Symplectic birational automorphisms of IHS manifolds

Construction and classification of finite group actions

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Setting

- $\mathcal{T}$: known deformation type of IHS manifolds
- Λ: abstract BBF lattice
- $\mathrm{Mon}^2(\Lambda)$: numerical monodromy group
- $\mathcal{W}^{pex}(\Lambda)$: numerical prime exceptional divisors
- M: associated (extended) Mukai lattice
- U: hyperbolic plane lattice
- $\mathbb{L}$: Leech lattice
- A_n, D_n, E_n: negative definite root lattices
- $O^+(-)$: isometries with trivial spinor norm
- $O^\#(-)$: stable isometries, i.e. acting trivially on $D_{(-)}$
- $SO(-)$: isometries of determinant +1

$\mathcal{T}$	b_2	sign	Λ	D_Λ	$\mathrm{Mon}^2(\Lambda)$	M
Kum_n	7	$(3,4)$	$U^{\oplus 3}\oplus A_1(n+1)$	$\mathbb{Z}/(2n+2)\mathbb{Z}$	$\{g\in O^+(\Lambda)\ :\ \det(g)D_g=\mathrm{id}\}$	$U^{\oplus 4}$
OG6	8	$(3,5)$	$U^{\oplus 3}\oplus A_1^{\oplus 2}$	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$	$O^+(\Lambda)$	$U^{\oplus 5}$
K3	22	$(3,19)$	$U^{\oplus 3}\oplus E_8^{\oplus 2}$	$\{0\}$	$O^+(\Lambda)$	$U^{\oplus 3}\oplus E_8^{\oplus 2}$
$\mathrm{K3}^{[n]}$	23	$(3,20)$	$U^{\oplus 3}\oplus E_8^{\oplus 2}\oplus A_1(n-1)$	$\mathbb{Z}/(2n-2)\mathbb{Z}$	$\{g\in O^+(\Lambda)\ :\ D_g=\pm\mathrm{id}\}$	$U^{\oplus 4}\oplus E_8^{\oplus 2}$
OG10	24	$(3,21)$	$U^{\oplus 3}\oplus E_8^{\oplus 2}\oplus A_2$	$\mathbb{Z}/3\mathbb{Z}$	$O^+(\Lambda)$	$U^{\oplus 5}\oplus E_8^{\oplus 2}$

Stable symplectic groups

Symplectic automorphisms of finite order <u>are stable</u> (L. Giovenzana, Grossi, Mongardi, Onorati, Tari, Veniani, Wandel).

→ Classified for: K3 [Mukai '88],
K3[2] [Höhn–Mason '19],
OG6 [Grossi–Onorati–Veniani '23],
OG10 [Giovenzana–Grossi–Onorati–Veniani '24], and
K3[3] [Billi–Muller–Wawak '24].

Symplectic **birational** automorphisms of finite order <u>can be nonstable</u>.

→ **By studying birational automorphisms of IHS manifolds, need to consider a new s.e.s as shown vertically.**

Definition

We call a primitive sublattice $C\subseteq\Lambda$ a **heart** if C is negative definite, $S(C)\curvearrowright C$ is free and $C\cap\mathcal{W}^{pex}(\Lambda)=\emptyset$ where $S(C):=SO^\#(C)$ if $\mathcal{T}=\mathrm{Kum}_n$ and $S(C):=O^\#(C)$ otherwise.

Proposition 1

There is a bijection

$$\mathrm{Mon}^2(\Lambda)\backslash\{H\leq\mathrm{Mon}^2(\Lambda)\text{ stable symplectic with }H=S(\Lambda_H)\}\overset{1:1}{\longleftrightarrow}\mathrm{Mon}^2(\Lambda)\backslash\{C\subseteq\Lambda\text{ heart}\}$$

→ **Effective classification possible if $\mathrm{Mon}^2(\Lambda)=O^+(\Lambda)$**

What are the hearts C for each $\mathcal{T}$?

Potential hearts C for known $\mathcal{T}$

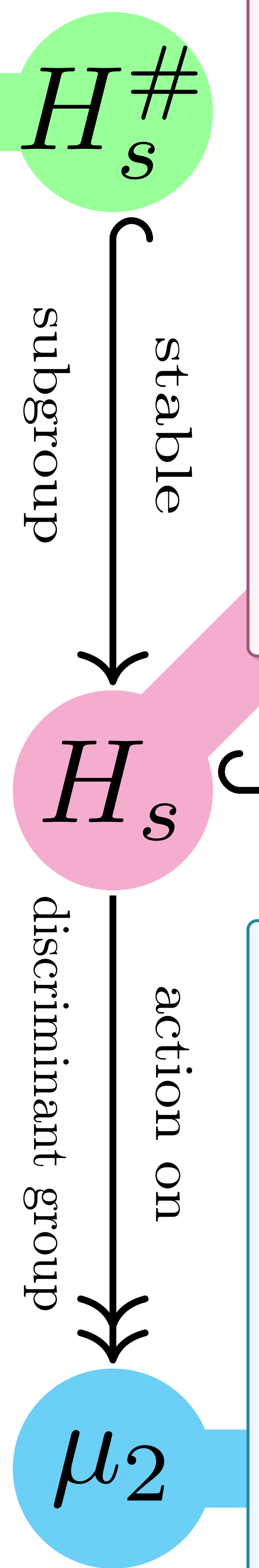
- $\mathcal{T}=\mathrm{Kum}_n$: $C\hookrightarrow E_8$ primitively. [Mongardi '13]
- $\mathcal{T}=\mathrm{OG6}$: $C\hookrightarrow E_8$ primitively, except if $(\mathrm{rk}(C),l_2(D_C))\in\{(5,5),(5,4)\}$. [Grossi '20]
- $\mathcal{T}=\mathrm{K3}^{[n]}$: $C\hookrightarrow\mathbb{L}$ primitively. [Mongardi '13]
- $\mathcal{T}=\mathrm{OG10}$: $C\hookrightarrow\mathbb{L}$ primitively, except if $(\mathrm{rk}(C),l_3(D_C))\in\{(r,25-r)\mid 13\leq r\leq 21\}$. [Marquand–Muller '24]

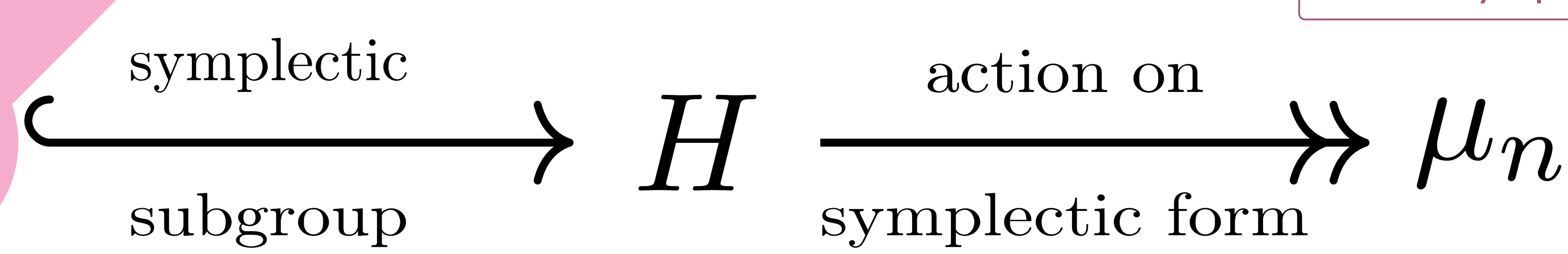
Hearts embedding primitively into $\mathbb{L}$ and E_8 are known and classified [Höhn–Mason '16].

→ **Need to determine the missing ones for OG6 and OG10.**

Theorem 1 [Marquand–Muller '24]

For $\mathcal{T}=\mathrm{OG10}$, there exist exactly 185 conjugacy classes of symplectic finite subgroups $H\leq O^{+,\#}(\Lambda)$ such that $H=O^\#(\Lambda_H)$ and Λ_H embeds primitively into the Leech lattice.





Stable symplectic to symplectic

Case study: $\mathrm{Mon}^2(\Lambda)=O^+(\Lambda)\to$ Applicable to $\mathcal{T}=$ OG10 and $\mathcal{T}=\mathrm{K3}^{[p^k+1]}$, with p prime and $k\geq 1$.

Goal

Compute and classify all $H_s\leq O^+(\Lambda)$ finite symplectic with given $H_s^\#\leq O^{+,\#}(\Lambda)$, in the case $H_s^\#=O^\#(\Lambda_{H_s^\#})$.

By Proposition 1, any such $H_s^\#\leq O^{+,\#}(\Lambda)$ is uniquely determined by the heart $\Lambda_{H_s^\#}\subseteq\Lambda$, up to the action of $O^+(\Lambda)$.

Strategy

1. Choose a heart $C\subseteq\Lambda$; *(Theorem 1)*
2. Enumerate classes of pairs (F,a) with $F=C^\perp$, $a^2=\mathrm{id}_F$ and F_a negative definite; *[Brandhorst–Hofmann '23]*
3. Classify equivariant primitive extensions $(F,a)\oplus(C,b)\subseteq(\Lambda,h)$ with $D_h\neq\mathrm{id}$; *[Nikulin '80]*
4. Collect the groups $H_s:=\langle O^\#(C),h\rangle$ such that $\Lambda_{H_s}\cap\mathcal{W}^{pex}(\Lambda)=\emptyset$.

→ **Steps (1)–(4) are effective and computationally accessible.**

Theorem 2 [Marquand–Muller '24]

For $\mathcal{T}=$ OG10, there exist exactly 921 conjugacy classes of finite symplectic subgroups $H_s\leq O^+(\Lambda)$ such that $H_s^\#$ is given in Theorem 1.

Nonstable symplectic involutions

Nonstable symplectic isometries only exist if $\mathcal{T}\neq$ K3, K3[2], OG6 → **Case study: $\mathcal{T}=$ OG10**

Lemma 1

Let $i\in O^+(\Lambda)$ be a nonstable symplectic involution. Then $D_{\Lambda^i}\cong(\mathbb{Z}/2\mathbb{Z})^a$ and $D_{\Lambda_i}\cong\mathbb{Z}/3\mathbb{Z}\times D_{\Lambda^i}$.

Strategy

1. Determine the possible genera g for Λ_i; *(Lemma 1)*
2. Enumerate each such g; *[Kneser '02]*
3. For all $C'\in g$, classify primitive embeddings $C'\hookrightarrow\Lambda$ such that $i(C')\cap\mathcal{W}^{pex}(\Lambda)=\emptyset$. *[Nikulin '80]*

Theorem 3 [Marquand–Muller '24]

For $\mathcal{T}=$ OG10, there exist exactly 4 conjugacy classes of nonstable symplectic involutions in $O^+(\Lambda)$.

→ For the remaining $\mathcal{T}=\mathrm{K3}^{[n]}$, Kum_n, can unify the study and classify involutions of the associated *Mukai lattices*?

Comment on classification

Regarding some moduli problems, we want to classify (groups of) isometries up to monodromy conjugation.

→ Case with maximal monodromy $\mathrm{Mon}^2(\Lambda)=O^+(\Lambda)$ are the easiest to handle.

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