Degree Formula for Lagrangian Fibrations

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What could be the discriminant divisor?

Let M be an irreducible holomorphic symplectic manifold of complex dimension $2n$, and suppose

$$f : M \to \mathbb{P}^n$$

is a Lagrangian fibrations. The *discriminant locus* is the set

$$\Delta = \{ b \in \mathbb{P}^n : f^{-1}(b) \text{ is singular} \}.$$

Hwang and Oguiso proved that Δ has codimension one[1]. One may wonder if this is the support of a somewhat natural divisor. In other words, what is a natural way to assign weights $w_i \in \mathbb{N}$ to the irreducible components $\Delta_i \subset \Delta$?

The ΩT-complex

The ΩT-complex associated to f is the cochain complex

$$f^*\Omega_{\mathbb{P}^n} \to (\Omega_M \cong T_M) \to f^*T_{\mathbb{P}^n}. \qquad (\Omega T)$$

The first map is injective, so there are two cohomology sheaves

$$H^1(\Omega T) \quad \text{and} \quad H^2(\Omega T),$$

which are supported on $f^{-1}(\Delta)$.

It is noteworthy that dualizing reproduces the ΩT-complex up to a sign in the isomorphism $\Omega_M \cong T_M$. This can be exploited to prove

$$H^1(\Omega T) \cong \mathcal{E}xt^1_M(H^2(\Omega T), \mathcal{O}_M). \qquad (1)$$

The definition of weights

Choose a Kähler class $\omega \in H^2(M, \mathbb{C})$, which restricts to an integral, non-divisible cohomology class of type $(d_1, \dots, d_n)$ on the smooth fibers. Let F_i be a general singular fiber over the component $\Delta_i \subset \Delta$, and define

$$w_i = n(d_1 \cdots d_n)^{-1} \int_N (c_2(H^1(\Omega T|_{F_i})) - c_2(H^2(\Omega T|_{F_i})))\omega^{n-1},$$

where $N = f^{-1}(\ell)$ is the preimage of a general line $\ell \subset \mathbb{P}^n$ which contains $f(F_i)$. Following techniques of Sawon[4], C. Lehn proved in his PhD thesis[2]

$$\sum_i w_i \deg(\Delta_i) = 24\left(\frac{n! \int_M \sqrt{\hat{A}(M)}}{d_1 \cdots d_n}\right)^{\frac{1}{n}}.$$

Main Problem

Does this definition depend on the choice of ω?

Independence for smooth fibers

Note that the polarisation type $(d_1, \dots, d_n)$ does not depend on the choice of ω. This is due to a theorem of Voisin[5], combined with Matsushita's result that Lagrangian fibrations deform in codimension one[3].

Theorem 1. *Let $F \subset M$ be a smooth fiber. Then the restriction map*

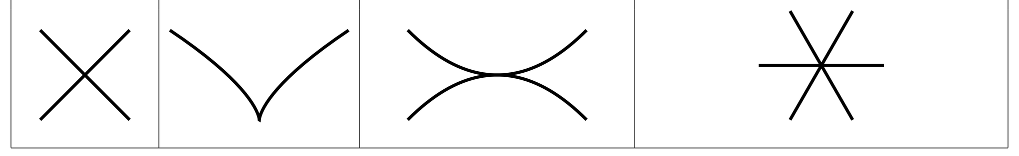
$$H^2(M, \mathbb{Q}) \to H^2(F, \mathbb{Q})$$

has rank one.

Local analysis of the $H^i(\Omega T)$

Hwang and Oguiso also proved that a general singular fiber looks locally like a $(n-1)$-dimensional complex disc times one of the four curve singularities depicted in the following table[1].

Node	Cusp	Tangent	Triple intersection
I	II	III	IV
$xy = 0$	$x^3 = y^2$	$y(y - x^2) = 0$	$x(x - y)y = 0$



The sheaf $H^2(\Omega T|_{F_i})$ is the cokernel of the differential map $T_N \to f^*T_\ell$. Since f^*T_ℓ is trivial in a neighbourhood of F_i, $H^2(\Omega T|_{F_i})$ is the structure sheaf of its support. Using the local picture of Hwang and Oguiso one can then fully describe its second Chern class in N. Combining this with (1), one obtains Theorem 2.

The singular locus $\mathrm{Sing}(F_i)_{\mathrm{red}}$ is a disjoint union of $(n-1)$-dimensional tori $Z_{i1}, \dots, Z_{ir}$. The tori Z_{ik} are called *singular tori*. Let μ_k be the Milnor number of the curve singularity at Z_k.

Theorem 2. *The weight w_i associated to F_i satisfies*

$$w_i = \left(d_1 \cdots d_n\right)^{-1} \cdot n \cdot \sum_{k=1}^{r} a_k \int_{Z_{ik}} \omega^{n-1},$$

where

$$a_k = \begin{cases} 1 + \frac{(a-b)^2}{ab} & \text{if } Z_k \text{ is an intersection of type } I \text{ with local multiplicities } a, b \\ \mu_k & \text{if } Z_k \text{ is of type } II, III \text{ or } IV. \end{cases}$$

Moving singular tori to smooth fibers

Theorem 2 allows us to solve the Main Problem, by proving:

Theorem 3. *The restriction map $H^2(M, \mathbb{Q}) \to H^2(Z, \mathbb{Q})$ has rank one.*

The strategy is to write down a homotopy which moves Z into a smooth fiber. Then Theorem 1 implies 3.

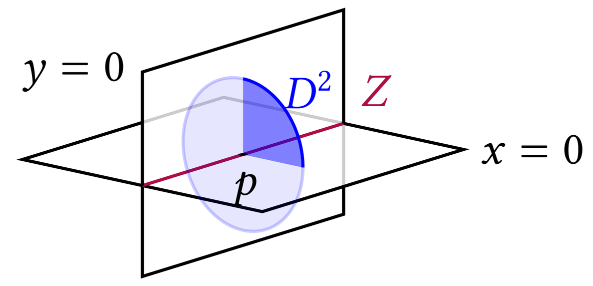
Let $X_1, \dots, X_{n-1}$ be the Hamiltonian vector fields in a neighbourhood N_i of F_i, which give a flow

$$\Phi : \mathbb{C}^{n-1} \times N_i \to N_i.$$

Pick a base point $p \in Z$, to get the universal covering $\nu : \mathbb{C}^{n-1} \to Z$, and the *period lattice*

$$\Lambda = \nu^{-1}(p) \subset \mathbb{C}^{n-1}.$$

Choose a complex two-dimensional disc $D = D^2 \subset N$, centered at p and transverse to Z.



By the theorem on implicit functions, there is for each period $\lambda \in \Lambda$ a map of germs $\tilde{\lambda} : (D, p) \to (\mathbb{C}^{n-1}, \lambda)$, such that $\Phi(\tilde{\lambda}(q), q) \in D$ for each $q \in D$. This induces an isomorphism

$$\hat{\lambda} : (D, p) \to (D, p),$$

which preserves the fibers of f. On local rings one gets an isomorphism

$$\hat{\lambda}^* : \mathbb{C}\{x, y\} \to \mathbb{C}\{x, y\}, \quad x \mapsto \xi, y \mapsto \eta$$

which satisfies $f(x, y) = f(\xi, \zeta)$. For example if Z has type I, then

$$\xi^k \eta^\ell = x^k y^\ell.$$

The next step is to write down a homotopy $H_t(x, y)$ from $H_0 = \hat{\lambda}^*$ to $H_1 = \mathrm{id}$. Before this can be done, one has to replace λ by a multiple, because $\hat{\lambda}$ could act non-trivially on the connected components of the fiber in D. For example the coordinate change $x \mapsto y, y \mapsto x$ cannot be deformed to the identity while preserving fibers of f.

In the case of type I singularities, one can define the first half of the homotopy $H_t = (\xi_t, \eta_t)$ for $0 \le t \le \frac{1}{2}$ by

$$\xi_t = \frac{\xi((1-2t)x, (1-2t)y)}{1 - 2t}, \qquad \eta_t = \frac{\eta((1-2t)x, (1-2t)y)}{1 - 2t}.$$

Then $\xi_0 = \xi, \eta_0 = \eta$, and

$$(\xi_{1/2}, \eta_{1/2}) = (ax, by)$$

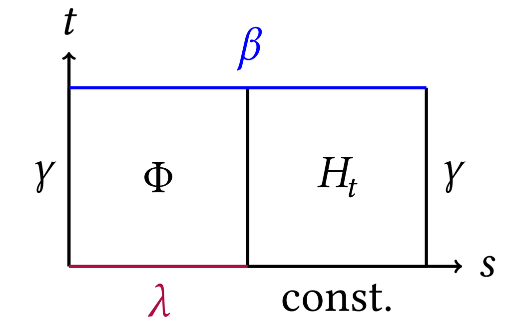
where $a, b \in \mathbb{C}$ satisfy $a^k b^\ell = 1$. Since $\hat{\lambda}$ preserves the connected components of $\{x^k y^\ell = 1\}$, the pair (a, b) is in the same component as $(1, 1)$. A path from (a, b) to $(1, 1)$ inside $\{x^k y^\ell = 1\}$ completes the homotopy.

Moving cycles to smooth fibers cont'd

Using the homotopy H one can easily move 1-cycles from Z to a smooth fiber. Each 1-cycle in Z corresponds to a period $\lambda \in \Lambda$. Assume λ admits a homotopy H as before. Consider the path $\gamma(t) = (t, t)$, in D, which starts at p and moves into a smooth fiber F_{sm}. The singular square

$$\sigma : [0,1]^2 \to M, \sigma(s,t) = \begin{cases} \Phi(2t\tilde{\lambda}(\gamma(s)), \gamma(s)) & \text{for } t \le \frac{1}{2} \\ H_{2t-1}(\hat{\lambda}(\gamma(s))) & \text{for } t \ge \frac{1}{2} \end{cases}$$

then moves λ into F_{sm}.



Using more complicated higher-dimensional singular cubes one can also move higher-dimensional cycles which are obtained by combining multiple periods.

Weights via the characteristic cycle

Theorem 4. *If the characteristic cycle Θ_i of F_i is compact, then*

$$w_i = \frac{\chi(\Theta_i)}{\int_{\Theta_i} \omega}.$$

The main ingredients here are of course the classification of characteristic cycles and the fact that singular tori Z have numerically trivial normal bundle in the components of F_i.

References

[1] Jun-Muk Hwang and Keiji Oguiso. "Characteristic foliation on the discriminant hypersurface of a holomorphic Lagrangian fibration". In: *Amer. J. Math.* 131.4 (2009), pp. 981–1007. ISSN: 0002-9327. DOI: 10.1353/ajm.0.0062. URL: https://doi.org/10.1353/ajm.0.0062.

[2] Christian Lehn. "Symplectic Lagrangian Fibrations". PhD thesis. Johannes Gutenberg-Universität Mainz, 2011. DOI: http://doi.org/10.25358/openscience-3153.

[3] Daisuke Matsushita. "On Deformations of Lagrangian Fibrations". In: *K3 Surfaces and Their Moduli* (Jan. 2016). DOI: 10.1007/978-3-319-29959-4_9. URL: http://dx.doi.org/10.1007/978-3-319-29959-4_9.

[4] Justin Sawon. "On the discriminant locus of a Lagrangian fibration". In: *Mathematische Annalen* 341.1 (2008), pp. 201–221. ISSN: 0025-5831. DOI: 10.1007/s00208-007-0189-9.

[5] Claire Voisin. "Sur la stabilité des sous-variétés lagrangiennes des variétés symplectiques holomorphes". In: *Complex projective geometry (Trieste, 1989/Bergen, 1989)*. Vol. 179. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 1992, pp. 294–303. DOI: 10.1017/CBO9780511662652.022.