Nonstandard Combinatorics

Lesson 1 - 13 May 2018

One of the interesting features of combinatorial number theory is that results in this area can be proven using several techniques coming from various different areas. In this very short course we will focus on two of these areas: ultrafilters and nonstandard analysis.

Before starting with ultrafilters, let me give an idea of the kind of results we are aiming for:

① Schur: For every finite partition $\mathbb{N} = A_1 \cup \ldots \cup A_k$ $\exists i \leq k$, $x, y \in A_i$ s.t. $x+y \in A_i$.

② Van der Waerden: For every finite partition $\mathbb{N} = A_1 \cup \ldots \cup A_k$, $\forall r \in \mathbb{N}$ $\exists i \leq k$, $x, y \in A_i$ s.t.

$$x, x+y, x+2y, \ldots, x+ry \in A_i.$$

③ Hindman: For every finite partition $\mathbb{N} = A_1 \cup \ldots \cup A_k$ $\exists i \leq k$ $\exists X \subseteq A_i$, X infinite s.t.

$$FS(X) = \left\{ \sum_{i=1}^{n} x_i \mid n \in \mathbb{N}, x_1 < \ldots < x_n \in X \right\} \subseteq A_i.$$

In particular, in this course we will be interested in the so-called partition regular equations:

Def: Let $P(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]$. We say that the equation $P(x_1, \ldots, x_n) = 0$ is PARTITION REGULAR (PR from now on) on $\mathbb{N}$ if for every finite partition $\mathbb{N} = A_1 \cup \ldots \cup A_k$ $\exists i \leq k$, $a_1, \ldots, a_n \in A_i$ s.t. $P(a_1, \ldots, a_n) = 0$.

E.g., Schur's theorem can be rephrased by saying that the equation $x+y=z$ is PR.

The linear case was settled by Richard Rado a century ago (more or less):

④ Rado: Let $n \in \mathbb{N}$, $c_1, \ldots, c_n \in \mathbb{Z} \setminus \{0\}$. The equation $c_1 x_1 + \ldots + c_n x_n = 0$ is PR iff $\exists I \neq \emptyset$, $I \subseteq \{1, \ldots, n\}$ s.t. $\sum_{i \in I} c_i = 0$.

This last condition is so important that we will refer to it (and to its modifications) as Rado's condition.

Rado settled not only the case of single equations, but that of finite systems of linear equations (notice that Hindman settles the case of an infinite system).

However, later on we will focus on a case where little is known: the nonlinear case.

Ultrafilters

Def 1: Let $I \neq \emptyset$ be a set. Let $\emptyset \neq \mathcal{F} \subseteq \mathcal{P}(I)$. Then $\mathcal{F}$ is a FILTER on I if it satisfies the following properties:

① $\forall A, B \in \mathcal{F}$ $A \cap B \in \mathcal{F}$;

② $\forall A \in \mathcal{F}, \forall B \in \mathcal{P}(I)$ if $A \subseteq B$ then $B \in \mathcal{F}$.

The filter is proper if

③ $\emptyset \notin \mathcal{F}$.

← Notice that it can NEVER be that $A, A^c \in \mathcal{F}$ for a proper filter. However, it can be that $A, A^c \notin \mathcal{F}$.

Finally, the filter is an ultrafilter iff

④ $\forall A \in \mathcal{P}(I)$ $A \in \mathcal{F} \iff I \setminus A \notin \mathcal{F}$.

Exercise 1: (1.1) Prove that any filter is closed w.r.t. arbitrary finite intersections; what about infinite intersections?

(1.2) Prove that $\emptyset \in \mathcal{F} \iff \mathcal{F} = \mathcal{P}(I)$.

(1.3) Let $\mathcal{U}$ be an ultrafilter. Prove that, $\forall A_1, \dots, A_n \in \mathcal{P}(I)$, $A_1 \cup \dots \cup A_n \in \mathcal{U} \iff \exists i \leq n$ $A_i \in \mathcal{U}$.

Def 2: Let I be a set. For any set $J \subseteq \mathcal{P}(I)$ let $\mu_J : \mathcal{P}(I) \to \{0,1\}$ be defined as

$$\forall A \in \mathcal{P}(I) \ \mu_J(A) = 1 \iff A \in J$$

(i.e, μ_J is the characteristic function of J).

Conversely, given any function $\mu : \mathcal{P}(I) \to \{0,1\}$ we let $I_\mu \subseteq \mathcal{P}(I)$ be the set such that

$$\forall A \in \mathcal{P}(I) \ A \in I_\mu \iff \mu(A) = 1.$$

Measures with values in $\{0,1\}$ and ultrafilters are basically the same.

Prop 3: ① $\forall \mu : \mathcal{P}(I) \to \{0,1\}$ $\mu_{I_\mu} = \mu$ and $\forall J \subseteq \mathcal{P}(I)$ $I_{\mu_J} = J$;

② μ is a non trivial measure iff. μ_J is an ultrafilter; i.e, J is an ultrafilter iff μ_J is a non trivial measure.

PROOF: ① is trivial.

②: this is the same as the proof of Thm 2 in Lesson 3: let μ be a non trivial measure.

Then: $I_\mu \neq \emptyset$ as μ is non trivial; $I \in I_\mu$ as $\mu(I)=1$ for every non trivial measure;

As μ is a measure, we have that $\forall A, B \in \mathcal{P}(I)$ with $A \cap B = \emptyset$ we have $\mu(A \cup B) = \mu(A) + \mu(B)$.

But then: I_μ is closed w.r.t. superset as, if $A \in I_\mu$ and $A \subseteq B$, then $B = A \cup (B \setminus A)$, and so $\mu(B) = \mu(A) + \mu(B \setminus A) \geq 1$.

I_μ is closed w.r.t. intersection: by contrast, if $A, B \in I_\mu$ are such that $A \cap B \notin I_\mu$, then, as $A = (A \setminus B) \cup (A \cap B)$, since $1 = \mu(A) = \mu(A \setminus B) + \underset{0}{\mu(A \cap B)}$ we get that $A \setminus B \in I_\mu$, hence $I \setminus B \in I_\mu$, which is absurd: $1 = \mu(I) = \underset{1}{\mu(I \setminus B)} + \underset{1}{\mu(B)} = 2$ ↯.

Finally, I_μ is an ultrafilter: let $A \in \mathcal{P}(I)$. Then $1 = \mu(I) = \mu(A) + \mu(A^c)$, and so $\mu(A) = 1 \iff \mu(A^c) = 0$.

The converse is similar. □

As a consequence, we get that ultrafilters can be used to define a notion of "largeness": if we fix an ultrafilter $\mathcal{U}$, we can say that a set $A \subseteq I$ is large (or, more precisely, $\mathcal{U}$-large) iff. $A \in \mathcal{U}$.

From now on, we let I be an INFINITE set.

EXAMPLE: ① Let $i \in I$. Let $\mathcal{U}_i := \{A \in \mathcal{P}(I) \mid i \in A\}$. Then $\mathcal{U}_i$ is trivially an ultrafilter, called the PRINCIPAL ultrafilter on i.

② Let $\mathfrak{Fr} = \{A \in \mathcal{P}(I) \mid A^c \text{ is finite}\}$. Then $\mathfrak{Fr}$ is a filter on I, called the Fréchet filter. Notice that $\mathfrak{Fr}$ is NOT an ultrafilter and that, $\forall i \in I$, $\mathfrak{Fr} \nsubseteq \mathcal{U}_i$.

DEF 4: We say that an ultrafilter $\mathcal{U}$ on I is NON PRINCIPAL iff $\mathcal{U} \neq \mathcal{U}_i$ $\forall i \in I$.

An important fact about non principal ultrafilters that we will use in the following is this:

PROP 5: Let $\mathcal{U}$ be an ultrafilter on I. Then $\mathcal{U}$ is non principal iff $\mathfrak{Fr} \subseteq \mathcal{U}$.

Proof: We already noticed that $\mathfrak{Fr} \nsubseteq \mathcal{U}$ if $\mathcal{U}$ is principal. Conversely, let $\mathcal{U}$ be non principal. Suppose that there is $A \in \mathfrak{Fr}$ s.t. $A \notin \mathcal{U}$. Then $A^c \in \mathcal{U}$. But A^c is finite, so $A^c = \{i_1, \dots, i_n\} = \{i_1\} \cup \dots \cup \{i_n\}$. But then $\exists j \leq n$ s.t. $\{i_j\} \in \mathcal{U}$, and so $\mathcal{U} = \mathcal{U}_{i_j}$. □

Do nonprincipal ultrafilters exist?

Oss: $\mathcal{U}$ is an ultrafilter $\iff$ $\mathcal{U}$ is maximal in the set $F = \{\mathcal{F} \subseteq \mathcal{P}(I) \mid \mathcal{F} \text{ is a filter}\}$, ordered by inclusion.

EXERCISE 3: Prove the previous observation.

DEF 6: Let $H \subseteq \mathcal{P}(I)$. We say that H has the finite intersection property (F.I.P.) if $\forall n \geq 1$, $\forall B_1, \dots, B_n \in H$, $B_1 \cap \dots \cap B_n \neq \emptyset$.

EXERCISE 4: (4.1) Every filter has the FIP, but not every family with the FIP is a filter.

(4.2) $\mathcal{F}_r$ has the FIP.

(4.3) If H has the FIP then $\forall A \subseteq I$ at least one between $H \cup \{A\}$ and $H \cup \{A^c\}$ has the FIP.

We are now ready to conclude:

THM 7: Let $H \subseteq \mathcal{P}(I)$ be a family with the FIP. Then there exists an ultrafilter $\mathcal{U}$ on I s.t. $H \subseteq \mathcal{U}$.

PROOF: Let $\mathcal{F}^H = \{A \subseteq I \mid A \supseteq B_1 \cap \dots \cap B_n \text{ for some } n \geq 1 \text{ and some } B_1, \dots, B_n \in H\}$.

As H has the FIP, $\mathcal{F}^H$ is a proper filter: it is non empty as it includes H; $\emptyset \notin \mathcal{F}^H$ as, otherwise, there would be $B_1, \dots, B_n \in H$ s.t. $B_1 \cap \dots \cap B_n = \emptyset$. It is closed under intersection as, if $A_1 \supseteq B_1 \cap \dots \cap B_n$ and $A_2 \supseteq C_1 \cap \dots \cap C_m$ then $A_1 \cap A_2 \supseteq B_1 \cap \dots \cap B_n \cap C_1 \cap \dots \cap C_m$. Finally, it is clearly closed under superset.

Now let $P = \{\mathcal{F} \subseteq \mathcal{P}(I) \mid \mathcal{F} \text{ is a proper filter}, \mathcal{F}^H \subseteq \mathcal{F}\}$ Every chain w.r.t. $\subseteq$ has a sup, which is just the union of the elements of the chain. Hence by Zorn's lemma there is a maximal element in P which, by construction, is an ultrafilter that extends H. $\square$

COR 8: On every infinite set I there exists a non principal ultrafilter $\mathcal{U}$.

PROOF: Just set $H = \mathcal{F}_r$ in Thm 7. $\square$

EXERCISE 5: (5.1) If $A \neq \emptyset$, $A \subseteq I$ then $\exists \mathcal{U}$ ultrafilter on I s.t. $A \in \mathcal{U}$. If A is infinite, $\mathcal{U}$ can be chosen to be nonprincipal.

(5.2) If I is finite then every ultrafilter on I is principal.

From now on, we let $I=\mathbb{N}$. We let

$$\beta\mathbb{N}=\{\mathcal{U}\subseteq\mathcal{P}(\mathbb{N})\mid \mathcal{U} \text{ is an ultrafilter}\}.$$

$\beta\mathbb{N}$ is called the Stone-Čech compactification of $\mathbb{N}$. $\mathbb{N}$ can be identified as a subset of $\beta\mathbb{N}$ by identifying every natural number $n\in\mathbb{N}$ with

$$\mathcal{U}_n=\{A\subseteq\mathbb{N}\mid n\in A\}.$$

DEF 9: We can extend the sum operation $+:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ to a sum operation $\oplus:\beta\mathbb{N}\times\beta\mathbb{N}\to\beta\mathbb{N}$, defined as follows: $\forall\mathcal{U},\mathcal{V}\in\beta\mathbb{N}$, $\forall A\subseteq\mathbb{N}$

$$A\in\mathcal{U}\oplus\mathcal{V}\Longleftrightarrow\{n\in\mathbb{N}\mid\{m\in\mathbb{N}\mid n+m\in A\}\in\mathcal{V}\}\in\mathcal{U}.$$

Similarly, we can extend the product operation $\cdot:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ to $\odot:\beta\mathbb{N}\times\beta\mathbb{N}\to\beta\mathbb{N}$ by letting $\forall\mathcal{U},\mathcal{V}\in\beta\mathbb{N}$, $\forall A\subseteq\mathbb{N}$

$$A\in\mathcal{U}\odot\mathcal{V}\Leftrightarrow\{n\in\mathbb{N}\mid\{m\in\mathbb{N}\mid n\cdot m\in A\}\in\mathcal{V}\}\in\mathcal{U}.$$

EXERCISE 6: Prove that $\mathcal{U}\oplus\mathcal{V}$, $\mathcal{U}\odot\mathcal{V}$ are actually ultrafilters.

Is this extension nice? Well, it depends on your definition of being nice:

PROPERTIES OF $\oplus$: ① $\forall a\in\mathbb{N},\forall\mathcal{U}\in\beta\mathbb{N}$ $\mathcal{U}_a\oplus\mathcal{U}=\mathcal{U}\oplus\mathcal{U}_a$ ← I will often just write $a\oplus\mathcal{U}$

PROOF: $A\in a\oplus\mathcal{U}\Leftrightarrow\{n\in\mathbb{N}\mid\{m\in\mathbb{N}\mid n+m\in A\}\in\mathcal{U}\}\in\mathcal{U}_a\Leftrightarrow$

$\Leftrightarrow\{m\in\mathbb{N}\mid a+m\in A\}\in\mathcal{U}\Leftrightarrow\{m\in\mathbb{N}\mid\{n\in\mathbb{N}\mid m+n\in A\}\in\mathcal{U}_a\}\in\mathcal{U}$. □

② $\forall\mathcal{U},\mathcal{V},\mathcal{W}$ $(\mathcal{U}\oplus\mathcal{V})\oplus\mathcal{W}=\mathcal{U}\oplus(\mathcal{V}\oplus\mathcal{W})$.

PROOF: $A\in(\mathcal{U}\oplus\mathcal{V})\oplus\mathcal{W}\Leftrightarrow\{n\in\mathbb{N}\mid\{m\in\mathbb{N}\mid n+m\in A\}\in\mathcal{W}\}\in\mathcal{U}\oplus\mathcal{V}\Leftrightarrow$

$\Leftrightarrow\{r\in\mathbb{N}\mid\{s\in\mathbb{N}\mid r+s\in\{n\in\mathbb{N}\mid\{m\in\mathbb{N}\mid n+m\in A\}\in\mathcal{W}\}\}\in\mathcal{V}\}\in\mathcal{U}\Leftrightarrow$

$\Leftrightarrow\{r\in\mathbb{N}\mid\{s\in\mathbb{N}\mid\{m\in\mathbb{N}\mid r+s+m\in A\}\in\mathcal{W}\}\in\mathcal{V}\}\in\mathcal{U}\Leftrightarrow$

$\Leftrightarrow\{r\in\mathbb{N}\mid\{n\in\mathbb{N}\mid r+n\in A\}\in\mathcal{V}\oplus\mathcal{W}\}\in\mathcal{U}\Leftrightarrow A\in\mathcal{U}\oplus(\mathcal{V}\oplus\mathcal{W})$. □

Therefore $(\beta\mathbb{N},\oplus)$ is a semigroup. Is it commutative?

No! Actually, the following holds:

Prop 10: $\exists A \in \mathbb{N}\ \forall \mathcal{U} \in \beta\mathbb{N} \setminus \mathbb{N}\ \exists \mathcal{V} \in \beta\mathbb{N} \setminus \mathbb{N}$ s.t. $A \in \mathcal{U} \oplus \mathcal{V} \Leftrightarrow A^c \in \mathcal{V} \oplus \mathcal{U}$.

We will give a nonstandard proof of this fact in the next lecture.

$\beta\mathbb{N}$ can be endowed with a useful topology:

Def 11: For every $A \in \mathcal{P}(\mathbb{N})$ we let

$$\Theta_A := \{\mathcal{U} \in \beta\mathbb{N} \mid A \in \mathcal{U}\}.$$

Notice that $\Theta_{A_1} \cap \dots \cap \Theta_{A_n} = \Theta_{A_1 \cap \dots \cap A_n}$ and that $\Theta_A^c = \Theta_{A^c}$.

We endow $\beta\mathbb{N}$ with the topology τ generated by $\{\Theta_A\}_{A \in \mathcal{P}(\mathbb{N})}$.

Thm 12: $(\beta\mathbb{N}, \tau)$ has the following property:

(i) it is Hausdorff;

(ii) it is compact;

(iii) every Θ_A is clopen (hence the space is extremally disconnected [the closure of every open subset is open.]);

(iv) $\mathbb{N}$ is dense in $\beta\mathbb{N}$ (and $\mathbb{N}$ is the set of isolated points of $(\beta\mathbb{N}, \tau)$).

Proof: (i) Let $\mathcal{U} \neq \mathcal{V}$. Then $\exists A \in \mathcal{P}(\mathbb{N})$ s.t. $A \in \mathcal{U}$, $A \notin \mathcal{V}$. So $A^c \in \mathcal{V}$, and then $\mathcal{U} \in \Theta_A$, $\mathcal{V} \in \Theta_{A^c}$ and $\Theta_A \cap \Theta_{A^c} = \emptyset$.

(ii) We use the following characterisation of compact sets: X is compact iff every family of closed sets with the FIP has a nonempty intersection. As $\{\Theta_A\}_{A \in \mathcal{P}(\mathbb{N})}$ is also a basis for closed sets, we can restrict to consider families of such sets.

Let $\{\Theta_A\}_{A \in \mathcal{F}}$ be such a family. If it has the FIP then necessarily the family $\mathcal{F}$ has the FIP. But then by Thm 7 there exists $\mathcal{U} \supseteq \mathcal{F}$, and trivially $\mathcal{U} \in \bigcap_{A \in \mathcal{F}} \Theta_A$.

(iii) $(\Theta_A)^c = \Theta_{(A^c)}$.

(iv) Let $A \in \mathcal{P}(\mathbb{N}) \setminus \{\emptyset\}$. Let $n \in A$. Then $\mathcal{U}_n \equiv n \in \Theta_A$, as $A \in \mathcal{U}_n$. □

For every $\mathcal{U} \in \beta\mathbb{N}$, we let $\rho_{\mathcal{U}}: \beta\mathbb{N} \to \beta\mathbb{N}$ be the function such that

$$\forall \mathcal{V} \in \beta\mathbb{N} \quad \rho_{\mathcal{U}}(\mathcal{V}) = \mathcal{V} \oplus \mathcal{U}.$$

Thm 13: $\forall \mathcal{U} \in \beta\mathbb{N}$ $\rho_{\mathcal{U}}$ is continuous in $(\beta\mathbb{N}, \tau)$. Hence $(\beta\mathbb{N}, \oplus, \tau)$ is a right compact topological semigroup.

Proof: Let $A \in \mathcal{P}(\mathbb{N})$. Then $\rho_{\mathcal{U}}^{-1}(A) = \{\mathcal{V} \in \beta\mathbb{N} \mid A \in \mathcal{V} \oplus \mathcal{U}\} =$

$$\{\mathcal{V}\in\beta\mathbb{N} \mid \{n\in\mathbb{N} \mid \{m\in\mathbb{N} \mid n+m\in A\}\in\mathcal{U}\}\in\mathcal{V}\} =$$

$$= \{\mathcal{V}\in\beta\mathbb{N} \mid \{n\in\mathbb{N} \mid A-n\in\mathcal{U}\}\in\mathcal{V}\} = \Theta_B, \text{ where } B=\{n\in\mathbb{N} \mid A-n\in\mathcal{U}\}. \quad \square$$

EXERCISE 7: Let $\mathcal{U}\in\beta\mathbb{N}$ and $\sigma_{\mathcal{U}}:\beta\mathbb{N}\to\beta\mathbb{N}$ be s.t. $\sigma_{\mathcal{U}}(\mathcal{V}):=\mathcal{U}\oplus\mathcal{V}$. When is $\sigma_{\mathcal{U}}$ continuous?

Very similar proofs can be used to show that also $(\beta\mathbb{N},\odot,\tau)$ is a compact right topological semigroup.

Being compact right topological semigroups, $(\beta\mathbb{N},\oplus)$ and $(\beta\mathbb{N},\odot)$ have many useful properties. For our purposes, two are the important properties:

FACT 14: $(\beta\mathbb{N},\oplus)$ and $(\beta\mathbb{N},\odot)$ contain non trivial idempotents, namely $\exists\,\mathcal{U}\in\beta\mathbb{N}\setminus\{0\}$, $\mathcal{V}\in\beta\mathbb{N}\setminus\{0,1\}$ s.t. $\mathcal{U}\oplus\mathcal{U}=\mathcal{U}$ and $\mathcal{V}\odot\mathcal{V}=\mathcal{V}$.

The above fact is just a particular instance of Ellis' Theorem:

THM (ELLIS): Let S be a compact right topological semigroup. Then S contains idempotents.

FACT 15: $(\beta\mathbb{N},\oplus)$ has a minimal bilateral ideal $K(\beta S,\oplus)$. Moreover,

$$K(\beta S,\oplus)=\bigcup\{L \mid L \text{ is a minimal left ideal of } S\}.$$

An analogous property holds for $(\beta\mathbb{N},\odot)$. Moreover:

$$K(\beta\mathbb{N},\oplus)\cap K(\beta\mathbb{N},\odot)=\emptyset, \text{ but } \overline{K(\beta\mathbb{N},\oplus)}\cap\overline{K(\beta\mathbb{N},\odot)}\neq\emptyset.$$

The first fact is a general property of CRTS. The second one is due to Dona Strauss.

APPLICATIONS

Let us prove Schur's Theorem.

PROOF: Let $\mathcal{U}=\mathcal{U}\oplus\mathcal{U}$. Let the partition $\mathbb{N}=A_1\cup\ldots\cup A_K$ be given. Let $i\leq K$ be s.t. $A_i\in\mathcal{U}$ ("A"). Then

$$I=\{n\in\mathbb{N} \mid \{m\in\mathbb{N} \mid n+m\in A\}\in\mathcal{U}\}\in\mathcal{U}.$$

As $I\in\mathcal{U}$, also $I\cap A\in\mathcal{U}$. If $n\in I\cap A$, let $J_n=\{m\in\mathbb{N} \mid n+m\in A\}$. Then $J_n\in\mathcal{U}$, hence also $J_n\cap A\in\mathcal{U}$. If $m\in J_n\cap A$, by construction $n,m,n+m\in A$. $\square$

EXERCISE 8: Prove that $x+y-2z=0$ is PR. Find $c_1,c_2\in\mathbb{N}$ s.t. $(c_1\odot\mathcal{U})\oplus(c_2\odot\mathcal{U})\models x+y-2z=0$.

By recursively doing the above proof, we can prove Hindman's Theorem (whose original proof is more of 20 pages of rather complicated combinatorial arguments)

PROOF (HINDMAN): As above, we take $\mathcal{U}=\mathcal{U}\oplus\mathcal{U}$. Given $\mathbb{N}=A_1\cup\ldots\cup A_k$, we let $i\leq k$ be such that $A_i:=C_1\in\mathcal{U}$. We now construct $X=\{x_n\mid n\in\mathbb{N}\}$ inductively.

STEP 1: Let $B_1=\{n\in\mathbb{N}\mid C_1-n\in\mathcal{U}\}$. As $\mathcal{U}\oplus\mathcal{U}=\mathcal{U}$, $B_1\in\mathcal{U}$. Pick $x_1\in B_1\cap C_1$. Let $C_2=B_1\cap C_1$. Then $C_2\in\mathcal{U}$.

STEP n: Assume to have inductively defined $B_1,\ldots,B_{n-1}\in\mathcal{U}$, $C_1,\ldots,C_n\in\mathcal{U}$ and $x_1<\ldots<x_{n-1}\in A_i$ s.t. $FS(\{x_1,\ldots,x_{n-1}\})\subseteq A_i$.

Let $B_n=\{x\in S\mid C_n-x\in\mathcal{U}\}$. Then $B_n\in\mathcal{U}\Rightarrow B_n\cap C_n\in\mathcal{U}$.

Let $x_n>x_{n-1}\in B_n\cap C_n$ and let $C_{n+1}=C_n\cap(C_n-x_n)$.

Notice that, by construction, $C_1\supseteq C_2\supseteq C_3\supseteq\ldots$

Let $\{x_i\mid i\in F\}\subseteq X$. We proceed on induction on $|F|$ and show that $\sum_{i\in F}x_i\in C_{\min F}$.

If $|F|=1$ then $\{x_i\mid i\in F\}=\{x_m\}$ and we are done as $x_m\in C_m\subseteq C_1=A_1$.

If $|F|>1$ let $m=\min F$. Let $G=F\setminus\{m\}$. Let $r=\min G>m$. Then

$$\sum_{n\in G}x_n\in C_k\subseteq C_{m+1}\subseteq C_m-x_m\Rightarrow x_m+\sum_{n\in G}x_n\in C_m\subseteq C_1=A_1.\ \square$$

Finally, let us prove three results that will be very useful later on.

First, a definition:

DEF 16: Let $P(x_1,\ldots,x_n)\in\mathbb{Z}[x_1,\ldots,x_n]$. We say that $\mathcal{U}$ witnesses the PR of the equation $P(x_1,\ldots,x_n)=0$ iff $\forall A\in\mathcal{U}\ \exists a_1,\ldots,a_n\in A\ P(a_1,\ldots,a_n)=0$. In this case, we use the notation $\mathcal{U}\models P(x_1,\ldots,x_n)=0$.

The first result explains why ultrafilters are so important in this area:

THM 17: Let $P(\vec{x})\in\mathbb{Z}[\vec{x}]$. The following facts are equivalent:

(i) $P(\vec{x})$ is PR;

(ii) $\exists\mathcal{U}\in\beta\mathbb{N}\ \mathcal{U}\models P(\vec{x})=0$.

PROOF: (i)⇒(ii) Let $\mathcal{G}=\{A\subseteq\mathbb{N}\mid\exists\vec{a}\in A\ P(\vec{a})=0\}$. Let $\mathcal{J}\subseteq\mathcal{G}$ be the family

$$\mathcal{J}=\{A\in\mathcal{G}\mid A^c\notin\mathcal{G}\}.$$

Notice that $\emptyset\notin\mathcal{J}$ and $\mathcal{J}$ is closed under superset. Moreover, if $A,B\in\mathcal{J}$ then $A\cap B\in\mathcal{G}$:

in fact, $\mathbb{N} = (A\cap B)\cup A^c \cup (B^c \setminus A^c)$ and, as $P(\vec{x})$ is PR, at least one of these sets is in $\mathcal{G}$. As $A^c \notin \mathcal{G}$, $B^c\notin\mathcal{G}$ it must be $(A\cap B)\in\mathcal{G}$. Analogously one can prove that if $A_1,\dots, A_k \in \mathcal{F}$ then $A_1\cap\dots\cap A_k\in\mathcal{G}$.

So $\mathcal{H} = \{\bigcap_{i=1}^{k} A_i \mid k\in\mathbb{N},\ A_i\in\mathcal{F}\}\subseteq\mathcal{G}$ is a filter. Hence it can be extended to an ultrafilter $\mathcal{U}$. Now assume that $A\in\mathcal{U}$ is such that $\forall \vec{a}\in A$ $P(\vec{a})\neq 0$. Then (as P is PR) $A^c\in\mathcal{G}$, so $A^c\in\mathcal{F}$, hence $A^c\in\mathcal{U}$ ↯. This shows that $\mathcal{U}$ is the desired ultra.

(ii) ⇒ (i): Trivial. □

The next theorem shows that ultrafilters in $\overline{K(\beta\mathbb{N},\odot)}$ have nice properties when it comes to PR equations:

THM 18: Let $P(\vec{x})\in\mathbb{Z}[\vec{x}]$ be homogeneous. The following facts are equivalent:

(i) $P(\vec{x})$ is PR;

(ii) $\forall\mathcal{U}\in\overline{K(\beta\mathbb{N},\odot)}$ $\mathcal{U}\models P(\vec{x})=0$.

PROOF: (ii) ⇒ (i) is trivial.

(i) ⇒ (ii) Let $I_P=\{\mathcal{U}\in\beta\mathbb{N}\mid\mathcal{U}\models P(\vec{x})=0\}$. By Thm 17, $I_P\neq\emptyset$.

CLAIM: I_P is a closed bilateral ideal in $(\beta\mathbb{N},\odot,\tau)$.

If we have the claim we are done, as $\overline{K(\beta\mathbb{N},\odot)}$ is the minimal closed bilateral ideal, hence $\overline{K(\beta\mathbb{N},\odot)}\subseteq I_P$.

Closure: Let $\mathcal{U}\notin I_P$. Then $\exists A\in\mathcal{U}$ $\forall\vec{a}\in A$ $P(\vec{a})\neq 0$. Then $\Theta_A\cap I_P=\emptyset$.

Ideal: Let $\mathcal{U}\in I_P$, $\mathcal{V}\in\beta\mathbb{N}$. Then $\mathcal{V}\odot\mathcal{U}\in I_P$: in fact

$$A\in\mathcal{V}\odot\mathcal{U} \iff \underbrace{\{n\in\mathbb{N}\mid\{m\in\mathbb{N}\mid m\cdot n\in A\}\in\mathcal{U}\}}_{J}\in\mathcal{V}.$$

Let $n\in J$ and let $J_n=\{m\in\mathbb{N}\mid m\cdot n\in A\}$. $J_n\in\mathcal{U}\Rightarrow\exists\vec{j}\in J_n$ $P(\vec{j})=0$. As P is homogeneous, also $P(n\cdot\vec{j})=0$, and we conclude as $n\cdot\vec{j}\in A$.

The result for $\mathcal{U}\odot\mathcal{V}$ is similar. □

THM 19: Let $P(x_1,\dots,x_n)\in\mathbb{Z}[x_1,\dots,x_n]$. Let $\mathcal{U}\models P(\vec{x})=0$. Then $\forall A\in\mathcal{U}$

$$\{a\in A\mid\exists a_2,\dots,a_n\in A\ P(a,a_1,\dots,a_n)=0\}\in\mathcal{U}.$$

PROOF: If not, then $B=\{a\in A\mid\forall a_2,\dots,a_n\in A\ P(a,a_1,\dots,a_n)\neq 0\}\in\mathcal{U}$. And in B there are no solutions to the desired equation, hence $\mathcal{U}\not\models P(\vec{x})=0$, ↯. □

Exercise 9: Prove that if $\mathcal{U} \models P(\vec{x}) = z$ and $\mathcal{U} \models Q(\vec{y}) = t$ then $\mathcal{U} \models P(\vec{x}) = Q(\vec{y})$.

Our final result is the following:

Thm 20: The equation $x+y=tz$ is PR.

Proof: Let $\mathcal{U} \in \overline{K(\beta\mathbb{N},\odot)}$ be such that $\mathcal{U}\odot\mathcal{U}=\mathcal{U}$. (i)

As $\mathcal{U}\odot\mathcal{U}=\mathcal{U}$, we have that $\mathcal{U}\models t\cdot z = r$. (ii)

Moreover, by Theorem 18 and Schur's, $\mathcal{U}\models x+y=s$.

Therefore we conclude by Exercise 9. □

Exercise 10: Prove (i), (ii).

Exercise 11: Prove or disprove that the following equations are PR:

1. $x+2y=tz$;
2. $x+y=2tz$;
3. $x_1x_2+y_1y_2=z$;
4. $x_1x_2x_3+y_1=z_1z_2$;
5. $xz+yz=xy$;
6. $xz+yz=2xy$;
7. $x_1x_2^2=y$;
8. $x_1=y_1^2$;
9. $x_1^2x_3^4=y_1y_2^5y_3^3$;
10. $x+y^2=z$;
11. $x+y=z^2$;
12. $x^2+y^2=z^2$;
13. $x^2+y^2=3z^2$;
14. $x^2+y^3=z$.

Fact: some of these equations are PR, some are not, some are simple to solve, some are not; one of them is an open problem.