Coble duality and Kummer fourfolds

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Coble cubic and duality

MAIN OBJECTS: C, $A = JC$, $\mathcal{C}_3 \subset \mathbb{P}^8$, $\mathcal{C}_6 \subset (\mathbb{P}^8)^\vee$.

Let (A, Θ) be a (very general) principally polarized abelian variety. The map associated to the linear system $|3\Theta|$ is an embedding

$$\phi_{|3\Theta|} : A \hookrightarrow |3\Theta|^\vee \cong \mathbb{P}^8$$

- *(Coble, 1917 - Beauville, 2003).* There is exactly one cubic $\mathcal{C}_3 \subset |3\Theta|^\vee$ containing $\phi_{|3\Theta|}(A)$, singular along $\phi_{|3\Theta|}(A)$ itself.

There is a curve C of genus 2 whose Jacobian is A, $JC \cong A$. The moduli space of rank 3 semi-stables vector bundles over C is $\mathcal{SU}_C(3)$.

- *(Dolgachev, 1996).* There is a natural double cover $r : \mathcal{SU}_C(3) \longrightarrow |3\Theta|$, ramified along a sextic $\mathcal{C}_6$.
- *(Dolgachev, 1996).* **QUESTION**: is $\mathcal{C}_6 \subset |3\Theta|$ projectively dual to $\mathcal{C}_3 \subset |3\Theta|^\vee$?
- *(Ortega, 2005 - Nguyen, 2007).* **YES!** The image of $\mathcal{C}_3$ via the associated Gauss map is indeed $\mathcal{C}_6$.

$$|3\Theta|^\vee \supset \mathcal{C}_3 \xrightarrow{Gauss} \mathcal{C}_6 \subset |3\Theta| \tag{1}$$

And the story is not finished yet...

How about $\mathrm{Kum}_2\, A$??

GOAL: study projective models of Kummer fourfolds. Very few are known!

$\mathrm{Kum}_2\, A$ admits two big+nef, non-ample divisors [Y]:
$D_1 = \Theta_2$ and $D_1 = 2\Theta_2 - \delta$,
whose square is minimal ($q_{BB}(D_1) = q_{BB}(D_2) = 2$).

The linear system $|D_1|$ [BMT]

There is a natural map $f : \mathrm{Kum}_2\, A \to \mathcal{SU}_C(3)$. The composition with the double cover is identified with $\phi_{|D_1|}$,

$$\phi_{|D_1|} = r \circ f : \mathrm{Kum}_2\, A \to \mathcal{SU}_C(3) \to |3\Theta| \cong |D_1|^\vee. \tag{2}$$

The image of the Kummer fourfold lies in the sextic $\mathcal{C}_6$!

The linear system $|D_2|$

Our main tool is the action of $A[3] \cong \left(\frac{\mathbb{Z}}{3\mathbb{Z}}\right)^4$ (the 3-torsion points of A) over $\mathrm{Kum}_2\, A$. The 3-torsion $A[3]$ acts trivially on the second cohomology group of $\mathrm{Kum}_2\, A$ and the maps associated to the linear systems $|D_2|$ and $|D_1|$ are equivariant with respect to the action of $A[3]$.

Proposition 1. *(A-GR) The linear system $|D_2|$ is base-point-free.*

The 3-torsion acts also over $|3\Theta|$. Both actions lifts to the action of a central extension H_3 on $H^0(A, 3\Theta)$ and $H^0(\mathrm{Kum}_2\, A, 2\Theta_2 - \delta)$,

$$1 \longrightarrow \mu_3 \longrightarrow H_3 \longrightarrow A[3] \longrightarrow 1.$$

In both cases, it is the the so-called *irreducible Schrödinger representation* (classical for A, very recently proved by O'Grady for $\mathrm{Kum}_2\, A$ [OG]). We are able to make O'Grady's result effective, by finding a copy of A inside the image of $\mathrm{Kum}_2\, A$.

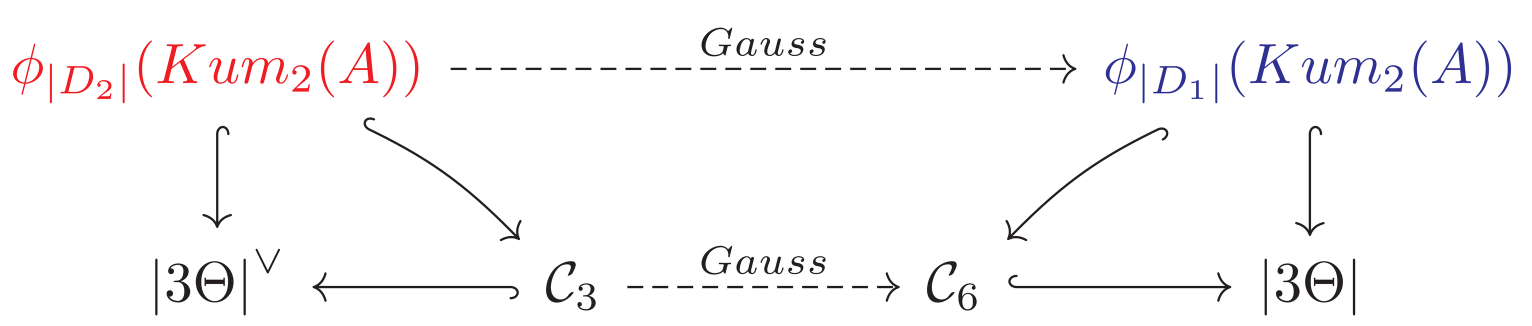
Proposition 2. *(A-GR) The image $\phi_{|D_2|}(\mathrm{Kum}_2\, A)$ contains A, projectively embedded via the linear system $|3\Theta|$. This yields a natural identification*

$$\phi_{|D_2|} : \mathrm{Kum}_2\, A \longrightarrow |D_2|^\vee \cong |3\Theta|^\vee. \tag{3}$$

QUESTIONS. Pulling together (1), (2) and (3): Does the Gauss map send $\phi_{|D_2|}(\mathrm{Kum}_2\, A)$ to $\phi_{|D_1|}(\mathrm{Kum}_2\, A)$? Does $\phi_{|D_2|}(\mathrm{Kum}_2\, A) \subset \mathcal{C}_3$?

Duality

Theorem 1. *(A-GR) The composition of $\phi_{|D_2|}$ with the Gauss map associated to $\mathcal{C}_3$ is $\phi_{|D_1|}$,*



One step further

Corollary 1. *(A-GR) $\phi_{|D_2|}(\mathrm{Kum}_2\, A)$ is contained in the cubic $\mathcal{C}_3$.*
Corollary 2. *(A-GR) The map $\phi_{|D_2|}$ is birational onto its image.*

Some interesting upcoming questions:

- Let (X, H) be a general deformation of $(\mathrm{Kum}_2\, A, D_2)$. Is $\phi_{|H|}$ still birational?
- Can we find explicit equations for $\phi_{|D_2|}(Kum_2(A))$?
- Can we describe the image in $|3\Theta|$ to the point of having a good grasp on the image of $\phi_{|H|}$ as above?

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