On the geometry of some quiver zero loci Fano fourfolds

Federico Tufo
federico.tufo2@unibo.it
Universitá di Bologna

Fano manifolds

Classification of Fano manifolds is an active area of Algebraic Geometry. One of the motivating result is the following:

Theorem [J. Kollár, V. A. Iskovskikh, Y. Miyaoka, S. Mori, S. Mukai]

There is a finite number of deformation families of Fano manifolds in each dimension. Moreover:
- in dimension one we have only $\mathbb{P}^1$;
- in dimension two we have 10 families of Fano surfaces, the Del Pezzo surfaces;
- in dimension three we have 105 families of Fano threefolds.

From dimension five and above the classification is still unknown, while in dimension four there are some tries to crack the problem.

The big numbers approach

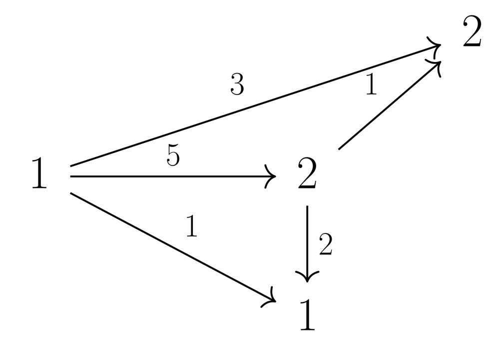
An approach to classifying the deformation families of Fano fourfolds is to produce databases of manifolds and look for the ones that satisfy the Fano condition, i.e. the ampleness of the anticanonical bundle.

Some databases

We recall some of the main databases of Fano fourfolds. Each of them is obtained in an automatized way.
- T. Coates, A. Kasprzyk, T. Prince produced a database of 738 Fano fourfolds which are complete intersections in toric varieties;
- E. Kalashnikov produced a database of 749 Fano fourfolds which are quiver flag zero loci;
- M. Bernardara, E. Fatighenti, L. Manivel, F. Tanturri produced a database of around 640 Fano fourfolds that are section zero loci in product of Grassmannians.

Quiver flag manifolds and Grassmannians

A quiver is a acyclic graph with one source as the following image



A quiver moduli space is a moduli space $\mathcal{M} = \mathcal{M}_\theta(Q, r)$ of quiver via a specific stability condition. In the case of quiver flag variety this stability condition makes $\mathcal{M}$ behave as a flag variety. Following *Quiver flag varieties and multigraded linear series*, by A. Craw, is possible to translate the language of quiver flag varieties into products of Grassmannians.

Rosetta Stone

- If a vertex is a head only of n_i arrows coming from the source, and its space associated has dimension r_i, then the zero loci of the product of Grassmannians corresponding to the subquiver obtained from the source and that vertex is $\mathbf{Gr}(r_i, n_i)$;
- if a vertex is the head of n_a arrows coming from the source and n_b arrows coming from another vertex of the previous type, and its space associated has dimension r_j, then the zero locus of product of Grassmannians corresponding to the subquiver obtained from the vertices involved is $\mathcal{Z}((\mathcal{Q}_{\mathbf{Gr}(r_i,n_i)} \boxtimes \mathcal{U}^\vee_{\mathbf{Gr}(r_j,n_i\times n_b+n_a)})^{\oplus n_b}) \subset \mathbf{Gr}(r_i, n_i) \times \mathbf{Gr}(r_j, n_i \times n_b + n_a)$;
- the bundle E is rewritten in terms of duals of the corresponding Schur powers of tautological vector bundles defined on the Grassmannians.

Note that following this algorithm the quiver flag manifold associated to the above quiver can be described as: $\mathcal{Z}(\mathcal{Q}_{Gr(2,5)} \boxtimes \mathcal{O}_{\mathbb{P}^{10}}(1)^{\oplus 2} \oplus \mathcal{Q}_{Gr(2,5)} \boxtimes \mathcal{U}^{vee}_{Gr(2,8)}) \subset Gr(2,5) \times \mathbb{P}^{10} \times Gr(2,8)$.

A small Fanodex (joint with E. Fatighenti, F. Tanturri)

The three databases mentioned above are related. In particular the Kalashnicov one add 141 new families of Fano fourfolds to the one built by Coates, Kasprzyk, Prince. There are also 29 families already known, but they were not found in toric quiver flag varieties (possibly appearing in another way as a toric complete intersection). Moreover the language of quiver flag varieties is deeply linked to the language of zero loci in products of Grassmannians. Hence we started translating this 170 families into the Grassmannians language. In this way we could describe them biregurarly and obtain the following result.

Theorem [E. Fatighenti, F. Tanturri, F. Tufo]

The 170 families of Fano fourfolds mentioned above can be subdivided into subsets as follows:
- 9 have Picard rank $\rho = 1$ and are classical;
- 124 are blow-ups of smooth Fano fourfolds (from the list of the 749) in smooth centers;
- 8 are products of lower dimensional Fano manifolds;
- 3 are projective bundles over Fano threefolds;
- 8 are conic bundles;
- 2 are stratified projective bundles over Fano threefolds;
- 16 are small resolutions of singular fourfolds.

Next directions

The possibility of translating the language of quiver flag variety into the language of Grassmannians open the possibility of translate each of the variety in the database of Kalashnikov into the language of section zero loci in products of Grassmannians.

Why product of Grassmannians?

- Comfortable setting to compute various deformation invariants, such as Hodge numbers, the volume, the h^0 of the anticanonical bundle and the characteristic of the tangent bundle.
- Comfortable setting to describe the varieties biregulary via degeneracy loci theory.
- Comparing the databases since they are not contained in each other.

This dictionary is also useful the other way around, in fact, another possible direction is to use the combinatorial structure of the quiver flag manifold to compute boundaries for the number of family of strictly Fano fourfoulds obtained as section zero loci.

Some problems

- Prove that the 64 families of Fano fourfolds of K3-type found in *Fano fourfolds of K3 type*, by M. Bernardara, E. Fatighenti, L. Manivel, F. Tanturri are the only one obtained as section zero loci.
- Understanding the link between the graph representation and the birational geometry of the quiver flag varieties.
- Finding new equivalent models for section zero loci varieties with the same deformation invariants by looking at the associated quiver.