Locally trivial monodromy of moduli spaces of sheaves on abelian surfaces

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Setting

Irreducible symplectic varieties from moduli spaces

We fix a triple (S, v, H), with S be a projective K3 or abelian surface, $v \in H^{even}(S,\mathbb{Z})$ a Mukai vector such that $v^2 > 0$ and $H \in Amp(S)$ a $v-$generic polarization.

$\rightarrow$ $M_v(S,H) :=$ **moduli space** of Gieseker $H-$semistable sheaves F on S such that

$$\mathrm{ch}(F)\sqrt{\mathrm{td}(S)} = v.$$

If S is abelian, let $a_v \colon M_v(S,H) \longrightarrow S \times \hat{S}$ be the associated *Yoshioka fibration* ([Yos01],[Yos99]).

$\rightarrow$ $K_v(S,H) := a_v^{-1}(0_S, \mathcal{O}_S)$.

If v is primitive, $M_v(S,H)$ and $K_v(S,H)$ are irreducible holomorphic symplectic manifolds ([Muk84], [Yos00],[Yos01]).

If v is not primitive, $M_v(S,H)$ and $K_v(S,H)$ are irreducible symplectic varieties ([PR23], [KLS06]).

Locally trivial monodromy operators

Let X, X_1, X_2 be ISVs.

An isometry $g \in O(H^2(X_1,\mathbb{Z}), H^2(X_2,\mathbb{Z}))$ is a **locally trivial parallel transport operator** if there exists a locally trivial family $p \colon \mathcal{X} \longrightarrow T$ and two points $t_1, t_2 \in T$, with $X_i = p^{-1}(t_i)$ for $t_i = 1, 2$, s.t. g is the parallel transport along a path in T from t_1 to t_2 in the local system $R^2p_*\mathbb{Z}$.

$\rightarrow$ $\mathrm{PT}^2_{\mathrm{lt}}(X_1, X_2) := \{$l.t. parallel transport operators from X_1 to $X_2\} \subseteq O(H^2(X_1,\mathbb{Z}), H^2(X_2,\mathbb{Z}))$.

$\rightarrow$ **Monodromy group of** X: $\mathrm{Mon}^2_{\mathrm{lt}}(X) := \mathrm{PT}^2(X,X) \leq O^+(H^2(X,\mathbb{Z}))$ (of finite index).

Theorem 1 Hodge-theoretic Global Torelli Thm, singular version, [BL22]

Let X_1, X_2 be two ISVs. Then X_1 and X_2 are bimeromorphic if and only if there exists a l.t parallel transport operator $g \colon H^2(X_1,\mathbb{Z}) \longrightarrow H^2(X_2,\mathbb{Z})$ which is a Hodge isometry.

Starting point/strategy

The non primitive $K3$ case
(Onorati-Perego-Rapagnetta, 2023, [OPR24])

Let (S, v, H) a triple as above, with S a $K3$ surface, $v = mw$ with $m > 1$ and w primitive and such that $w^2 \geq 2$.

- If X is an ISV l.t. deformation equivalent to $M_v(S,H)$, then its most singular locus $Y \subseteq X$ is an IHSM deformation equivalent to $M_w(S,H)$. The closed embedding $\iota_{Y,X} \colon Y \longrightarrow X$ induces an isomorphism
$$\iota^\#_{Y,X} \colon \mathrm{Mon}^2_{\mathrm{lt}}(X) \longrightarrow \mathrm{Mon}^2(Y).$$
- If Y is a IHSM deformation equivalent to $M_w(S,H)$, by [Mar08] and [Mar10], $\mathrm{Mon}^2(Y) \simeq \mathrm{W}(Y)$, where the latter is the group of orientation preserving isometries of $H^2(Y,\mathbb{Z})$ acting as $\pm\mathrm{id}$ on the discriminant group.

The primitive abelian case
(Markman, 2018, [Mar22] - Mongardi, 2014, [Mon16])

Let Y be an irreducible holomorphic symplectic manifold deformation equivalent to $M_w(S,H)$, with S an abelian surface and w primitive such that $w^2 \geq 6$. Then there is an isomorphism

$$\mathrm{Mon}^2(Y) \simeq \mathrm{W}(Y)^{\det\cdot\chi} := \ker(\det\cdot\chi \colon \mathrm{W}(Y) \longrightarrow \mathbb{C}^\times),$$

where $\chi \colon \mathrm{W}(Y) \longrightarrow \{\pm 1\}$ is the character associated to $\mathrm{W}(Y)$.

An injective morphism $\mathrm{Mon}^2_{\mathrm{lt}}(K_v(S,H)) \longrightarrow \mathrm{Mon}^2(K_w(S,H))$

Let (S, v, H) be a triple as above with S an abelian surface, $v = mw$ with $m > 1$ and w primitive such that $w^2 \geq 6$.

The embedding $K_w(S,H) \longrightarrow K_v(S,H)$

1. By [Kal06], M_v admits a stratification of the singularities $M_v \supseteq X_1 \supseteq \cdots \supseteq X_l =: M_v^{singmax}$, whose smaller stratus, by [KLS06], is $M_v^{singmax} = \{E^{\oplus m} \in M_v \colon E \in M_w\} \simeq M_w$.
2. This description is compatible with the Yoshioka fibrations involved and we have

$$\begin{aligned} K_v^{singmax} = M_v^{singmax} \cap K_v &= \{E^{\oplus m} \in M_v \colon E \in M_w, \quad a_v(E^{\oplus m}) = (0_S, \mathcal{O}_S)\} = \\ &= \{E^{\oplus m} \in M_v \colon E \in M_w, \quad a_w(E)^m = (0_S, \mathcal{O}_S)\} \simeq \bigcup_{(p,L)\in S[m]\times\hat{S}[m]} a_w^{-1}(p,L) \end{aligned}$$

Proposition 2

For all $(p, L) \in S[m] \times \hat{S}[m]$ there is a closed embedding

$$\begin{aligned} \tau_{(p,L)} \colon K_w &\longrightarrow K_v^{singmax} \subseteq K_v \\ E &\longmapsto (\tau_{p,*}(E) \otimes L)^{\oplus m}. \end{aligned}$$

that behaves well in locally trivial families.

Action on the second integral cohomology

Set $\iota_{w,m} := \tau_{(0_S,\mathcal{O}_S)}$ and let us compare (by [PR24])

$$\begin{tikzcd} H^2(K_v,\mathbb{Z}) \arrow[r, "\sim", "\lambda^{-1}_{(S,v,H)}"'] \arrow[rrr, bend left, "\iota^*_{w,m}"] & v^\perp \arrow[r, equal] & w^\perp \arrow[r, "\sim", "\lambda_{(S,w,H)}"'] & H^2(K_w,\mathbb{Z}) \end{tikzcd}$$

Proposition 3

The morphism $\iota^*_{w,m} \colon H^2(K_v,\mathbb{Z}) \longrightarrow H^2(K_w,\mathbb{Z})$ is a similitude of lattices satisfying

$$\iota^*_{w,m} = m\lambda_{(S,w,H)} \circ \lambda^{-1}_{(S,v,H)}.$$

By conjugation (and passing through $\mathbb{Q}-$linear extensions), we get the following action on integral isometries:

Proposition 4

The morphism $\iota^*_{w,m}$ induces an isomorphism

$$\iota^\#_{w,m} \colon O(H^2(K_v,\mathbb{Z})) \longrightarrow O(H^2(K_w,\mathbb{Z}))$$

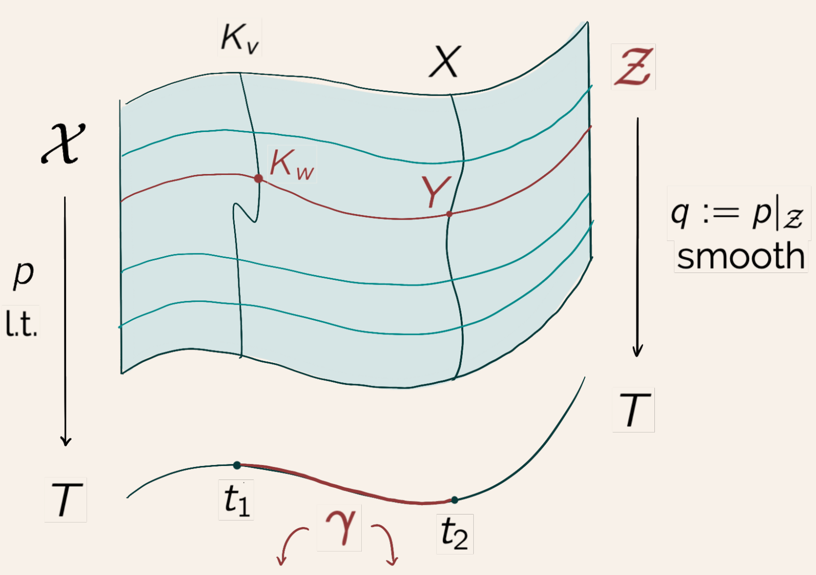
satisfying the identity $\iota^\#_{w,m} = \lambda^\#_{(S,w,H)} \circ (\lambda^\#_{(S,v,H)})^{-1}$.

Theorem 5

The isomorphism $\iota^\#_{w,m}$ restricts to an injective morphism

$$\iota^\#_{w,m} \colon \mathrm{Mon}^2_{\mathrm{lt}}(K_v(S,H)) \longrightarrow \mathrm{Mon}^2(K_w(S,H)).$$

Sketch.



$\mathrm{pt}_p(\gamma) \colon H^2(K_v,\mathbb{Z}) \to H^2(X,\mathbb{Z})$ $\xmapsto{\iota^\#_{w,m}}$ $\mathrm{pt}_q(\gamma) \colon H^2(K_w,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$

l.t. parallel transp. — parallel transp.

More generally, if (S_1, v_1, H_1) and (S_2, v_2, H_2) are two triples as above, with $v_i = mw_i$ for $i = 1, 2$ and $w_1^2 = w_2^2$, there is an injective map

$$\iota^\#_{w_1,w_2,m} \colon : \mathrm{PT}^2_{lt}(K_{v_1}(S_1,H_1), K_{v_2}(S_2,H_2)) \longrightarrow \mathrm{PT}^2(K_{w_1}(S_1,H_1), K_{w_2}(S_2,H_2)).$$

Corollary 6

If X is an ISV l.t. deformation equivalent to $K_v(S,H)$ and Y is a connected component of the most singular locus, the embedding $\iota_{Y,X} \colon Y \longrightarrow X$ induces an injective morphism

$$\iota^\#_{Y,X} \colon \mathrm{Mon}^2_{\mathrm{lt}}(X) \longrightarrow \mathrm{Mon}^2(Y).$$

Ideas to prove surjectivity (following [OPR24])

WLOG we may assume that $X = K_v(S,H)$ and $Y = K_w(S,H)$.

$$\begin{tikzcd} \mathrm{Mon}^2_{\mathrm{lt}}(K_v) \arrow[rrr, "\iota^\#_{w,m}"] & & & \mathrm{Mon}^2(K_w) \arrow[d, equal, "\wr\ {[Mar22], [Mon16]}"] \\ \mathrm{W}(K_v)^{\det\cdot\chi} \arrow[u, dashed, "?"] \arrow[r, "{\lambda^\#_{(S,v,H)}}^{-1}", "\sim"'] \arrow[rrr, bend right, "\iota^\#_{w,m}"'] & \mathrm{W}(v^\perp)^{\det\cdot\chi} \arrow[r, equal] & \mathrm{W}(w^\perp)^{\det\cdot\chi} \arrow[r, "\lambda^\#_{(S,w,H)}", "\sim"'] & \mathrm{W}(K_w)^{\det\cdot\chi} \end{tikzcd}$$

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