Non-symplectic automorphisms of prime order of OG10 and cubic fourfolds

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Abstract

We give a lattice-theoretic classification of non-symplectic automorphisms of prime order of irreducible holomorphic symplectic manifolds of OG10-type. We determine which automorphisms are induced by a non-symplectic automorphism of prime order of a cubic fourfold on the associated Laza–Saccà–Voisin manifolds, giving a geometric and lattice-theoretic description of the algebraic and transcendental lattices of the cubic fourfold. As an application we discuss the rationality conjecture for a general cubic fourfold with a non-symplectic automorphism of prime order.

Non-symplectic automorphisms on manifolds of OG10-type

An irreducible holomorphic symplectic (IHS) manifold X is a compact Kähler manifold such that
- $\pi_1(X) = \{\mathrm{id}\}$,
- $\mathrm{H}^2(X, \Omega_X^2) \cong \mathbb{C}\sigma_X$ with σ_X a symplectic form,

it is called OG10-type if it is deformation equivalent to O'Grady's ten dimensional example [5].

An automorphism $f \in \mathrm{Aut}(X)$ is called symplectic if $f^*(\sigma_X) = \sigma_X$, it is called non-symplectic otherwise. Similarly, if Y is a smooth cubic fourfold, an automorphism $f\phi \in \mathrm{Aut}(Y)$ is called symplectic if $\phi^*(\sigma_Y) = \sigma_Y$ and non-symplectic otherwise, where $\mathbb{C}\sigma_Y \cong \mathrm{H}^1(Y, \Omega_Y^3)$.

The lattice of OG10 manifolds

Let X be a manifold of OG10-type, then there is the following lattice isometry

$$\mathrm{H}^2(X, \mathbb{Z}) \cong \mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U}^{\oplus 3} \oplus \mathbf{A}_2(-1) =: \mathbf{L}$$

where $\mathrm{H}^2(X, \mathbb{Z})$ is endowed with the Beauville-Bogomolov-Fujiki form. The lattice $\mathbf{L}$ has a unique embedding in the unimodular lattice

$$\mathbf{\Lambda} := \mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U}^{\oplus 5}$$

with orthogonal complement isometric to $\mathbf{A}_2$.

The lattice of smooth cubic fourfolds

Let $Y \subset \mathbb{P}^5$ be a smooth cubic fourfold, then the lattice $\mathrm{H}^4(Y, \mathbb{Z})$ given by the Poincaré pairing is unimodular, more precisely there is an isometry

$$\mathrm{H}^4(Y, \mathbb{Z}) \cong [1]^{\oplus 21} \oplus [-1]^{\oplus 2}.$$

Let $\eta_Y \in \mathrm{H}^4(Y, \mathbb{Z})$ be the class of a hyperplane section and let $\mathrm{H}^4_p(Y, \mathbb{Z})$ be its othogonal complement, then there is an isometry

$$\mathrm{H}^4_p(Y, \mathbb{Z}) \cong \mathbf{U}^{\oplus 2} \oplus \mathbf{E}_8^{\oplus 2} \oplus \mathbf{A}_2 =: \mathbf{F}\,.$$

The classification for OG10

An isometry of the lattice $\mathbf{L}$ can be extended to an isometry of the lattice $\mathbf{\Lambda}$, which is unimodular. In [1], the authors give numerical criteria for prime-order isometries on unimodular lattices.

Theorem 1. *We classify all the pairs $(\mathbf{L}^f, \mathbf{L}_f)$ of invariant and coinvariant lattices for a non-symplectic automorphism of prime order $f \in \mathrm{Aut}(X)$ with X of OG10-type.*

We recall that if X is general of OG10-type with an non-symplectic automorphism $f \in \mathrm{Aut}(X)$, then we have $\mathrm{H}^{1,1}(X, \mathbb{Z}) \cong \mathbf{L}^f$.

The classification for cubic fourfolds

Cubic fourfolds with an automorphism of prime order are classified in [2], the possible non-symplectic automorphisms have order two or three.

- ϕ_2^1: $p = 2$, $F = L_3(x_0, \ldots, x_4) + x_5^2 L_1(x_0, \ldots, x_4)$,
- ϕ_2^3: $p = 2$, $F = L_3(x_0, x_1, x_2) + x_0 L_2(x_3, x_4, x_5) + x_1 M_2(x_3, x_4, x_5) + x_2 N_2(x_3, x_4, x_5)$,
- ϕ_3^1: $p, = 3$ $F = L_3(x_0, \ldots, x_4) + x_5^3$,
- ϕ_3^2: $p = 3$, $F = L_3(x_0, \ldots, x_3) + M_3(x_4, x_5)$,
- ϕ_3^5: $p = 3$, $F = L_3(x_0, x_1, x_2) + M_3(x_3, x_4) + x_5^3 + x_3 x_5 L_1(x_0, x_1, x_2) + x_4 x_5 M_1(x_0, x_1, x_2)$,
- ϕ_3^7: $p = 3$, $F = x_2 L_2(x_0, x_1) + x_3 M_2(x_0, x_1) + x_4^2 L_1(x_0, x_1) + x_4 x_5 M_1(x_0, x_1) + x_5^2 N_1(x_0, x_1) + x_4 N_2(x_2, x_3) + x_5 O_2(x_2, x_3)$

where L_i, M_i, N_i and O_i are homogeneous polynomials of degree i.

Recall that if Y is a general cubic fourfold with a non-symplectic automorphism $\phi \in \mathrm{Aut}(Y)$, then we have $\mathrm{H}^{2,2}_p(Y, \mathbb{Z}) \cong \mathbf{F}^\phi$.

Theorem 2. *We classify all the pairs $(\mathbf{F}^\phi, \mathbf{F}_\phi)$ of invariant and coinvariant lattices for a non-symplectic automorphism of prime order $\phi \in \mathrm{Aut}(Y)$ of a cubic fourfold Y.*

The case of involutions was already studied by Marquand in [4].

Corollary 3. *A general cubic fourfold with automorphism ϕ_3^1 or ϕ_3^5 has no associated K3. A general cubic fourfold with automorphism ϕ_3^2 or ϕ_3^7 has an associated K3 and it is rational.*

LSV manifolds

Let Y be a cubic fourfold and $U \subset (\mathbb{P}^5)^\vee$ the open set parametrizing the smooth hyperplane sections of Y. Consider the fibration

$$J_U(Y) \to U$$

whose fiber over a hyperplane H is give by the intermediate Jacobian of the hyperplane section $Y_H = Y \cap H$. Consider $J_U^t(Y) \to U$ the fibration of twisted intermediate Jacobians.

Theorem 4 (Laza-Saccà-Voisin,Voisin)**.** *There are compactifications $J(Y) \to \mathbb{P}^5$ and $J^t(Y) \to \mathbb{P}^5$ of $J_U(Y) \to U$ with $J(Y)$ and $J^t(Y)$ of OG10-type.*

The manifold $J(Y)$ is called LSV manifold and $J^t(Y)$ is called twisted LSV manifold. We have a lattice $\mathbf{U}_Y \subseteq \mathrm{NS}(J(Y))$ isometric to $\mathbf{U}$ and a lattice $\mathbf{U}_Y^t \subseteq \mathrm{NS}(J^t(Y))$ isometric to $\mathbf{U}(3)$.

Induced action on the LSV manifolds

Let $\phi \in \mathrm{Aut}(Y)$ be an automorphism of the cubic fourfold, then ϕ is the restriction of a linear transformation of $\mathbb{P}^5$, this induces an action on the hyperplane sections of the cubic fourfold and hence on the fibration $J_U(Y) \to U$. There are induced birationalities

$$f \in \mathrm{Bir}(J(Y)),\ f^t \in \mathrm{Bir}(J^t(Y)).$$

Proposition 5. *Let Y be a general cubic fourfold with a non-symplectic automorphism of prime order $\phi \in \mathrm{Aut}(Y)$, then the induced birational tranformations extend to automorphisms $f \in \mathrm{Aut}(J(Y))$, $f^t \in \mathrm{Aut}(J^t(Y))$.*

The twisted LSV

There is a Hodge isometry

$$\mathrm{H}^4(Y, \mathbb{Z})_{prim}(-1) \cong (\mathbf{U}_Y^t)^\perp \subset \mathrm{H}^2(J^t(Y), \mathbb{Z}),$$

combining this with the Torelli Theorem for cubic fourfold [3] we get the following criterion.

Theorem 6. *A manifold X of OG10-type is birational to $J^t(Y)$ for a cubic fourfold Y if and only if:*

- *there is an embeddng $\mathbf{U}(3) \subset \mathrm{NS}(X)$ such that $\mathbf{U}(3)$ contains a vector of divisibility 3 in $\mathrm{H}^2(X, \mathbb{Z})$.*
- *$\mathbf{U}(3)^\perp \subset \mathrm{NS}(X)$ cointains no vectors of square -2 and no vectors of square -6 and divisibility 3.*

This leads to the following:

Corollary 7. *Let Y be a general cubic fourfold with a non-symplectic automorphism of prime order and consider the induced automorphism on $J^t(Y)$. Then the possible invariant lattices for $J^t(Y)$ are*

- *$\mathbf{U} \oplus \mathbf{E}_6(-2)$ or $[2] \oplus [-2] \oplus \mathbf{E}_6(-2) \oplus \mathbf{D}_4(-1)$ for order two,*
- *$\mathbf{U}(3)$, $\mathbf{U}(3) \oplus \mathbf{E}_6^*(-3)$, $\mathbf{U}(3) \oplus \mathbf{E}_6^*(-3) \oplus \mathbf{A}_2(-1)$ or $\mathbf{U}(3) \oplus \mathbf{E}_6(-1) \oplus \mathbf{A}_2(-1)^{\oplus 3}$ for order three.*

Viceversa, non-symplectic automorphisms or prime order of a manifold of OG10-type with the above invariant lattices are induced by a non-symplectic automorphism on a cubic fourfold.

Questions

Question 8 (Laza)**.** What is the maximum order $2^l 3^k$ of a non-symplectic automorphism of a cubic fourfold (and hence of an induced automorphism on the LSV manifolds)?

Question 9. What about cubic fourfolds with a symplectic automorphism? Work in progress with A. Grossi and L. Marquand.

References

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