm{Ext}^1_{\mathcal{M}}(G, L'[1])$ is finite, it suffices to show that $\varinjlim_n \mathrm{Ext}^1_k({}_nG, L')\otimes \mathbb{Z}[1/p]$ is finite. Let L'_{tor} be the torsion subgroup of L' and L'_{fr} be L'/L'_{tor}. Then we have an exact sequence

$$\to \varinjlim_n \mathrm{Ext}^1_k({}_nG, L'_{\mathrm{tor}})\otimes\mathbb{Z}[1/p] \to \varinjlim_n \mathrm{Ext}^1_k({}_nG, L')\otimes\mathbb{Z}[1/p] \to \varinjlim_n \mathrm{Ext}^1_k({}_nG, L'_{\mathrm{fr}})\otimes\mathbb{Z}[1/p].$$

The term on the left hand side equals $H^1(k, \mathrm{Hom}_{\bar{k}}({}_NG, L'_{\mathrm{tor}})) \otimes \mathbb{Z}[1/p]$ for some positive integer N with $N \cdot L'_{\mathrm{tor}} = 0$ by the argument in I.26, hence it is finite. So we can assume L' is torsion-free. Consider the exact sequence

$$\begin{aligned} 0 \to H^1(k, \mathrm{Hom}_{\bar{k}}({}_nG, L')) \to \mathrm{Ext}^1_k({}_nG, L') \to H^0(k, \mathrm{Ext}^1_{\bar{k}}({}_nG, L')) \\ \to H^2(k, \mathrm{Hom}_{\bar{k}}({}_nG, L')). \end{aligned}$$

Then $\mathrm{Hom}_{\bar{k}}({}_nG, L')) = 0$ implies that $\mathrm{Ext}^1_k({}_nG, L') \cong H^0(k, \mathrm{Ext}^1_{\bar{k}}({}_nG, L'))$. Take a finite extension k'/k such that $L'_{k'} \cong \mathbb{Z}^r$ for some integer r. We have that $\mathrm{Ext}^1_k({}_nG, L')) = \mathrm{Ext}^1_{k'}({}_nG, L')^{\mathrm{Gal}(k'/k)}$, and

$$\begin{aligned} \varinjlim_n \mathrm{Ext}^1_k({}_nG, L') \otimes \mathbb{Z}[1/p] &= \varinjlim_n \mathrm{Ext}^1_{k'}({}_nG, L')^{\mathrm{Gal}(k'/k)} \otimes \mathbb{Z}[1/p] \\ &= (\varinjlim_n \mathrm{Ext}^1_{k'}({}_nG, L') \otimes \mathbb{Z}[1/p])^{\mathrm{Gal}(k'/k)}. \end{aligned}$$

By III.23, the group $\varinjlim_n \mathrm{Ext}^1_{k'}({}_nG, L')\otimes\mathbb{Z}[1/p]$ is finite, hence so is $\varinjlim_n \mathrm{Ext}^1_k({}_nG, L')$.

By III.9 (c), in order to show that $\mathrm{Ext}^1_{\mathcal{M}}(G, G')$ is finite, it is enough to show that $\mathrm{Ext}^1_k(G, G')$ is finite. Moreover, it is enough to show that the groups $\mathrm{Ext}^1_k(T, T')$, $\mathrm{Ext}^1_k(T, A')$, $\mathrm{Ext}^1_k(A, T')$ and $\mathrm{Ext}^1_k(A, A')$ are all finite. The finiteness of $\mathrm{Ext}^1_k(T, T')$ follows from the corresponding result for their character groups. The finiteness of

$\mathrm{Ext}^1_k(A, T')$ is a consequence of the Weil-Barsotti formula with the help of the claim. The finiteness of $\mathrm{Ext}^1_k(A, A')$ is given in [20, theorem 3]. The finiteness of $\mathrm{Ext}^1_k(T, A')$ follows from III.22 with the help of the claim. □

Before going to the number field case, let's first look at the p-adic field case. Let k be a p-adic field, i.e. a finite field extension of $\mathbb{Q}_p$. By [33, Chap. II, Prop. 15], the absolute Galois group $\mathrm{Gal}(\bar{k}/k)$ has cohomological dimension 2 (actually even the strict cohomological dimension is 2, but we don't need it here). Hence by III.1, the homological dimension of the abelian category of 1-motives over k is 3. Now we turn to the case of k being a number field. Number fields are almost as good as p-adic fields. The absolute Galois group of k has l-cohomological dimension 2 for all prime number $l \neq 2$, but might have infinite 2-cohomological dimension depending on whether it is totally imaginary or not. Hence the cohomological dimension of $\mathrm{Gal}(\bar{k}/k)$ might be infinite. So we cannot use theorem III.1 directly to determine the homological dimension of the category $\mathcal{M}$. However, if we formally make the multiplication by 2 map invertible, i.e. kill the 2-torsion parts of the Hom groups, we can have that the homological dimension of the abelian category $\mathcal{M} \otimes_{\mathbb{Z}} \mathbb{Z}[1/2]$ is equal to 3, by using a modified version of III.1. Here the category $\mathcal{M} \otimes_{\mathbb{Z}} \mathbb{Z}[1/2]$ has the same objects as $\mathcal{M}$, but with

$$\mathrm{Hom}_{\mathcal{M}\otimes_{\mathbb{Z}}\mathbb{Z}[1/2]}(-,-) = \mathrm{Hom}_{\mathcal{M}}(-,-) \otimes_{\mathbb{Z}} \mathbb{Z}[1/2].$$

Theorem III.25. *Let k be a number field.*

(a) *If k is totally imaginary, then the category $\mathcal{M}$ has cohomological dimension* 3.

(b) *The category $\mathcal{M} \otimes_{\mathbb{Z}} \mathbb{Z}[1/2]$ over k has homological dimension* 3.

Proof. In case (a), the cohomological dimension of the absolute Galois group of k is 3, so (a) is just an easy conclusion of theorem III.1.

In case (b), since the multiplication by 2 map is an isomorphism in the abelian category $\mathcal{M} \otimes_{\mathbb{Z}} \mathbb{Z}[1/2]$, the groups $\mathrm{Ext}^i_{\mathcal{M}}(M, M')$ have no 2-torsion part for any two 1-motives M, M' and $i \geq 0$. At the same time, we still have that the groups $\mathrm{Ext}^i_{\mathcal{M}\otimes_{\mathbb{Z}}\mathbb{Z}[1/2]}(M, M')$ are all torsion for $i \geq 2$ by the same proof as for III.15, but without 2-torsion parts. Recall that the definition of p-cohomological dimension of a profinite group G in [33, Chap. I, 3.1], the p-cohomological dimension of G equals n if for every discrete torsion G-module A and every $q > n$, the p-primary component of $H^q(G, A)$ is null. Then it's easy to see that the proof of theorem III.1 also works here. $\square$

CHAPTER IV

Extensions of 1-motives and their l-adic realisations

Throughout this chapter, $M = [L \to G]$ and $M' = [L' \to G']$ will be two 1-motives over the base field k, and k will be either a number field or a finite field with $\Gamma = \mathrm{Gal}(\bar{k}/k)$ its absolute Galois group. We write $\mathcal{R}$ the abelian category of finitely generated $\mathbb{Z}_l$-module with continuous Galois action, with l some prime number which is different from the characteristic of the base field.

In section 1.3, we have defined the l-adic realisations of 1-motives, which lie in the category $\mathcal{R}$. Realisations are linearizations of geometric objects, they are easy to study and carry important information from the geometry. For example, there is a well-known theorem by Faltings (resp. by Tate in the finite field case), for reference see [8] (resp. [36]).

Theorem IV.1 (Faltings' Theorem)**.** *Let A, B be two abelian varieties defined over a field which is finitely generated over its prime subfield, let l be a prime number which is different from the characteristic of the base field. Then the canonical homomorphism*

$$\mathrm{Hom}_k(A, B) \otimes \mathbb{Z}_l \to \mathrm{Hom}_{\mathcal{R}}(T_l A, T_l B)$$

is an isomorphism.

We would like to know if the above result holds for 1-motives. Furthermore, what happens if we replace the Hom functor by the Ext^i for some positive integer i? In this chapter, we are going to investigate the homomorphisms

$$T_l : \mathrm{Ext}^i_{\mathcal{M}}(M, M') \otimes \mathbb{Z}_l \to \mathrm{Ext}^i_{\mathcal{R}}(T_l M, T_l M')$$

for $i = 0, 1$. The main results are the following two theorems.

Theorem IV.2. *The canonical homomorphism*

$$T_l : \mathrm{Hom}_{\mathcal{M}}(M, M') \otimes \mathbb{Z}_l \to \mathrm{Hom}_{\mathcal{R}}(T_l M, T_l M')$$

is an isomorphism.

Theorem IV.3. *The canonical homomorphism*

$$T_l : \mathrm{Ext}^1_{\mathcal{M}}(M, M') \otimes \mathbb{Z}_l \to \mathrm{Ext}^1_{\mathcal{R}}(T_l M, T_l M')$$

is injective.

We will first deal with four special cases in 4.1, 4.2, 4.3, and 4.4 respectively, then give the full proof of IV.2 and IV.3 in 4.5.

The last section is devoted to the case over finite fields, and the main result is theorem IV.23, in which we give an explicit description of the maps T_l for all Yoneda extension groups (actually just $i = 0, 1, 2$, since the $\mathrm{Ext}^i_{\mathcal{M}}$'s and the $\mathrm{Ext}^i_{\mathcal{R}}$'s vanish for $i > 2$).

4.1 The case $M = L[1]$ and $M' = L'[1]$

We begin with a lemma describing the structure of finite étale commutative group schemes over k.

Lemma IV.4. *Suppose X is a finite étale commutative group scheme over k. Then X can be written as $X = \oplus X(q)$, where q varies over all the prime numbers and $X(q)$ is a q-group scheme. In particular, $X(q)$ is the q-subgroup scheme of X.*

Proof. Since X is just a finite abelian group equipped with a Galois action, we can write X as $\oplus X(q)$ as an abelian group, with $X(q)$ the q-subgroup of X. Any automorphism of X as an abelian group must send $X(q)$ onto $X(q)$, hence X can be written as $\oplus X(q)$ as a finite group scheme. □

Lemma IV.5. *Let X, Y be two finitely generated locally constant sheaves for the étale topology over k, and X is finite. Then we have*

$$\mathrm{Ext}^i_k(X, Y) \cong \bigoplus_q \mathrm{Ext}^i_k(X(q), Y)$$

and

$$\mathrm{Ext}^i_k(Y,X) \cong \bigoplus_q \mathrm{Ext}^i_k(Y,X(q)),$$

hence we also have

$$\mathrm{Ext}^i_k(X,Y)\otimes \mathbb{Z}_l \cong \mathrm{Ext}^i_k(X(l),Y)$$

and

$$\mathrm{Ext}^i_k(Y,X)\otimes \mathbb{Z}_l \cong \mathrm{Ext}^i_k(Y,X(l))$$

for all $i \geq 0$.

Proof. Since the bifunctor $\mathrm{Ext}^i_k(-,-)$ commutes with direct sum, this is an immediate consequence of lemma IV.4. □

Proposition IV.6. *The canonical map*

$$\mathrm{Hom}_k(L,L')\otimes \mathbb{Z}_l \to \mathrm{Hom}_{\mathcal{R}}(L\otimes \mathbb{Z}_l, L'\otimes \mathbb{Z}_l)$$

is an isomorphism in the following cases:
(a)*Both* L *and* L' *are torsion;*
(b)L *is torsion-free, and* L' *is torsion;*
(c)*Both* L *and* L' *are torsion-free;*
(d)L' *is torsion-free.*

Proof. For (a), consider the following commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_k(L,L')\otimes \mathbb{Z}_l \arrow[r, "\cong"] \arrow[d] & \mathrm{Hom}_k(L(l),L'(l)) \arrow[d] \\
\mathrm{Hom}_{\mathcal{R}}(L\otimes \mathbb{Z}_l, L'\otimes \mathbb{Z}_l) \arrow[r, "\cong"] & \mathrm{Hom}_{\mathcal{R}}(L(l),L'(l)).
\end{tikzcd}$$

We have

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{R}}(L(l),L'(l)) &= (\mathrm{Hom}_{\mathbb{Z}_l}(L(l),L'(l)))^{\Gamma} \\
&= (\mathrm{Hom}_{\mathbb{Z}}(L(l),L'(l)))^{\Gamma} \\
&= \mathrm{Hom}_k(L(l),L'(l)),
\end{aligned}$$

then (a) follows from the above commutative diagram.

For (b), let N be a positive integer such that $N \cdot L' = 0$, then L being torsion-free gives a short exact sequence $0 \to L \xrightarrow{N} L \to L/NL \to 0$. So we get the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_k(L/NL, L') \otimes \mathbb{Z}_l \arrow[r] \arrow[d] & \mathrm{Hom}_k(L, L') \otimes \mathbb{Z}_l \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Hom}_{\mathcal{R}}(L/NL \otimes \mathbb{Z}_l, L' \otimes \mathbb{Z}_l) \arrow[r] & \mathrm{Hom}_{\mathcal{R}}(L \otimes \mathbb{Z}_l, L' \otimes \mathbb{Z}_l) \arrow[r] & 0.
\end{tikzcd}$$

Then the vertical map on the left hand side is an isomorphism by (a), hence so is the vertical map on the right hand side.

For (c), firstly it's easy to see that

$$\mathrm{Hom}_k(L, L') \otimes \mathbb{Z}_l = \mathrm{Hom}_{\mathbb{Z}}(L, L')^{\Gamma} \otimes \mathbb{Z}_l$$

and

$$\mathrm{Hom}_{\mathcal{R}}(L \otimes \mathbb{Z}_l, L' \otimes \mathbb{Z}_l) = (\mathrm{Hom}_{\mathbb{Z}_l}(L \otimes \mathbb{Z}_l, L' \otimes \mathbb{Z}_l))^{\Gamma}.$$

From the homological long exact sequence associated to the short exact sequence $0 \to M \xrightarrow{l^n} M \to M \otimes \mathbb{Z}/l^n\mathbb{Z} \to 0$, where M denotes the Galois module $\mathrm{Hom}_{\mathbb{Z}}(L, L')$, we get a short exact sequence

$$0 \to M^{\Gamma} \otimes \mathbb{Z}/l^n\mathbb{Z} \to (M \otimes \mathbb{Z}/l^n\mathbb{Z})^{\Gamma} \to {}_{l^n}(H^1(k, M)) \to 0$$

for each positive integer n. Since $M^{\Gamma} \otimes \mathbb{Z}/l^n\mathbb{Z}$ is finite for each positive integer n, the inverse system $\{M^{\Gamma} \otimes \mathbb{Z}/l^n\mathbb{Z}\}_{n \in \mathbb{N}}$ satisfies the Mittag-Leffler condition automatically. Hence passing to the projective limit, we get the following short exact sequence

$$0 \to M^{\Gamma} \otimes \mathbb{Z}_l \to (M \otimes \mathbb{Z}_l)^{\Gamma} \to \varprojlim_{l} {}_{l^n}(H^1(k, M)) \to 0.$$

Let k'/k be a finite Galois extension such that L and L' become constant over k', consider the inflation-restriction exact sequence

$$0 \to H^1(\mathrm{Gal}(k'/k), M) \to H^1(k, M) \to H^1(k', M)^{\mathrm{Gal}(k'/k)}.$$

Since M is a finitely generated free constant Galois module over k', we have $H^1(k', M)$ is zero. It follows that $H^1(k, M) \cong H^1(\mathrm{Gal}(k'/k), M)$ is a finite group. Hence we have $\varprojlim_l {}_{l^n}(H^1(k, M)) = 0$, and $\mathrm{Hom}_k(L, L') \otimes \mathbb{Z}_l = (\mathrm{Hom}_{\mathbb{Z}}(L, L') \otimes \mathbb{Z}_l)^{\Gamma}$. So it is left to show $(\mathrm{Hom}_{\mathbb{Z}}(L,L')\otimes\mathbb{Z}_l)^\Gamma \cong (\mathrm{Hom}_{\mathbb{Z}_l}(L\otimes\mathbb{Z}_l, L'\otimes\mathbb{Z}_l))^\Gamma$. Let $\{x_1,\cdots,x_m\}$ and $\{y_1,\cdots,y_n\}$ be bases of L and L' as abelian groups respectively. Then we have

$$\mathrm{Hom}_{\mathbb{Z}_l}(L\otimes\mathbb{Z}_l, L'\otimes\mathbb{Z}_l) = \bigoplus_{i,j}\mathbb{Z}_l f_{ij},$$

with $1\le i\le m$ and $1\le j\le n$, and f_{ij} is the homomorphism such that $f_{ij}(x_k) = \delta_{ik}y_j$ (here δ_{ik} denotes the Kronecker symbol). So we have a canonical isomorphism $\mathrm{Hom}_{\mathbb{Z}}(L,L')\otimes\mathbb{Z}_l \to \mathrm{Hom}_{\mathbb{Z}_l}(L\otimes\mathbb{Z}_l, L'\otimes\mathbb{Z}_l)$ of $\mathbb{Z}_l$-modules. Both sides of the map have natural Galois module structure inherited from L and L', by taking the Γ-invariants we get the isomorphism $\mathrm{Hom}_k(L,L')\otimes\mathbb{Z}_l \to \mathrm{Hom}_{\mathcal{R}}(L\otimes\mathbb{Z}_l, L'\otimes\mathbb{Z}_l)$.

For (d), consider the short exact sequence $0\to L_{\mathrm{tor}}\to L\to L_{\mathrm{tf}}\to 0$, where L_{tor} is the torsion subgroup of L and $L_{\mathrm{tf}} = L/L_{\mathrm{tor}}$. It's obvious that both $\mathrm{Hom}_k(L_{\mathrm{tor}}, L')$ and $\mathrm{Hom}_{\mathcal{R}}(L_{\mathrm{tor}}\otimes\mathbb{Z}_l, L'\otimes\mathbb{Z}_l)$ are zero. Then (d) is an easy consequence of (c) with the help of the following commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_k(L_{\mathrm{tf}}, L')\otimes\mathbb{Z}_l \arrow[r, hook] \arrow[d, "\cong"'] & \mathrm{Hom}_k(L,L')\otimes\mathbb{Z}_l \arrow[r] \arrow[d] & \mathrm{Hom}_k(L_{\mathrm{tor}}, L')\otimes\mathbb{Z}_l \arrow[d] \\
\mathrm{Hom}_{\mathcal{R}}(L_{\mathrm{tf}}\otimes\mathbb{Z}_l, L'\otimes\mathbb{Z}_l) \arrow[r, hook] & \mathrm{Hom}_{\mathcal{R}}(L\otimes\mathbb{Z}_l, L'\otimes\mathbb{Z}_l) \arrow[r] & \mathrm{Hom}_{\mathcal{R}}(L_{\mathrm{tor}}\otimes\mathbb{Z}_l, L'\otimes\mathbb{Z}_l).
\end{tikzcd}$$

□

Proposition IV.7. *Consider the canonical map*

$$\mathrm{Ext}^1_k(L,L')\otimes\mathbb{Z}_l \to \mathrm{Ext}^1_{\mathcal{R}}(L\otimes\mathbb{Z}_l, L'\otimes\mathbb{Z}_l).$$

We have the following.

(a) If both L and L' are torsion, then the above map is an isomorphism.

(b) If L is torsion free, and L' is torsion, then the above map is an isomorphism.

(c) If both L and L' are torsion-free, then the above map is injective.

(d) If L is torsion, and L' is torsion-free, then the above map is an isomorphism.

Proof. By lemma IV.5, we have

$$\mathrm{Ext}^1_k(L,L')\otimes\mathbb{Z}_l = \mathrm{Ext}^1_k(L(l), L'(l))$$

and

$$\mathrm{Ext}^1_{\mathcal{R}}(L\otimes\mathbb{Z}_l, L'\otimes\mathbb{Z}_l) = \mathrm{Ext}^1_{\mathcal{R}}(L(l), L'(l)).$$

Then (a) follows from the fact $\mathrm{Ext}^1_k(L(l), L'(l)) = \mathrm{Ext}^1_{\mathcal{R}}(L(l), L'(l))$.

For (b), we can assume L' to be an l-group scheme without loss of generality. Let r be some positive integer such that $l^r \cdot L' = 0$. The fact L being torsion-free gives a short exact sequence $0 \to L \xrightarrow{l^r} L \to L \otimes \mathbb{Z}/l^r\mathbb{Z} \to 0$, so we get a commutative diagram with exact rows

$$\begin{tikzcd}
\mathrm{Hom}_k(L, L') \otimes \mathbb{Z}/l^r\mathbb{Z} \arrow[r, hook] \arrow[d] & \mathrm{Ext}^1_k(L \otimes \mathbb{Z}/l^r\mathbb{Z}, L') \arrow[r, two heads] \arrow[d] & \mathrm{Ext}^1_k(L, L') \arrow[d] \\
\mathrm{Hom}_{\mathcal{R}}(L \otimes \mathbb{Z}_l, L') \otimes \mathbb{Z}/l^r\mathbb{Z} \arrow[r, hook] & \mathrm{Ext}^1_{\mathcal{R}}(L \otimes \mathbb{Z}_l, L') \arrow[r, two heads] & \mathrm{Ext}^1_{\mathcal{R}}(L \otimes \mathbb{Z}_l, L').
\end{tikzcd}$$

By proposition IV.6 (b), the vertical map on the left hand side is an isomorphism, and by (a) the vertical map in the middle is an isomorphism, so is the vertical one on the right hand side.

Now we come to the proof of (c). Since L is torsion-free, any extension of L by L' in the category of group schemes is just the abstract group $L \times L'$ with a $\mathrm{Gal}(\bar{k}/k)$-action which gives the corresponding Galois action of L and L' via restriction and quotient respectively. And this kind of action is classified by the Galois cohomology group $H^1(k, \mathrm{Hom}_k(L, L'))$. We also have $\mathrm{Ext}^1_{\mathcal{R}}(L \otimes \mathbb{Z}_l, L' \otimes \mathbb{Z}_l)$ is isomorphic to the continuous cochain cohomology

$$H^1_{\mathrm{cts}}(\Gamma, \mathrm{Hom}_{\mathcal{R}}(L \otimes \mathbb{Z}_l, L' \otimes \mathbb{Z}_l)) = H^1_{\mathrm{cts}}(\Gamma, \mathrm{Hom}_k(L, L') \otimes \mathbb{Z}_l)$$

by proposition II.21. The functor T_l sends a cocycle $(f_\sigma)_{\sigma \in \Gamma} \in H^1(k, \mathrm{Hom}_k(L, L'))$ to the continuous cocycle $(f_\sigma \otimes 1)_{\sigma \in \Gamma} \in H^1_{\mathrm{cts}}(\Gamma, \mathrm{Hom}_k(L, L') \otimes \mathbb{Z}_l)$. Let k'/k be a finite Galois extension of fields such that both L and L' are constant over k', then we get a commutative diagram with two rows coming from the inflation-restriction exact sequence

$$\begin{tikzcd}
0 \arrow[r] & H^1(H, S) \otimes \mathbb{Z}_l \arrow[r] \arrow[d] & H^1(k, S) \otimes \mathbb{Z}_l \arrow[r] \arrow[d] & H^1(k', S)^H \otimes \mathbb{Z}_l \arrow[d] \\
0 \arrow[r] & H^1(H, S \otimes \mathbb{Z}_l) \arrow[r] & H^1_{\mathrm{cts}}(\Gamma, S \otimes \mathbb{Z}_l) \arrow[r] & H^1_{\mathrm{cts}}(\mathrm{Gal}(\bar{k}/k'), S \otimes \mathbb{Z}_l)^H,
\end{tikzcd}$$

where S denotes the Galois module $\mathrm{Hom}_{\bar{k}}(L, L') = \mathrm{Hom}_{\mathbb{Z}}(L, L')$ and H denotes the Galois group $\mathrm{Gal}(k'/k)$. We need to show the vertical map in the middle is injective. Since we have $H^1(k', S) = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Gal}(\bar{k}/k'), S) = 0$, it's enough to show the vertical map on the left side is injective by the five lemma. It's a standard result that

$H^1(H, S)$ is finite, and $H^1(H, S \otimes \mathbb{Z}_l)$ can be shown to be finite by the similar reason in the context of $\mathbb{Z}_l$-modules instead of $\mathbb{Z}$-modules. Take some positive integer r such that l^r kills both $H^1(H, S) \otimes \mathbb{Z}_l$ and $H^1(H, S \otimes \mathbb{Z}_l)$, consider the short exact sequence $0 \to S \xrightarrow{l^r} S \to S \otimes \mathbb{Z}/l^r\mathbb{Z} \to 0$, then we get a commutative diagram with exact rows:

$$\begin{tikzcd}
S^{\mathrm{Gal}(k'/k)} \otimes \mathbb{Z}_l \arrow[r] \arrow[d] & (S \otimes \mathbb{Z}/l^r\mathbb{Z})^{\mathrm{Gal}(k'/k)} \otimes \mathbb{Z}_l \arrow[r] \arrow[d] & H^1(\mathrm{Gal}(k'/k), S) \otimes \mathbb{Z}_l \arrow[r] \arrow[d] & 0 \\
(S \otimes \mathbb{Z}_l)^{\mathrm{Gal}(k'/k)} \arrow[r] & (S \otimes \mathbb{Z}_l \otimes \mathbb{Z}/l^r\mathbb{Z})^{\mathrm{Gal}(k'/k)} \arrow[r] & H^1_{\mathrm{cts}}(\mathrm{Gal}(k'/k), S \otimes \mathbb{Z}_l) \arrow[r] & 0.
\end{tikzcd}$$

The vertical map on the left hand side is an isomorphism by proposition IV.6 (c), the vertical map in the middle is obviously an isomorphism, hence so is the vertical map on the right hand side by the five lemma. This finish the proof of (c).

For (d), without loss of generality we can assume that $l^n \cdot L = 0$ for some positive integer n. L' being torsion-free gives rise to a short exact sequence

$$0 \to L' \xrightarrow{l^n} L' \to L' \otimes \mathbb{Z}/l^n\mathbb{Z} \to 0.$$

So we get a commutative diagram with exact rows

$$\begin{tikzcd}
\mathrm{Hom}_k(L, L') \otimes \mathbb{Z}_l \arrow[r] \arrow[d] & \mathrm{Hom}_k(L, L' \otimes \mathbb{Z}/l^n\mathbb{Z}) \otimes \mathbb{Z}_l \arrow[r, two heads] \arrow[d] & \mathrm{Ext}^1_k(L, L') \otimes \mathbb{Z}_l \arrow[d] \\
\mathrm{Hom}_{\mathcal{R}}(L \otimes \mathbb{Z}_l, L' \otimes \mathbb{Z}_l) \arrow[r] & \mathrm{Hom}_{\mathcal{R}}(L \otimes \mathbb{Z}_l, L' \otimes \mathbb{Z}_l/l^n\mathbb{Z}) \arrow[r, two heads] & \mathrm{Ext}^1_{\mathcal{R}}(L \otimes \mathbb{Z}_l, L' \otimes \mathbb{Z}_l).
\end{tikzcd}$$

The vertical map on the left hand side (resp. in the middle) is an isomorphism by proposition IV.6 (d) (resp. (a)), hence the vertical map on the right hand side is an isomorphism too. $\square$

Theorem IV.8. *Theorem IV.2 and theorem IV.3 are true for* $M = L[1], M' = L'[1]$.

Proof. First we remark that there is no need to distinguish the positive characteristic case from characteristic zero case, since we have

$$\mathrm{Hom}_{\mathcal{M}}(L[1], L'[1]) \otimes \mathbb{Z}_l = \mathrm{Hom}_k(L, L') \otimes \mathbb{Z}[1/p] \otimes \mathbb{Z}_l = \mathrm{Hom}_k(L, L') \otimes \mathbb{Z}_l$$

and

$$\mathrm{Ext}^1_{\mathcal{M}}(L[1], L'[1]) \otimes \mathbb{Z}_l = \mathrm{Ext}^1_k(L, L') \otimes \mathbb{Z}[1/p] \otimes \mathbb{Z}_l = \mathrm{Ext}^1_k(L, L') \otimes \mathbb{Z}_l$$

in the positive characteristic case.

We are going to use the five lemma repeatedly. Let L_{tor} (resp. L'_{tor}) be the torsion subgroup of L (resp. L'), and L_{tf} (resp. L'_{tf}) be L/L_{tor} (resp. L'/L'_{tor}). The short exact sequence $0 \to L_{\mathrm{tor}} \to L \to L_{\mathrm{tf}} \to 0$ gives the following commutative diagram with exact rows:

$$\begin{array}{ccccc}
0 \longrightarrow & \mathrm{Hom}_{\mathcal{M}}(L_{\mathrm{tf}}[1], L'_{\mathrm{tor}}[1]) \otimes \mathbb{Z}_l & \longrightarrow & \mathrm{Hom}_{\mathcal{M}}(L[1], L'_{\mathrm{tor}}[1]) \otimes \mathbb{Z}_l \\
& \downarrow{\scriptstyle (1)} & & \downarrow{\scriptstyle (2)} \\
0 \longrightarrow & \mathrm{Hom}_{\mathcal{R}}(T_l L_{\mathrm{tf}}[1], T_l L'_{\mathrm{tor}}[1]) & \longrightarrow & \mathrm{Hom}_{\mathcal{R}}(T_l L[1], T_l L'_{\mathrm{tor}}[1])
\end{array}$$

$$\begin{array}{ccccc}
\longrightarrow & \mathrm{Hom}_{\mathcal{M}}(L_{\mathrm{tor}}[1], L'_{\mathrm{tor}}[1]) \otimes \mathbb{Z}_l & \longrightarrow & \mathrm{Ext}^1_{\mathcal{M}}(L_{\mathrm{tf}}[1], L'_{\mathrm{tor}}[1]) \otimes \mathbb{Z}_l \\
& \downarrow{\scriptstyle (3)} & & \downarrow{\scriptstyle (4)} \\
\longrightarrow & \mathrm{Hom}_{\mathcal{R}}(T_l L_{\mathrm{tor}}[1], T_l L'_{\mathrm{tor}}[1]) & \longrightarrow & \mathrm{Ext}^1_{\mathcal{R}}(T_l L_{\mathrm{tf}}[1], T_l L'_{\mathrm{tor}}[1])
\end{array}$$

$$\begin{array}{ccccc}
\longrightarrow & \mathrm{Ext}^1_{\mathcal{M}}(L[1], L'_{\mathrm{tor}}[1]) \otimes \mathbb{Z}_l & \longrightarrow & \mathrm{Ext}^1_{\mathcal{M}}(L_{\mathrm{tor}}[1], L'_{\mathrm{tor}}[1]) \otimes \mathbb{Z}_l \\
& \downarrow{\scriptstyle (5)} & & \downarrow{\scriptstyle (6)} \\
\longrightarrow & \mathrm{Ext}^1_{\mathcal{R}}(T_l L[1], T_l L'_{\mathrm{tor}}[1]) & \longrightarrow & \mathrm{Ext}^1_{\mathcal{R}}(T_l L_{\mathrm{tor}}[1], T_l L'_{\mathrm{tor}}[1]).
\end{array}$$

The homomorphism (1) and (3) are isomorphisms by proposition IV.6, the homomorphism (4) and (6) are also isomorphisms by proposition IV.7, hence by the five lemma (2) is an isomorphism and (5) is monomorphism. The short exact sequence $0 \to L'_{\mathrm{tor}} \to L' \to L'_{\mathrm{tf}} \to 0$ gives the following commutative diagram with exact rows:

$$\begin{array}{ccccc}
0 \longrightarrow & \mathrm{Hom}_{\mathcal{M}}(L[1], L'_{\mathrm{tor}}[1]) \otimes \mathbb{Z}_l & \longrightarrow & \mathrm{Hom}_{\mathcal{M}}(L[1], L'[1]) \otimes \mathbb{Z}_l \\
& \downarrow{\scriptstyle (2)} & & \downarrow{\scriptstyle (7)} \\
0 \longrightarrow & \mathrm{Hom}_{\mathcal{R}}(T_l L[1], T_l L'_{\mathrm{tor}}[1]) & \longrightarrow & \mathrm{Hom}_{\mathcal{R}}(T_l L[1], T_l L'[1])
\end{array}$$

$$\begin{array}{ccccc}
\longrightarrow & \mathrm{Hom}_{\mathcal{M}}(L[1], L'_{\mathrm{tf}}[1]) \otimes \mathbb{Z}_l & \longrightarrow & \mathrm{Ext}^1_{\mathcal{M}}(L[1], L'_{\mathrm{tor}}[1]) \otimes \mathbb{Z}_l \\
& \downarrow{\scriptstyle (8)} & & \downarrow{\scriptstyle (5)} \\
\longrightarrow & \mathrm{Hom}_{\mathcal{R}}(T_l L[1], T_l L'_{\mathrm{tf}}[1]) & \longrightarrow & \mathrm{Ext}^1_{\mathcal{R}}(T_l L[1], T_l L'_{\mathrm{tor}}[1])
\end{array}$$

$$\begin{array}{ccccc}
\longrightarrow & \mathrm{Ext}^1_{\mathcal{M}}(L[1], L'[1]) \otimes \mathbb{Z}_l & \longrightarrow & \mathrm{Ext}^1_{\mathcal{M}}(L[1], L'_{\mathrm{tf}}[1]) \otimes \mathbb{Z}_l \\
& \downarrow{\scriptstyle (9)} & & \downarrow{\scriptstyle (10)} \\
\longrightarrow & \mathrm{Ext}^1_{\mathcal{R}}(T_l L[1], T_l L'[1]) & \longrightarrow & \mathrm{Ext}^1_{\mathcal{R}}(T_l L[1], T_l L'_{\mathrm{tf}}[1]).
\end{array}$$

The homomorphism (8) is an isomorphism by proposition IV.6, we already know that the homomorphism (5) is a monomorphism, and the homomorphism (10) is a monomorphism given by the five lemma with the help of proposition IV.6 (d), proposition IV.7 (c) and proposition IV.7 (d). It follows that the homomorphism (7) is an isomorphism and the homomorphism (9) is an monomorphism. □

4.2 The case $M = L[1]$ and $M' = G'$

Theorem IV.9. *Theorem IV.2 is true for $M = L[1], M' = G'$.*

Proof. We have $\mathrm{Hom}_{\mathcal{M}}(L[1], G') = 0$ by proposition I.24 (b), so need to show that $\mathrm{Hom}_{\mathcal{R}}(\mathbb{Z} \otimes \mathbb{Z}_l, T_l G) = 0$. Firstly, let's consider the case where L equals $\mathbb{Z}$. We have $\mathrm{Hom}_{\mathcal{R}}(\mathbb{Z} \otimes \mathbb{Z}_l, T_l G) = (T_l G)^\Gamma$, and $(T_l G)^\Gamma$ equals zero due to the fact that k is either a number field or a finite field. The general case can be deduced from this case after a finite Galois field extension of k such that L becomes constant. □

Lemma IV.10. *The canonical map*

$$T_l : \mathrm{Ext}^1_{\mathcal{M}}(\mathbb{Z}^r[1], G') \otimes \mathbb{Z}_l \to \mathrm{Ext}^1_{\mathcal{R}}(\mathbb{Z}_l^r, T_l G')$$

is injective, here r is some positive integer.

Proof. Without loss of generality, we can assume $r = 1$. We have

$$\mathrm{Ext}^1_{\mathcal{M}}(\mathbb{Z}[1], G') \otimes \mathbb{Z}_l \cong G'(k) \otimes \mathbb{Z}_l$$

by proposition I.25 (b) in characteristic zero case and by proposition III.9 (b) in positive characteristic case. We also have $\mathrm{Ext}^1_{\mathcal{R}}(\mathbb{Z}_l, T_l G') \cong H^1_{\mathrm{cts}}(k, T_l G')$ by the same reason as in the proof of (c) of proposition IV.7. Now we turn to give an explicit description of the map T_l under these two identifications. Given a point $P \in G'(k)$, the Tate module of the 1-motive $[\mathbb{Z} \to G']$ corresponding to P is

$$T_P := T_l([\mathbb{Z} \to G']) = \frac{\{(P_n, y_n)_{n \in \mathbb{N}}\}}{\{(P, l^n)_{n \in \mathbb{N}}\}},$$

where $P_n \in G'(k^s), y_n \in \mathbb{Z}$ satisfying

$$\begin{aligned} l^n P_n &= y_n P \\ l P_{n+1} - P_n &= z_n P \\ y_{n+1} &= y_n + l^n z_n \end{aligned}$$

for some $z_n \in \mathbb{Z}$. And T_P fits into the following short exact sequence

$$0 \to T_l G' \to T_P \to \mathbb{Z}_l \to 0.$$

This short exact sequence splits in the category of Z_l-modules, and a splitting can be chosen by a section $s : \mathbb{Z}_l \to T_P$, $(1)_{n\in\mathbb{N}} \mapsto (Q_n, 1)_{n\in\mathbb{N}}$, where $Q_n \in G'(k^s)$ are chosen such that $l^n Q_n = P$, $lQ_{n+1} = Q_n$ for all $n \in \mathbb{N}$. Under such a splitting, for any $\sigma \in \Gamma$ and $(y_n)_{n\in\mathbb{N}} \in \mathbb{Z}_l$, we have

$$\begin{aligned} \sigma \cdot s((y_n)_{n\in\mathbb{N}}) &= \sigma \cdot (y_n Q_n, y_n)_{n\in\mathbb{N}} \\ &= (y_n(\sigma \cdot Q_n - Q_n), 0) + (y_n)_{n\in\mathbb{N}}(Q_n, 1)_{n\in\mathbb{N}}. \end{aligned}$$

It follows that the cocycle corresponding to the extension T_P can be represented by $((\sigma \cdot Q_n - Q_n)_n)_{\sigma\in\Gamma}$. In other words, the map T_l sends a point $P \in G'(k)$ to the 1-cocycle $((\sigma \cdot Q_n - Q_n)_n)_{\sigma\in\Gamma}$. Write $G'(k)$ as $F_l \oplus F_{l'} \oplus S$, where F_l (resp. $F_{l'}$, resp. S) is the l subgroup (resp. l-prime subgroup, resp. free subgroup) of $G'(k)$. To show the map T_l is injective, it suffices to show that any point $P \in F_{l'} \oplus S$ satisfying $T_l(P) = 0$ has to be zero. If P is not zero, then $T_l(P) = 0$ implies that

$$((\sigma \cdot Q_n - Q_n)_n)_{\sigma\in\Gamma} = ((\sigma \cdot R_n - R_n)_n)_{\sigma\in\Gamma}$$

for some $(R_n)_{n\in\mathbb{N}} \in T_l G'$. Hence we get a set of points $\{Q_n - R_n | \, n \in \mathbb{N}\}$ satisfying $\sigma \cdot (Q_n - R_n) = Q_n - R_n$, $l(Q_{n+1} - R_{n+1}) = Q_n - R_n$ and $l^n(Q_n - R_n) = P$. Let $l^{-\infty}(P)$ denotes the subgroup of $G'(k)$ generated by $\{Q_n - R_n | \, n \in \mathbb{N}\} \cup \{P\}$, then $l^{-\infty}(P)$ is a non-trivial l-divisible subgroup of $G'(k)$. But the group $G'(k)$ is finitely generated by Mordell-Weil theorem in the number field case, and it is finite in the finite field case. We get a contradiction, it follows that P has to be zero. $\square$

Corollary IV.11. *Suppose L is torsion-free, then the canonical map*

$$T_l : \mathrm{Ext}^1_{\mathcal{M}}(L[1], G') \otimes \mathbb{Z}_l \to \mathrm{Ext}^1_{\mathcal{R}}(L \otimes \mathbb{Z}_l, T_l G')$$

is injective.

Proof. Firstly, we have

$$\mathrm{Ext}^1_{\mathcal{M}}(L[1], G') \otimes \mathbb{Z}_l \cong \mathrm{Hom}_k(L, G') \otimes \mathbb{Z}_l$$

and

$$\mathrm{Ext}^1_{\mathcal{R}}(L \otimes \mathbb{Z}_l, T_l G') \cong H^1(k, \mathrm{Hom}_{\mathbb{Z}_l}(L \otimes \mathbb{Z}_l, T_l G'))$$

Let k'/k be a finite Galois extension such that L becomes constant over k', H be the Galois group $\mathrm{Gal}(k'/k)$, S be the module $\mathrm{Hom}_{\mathbb{Z}_l}(L \otimes \mathbb{Z}_l, T_l G') \in \mathcal{R}$. Then we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_k(L, G') \otimes \mathbb{Z}_l \arrow[r, equal] \arrow[d] & \mathrm{Hom}_{k'}(L, G')^H \otimes \mathbb{Z}_l \arrow[d] \\
H^1_{\mathrm{cts}}(\mathrm{Gal}(\bar{k}/k), S) \arrow[r, "\mathrm{Res}"'] & H^1_{\mathrm{cts}}(\mathrm{Gal}(\bar{k}/k'), S)^H
\end{tikzcd}$$

where the map Res denotes the restriction map for continuous cochain cohomology. The vertical map on the right hand side is injective by lemma IV.10, hence so is the vertical map on the left hand side. This finishes the proof. $\square$

Now we come to the main result of this section.

Theorem IV.12. *Theorem IV.3 is true for $M = L[1], M' = G'$.*

Proof. Firstly, we have $\mathrm{Ext}^1_{\mathcal{M}}(L[1], G') \otimes \mathbb{Z}_l = \mathrm{Hom}_k(L, G') \otimes \mathbb{Z}_l$. Let L_{tor} be the torsion subgroup of L, and L_{tf} be the quotient L/L_{tor}, so we get a short exact sequence $0 \to L_{\mathrm{tor}}[1] \to L[1] \to L_{\mathrm{tf}}[1] \to 0$ in $\mathcal{M}$. This exact sequence gives us a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_k(L_{\mathrm{tf}}, G') \otimes \mathbb{Z}_l \arrow[r] \arrow[d, "(1)"'] & \mathrm{Hom}_k(L, G') \otimes \mathbb{Z}_l \arrow[d, "(2)"'] \\
\mathrm{Hom}_{\mathcal{R}}(L_{\mathrm{tor}} \otimes \mathbb{Z}_l, T_l G') \arrow[r] & \mathrm{Ext}^1_{\mathcal{R}}(L_{\mathrm{tf}} \otimes \mathbb{Z}_l, T_l G') \arrow[r] & \mathrm{Ext}^1_{\mathcal{R}}(L \otimes \mathbb{Z}_l, T_l G')
\end{tikzcd}$$

$$\begin{tikzcd}
\arrow[r] & \mathrm{Hom}_k(L_{\mathrm{tor}}, G') \otimes \mathbb{Z}_l \arrow[d, "(3)"'] \\
\arrow[r] & \mathrm{Ext}^1_{\mathcal{R}}(L_{\mathrm{tor}} \otimes \mathbb{Z}_l, T_l G').
\end{tikzcd}$$

It's obvious that $\mathrm{Hom}_{\mathcal{R}}(L_{\mathrm{tor}} \otimes \mathbb{Z}_l, T_l G')$ equals zero, and the map (1) is injective by IV.11. Hence to show the injectivity of the map (2), it suffices to show the injectivity of the map (3). Let r be some positive integer such that $rL_{\mathrm{tor}} = 0$, and consider the short exact sequence $0 \to G' \xrightarrow{r} G' \to {}_rG'[1] \to 0$. Then we have the following commutative diagram with exact rows

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{M}}(L_{\mathrm{tor}}[1], G') \otimes \mathbb{Z}_l \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{M}}(L_{\mathrm{tor}}[1], {}_rG'[1]) \otimes \mathbb{Z}_l \arrow[r, two heads] \arrow[d] & \mathrm{Ext}^1_{\mathcal{M}}(L_{\mathrm{tor}}[1], G') \otimes \mathbb{Z}_l \arrow[d, "(4)"'] \\
\mathrm{Hom}_{\mathcal{R}}(L_{\mathrm{tor}} \otimes \mathbb{Z}_l, T_l G') \arrow[r] & \mathrm{Hom}_{\mathcal{R}}(L_{\mathrm{tor}} \otimes \mathbb{Z}_l, {}_rG' \otimes \mathbb{Z}_l) \arrow[r, two heads] & \mathrm{Ext}^1_{\mathcal{R}}(L_{\mathrm{tor}} \otimes \mathbb{Z}_l, T_l G').
\end{tikzcd}$$

Both $\mathrm{Hom}_{\mathcal{M}}(L_{\mathrm{tor}}[1], G')$ and $\mathrm{Hom}_{\mathcal{R}}(L_{\mathrm{tor}} \otimes \mathbb{Z}_l, T_l G')$ are zero, and the vertical map in the middle is an isomorphism by IV.6 (a), hence so is the map (4). And the map (4) is nothing else but the map (3). $\square$

4.3 The case $M = G$ and $M' = L'[1]$

Theorem IV.13. *Theorem IV.2 is true for $M = G, M' = L'[1]$.*

Proof. Let L'_{tor} be the torsion subgroup of L', and L'_{tf} be the quotient L'/L'_{tor}. Then we have the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathcal{M}}(G, L'_{\mathrm{tor}}[1]) \otimes \mathbb{Z}_l \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{M}}(G, L'[1]) \otimes \mathbb{Z}_l \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{M}}(G, L'_{\mathrm{tf}}[1]) \otimes \mathbb{Z}_l \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}_{\mathcal{R}}(T_l G, L'_{\mathrm{tor}} \otimes \mathbb{Z}_l) \arrow[r] & \mathrm{Hom}_{\mathcal{R}}(T_l G, L' \otimes \mathbb{Z}_l) \arrow[r] & \mathrm{Hom}_{\mathcal{R}}(T_l G, L'_{\mathrm{tf}} \otimes \mathbb{Z}_l).
\end{tikzcd}$$

Since both $\mathrm{Hom}_{\mathcal{M}}(G, L'_{\mathrm{tf}}[1]) \otimes \mathbb{Z}_l$ and $\mathrm{Hom}_{\mathcal{R}}(T_l G, L'_{\mathrm{tf}} \otimes \mathbb{Z}_l)$ are zero, it's enough to prove the theorem for the case L' being torsion. Moreover L' can be assumed to be an l-group. Let r be a positive integer such that l^r kills L'. The short exact sequence $0 \to G \xrightarrow{l^r} G \to {}_{l^r}G[1] \to 0$ gives a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathcal{M}}({}_{l^r}G[1], L'[1]) \otimes \mathbb{Z}_l \arrow[r] \arrow[d, "(1)"] & \mathrm{Hom}_{\mathcal{M}}(G, L'[1]) \otimes \mathbb{Z}_l \arrow[r, "l^r"] \arrow[d, "(2)"] & \mathrm{Hom}_{\mathcal{M}}(G, L'[1]) \otimes \mathbb{Z}_l \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}_{\mathcal{R}}({}_{l^r}G, L' \otimes \mathbb{Z}_l) \arrow[r] & \mathrm{Hom}_{\mathcal{R}}(T_l G, L' \otimes \mathbb{Z}_l) \arrow[r, "l^r"] & \mathrm{Hom}_{\mathcal{R}}(T_l G, L' \otimes \mathbb{Z}_l).
\end{tikzcd}$$

The two multiplication-by-l^r maps on the right hand side are zero, and (1) is an isomorphism by proposition IV.6 (a), hence (2) is an isomorphism. $\square$

Theorem IV.14. *Theorem IV.3 is true for $M = G, M' = L'[1]$.*

Proof. By proposition I.26 and proposition III.10, we have an isomorphism

$$\Phi : \varinjlim_{r} \mathrm{Ext}^1_k({}_{l^r}G, L') = \varinjlim_{n} \mathrm{Ext}^1_k({}_nG, L') \otimes \mathbb{Z}_l \to \mathrm{Ext}^1_{\mathcal{M}}(G, L'[1]) \otimes \mathbb{Z}_l.$$

Note that here we abuse the notation Φ, which should be $\Phi\otimes 1$ according to propositions I.26 and III.10. Since the group $\mathrm{Ext}^1_{\mathcal{M}}(G,L'[1])\otimes\mathbb{Z}_l$ is torsion, its image under the map T_l lies in the torsion subgroup of $\mathrm{Ext}^1_{\mathcal{R}}(T_lG,L'\otimes\mathbb{Z}_l)$, which can be expressed as $\mathrm{Ext}^1_{\mathcal{R}}(T_lG,L'\otimes\mathbb{Z}_l)(l):=\varinjlim_r {}_{l^r}\mathrm{Ext}^1_{\mathcal{R}}(T_lG,L'\otimes\mathbb{Z}_l)$. The short exact sequence $0\to T_lG\xrightarrow{l^r}T_lG\to {}_{l^r}G[1]\to 0$ in $\mathcal{R}$ gives a short exact sequence

$$0\to\mathrm{Hom}_{\mathcal{R}}(T_lG,L'\otimes\mathbb{Z}_l)\otimes\mathbb{Z}/l^r\mathbb{Z}\to\mathrm{Ext}^1_{\mathcal{R}}({}_{l^r}G,L'\otimes\mathbb{Z}_l)\to {}_{l^r}\mathrm{Ext}^1_{\mathcal{R}}(T_lG,L'\otimes\mathbb{Z}_l)\to 0.$$

Passing to the direct limit, we get a short exact sequence fitting into the following commutative diagram with exact rows

$$\begin{tikzcd}
 & & \varinjlim_r \mathrm{Ext}^1_k({}_{l^r}G,L') \arrow[r, "\Phi", "\cong"'] \arrow[d, "(1)"'] & \mathrm{Ext}^1_{\mathcal{M}}(G,L'[1])\otimes\mathbb{Z}_l \arrow[r] \arrow[d, "T_l"] & 0 \\
0 \arrow[r] & B \arrow[r] & \varinjlim_r \mathrm{Ext}^1_{\mathcal{R}}({}_{l^r}G,L'\otimes\mathbb{Z}_l) \arrow[r] & \mathrm{Ext}^1_{\mathcal{R}}(T_lG,L'\otimes\mathbb{Z}_l)(l) \arrow[r] & 0,
\end{tikzcd}$$

where B denotes the group $\mathrm{Hom}_{\mathcal{R}}(T_lG,L'\otimes\mathbb{Z}_l)\otimes\mathbb{Q}_l/\mathbb{Z}_l$. The group $\mathrm{Hom}_{\mathcal{R}}(T_lG,L'\otimes\mathbb{Z}_l)$ ia actually a finite group due to the fact $\mathrm{Hom}_{\mathcal{R}}(T_lG,\mathbb{Z}_l)=0$, hence the group B is zero. So T_l being injective is equivalent to the map (1) being injective. For the cases L being torsion and being torsion free, we have $\mathrm{Ext}^1_k({}_{l^r}G,L')\cong\mathrm{Ext}^1_{\mathcal{R}}({}_{l^r}G,L'\otimes\mathbb{Z}_l)$ by proposition IV.7 (a) and (d), hence the map (1) is actually an isomorphism in both cases.

In the general case, let L'_{tor} be the torsion subgroup of L' and L'_{tf} be the quotient L'/L'_{tor}, then we have a commutative diagram with exact rows

$$\begin{tikzcd}
{} \arrow[r] & \mathrm{Hom}_{\mathcal{M}}(G,L'_{\mathrm{tf}}[1])\otimes\mathbb{Z}_l \arrow[r] \arrow[d] & \mathrm{Ext}^1_{\mathcal{M}}(G,L'_{\mathrm{tor}}[1])\otimes\mathbb{Z}_l \arrow[r] \arrow[d, "(2)"'] & \mathrm{Ext}^1_{\mathcal{M}}(G,L'[1])\otimes\mathbb{Z}_l \arrow[d, "(3)"'] \\
{} \arrow[r] & \mathrm{Hom}_{\mathcal{R}}(T_lG,L'_{\mathrm{tf}}\otimes\mathbb{Z}_l) \arrow[r] & \mathrm{Ext}^1_{\mathcal{R}}(T_lG,L'_{\mathrm{tor}}\otimes\mathbb{Z}_l) \arrow[r] & \mathrm{Ext}^1_{\mathcal{R}}(T_lG,L'\otimes\mathbb{Z}_l)
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r] & \mathrm{Ext}^1_{\mathcal{M}}(G,L'_{\mathrm{tf}}[1])\otimes\mathbb{Z}_l \arrow[r] \arrow[d, "(4)"'] & \mathrm{Ext}^2_{\mathcal{M}}(G,L'_{\mathrm{tor}}[1])\otimes\mathbb{Z}_l \arrow[d] \\
{} \arrow[r] & \mathrm{Ext}^1_{\mathcal{R}}(T_lG,L'_{\mathrm{tf}}\otimes\mathbb{Z}_l) \arrow[r] & \mathrm{Ext}^2_{\mathcal{R}}(T_lG,L'_{\mathrm{tor}}\otimes\mathbb{Z}_l).
\end{tikzcd}$$

Both groups $\mathrm{Hom}_{\mathcal{M}}(G,L'_{\mathrm{tf}}[1])$ and $\mathrm{Hom}_{\mathcal{R}}(T_lG,L'_{\mathrm{tf}}\otimes\mathbb{Z}_l)$ are zero, and we have proven both map (2) and map (4) are isomorphism, hence the map (3) is injective. $\square$

4.4 The case $M = G$ and $M' = G'$

Let T (resp. T') be the torus part of the semiabelian variety G (resp. G'), and A (resp A') be the corresponding abelian quotient. So we get two short exact sequences of k-group schemes $0 \to T \to G \to A \to 0$ and $0 \to T' \to G' \to A' \to 0$. In order to prove theorems IV.2 and IV.3 for the case $M = G$, $M' = G'$, we first deal with some special cases.

Lemma IV.15. *Theorem IV.2 is true for the following cases.*
(a)$G' = A'$;
(b)$G = T, G' = T'$;
(c)$G = A, G' = T'$.

Proof. First we have $\mathrm{Ext}^1_{\mathcal{M}}(G, G') \otimes \mathbb{Z}_l \cong \mathrm{Ext}^1_k(G, G') \otimes \mathbb{Z}_l$ by proposition I.25 (c) in characteristic zero case and by proposition III.9 (c) in positive characteristic case.

The short exact sequence $0 \to T \to G \to A \to 0$ gives us a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_k(A, A') \otimes \mathbb{Z}_l \arrow[r] \arrow[d, "(1)"'] & \mathrm{Hom}_k(G, A') \otimes \mathbb{Z}_l \arrow[r] \arrow[d] & \mathrm{Hom}_k(T, A') \otimes \mathbb{Z}_l \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}_{\mathcal{R}}(T_l A, T_l A') \arrow[r] & \mathrm{Hom}_{\mathcal{R}}(T_l G, T_l A') \arrow[r] & \mathrm{Hom}_{\mathcal{R}}(T_l T, T_l A').
\end{tikzcd}$$

The group $\mathrm{Hom}_k(T, A')$ is obviously trivial. By using Cartier duality, we have

$$\mathrm{Hom}_{\mathcal{R}}(T_l \mathbb{G}_m, T_l A') \cong \mathrm{Hom}_{\mathcal{R}}(T_l \hat{A}', \mathbb{Z} \otimes \mathbb{Z}_l) = (T_l \hat{A'}^{\vee})^{\Gamma} \cong T_l \hat{A'}^{\Gamma} = 0,$$

where $\hat{A}'$ denotes the dual abelian variety of A' and $(-)^{\vee}$ denotes the dual continuous Galois module. Hence it's easy to see that the group $\mathrm{Hom}_{\mathcal{R}}(T_l T, T_l A')$ is also trivial. The map (1) is an isomorphism by theorem IV.1, then case (a) follows.

Case (b) actually follows from proposition IV.6 (d) with the help of the fact that tori are dual to lattices under Cartier duality. More explicitly, this follows from the following results

$$\mathrm{Hom}_k(T, T') = \mathrm{Hom}_k(X(T'), X(T)),$$

$$T_l T = \mathrm{Hom}_{\mathcal{R}}(X(T) \otimes \mathbb{Z}_l, \mathbb{Z}_l(1)),$$

and

$$\mathrm{Hom}_{\mathcal{R}}(T_l T, T_l T') = \mathrm{Hom}_{\mathcal{R}}(X(T') \otimes \mathbb{Z}_l, X(T) \otimes \mathbb{Z}_l).$$

Here $X(T)$ denotes the group of characters of the torus T.

By using Cartier duality, we have

$$\mathrm{Hom}_{\mathcal{R}}(T_lA, T_l\mathbb{G}_m) = (\mathrm{Hom}_{\mathbb{Z}_l}(T_lA, T_l\mathbb{G}_m))^{\Gamma} \cong (T_l\hat{A})^{\Gamma} = 0.$$

Then case (c) follows since we have $\mathrm{Hom}_k(A, T') = 0 = \mathrm{Hom}_{\mathcal{R}}(T_lA, T_lT')$. $\square$

Theorem IV.16. *Theorem IV.3 is true for $M = G, M' = G'$.*

Proof. Consider the short exact sequence $0 \to G' \xrightarrow{l^n} G' \to {}_{l^n}G' \to 0$ in $\mathcal{M}$, then we have the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}^1_{\mathcal{M}}(G, G') \otimes \mathbb{Z}_l/l^n\mathbb{Z}_l \arrow[r] \arrow[d, "(1)"'] & \mathrm{Ext}^1_{\mathcal{M}}(G, {}_{l^n}G'[1]) \otimes \mathbb{Z}_l \arrow[r] \arrow[d, "(2)"'] & {}_{l^n}\mathrm{Ext}^2_{\mathcal{M}}(G, G') \arrow[d] \\
0 \arrow[r] & \mathrm{Ext}^1_{\mathcal{R}}(T_lG, T_lG') \otimes \mathbb{Z}_l/l^n\mathbb{Z}_l \arrow[r] & \mathrm{Ext}^1_{\mathcal{R}}(T_lG, {}_{l^n}G') \arrow[r] & {}_{l^n}\mathrm{Ext}^2_{\mathcal{R}}(T_lG, T_lG').
\end{tikzcd}$$

The map (2) is injective by theorem IV.13, so is the map (1) by the five lemma. Passing to the projective limit, we get an injection

$$\varprojlim_n \mathrm{Ext}^1_{\mathcal{M}}(G, G') \otimes \mathbb{Z}_l/l^n\mathbb{Z}_l \longrightarrow \varprojlim_n \mathrm{Ext}^1_{\mathcal{R}}(T_lG, T_lG') \otimes \mathbb{Z}_l/l^n\mathbb{Z}_l.$$

It follows that the canonical morphism $\mathrm{Ext}^1_{\mathcal{M}}(G, G') \otimes \mathbb{Z}_l \to \mathrm{Ext}^1_{\mathcal{R}}(T_lG, T_lG')$ is injective. $\square$

Theorem IV.17. *Theorem IV.2 is true for $M = G, M' = G'$.*

Proof. The short exact sequence $0 \to T \to G \to A \to 0$ gives a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathcal{M}}(A, T') \otimes \mathbb{Z}_l \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{M}}(G, T') \otimes \mathbb{Z}_l \arrow[r] \arrow[d, "(1)"'] & \mathrm{Hom}_{\mathcal{M}}(T, T') \otimes \mathbb{Z}_l \arrow[d, "(2)"'] \\
0 \arrow[r] & \mathrm{Hom}_{\mathcal{R}}(T_lA, T_lT') \arrow[r] & \mathrm{Hom}_{\mathcal{R}}(T_lG, T_lT') \arrow[r] & \mathrm{Hom}_{\mathcal{R}}(T_lT, T_lT')
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r] & \mathrm{Ext}^1_{\mathcal{M}}(A, T') \otimes \mathbb{Z}_l \arrow[r] \arrow[d, "(3)"'] & \mathrm{Ext}^1_{\mathcal{M}}(G, T') \otimes \mathbb{Z}_l \arrow[d] \\
{} \arrow[r] & \mathrm{Ext}^1_{\mathcal{R}}(T_lA, T_lT') \arrow[r] & \mathrm{Ext}^1_{\mathcal{R}}(T_lG, T_lT').
\end{tikzcd}$$

Both groups $\mathrm{Hom}_{\mathcal{M}}(A, T')$ and $\mathrm{Hom}_{\mathcal{R}}(T_lA, T_lT')$ are zero, the map (2) is an isomorphism by lemma IV.15 (b), and the map (3) is injective by theorem IV.16, hence the map (1) is an isomorphism by the five lemma. The short exact sequence $0 \to T' \to G' \to A' \to 0$ gives another commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathcal{M}}(G,T')\otimes\mathbb{Z}_l \arrow[r] \arrow[d, "(1)"'] & \mathrm{Hom}_{\mathcal{M}}(G,G')\otimes\mathbb{Z}_l \arrow[r] \arrow[d, "(4)"'] & \mathrm{Hom}_{\mathcal{M}}(G,A')\otimes\mathbb{Z}_l \arrow[d, "(5)"'] \\
0 \arrow[r] & \mathrm{Hom}_{\mathcal{R}}(T_lG,T_lT') \arrow[r] & \mathrm{Hom}_{\mathcal{R}}(T_lG,T_lG') \arrow[r] & \mathrm{Hom}_{\mathcal{R}}(T_lG,T_lA')
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r] & \mathrm{Ext}^1_{\mathcal{M}}(G,T')\otimes\mathbb{Z}_l \arrow[r] \arrow[d, "(6)"'] & \mathrm{Ext}^1_{\mathcal{M}}(G,G')\otimes\mathbb{Z}_l \arrow[d] \\
{} \arrow[r] & \mathrm{Ext}^1_{\mathcal{R}}(T_lG,T_lT') \arrow[r] & \mathrm{Ext}^1_{\mathcal{R}}(T_lG,T_lG').
\end{tikzcd}$$

The map (5) are an isomorphisms by lemma IV.15 (a), the map (6) is injective by theorem IV.16, hence the map (4) is an isomorphism by the five lemma. $\square$

4.5 Proof of IV.2 and IV.3

After a long preparation, now we come to the final proofs of theorem IV.2 and theorem IV.3.

Proof of IV.2 and IV.3: The canonical exact sequence $0 \to G' \to M' \to L'[1] \to 0$ in $\mathcal{M}$ gives the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathcal{M}}(L[1],G')\otimes\mathbb{Z}_l \arrow[r] \arrow[d, "(1)"'] & \mathrm{Hom}_{\mathcal{M}}(L[1],M')\otimes\mathbb{Z}_l \arrow[r] \arrow[d, "(2)"'] & \mathrm{Hom}_{\mathcal{M}}(L[1],L'[1])\otimes\mathbb{Z}_l \arrow[d, "(3)"'] \\
0 \arrow[r] & \mathrm{Hom}_{\mathcal{R}}(T_lL[1],T_lG') \arrow[r] & \mathrm{Hom}_{\mathcal{R}}(T_lL[1],T_lM') \arrow[r] & \mathrm{Hom}_{\mathcal{R}}(T_lL[1],T_lL'[1])
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r] & \mathrm{Ext}^1_{\mathcal{M}}(L[1],G')\otimes\mathbb{Z}_l \arrow[r] \arrow[d, "(4)"'] & \mathrm{Ext}^1_{\mathcal{M}}(L[1],M')\otimes\mathbb{Z}_l \arrow[r] \arrow[d, "(5)"'] & \mathrm{Ext}^1_{\mathcal{M}}(L[1],L'[1])\otimes\mathbb{Z}_l \arrow[d, "(6)"'] \\
{} \arrow[r] & \mathrm{Ext}^1_{\mathcal{R}}(T_lL[1],T_lG') \arrow[r] & \mathrm{Ext}^1_{\mathcal{R}}(T_lL[1],T_lM') \arrow[r] & \mathrm{Ext}^1_{\mathcal{R}}(T_lL[1],T_lL'[1]).
\end{tikzcd}$$

The map (1) and the map (3) are isomorphisms by theorem IV.9 and theorem IV.8, respectively. And the map (4) and the map (6) are injective by theorem IV.12 and theorem IV.8, respectively. Hence by the five lemma, we have that the map (2) is an isomorphism, and the map (5) is injective.

The short exact sequence $0 \to G' \to M' \to L'[1] \to 0$ in $\mathcal{M}$ also gives another commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathcal{M}}(G, G') \otimes \mathbb{Z}_l \arrow[r] \arrow[d, "(7)"'] & \mathrm{Hom}_{\mathcal{M}}(G, M') \otimes \mathbb{Z}_l \arrow[r] \arrow[d, "(8)"'] & \mathrm{Hom}_{\mathcal{M}}(G, L'[1]) \otimes \mathbb{Z}_l \arrow[d, "(9)"'] \\
0 \arrow[r] & \mathrm{Hom}_{\mathcal{R}}(T_lG, T_lG') \arrow[r] & \mathrm{Hom}_{\mathcal{R}}(T_lG, T_lM') \arrow[r] & \mathrm{Hom}_{\mathcal{R}}(T_lG, T_lL'[1])
\end{tikzcd}$$

$$\begin{tikzcd}
\arrow[r] & \mathrm{Ext}^1_{\mathcal{M}}(G, G') \otimes \mathbb{Z}_l \arrow[r] \arrow[d, "(10)"'] & \mathrm{Ext}^1_{\mathcal{M}}(G, M') \otimes \mathbb{Z}_l \arrow[r] \arrow[d, "(11)"'] & \mathrm{Ext}^1_{\mathcal{M}}(G, L'[1]) \otimes \mathbb{Z}_l \arrow[d, "(12)"'] \\
\arrow[r] & \mathrm{Ext}^1_{\mathcal{R}}(T_lG, T_lG') \arrow[r] & \mathrm{Ext}^1_{\mathcal{R}}(T_lG, T_lM') \arrow[r] & \mathrm{Ext}^1_{\mathcal{R}}(T_lG, T_lL'[1]).
\end{tikzcd}$$

The map (7) and the map (9) are isomorphisms by theorem IV.17 and theorem IV.13 respectively, and the map (10) and the map (12) are injective by theorem IV.16 and theorem IV.14, hence the map (8) is an isomorphism by the five lemma, and the map (11) is injective.

Now consider another short exact sequence $0 \to G \to M \to L[1] \to 0$ in $\mathcal{M}$, we get the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathcal{M}}(L[1], M') \otimes \mathbb{Z}_l \arrow[r] \arrow[d, "(2)"'] & \mathrm{Hom}_{\mathcal{M}}(M, M') \otimes \mathbb{Z}_l \arrow[r] \arrow[d, "(13)"'] & \mathrm{Hom}_{\mathcal{M}}(G, M') \otimes \mathbb{Z}_l \arrow[d, "(8)"'] \\
0 \arrow[r] & \mathrm{Hom}_{\mathcal{R}}(T_lL[1], T_lM') \arrow[r] & \mathrm{Hom}_{\mathcal{R}}(T_lM, T_lM') \arrow[r] & \mathrm{Hom}_{\mathcal{R}}(T_lG, T_lM')
\end{tikzcd}$$

$$\begin{tikzcd}
\arrow[r] & \mathrm{Ext}^1_{\mathcal{M}}(L[1], M') \otimes \mathbb{Z}_l \arrow[r] \arrow[d, "(5)"'] & \mathrm{Ext}^1_{\mathcal{M}}(M, M') \otimes \mathbb{Z}_l \arrow[r] \arrow[d, "(14)"'] & \mathrm{Ext}^1_{\mathcal{M}}(G, M') \otimes \mathbb{Z}_l \arrow[d, "(11)"'] \\
\arrow[r] & \mathrm{Ext}^1_{\mathcal{R}}(T_lL[1], T_lM') \arrow[r] & \mathrm{Ext}^1_{\mathcal{R}}(T_lM, T_lM') \arrow[r] & \mathrm{Ext}^1_{\mathcal{R}}(T_lG, T_lM').
\end{tikzcd}$$

Then the map (13) being an isomorphism and the injectivity of the map (14) are just easy consequences of the five lemma. $\square$

4.6 The image of T_l over a finite field

Through out this section, k will be a finite field.

Theorem IV.3 tells us that the map $T_l : \mathrm{Ext}^1_{\mathcal{M}}(M, M') \otimes \mathbb{Z}_l \to \mathrm{Ext}^1_{\mathcal{R}}(T_lM, T_lM')$ is injective over both finite fields and number fields. In general the arithmetic over finite fields is much easier than the arithmetic over number fields. So it's natural to ask what else can be read off beyond theorem IV.3 over finite fields. We have proven that the group $\mathrm{Ext}^1_{\mathcal{M}}(M, M')$ is a finite group over finite fields, but the group $\mathrm{Ext}^1_{\mathcal{R}}(T_l M, T_l M')$ is not necessary a finite group (actually it may not even be a torsion group). However we may expect the image of T_l to be the torsion subgroup of $\mathrm{Ext}^1_{\mathcal{R}}(T_l M, T_l M')$.

Theorem IV.18. *The image of the natural map*

$$T_l : \mathrm{Ext}^1_{\mathcal{M}}(M, M') \otimes \mathbb{Z}_l \to \mathrm{Ext}^1_{\mathcal{R}}(T_l M, T_l M')$$

is the torsion subgroup $\mathrm{Ext}^1_{\mathcal{R}}(T_l M, T_l M')_{\mathrm{tor}}$ *of* $\mathrm{Ext}^1_{\mathcal{R}}(T_l M, T_l M')$*.*

As usual, before going to the proof of theorem IV.18, we first deal with some special cases.

Lemma IV.19. *Suppose M' is torsion-free, then the canonical map*

$$T_l : \mathrm{Ext}^1_{\mathcal{M}}(M, M') \otimes \mathbb{Z}_l \to \mathrm{Ext}^1_{\mathcal{R}}(T_l M, T_l M')_{\mathrm{tor}}$$

is an isomorphism.

Proof. Firstly the group $\mathrm{Ext}^1_{\mathcal{M}}(M, M')$ is torsion by theorem III.21, hence the image of T_l lies in $\mathrm{Ext}^1_{\mathcal{R}}(T_l M, T_l M')_{\mathrm{tor}}$. The map T_l is injective by theorem IV.3, so it is left to show that T_l has image $\mathrm{Ext}^1_{\mathcal{R}}(T_l M, T_l M')_{\mathrm{tor}}$. Given any element $\alpha \in \mathrm{Ext}^1_{\mathcal{R}}(T_l M, T_l M')_{\mathrm{tor}}$, we need to find a preimage of α. Let r be some positive integer such that $l^r \cdot \alpha$ equals zero. Since M' is torsion-free, we have a canonical short exact sequence $0 \to M' \xrightarrow{l^r} M' \to L'_r[1] \to 0$ in $\mathcal{M}$, where L'_r is a finite k-group scheme such that M'/l^r is $L'_r[1]$. Then we have a commutative diagram with exact rows

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{M}}(M, L'_r[1]) \otimes \mathbb{Z}_l \arrow[r, "u"] \arrow[d, "T_l"'] & \mathrm{Ext}^1_{\mathcal{M}}(M, M') \otimes \mathbb{Z}_l \arrow[r, "l^r"] \arrow[d, "T_l"'] & \mathrm{Ext}^1_{\mathcal{M}}(M, M') \otimes \mathbb{Z}_l \arrow[d] \\
\mathrm{Hom}_{\mathcal{R}}(T_l M, T_l L'_r[1]) \arrow[r, "v"] & \mathrm{Ext}^1_{\mathcal{R}}(T_l M, T_l M') \arrow[r, "l^r"] & \mathrm{Ext}^1_{\mathcal{R}}(T_l M, T_l M').
\end{tikzcd}$$

The fact $l^r \alpha = 0$ implies $\alpha = v(\beta)$ for some $\beta \in \mathrm{Hom}_{\mathcal{R}}(T_l M, T_l L'_r)$. The map T_l on the left hand side is an isomorphism by theorem IV.2, so there exists a $\delta \in \mathrm{Hom}_{\mathcal{M}}(M, L'_r[1]) \otimes \mathbb{Z}_l$ such that $T_l(\delta)$ equals β. It follows that we have $\alpha = T_l(u(\delta))$. $\square$

Lemma IV.20. *Let X' be a finite étale k-group scheme, then the following two canonical maps*

$$T_l : \mathrm{Ext}^1_{\mathcal{M}}(M, X'[1]) \otimes \mathbb{Z}_l \to \mathrm{Ext}^1_{\mathcal{R}}(T_l M, X' \otimes \mathbb{Z}_l)$$

and

$$T_l : \mathrm{Ext}^2_{\mathcal{M}}(M, X'[1]) \otimes \mathbb{Z}_l \to \mathrm{Ext}^2_{\mathcal{R}}(T_l M, X' \otimes \mathbb{Z}_l)$$

are both isomorphisms.

Proof. Without loss of generality, we can assume $l^r \cdot X' = 0$, with r some positive integer. We have the canonical short exact sequence $0 \to X[1] \to M \to M_{\mathrm{tf}} \to 0$ associated to M, where X is a finite étale k-group scheme satisfying $M_{\mathrm{tor}} = X[1]$. Consider the short exact sequence $0 \to M_{\mathrm{tf}} \xrightarrow{l^r} M_{\mathrm{tf}} \to L_r[1] \to 0$, where L_r is a finite étale k-group scheme such that $M_{\mathrm{tf}}/l^r = L_r[1]$, then we get a commutative diagram with exact rows

$$\begin{tikzcd}
{} \arrow[r] & \mathrm{Hom}_{\mathcal{M}}(M_{\mathrm{tf}}, X'[1]) \otimes \mathbb{Z}_l \arrow[r, "l^r"] \arrow[d] & \mathrm{Hom}_{\mathcal{M}}(M_{\mathrm{tf}}, X'[1]) \otimes \mathbb{Z}_l \arrow[r] \arrow[d, "(1)"'] & \mathrm{Ext}^1_{\mathcal{M}}(L_r[1], X'[1]) \otimes \mathbb{Z}_l \arrow[d, "(2)"'] \\
{} \arrow[r] & \mathrm{Hom}_{\mathcal{R}}(T_l M_{\mathrm{tf}}, T_l X'[1]) \arrow[r, "l^r"] & \mathrm{Hom}_{\mathcal{R}}(T_l M_{\mathrm{tf}}, T_l X'[1]) \arrow[r] & \mathrm{Ext}^1_{\mathcal{R}}(T_l L_r[1], T_l X'[1])
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r] & \mathrm{Ext}^1_{\mathcal{M}}(M_{\mathrm{tf}}, X'[1]) \otimes \mathbb{Z}_l \arrow[r, "l^r"] \arrow[d, "(3)"'] & \mathrm{Ext}^1_{\mathcal{M}}(M_{\mathrm{tf}}, X'[1]) \otimes \mathbb{Z}_l \arrow[r] \arrow[d, "(4)"'] & \mathrm{Ext}^2_{\mathcal{M}}(L_r[1], X'[1]) \otimes \mathbb{Z}_l \arrow[d, "(5)"'] \\
{} \arrow[r] & \mathrm{Ext}^1_{\mathcal{R}}(T_l M_{\mathrm{tf}}, T_l X'[1]) \arrow[r, "l^r"] & \mathrm{Ext}^1_{\mathcal{R}}(T_l M_{\mathrm{tf}}, T_l X'[1]) \arrow[r] & \mathrm{Ext}^2_{\mathcal{R}}(T_l L_r[1], T_l X'[1])
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r] & \mathrm{Ext}^2_{\mathcal{M}}(M_{\mathrm{tf}}, X'[1]) \otimes \mathbb{Z}_l \arrow[r, "l^r"] \arrow[d] & \mathrm{Ext}^2_{\mathcal{M}}(M_{\mathrm{tf}}, X'[1]) \otimes \mathbb{Z}_l \arrow[r] \arrow[d] & \mathrm{Ext}^3_{\mathcal{M}}(L_r[1], X'[1]) \otimes \mathbb{Z}_l \arrow[d] \\
{} \arrow[r] & \mathrm{Ext}^2_{\mathcal{R}}(T_l M_{\mathrm{tf}}, T_l X'[1]) \arrow[r, "l^r"] & \mathrm{Ext}^2_{\mathcal{R}}(T_l M_{\mathrm{tf}}, T_l X'[1]) \arrow[r] & \mathrm{Ext}^3_{\mathcal{R}}(T_l L_r[1], T_l X'[1]).
\end{tikzcd}$$

Since X' is killed by l^r, all the multiplication-by-l^r maps in the diagram are zero. So the rows break down into short exact sequences. The maps (1) and (2) are isomorphisms by theorem IV.2 and proposition IV.7 (a), respectively, so is the map (3) and hence so is the map (4). The groups $\mathrm{Ext}^3_{\mathcal{M}}(L_r[1], X'[1]) \otimes \mathbb{Z}_l$ and $\mathrm{Ext}^3_{\mathcal{R}}(T_l L_r[1], T_l X'[1])$ are zero by theorem III.21 and corollary II.36, respectively, so we conclude that the groups $\mathrm{Ext}^2_{\mathcal{M}}(M_{\mathrm{tf}}, X'[1]) \otimes \mathbb{Z}_l$ and $\mathrm{Ext}^2_{\mathcal{R}}(T_l M_{\mathrm{tf}}, T_l X'[1])$ are both zero. Then the map (4) being an isomorphism implies that the map (5) is an isomorphism.

Now we turn to consider the short exact sequence $0 \to X[1] \to M \to M_{\mathrm{tf}} \to 0$, and we get a commutative diagram with exact rows

$$
\begin{tikzcd}
{} \arrow[r] & \mathrm{Hom}_{\mathcal{M}}(M, X'[1]) \otimes \mathbb{Z}_l \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{M}}(X[1], X'[1]) \otimes \mathbb{Z}_l \arrow[r] \arrow[d, "(6)"'] & \mathrm{Ext}^1_{\mathcal{M}}(M_{\mathrm{tf}}, X'[1]) \otimes \mathbb{Z}_l \arrow[d, "(3)"'] \\
{} \arrow[r] & \mathrm{Hom}_{\mathcal{R}}(T_l M, T_l X'[1]) \arrow[r] & \mathrm{Hom}_{\mathcal{R}}(T_l X[1], T_l X'[1]) \arrow[r] & \mathrm{Ext}^1_{\mathcal{R}}(T_l M_{\mathrm{tf}}, T_l X'[1])
\end{tikzcd}
$$

$$
\begin{tikzcd}
{} \arrow[r] & \mathrm{Ext}^1_{\mathcal{M}}(M, X'[1]) \otimes \mathbb{Z}_l \arrow[r] \arrow[d, "(7)"'] & \mathrm{Ext}^1_{\mathcal{M}}(X[1], X'[1]) \otimes \mathbb{Z}_l \arrow[r] \arrow[d, "(8)"'] & \mathrm{Ext}^2_{\mathcal{M}}(M_{\mathrm{tf}}, X'[1]) \otimes \mathbb{Z}_l \arrow[d] \\
{} \arrow[r] & \mathrm{Ext}^1_{\mathcal{R}}(T_l M, T_l X'[1]) \arrow[r] & \mathrm{Ext}^1_{\mathcal{R}}(T_l X[1], T_l X'[1]) \arrow[r] & \mathrm{Ext}^2_{\mathcal{R}}(T_l M_{\mathrm{tf}}, T_l X'[1])
\end{tikzcd}
$$

$$
\begin{tikzcd}
{} \arrow[r] & \mathrm{Ext}^2_{\mathcal{M}}(M, X'[1]) \otimes \mathbb{Z}_l \arrow[r] \arrow[d, "(9)"'] & \mathrm{Ext}^2_{\mathcal{M}}(X[1], X'[1]) \otimes \mathbb{Z}_l \arrow[r] \arrow[d, "(10)"'] & \mathrm{Ext}^3_{\mathcal{M}}(M_{\mathrm{tf}}, X'[1]) \otimes \mathbb{Z}_l \arrow[d] \\
{} \arrow[r] & \mathrm{Ext}^2_{\mathcal{R}}(T_l M, T_l X'[1]) \arrow[r] & \mathrm{Ext}^2_{\mathcal{R}}(T_l X[1], T_l X'[1]) \arrow[r] & \mathrm{Ext}^3_{\mathcal{R}}(T_l M_{\mathrm{tf}}, T_l X'[1]).
\end{tikzcd}
$$

The maps (6) and (8) are isomorphisms by theorem IV.2 and proposition IV.7 (a), respectively. We already know the map (3) is an isomorphism, and the groups $\mathrm{Ext}^2_{\mathcal{M}}(M_{\mathrm{tf}}, X'[1]) \otimes \mathbb{Z}_l$ and $\mathrm{Ext}^2_{\mathcal{R}}(T_l M_{\mathrm{tf}}, T_l X'[1])$ are zero, hence the map (7) is an isomorphism by the five lemma. Both the group $\mathrm{Ext}^3_{\mathcal{M}}(M_{\mathrm{tf}}, X'[1])$ and the group $\mathrm{Ext}^3_{\mathcal{R}}(T_l M_{\mathrm{tf}}, T_l X'[1])$ are zero by the same reasons as in the above diagram. Then the map (9) is an isomorphism if and only if the map (10) is an isomorphism, and the latter is given by lemma IV.21. □

Lemma IV.21. *Let X, X' be two finite étale group schemes over k, then the canonical map*

$$T_l : \mathrm{Ext}^2_{\mathcal{M}}(X[1], X'[1]) \otimes \mathbb{Z}_l \to \mathrm{Ext}^2_{\mathcal{R}}(X \otimes \mathbb{Z}_l, X' \otimes \mathbb{Z}_l)$$

is an isomorphism.

Proof. Without loss of generality, we can assume both X and X' are killed by l^r for some positive integer r.

Claim. Any element in $\mathrm{Ext}^2_{\mathcal{R}}(X, X')$ can be represented by a 2-extension with all terms finite étale group schemes.

Indeed, let

$$0 \to X' \to Y_1 \xrightarrow{\alpha} Y_2 \to X \to 0$$

be a 2-extension in $\mathcal{R}$. Since Y_1 is finitely generated over $\mathbb{Z}_l$, we have $l^m Y_1 \cap X' = 0$ for some positive integer m big enough. We have the following commutative diagram with exact rows and column:

$$\begin{tikzcd}
 & & l^m Y_1 \arrow[d, hook] & & & \\
0 \arrow[r] & X' \arrow[r] \arrow[d, equal] & Y_1 \arrow[r, "\alpha"] \arrow[d] & Y_2 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X' \arrow[r] & Y_1/l^m Y_1 \arrow[r, "\bar{\alpha}"] & Y_2/\alpha(l^m Y_1) \arrow[r] & X \arrow[r] & 0.
\end{tikzcd}$$

Then the 2-extension

$$0 \to X' \to Y_1/l^m Y_1 \xrightarrow{\alpha} Y_2/\alpha(l^m Y_1) \to X \to 0$$

is equivalent to the original one and has all terms finite. This shows the claim.

The surjectivity follows from the claim.

Now we prove the injectivity. A similar argument shows that any element of the group $\mathrm{Ext}^2_{\mathcal{M}}(X[1], X'[1])$ can be represented by a 2-extension

$$0 \to X' \to Z_1 \xrightarrow{\beta} Z_2 \to X \to 0$$

with Z_1, Z_2 finite l-groups. This 2-extension is the splicing of the following two 1-extensions:

$$\mathcal{E} \quad 0 \to X' \to Z_1 \xrightarrow{\bar{\beta}} \mathrm{im}(\beta) \to 0 \qquad 0 \to \mathrm{im}(\beta) \xrightarrow{i} Z_2 \to X \to 0 \quad \mathcal{F}.$$

If this 2-extension represents the trivial element of $\mathrm{Ext}^2_{\mathcal{R}}(X, X')$, then by [23, chap. VII, lem. 4.1] the 1-extension $\mathcal{F}$ is the pushout of some 1-extension $\mathcal{F}' \in \mathrm{Ext}^1_{\mathcal{R}}(X, Z_1)$ along the morphism $\bar{\beta}$. Since any 1-extension of X by Z_1 in $\mathcal{R}$ also lies in the category of finite étale group schemes, it follows that the original 2-extension also represents the trivial element of $\mathrm{Ext}^2_{\mathcal{M}}(X[1], X'[1])$. This shows the injectivity. $\square$

After some preparation, we can go to the proof of theorem IV.18, which is just an easy consequence of the five lemma.

Proof of IV.18. Let's consider the short exact sequence

$$0 \to X'[1] \to M' \to M'_{\mathrm{tf}} \to 0.$$

We get the following commutative diagram with exact rows

(4.1)

$$\begin{tikzcd}
{} \arrow[r] & \mathrm{Hom}_{\mathcal{M}}(M, M') \otimes \mathbb{Z}_l \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{M}}(M, M'_{\mathrm{tf}}) \otimes \mathbb{Z}_l \arrow[r] \arrow[d, "(1)"'] & \mathrm{Ext}^1_{\mathcal{M}}(M, X'[1]) \otimes \mathbb{Z}_l \arrow[d, "(2)"'] \\
{} \arrow[r] & \mathrm{Hom}_{\mathcal{R}}(T_lM, T_lM') \arrow[r] & \mathrm{Hom}_{\mathcal{R}}(T_lM, T_lM'_{\mathrm{tf}}) \arrow[r] & \mathrm{Ext}^1_{\mathcal{R}}(T_lM, T_lX'[1])
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r] & \mathrm{Ext}^1_{\mathcal{M}}(M, M') \otimes \mathbb{Z}_l \arrow[r] \arrow[d, "(3)"'] & \mathrm{Ext}^1_{\mathcal{M}}(M, M'_{\mathrm{tf}}) \otimes \mathbb{Z}_l \arrow[r, "u"] \arrow[d, "(4)"'] & \mathrm{Ext}^2_{\mathcal{M}}(M, X'[1]) \otimes \mathbb{Z}_l \arrow[d, "(5)"'] \\
{} \arrow[r] & \mathrm{Ext}^1_{\mathcal{R}}(T_lM, T_lM') \arrow[r, "v"] & \mathrm{Ext}^1_{\mathcal{R}}(T_lM, T_lM'_{\mathrm{tf}}) \arrow[r] & \mathrm{Ext}^2_{\mathcal{R}}(T_lM, T_lX'[1]).
\end{tikzcd}$$

Since the group $\mathrm{Ext}^1_{\mathcal{R}}(T_lM, T_lX'[1])$ is torsion, the map v restricted to the free part of $\mathrm{Ext}^1_{\mathcal{R}}(T_lM, T_lM')$ has to be injective. It follows that the map v can be expressed as $v = v_{\mathrm{tor}} \oplus v_{\mathrm{fr}}$, with

$$v_{\mathrm{tor}} : \mathrm{Ext}^1_{\mathcal{R}}(T_lM, T_lM')_{\mathrm{tor}} \to \mathrm{Ext}^1_{\mathcal{R}}(T_lM, T_lM'_{\mathrm{tf}})_{\mathrm{tor}}$$
$$v_{\mathrm{fr}} : \mathrm{Ext}^1_{\mathcal{R}}(T_lM, T_lM')_{\mathrm{fr}} \to \mathrm{Ext}^1_{\mathcal{R}}(T_lM, T_lM'_{\mathrm{tf}})_{\mathrm{fr}},$$

where $\mathrm{Ext}^1_{\mathcal{R}}(T_lM, T_lM')_{\mathrm{tor}}$ (resp. $\mathrm{Ext}^1_{\mathcal{R}}(T_lM, T_lM')_{\mathrm{fr}}$) denotes the torsion (resp. free) subgroup of $\mathrm{Ext}^1_{\mathcal{R}}(T_lM, T_lM')$, and $\mathrm{Ext}^1_{\mathcal{R}}(T_lM, T_lM'_{\mathrm{tf}})_{\mathrm{tor}}$ (resp. $\mathrm{Ext}^1_{\mathcal{R}}(T_lM, T_lM'_{\mathrm{tf}})_{\mathrm{fr}}$) denotes the torsion (resp. free) subgroup of $\mathrm{Ext}^1_{\mathcal{R}}(T_lM, T_lM'_{\mathrm{tf}})$. We know the map (4) maps $\mathrm{Ext}^1_{\mathcal{M}}(M, M'_{\mathrm{tf}}) \otimes \mathbb{Z}_l$ bijectively to $\mathrm{Ext}^1_{\mathcal{R}}(T_lM, T_lM'_{\mathrm{tf}})_{\mathrm{tor}}$, hence the group $\mathrm{im}(u)$ goes to the group $\mathrm{coker}(v_{\mathrm{tor}})$ under the map (5). The torsionness of the groups $\mathrm{Ext}^1_{\mathcal{M}}(M, M')$ and $\mathrm{Ext}^1_{\mathcal{R}}(T_lM, T_lX'[1])$ implies that the map (3) and s have their images lying in $\mathrm{Ext}^1_{\mathcal{R}}(T_lM, T_lM')_{\mathrm{tor}}$, where s denotes the map

$$\mathrm{Ext}^1_{\mathcal{R}}(T_lM, T_lX'[1]) \to \mathrm{Ext}^1_{\mathcal{R}}(T_lM, T_lM').$$

So we get a new commutative diagram with exact rows out of the above diagram

$$\begin{tikzcd}
{} \arrow[r] & \mathrm{Hom}_{\mathcal{M}}(M,M')\otimes\mathbb{Z}_l \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{M}}(M,M'_{\mathrm{tf}})\otimes\mathbb{Z}_l \arrow[r] \arrow[d, "(1)"'] & \mathrm{Ext}^1_{\mathcal{M}}(M,X'[1])\otimes\mathbb{Z}_l \arrow[d, "(2)"'] \\
{} \arrow[r] & \mathrm{Hom}_{\mathcal{R}}(T_lM,T_lM') \arrow[r] & \mathrm{Hom}_{\mathcal{R}}(T_lM,T_lM'_{\mathrm{tf}}) \arrow[r] & \mathrm{Ext}^1_{\mathcal{R}}(T_lM,T_lX'[1])
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r] & \mathrm{Ext}^1_{\mathcal{M}}(M,M')\otimes\mathbb{Z}_l \arrow[r] \arrow[d, "(3)'"'] & \mathrm{Ext}^1_{\mathcal{M}}(M,M'_{\mathrm{tf}})\otimes\mathbb{Z}_l \arrow[r] \arrow[d, "(4)'"'] & \mathrm{im}(u) \arrow[d, "(5)'"'] \\
{} \arrow[r] & \mathrm{Ext}^1_{\mathcal{R}}(T_lM,T_lM')_{\mathrm{tor}} \arrow[r, "v_{\mathrm{tor}}"] & \mathrm{Ext}^1_{\mathcal{R}}(T_lM,T_lM'_{\mathrm{tf}})_{\mathrm{tor}} \arrow[r] & \mathrm{coker}(v_{\mathrm{tor}}).
\end{tikzcd}$$

The injectivity of the map (5) implies the map $(5)'$ is injective, the maps (1), (2) and $(4)'$ are isomorphisms by IV.2, IV.20 and IV.19 respectively, then the map $(3)'$ is an isomorphism by the five lemma. □

Next, we are going to give description to the map T_l for Ext^2 groups.

Theorem IV.22. *Suppose k is a finite field, M' is torsion-free, and g denotes the rank of the $\mathbb{Z}_l$-module $\mathrm{Ext}^1_{\mathcal{R}}(T_lM,T_lM')$. Then the canonical map*

$$T_l : \mathrm{Ext}^2_{\mathcal{M}}(M,M')\otimes\mathbb{Z}_l \to \mathrm{Ext}^2_{\mathcal{R}}(T_lM,T_lM')$$

is an epimorphism and has kernel isomorphic to $(\mathbb{Q}_l/\mathbb{Z}_l)^g$.

Proof. Notations as in IV.19. Let r be a positive integer such that l^r kills the group $\mathrm{Ext}^1_{\mathcal{M}}(M,M')\otimes\mathbb{Z}_l$. We enlarge the same commutative diagram with exact rows used in the proof of IV.19

$$\begin{tikzcd}
{} \arrow[r] & \mathrm{Hom}_{\mathcal{M}}(M, L'_r[1]) \otimes \mathbb{Z}_l \arrow[r] \arrow[d] & \mathrm{Ext}^1_{\mathcal{M}}(M, M') \otimes \mathbb{Z}_l \arrow[r, "l^r"] \arrow[d] & \mathrm{Ext}^1_{\mathcal{M}}(M, M') \otimes \mathbb{Z}_l \arrow[d] \\
{} \arrow[r] & \mathrm{Hom}_{\mathcal{R}}(T_lM, T_lL'_r[1]) \arrow[r] & \mathrm{Ext}^1_{\mathcal{R}}(T_lM, T_lM') \arrow[r, "l^r"] & \mathrm{Ext}^1_{\mathcal{R}}(T_lM, T_lM')
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r] & \mathrm{Ext}^1_{\mathcal{M}}(M, L'_r[1]) \otimes \mathbb{Z}_l \arrow[r] \arrow[d] & \mathrm{Ext}^2_{\mathcal{M}}(M, M') \otimes \mathbb{Z}_l \arrow[r, "l^r"] \arrow[d] & \mathrm{Ext}^2_{\mathcal{M}}(M, M') \otimes \mathbb{Z}_l \arrow[d] \\
{} \arrow[r] & \mathrm{Ext}^1_{\mathcal{R}}(T_lM, T_lL'_r[1]) \arrow[r] & \mathrm{Ext}^2_{\mathcal{R}}(T_lM, T_lM') \arrow[r, "l^r"] & \mathrm{Ext}^2_{\mathcal{R}}(T_lM, T_lM')
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r] & \mathrm{Ext}^2_{\mathcal{M}}(M, L'_r[1]) \otimes \mathbb{Z}_l \arrow[r] \arrow[d] & 0 \\
{} \arrow[r] & \mathrm{Ext}^2_{\mathcal{R}}(T_lM, T_lL'_r[1]) \arrow[r] & 0.
\end{tikzcd}$$

We have the multiplication-by-l^r map in the first row is zero, and by lemma IV.19 we have

$$\mathrm{Ext}^1_{\mathcal{R}}(T_lM, T_lM') \cong (\mathrm{Ext}^1_{\mathcal{M}}(M, M') \otimes \mathbb{Z}_l) \oplus \mathbb{Z}_l^g.$$

Then we get two commutative diagrams with exact rows

$$\begin{tikzcd}
\mathrm{Ext}^1_{\mathcal{M}}(M, M') \otimes \mathbb{Z}_l \arrow[r, hook] \arrow[d, "(1)"'] & \mathrm{Ext}^1_{\mathcal{M}}(M, L'_r[1]) \otimes \mathbb{Z}_l \arrow[r, two heads] \arrow[d, "(2)"'] & {}_{l^r}\mathrm{Ext}^2_{\mathcal{M}}(M, M') \arrow[d, "(3)"'] \\
\mathrm{Ext}^1_{\mathcal{R}}(T_lM, T_lM') \otimes \mathbb{Z}/l^r \arrow[r, hook] & \mathrm{Ext}^1_{\mathcal{R}}(T_lM, T_lL'_r[1]) \arrow[r, two heads] & {}_{l^r}\mathrm{Ext}^2_{\mathcal{R}}(T_lM, T_lM').
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}^2_{\mathcal{M}}(M, M') \otimes \mathbb{Z}/l^r \arrow[r] \arrow[d, "(4)"'] & \mathrm{Ext}^2_{\mathcal{M}}(M, L'_r[1]) \otimes \mathbb{Z}_l \arrow[r] \arrow[d, "(5)"'] & 0 \\
0 \arrow[r] & \mathrm{Ext}^2_{\mathcal{R}}(T_lM, T_lM') \otimes \mathbb{Z}/l^r \arrow[r] & \mathrm{Ext}^2_{\mathcal{R}}(T_lM, T_lL'_r[1]) \arrow[r] & 0.
\end{tikzcd}$$

Write

$$\mathrm{Ext}^2_{\mathcal{M}}(M, M') \otimes \mathbb{Z}_l \cong (\mathbb{Q}_l/\mathbb{Z}_l)^s \oplus S$$

and

$$\mathrm{Ext}^2_{\mathcal{R}}(T_lM, T_lM') \cong \mathbb{Z}_l^d \oplus (\mathbb{Q}_l/\mathbb{Z}_l)^t \oplus T,$$

where $(\mathbb{Q}_l/\mathbb{Z}_l)^t$ and $\mathbb{Z}_l^d$ are the l-divisible subgroup and the torsion-free subgroup of $\mathrm{Ext}^2_{\mathcal{R}}(T_lM, T_lM')$, respectively, $(\mathbb{Q}_l/\mathbb{Z}_l)^s$ is the l-divisible subgroup of $\mathrm{Ext}^2_{\mathcal{M}}(M, M')\otimes$

$\mathbb{Z}_l$, and S, T are two finite groups. The map (2) is an isomorphism by lemma IV.20, and the cokernel of the map (1) is isomorphic to $\mathbb{Z}_l^g \otimes \mathbb{Z}/l^r = (\mathbb{Z}/l^r)^g$, then the snake lemma gives a short exact sequence

$$0 \to (\mathbb{Z}/l^r)^g \to {}_{l^r}\mathrm{Ext}^2_{\mathcal{M}}(M, M') \to {}_{l^r}\mathrm{Ext}^2_{\mathcal{R}}(T_l M, T_l M') \to 0.$$

Taking the direct limit, we get

$$0 \to (\mathbb{Q}_l/\mathbb{Z}_l)^g \to (\mathbb{Q}_l/\mathbb{Z}_l)^s \oplus S \to (\mathbb{Q}_l/\mathbb{Z}_l)^t \oplus T \to 0.$$

Hence we have $s = t + g$ and $S \cong T$ under the map T_l. We also know the map (5) is an isomorphism by III.20, hence so is the map (4). Then we conclude $d = 0$, and this finishes the proof. □

Theorem IV.23. *Let*

$$0 \to X'[1] \to M' \to M'_{\mathrm{tf}} \to 0$$

be the canonical short exact sequence associated to the 1*-motive* M'*, with* $X'[1]$ *the torsion part and* M'_{tf} *the torsion-free part. Consider the diagram 4.1*

$$\begin{tikzcd}
{} \arrow[r] & \mathrm{Hom}_{\mathcal{M}}(M, M') \otimes \mathbb{Z}_l \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{M}}(M, M'_{\mathrm{tf}}) \otimes \mathbb{Z}_l \arrow[r] \arrow[d, "(1)"', "\cong"] & \mathrm{Ext}^1_{\mathcal{M}}(M, X'[1]) \otimes \mathbb{Z}_l \arrow[d, "(2)"', "\cong"] \\
{} \arrow[r] & \mathrm{Hom}_{\mathcal{R}}(T_l M, T_l M') \arrow[r] & \mathrm{Hom}_{\mathcal{R}}(T_l M, T_l M'_{\mathrm{tf}}) \arrow[r] & \mathrm{Ext}^1_{\mathcal{R}}(T_l M, T_l X'[1]) \\
{} \arrow[r] & \mathrm{Ext}^1_{\mathcal{M}}(M, M') \otimes \mathbb{Z}_l \arrow[r] \arrow[d, hook, "(3)"'] & \mathrm{Ext}^1_{\mathcal{M}}(M, M'_{\mathrm{tf}}) \otimes \mathbb{Z}_l \arrow[r] \arrow[d, hook, "(4)"'] & \mathrm{Ext}^2_{\mathcal{M}}(M, X'[1]) \otimes \mathbb{Z}_l \arrow[d, "(5)"', "\cong"] \\
{} \arrow[r] & \mathrm{Ext}^1_{\mathcal{R}}(T_l M, T_l M') \arrow[r] & \mathrm{Ext}^1_{\mathcal{R}}(T_l M, T_l M'_{\mathrm{tf}}) \arrow[r] & \mathrm{Ext}^2_{\mathcal{R}}(T_l M, T_l X'[1]) \\
{} \arrow[r] & \mathrm{Ext}^2_{\mathcal{M}}(M, M') \otimes \mathbb{Z}_l \arrow[r] \arrow[d, "(6)"'] & \mathrm{Ext}^2_{\mathcal{M}}(M, M'_{\mathrm{tf}}) \otimes \mathbb{Z}_l \arrow[r] \arrow[d, two heads, "(7)"'] & 0 \\
{} \arrow[r] & \mathrm{Ext}^2_{\mathcal{R}}(T_l M, T_l M') \arrow[r] & \mathrm{Ext}^2_{\mathcal{R}}(T_l M, T_l M'_{\mathrm{tf}}) \arrow[r] & 0,
\end{tikzcd}$$

then the canonical map

$$T_l = (6) : \mathrm{Ext}^2_{\mathcal{M}}(M, M') \otimes \mathbb{Z}_l \to \mathrm{Ext}^2_{\mathcal{R}}(T_l M, T_l M')$$

is surjective and its kernel fits into the exact sequence

$$0 \to \mathrm{coker}(3) \to \mathrm{coker}(4) \to \ker(6) \to \ker(7) \to 0.$$

Moreover, we have $\mathrm{coker}(3) \cong \mathrm{coker}(4) \cong \mathbb{Z}_l^g$ *and* $\ker(7) \cong (\mathbb{Q}_l/\mathbb{Z}_l)^g$*, with* g *being the rank of the* $\mathbb{Z}_l$*-module* $\mathrm{Ext}^1_{\mathcal{R}}(T_lM, T_lM'_{\mathrm{tf}})$*.*

Proof. First, we know the maps (1), (2), and (5) are isomorphisms by theorem IV.2 and lemma IV.20, the maps (3) and (4) are injective by theorem IV.18, and the map (7) is surjective by theorem IV.22. Then the map (6) is surjective by the five lemma.

Cut the above diagram into the following five small diagrams with exact rows

$$\begin{tikzcd} I: & {} \arrow[r] & - \arrow[r] \arrow[d, "(1)"', "\cong"] & - \arrow[r] \arrow[d, "(2)"', "\cong"] & - \arrow[r] \arrow[d, "(2.5)"] & 0 \\ & {} \arrow[r] & - \arrow[r] & - \arrow[r] & - \arrow[r] & 0 \end{tikzcd} \qquad \begin{tikzcd} II: & 0 \arrow[r] & - \arrow[r] \arrow[d, "(2.5)"'] & - \arrow[r] \arrow[d, hook, "(3)"'] & - \arrow[r] \arrow[d, "(3.5)"'] & 0 \\ & 0 \arrow[r] & - \arrow[r] & - \arrow[r] & - \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} III: & 0 \arrow[r] & - \arrow[r] \arrow[d, "(3.5)"'] & - \arrow[r] \arrow[d, hook, "(4)"'] & - \arrow[r] \arrow[d, "(4.5)"'] & 0 \\ & 0 \arrow[r] & - \arrow[r] & - \arrow[r] & - \arrow[r] & 0 \end{tikzcd} \qquad \begin{tikzcd} IV: & 0 \arrow[r] & - \arrow[r] \arrow[d, "(4.5)"'] & - \arrow[r] \arrow[d, "(5)"', "\cong"] & - \arrow[r] \arrow[d, "(5.5)"] & 0 \\ & 0 \arrow[r] & - \arrow[r] & - \arrow[r] & - \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} V: & 0 \arrow[r] & - \arrow[r] \arrow[d, "(5.5)"'] & - \arrow[r] \arrow[d, two heads, "(6)"'] & - \arrow[r] \arrow[d, two heads, "(7)"'] & 0 \\ & 0 \arrow[r] & - \arrow[r] & - \arrow[r] & - \arrow[r] & 0. \end{tikzcd}$$

In these diamgram, we just use the symbol "-" to indicate objects which can be read off from the maps, and the maps $(n.5)$ are the maps coming from cutting the original diagram along the horizontal arrows between map (n) and map $(n+1)$. From diagram I, we conclude that the map (2.5) is an isomorphism. From diagram II, we conclude that the map (3.5) is injective, and get $\mathrm{coker}(3) \cong \mathrm{coker}(3.5)$. From diagram IV, we conclude the maps (4.5) and (5.5) are injective and surjective respectively, and get $\ker(5.5) \cong \mathrm{coker}(4.5)$. From diagram III, we get an short exact sequence

$$0 \to \mathrm{coker}(3.5) \to \mathrm{coker}(4) \to \mathrm{coker}(4.5) \to 0.$$

From diagram V, we get a short exact sequence

$$0 \to \ker(5.5) \to \ker(6) \to \ker(7) \to 0.$$

To sum up, we get an exact sequence

$$0 \to \mathrm{coker}(3) \to \mathrm{coker}(4) \to \ker(6) \to \ker(7) \to 0.$$

We have $\mathrm{coker}(4) \cong \mathbb{Z}_l^g$ by the definition of g, hence we get $\mathrm{coker}(3) \cong \mathbb{Z}_l^g$. We also have $\ker(7) \cong (\mathbb{Q}_l/\mathbb{Z}_l)^g$ by theorem IV.22. $\square$

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