

Solutions to Primal and Dual Entropy Optimization Problems

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1 Introduction

Let $\mathcal{Q}$ be a convex set of probability measures defined on the probability space $(\Omega, \mathcal{F}, P)$ and absolutely continuous with respect to P . We say that Q^* is the optimal solution of the Entropy Minimization Problem ($\text{EMP}(\mathcal{Q})$) if

$$Q^* \in \mathcal{Q} \text{ satisfies } H(Q^*, P) = \min_{Q \in \mathcal{Q}} H(Q, P) < +\infty, \quad (\text{EMP}(\mathcal{Q}))$$

where $H(Q, P) = E_P \left[\frac{dQ}{dP} \ln \left(\frac{dQ}{dP} \right) \right]$ is the *relative entropy* of Q with respect to P . The strict convexity of $H(\cdot, P)$ assures the uniqueness of the optimal solution of the $\text{EMP}(\mathcal{M})$.

This problem has been widely studied in the past decades and many important contributions appear in the theoretic and applied probability literature as well in many different areas of applied science. An extensive list of references can be found in Gzyl [8]. The $\text{EMP}(\mathcal{Q})$ was discussed by Csiszar [3] (resp. Csiszar [4]) when $\mathcal{Q}$ is defined by equality (resp. inequality) constraints. In this paper we slightly improve some of the previous results in infinite dimensional setting with inequality constraints.

Given a convex cone F of P -a.s. finite random variables, we say that f^*

is the optimal solution of the Utility Maximization Problem (UMP(F)) if

$$f^* \in F \text{ satisfies } E_P[u(f^*)] = \max_{f \in F} E_P[u(f)] > -\infty, \quad (\text{UMP}(F))$$

where $u(x) = -e^{-x}$, $x \in \mathbb{R}$, is the exponential “utility” function.

Convex duality theory and the computation of Fenchel Lagrange transform suggest that $\text{EMP}(F_1^0)$ is *essentially* the dual of the $\text{UMP}(F)$, where F_1^0 is the set of probability densities belonging to the polar cone F^0 of F . However, in order to give to this sentence a precise and rigorous meaning one has to carefully select the topology and the cone F for which F^0 is a subset of $L^1(\Omega, \mathcal{F}, P)$ and the duality holds.

The relation between the two problems has been discussed in many papers in the Mathematical Finance literature, see Frittelli [6], Grandits-Rheinlander [7], Samperi [9], and in the applied probability literature, see Csiszar [5] and, for the finite dimensional setting, Borwein-Lewis [2].

In this paper we will study this relation and characterize the optimal solutions of both the primal and dual problem.

1.1 The Dual Problem: $\text{EMP}(\mathcal{M}(C))$

Let us now be more precise and denote with $L^0(P)$ (resp. $L^\infty(P)$, $L^1(P)$) the space of P -a.s. finite (resp. P -essentially bounded, P -integrable) random variables on $(\Omega, \mathcal{F})$, with $L_-^\infty(P)$ (resp. $L_+^\infty(P)$) the cone of P -a.s. non positive (resp. non negative) essentially bounded random variables, with $\text{co}\{C\}$ the positive convex cone generated by a set C , with $\overline{C}^Q$ the closure of a set $C \subseteq L^1(Q)$ in the $L^1(Q)$ norm topology, with $\overline{C}^{L^0}$ the closure of a set $C \subseteq L^0(P)$ in the topology of $L^0(P)$, i.e. the topology of convergence in probability. Given a set $C \subseteq L^0(P)$ we define:

$$\begin{aligned} \mathcal{M}(C) &= \{Q \ll P : C \subseteq L^1(Q) \text{ and } E_Q[f] \leq 0 \ \forall f \in C\}, \\ \mathcal{M}_0(C) &= \{Q \in \mathcal{M}(C) : H(Q, P) < +\infty\}. \end{aligned} \quad (1)$$

Note that Q_0 solves the $\text{EMP}(\mathcal{M}_0(C))$ if and only if it solves the $\text{EMP}(\mathcal{M}(C))$ and the two problems are equivalent.

Note that if $C \subseteq L^0(P)$ then

$$\mathcal{M}(C) = \mathcal{M}(\text{co}\{C\} - L_+^\infty(P)).$$

Therefore, without loss of generality, we will assume through out the paper that:

$$C \text{ is a convex cone and } -L_+^\infty(P) \subseteq C \subseteq L^0(P). \quad (2)$$

With the definition given in (1) and assumption (2) we can as well cover the case of equality constraints defined by an arbitrary set $G \subseteq L^0(P)$. Indeed if $C = \text{Lin}\{G\} - L_+^\infty(P)$, then:

$$\{Q \ll P : G \subseteq L^1(Q) \text{ and } E_Q[f] = 0 \forall f \in G\} = \mathcal{M}(C).$$

In the sequel of the paper we'll denote with $\mathcal{M}$ (resp. $\mathcal{M}_0$) the set $\mathcal{M}(C)$ (resp. $\mathcal{M}_0(C)$), where C is a fixed set satisfying assumption (2). Let $\mathbb{P}$ be the class of probabilities defined on $(\Omega, \mathcal{F})$ and equivalent to P . It is well known (see for example Csiszar [3]) that if

$$\mathcal{M}_0 \cap \mathbb{P} \neq \emptyset$$

then the optimal solution Q^* (whenever it exists) of the $\text{EMP}(\mathcal{M})$ is equivalent to P . It can be easily shown (Proposition 8) that the slightly weaker condition

$$\forall A \in \mathcal{F} : P(A) > 0 \exists Q^A \in \mathcal{M}_0 : Q^A(A) > 0 \quad (3)$$

is also (necessary) and sufficient for $Q^* \sim P$. In this paper condition (3) will be always assumed satisfied.

In [3] and [4] Csiszar provided one necessary and one sufficient conditions for the existence of the optimal solution of the $\text{EMP}(\mathcal{M}_0)$. We reformulate these results in the next

Theorem 1 (Csiszar (1975 and 1984)) *Let $Q_0 \in \mathcal{M}_0$ and*

$$f_0 = H(Q_0, P) - \ln\left(\frac{dQ_0}{dP}\right). \quad (4)$$

- a) If $f_0 \in C$ then Q_0 is the optimal solution of the $\text{EMP}(\mathcal{M}_0)$.*
- b) If Q_0 is the optimal solution of the $\text{EMP}(\mathcal{M}_0)$ then $f_0 \in \overline{C}^{Q_0}$.*

This theorem states that the density of the optimal solution has exponential form

$$\frac{dQ_0}{dP} = \frac{e^{-f_0}}{E_P[e^{-f_0}]}, \quad P - a.s., \quad (5)$$

where the exponent f_0 belongs to an appropriate set. However, for the optimality of Q_0 nor the sufficient condition (a) is necessary, nor the necessary condition (b) is sufficient, as it is shown in counterexample 3.5 in [3]. So it is natural to ask if it is possible to identify the “exact” set of random variables, which contains C and is contained in $\overline{C}^{Q_0}$, that provides a condition that is both necessary and sufficient. The answer is provided in the next theorem. Set

$$\widehat{C} = \bigcap_{Q \in \mathcal{M}_0} \overline{C}^Q,$$

and note that $\mathcal{M}_0 \triangleq \mathcal{M}_0(C) = \mathcal{M}_0(\widehat{C})$.

Theorem 2 *Let $Q_0 \in \mathcal{M}_0$ and $f_0 = H(Q_0, P) - \ln(\frac{dQ_0}{dP})$.*

Q_0 is the optimal solution of the EMP($\mathcal{M}_0$) if and only if $f_0 \in \widehat{C}$.

From Theorem 2 and the inclusions $C \subseteq \widehat{C} \subseteq \overline{C}^{Q_0}$, one can immediately deduce Csiszar’s Theorem 1. A particular case of Theorem 2 was stated in Frittelli [6] where the author studied the existence of a martingale probability measure of minimal relative entropy in the context of an incomplete financial market model.

From Theorem 2 we will also deduce the following:

Corollary 3 *The optimal solution Q_0 of the EMP($\mathcal{M}_0$) exists if and only if there exists $f_0 \in \widehat{C}$ satisfying:*

$$e^{-f_0} \in L^1(P) \text{ and } \forall f \in C \quad E_P[e^{-f_0} f] \leq E_P[e^{-f_0} f_0] = 0. \quad (\text{A})$$

1.2 The Primal Problem: UMP($\widehat{C}$)

By using convex duality argument and the properties of the dual system $(L^\infty(P), ba(\Omega, \mathcal{F}, P))$, in Bellini-Frittelli [1] (Theorem 1.b), it was shown that if $C \subseteq L^\infty(P)$ satisfies assumption (2) and if

$$\sup_{f \in C} E_P[u(f)] < 0 \quad (6)$$

then

$$u^{-1} \left(\sup_{f \in C} E_P[u(f)] \right) = u^{-1} \left(\min_{Q \in \mathcal{M}_0} \left\{ \min_{\lambda \in (0, +\infty)} -E[u^*(\lambda dQ/dP)] \right\} \right) = \min_{Q \in \mathcal{M}_0} H(Q, P),$$

where

$$u^*(z) = \inf_{x \in R} \{xz - u(x)\} = -z \ln z + z$$

is the concave conjugate of u . The stated Theorem shows that, when $C \subseteq L^\infty(P)$, the $\text{EMP}(\mathcal{M}_0)$ is indeed the dual of the $\text{UMP}(C)$ and that condition (6) guarantees the existence of the optimal solution of the $\text{EMP}(\mathcal{M}_0)$. However, in general the primal $\text{UMP}(C)$ does not admit an optimal solution.

In order to establish an existence theorem for optimal solutions to both the primal and dual problems it is essential to determine accurately the set F in the $\text{UMP}(F)$ i.e. to take $F = \widehat{C}$, where $\widehat{C}$ can be seen as the closure of C with respect to a properly defined topology.

We will analyze this problem also when C is not necessarily contained in $L^\infty(P)$. As it will be shown in counterexample 1, when we allow elements in C that are unbounded from above and from below nor condition 6 nor the stronger condition of the existence of the optimal solution of the $\text{UMP}(C)$ or of the $\text{UMP}(\overline{C}^{L^0})$ is any more sufficient for the existence of the optimal solution of the $\text{EMP}(\mathcal{M}_0)$. Hence to treat the $L^0(P)$ case we have to consider the complementary condition (B).

We say that the probability measure Q_0 satisfies condition (B) if

$$\forall f \in C \exists \alpha \in (0, +\infty) : E_{Q_0}[e^{-\alpha f}] < +\infty, \quad (\text{B})$$

where α may depend on f and Q_0 .

Theorem 4 .

(a) If Q_0 is the optimal solution to the $\text{EMP}(\mathcal{M}_0)$ then f_0 , given in (4), is the optimal solution to the $\text{UMP}(\widehat{C})$.

(b) If f_0 is the optimal solution to the $\text{UMP}(\widehat{C})$ and if Q_0 , given by (5), satisfies condition (B) then Q_0 is the optimal solution to the $\text{EMP}(\mathcal{M}_0)$.

If all random variables in C are bounded from below then any probability measure trivially satisfies condition (B). Hence from Theorem 4 we deduce:

Corollary 5 Suppose that all random variables in C are bounded from below, then f_0 is the optimal solution to the $\text{UMP}(\widehat{C})$ if and only if Q_0 is the optimal solution to the $\text{EMP}(\mathcal{M}_0)$.

In counterexample 1 we show that $f_0 \in C \not\subseteq L^\infty(P)$ is the optimal solution of the $\text{UMP}(\overline{C}^{L^0})$ (hence of the $\text{UMP}(C)$ and of the $\text{UMP}(\widehat{C})$) but there is no optimal solution of the $\text{EMP}(\mathcal{M}_0)$. Hence the reverse implication in sentence (a) of Theorem 4 is not true, even with the stronger assumption of the existence of the optimal solution of the $\text{UMP}(\overline{C}^{L^0})$.

In counterexample 2 we show that Q_0 is the optimal solution to the $\text{EMP}(\mathcal{M}_0)$, the corresponding f_0 is the optimal solution of $\text{UMP}(\widehat{C})$, but Q_0 doesn't satisfy condition (B). Hence the reverse implication in sentence (b) of Theorem 4 is not true. The counterexamples show also that Corollary 5 is false without the boundedness assumption on C .

Finally, a full duality relation between the two problems is shown in the next Theorem, where the topology τ will be defined in Section 2.1.

Theorem 6 *With respect to the topology τ we have: (a) $\widehat{C}$ is the closure of C , (b) $\widehat{C}$ and $\text{co}(\mathcal{M}_0)$ are one the polar of the other.*

2 Proofs

When we write $E_Q[f]$ it is implicitly assumed that the expectation with respect to Q is well defined i.e. that either f^+ or f^- are Q -integrable. A set C is a convex cone if $f_i \in C$, $\alpha_i \geq 0$, $i = 1, \dots, n$ implies that $\sum_{i=1}^n \alpha_i f_i \in C$.

The following Lemma is essentially Lemma 2.1 in Csiszar [3].

Lemma 7 *Let $H(Q_0, P) < +\infty$, $H(Q_1, P) < +\infty$ and set $Q_x = xQ_1 + (1-x)Q_0$, $x \in [0, 1]$. Then $E_{Q_1} \left[\ln\left(\frac{dQ_0}{dP}\right) \right]$ is well defined and:*

$$\frac{d}{dx} H(Q_x, P)|_{x=0} = E_{Q_1} \left[\ln\left(\frac{dQ_0}{dP}\right) \right] - H(Q_0, P). \quad (7)$$

Proof. Set $\varphi_1 = dQ_1/dP$, $\varphi_0 = dQ_0/dP$ and define $g = (\varphi_1 - \varphi_0) \ln \varphi_0 + (\varphi_1 - \varphi_0) \in L^0(P)$. It is elementary to check that:

$$g \leq \varphi_1 \ln \varphi_1 - \varphi_0 \ln \varphi_0 \quad P - a.s.$$

By assumption, $(\varphi_1 \ln \varphi_1 - \varphi_0 \ln \varphi_0) \in L^1(P)$, hence $g^+ \in L^1(P)$ and

$$E_P[g] = E_{Q_1} \left[\ln\left(\frac{dQ_0}{dP}\right) \right] - H(Q_0, P) \quad (8)$$

is well defined.

Set $H(x) = \phi(\frac{dQ_x}{dP})$, $x \in [0, 1]$, where $\phi(y) = y \ln y$. Since ϕ is convex, it is easy to check that $\left(\frac{H(x)-H(0)}{x}\right)$ is not increasing as $x \downarrow 0$. By monotone convergence and by the assumptions $H(Q_1, P) < \infty$ and $H(Q_0, P) < \infty$, the following limit can be exchanged with the integration operator and the right derivative is given by:

$$\begin{aligned} \frac{d}{dx}H(Q_x, P)|_{x=0} &= \lim_{x \downarrow 0} E_P \left[\frac{H(x) - H(0)}{x} \right] = E_P [H'(0)] \\ &= E_P \left[\phi'(\frac{dQ_0}{dP}) (\frac{dQ_1}{dP} - \frac{dQ_0}{dP}) \right] = E_P [g]. \quad \blacksquare \end{aligned} \quad (9)$$

Proposition 8 *If condition 3 is satisfied then the optimal solution Q_0 of the EMP($\mathcal{M}_0$) is equivalent to P .*

Proof. Suppose that $Q_0 \ll P$ is the optimal solution to the MEP($\mathcal{M}_0$). Let $Q_1 \in \mathcal{M}_0$, Q_x and g be defined as in the proof of Lemma 7. We necessarily have

$$\left(\frac{d}{dx}H(Q_x, P)|_{x=0} \right) \geq 0, \quad (10)$$

otherwise there would exist $x > 0$ such that $H(Q_x, P) < H(Q_0, P)$, which contradicts the optimality of Q_0 . Since $g^+ \in L^1(P)$, from equations (9) and (10) we deduce that $g \in L^1(P)$, and from equation (8) we get $\ln(\frac{dQ_0}{dP}) \in L^1(Q_1)$, $Q_1(\{\frac{dQ_0}{dP} = 0\}) = 0$, and $Q_1 \ll Q_0$. We conclude that $Q \ll Q_0$ for all $Q \in \mathcal{M}_0$.

Let $A \in \mathcal{F} : P(A) > 0$. Then, by assumption, there exists a $Q^A \in \mathcal{M}_0 : Q^A(A) > 0$, hence $Q_0(A) > 0$ and $P \ll Q_0$. $\blacksquare$

The proof of theorem 1 is based on the following lemma. Define:

$$\tilde{C} = \left\{ f \in \bigcap_{Q \in \mathcal{M}_0} L^1(Q) : E_Q[f] \leq 0 \quad \forall Q \in \mathcal{M}_0 \right\}.$$

Lemma 9

$$\hat{C} = \tilde{C}.$$

Proof. If $Q \in \mathcal{M}_0$ then for all $f \in C$ we have $E_Q[f] \leq 0$, so it is clear that $\tilde{C} \subseteq \overline{C^Q} \subseteq \tilde{C}$. To show the converse it is sufficient to prove that $\tilde{C} \subseteq \overline{C^Q}$, $\forall Q \in \mathcal{M}_0$. Let $Q \in \mathcal{M}_0$, $h \in \tilde{C}$ and suppose by contradiction that $h \notin \overline{C^Q}$. By the Hahn Banach Theorem there exists a continuous linear functional on $L^1(Q)$ that strictly separates h from the closed convex cone $\overline{C^Q}$, i.e. there exists $\eta \in L^\infty(Q)$ such that:

$$0 = \sup_{f \in C} E_Q[\eta f] < E_Q[\eta h]. \quad (11)$$

Since $-1_{\{\eta < 0\}} \in C$, we deduce that $\eta \geq 0$ Q -a.s. We then may normalize η and define a probability $Q_1 \ll Q$ by setting: $\frac{dQ_1}{dQ} = \eta$. Since $\sup_{f \in C} E_{Q_1}[f] = 0$ then $Q_1 \in \mathcal{M}$. Since $\eta \in L^\infty(Q)$ and $H(Q, P) < +\infty$, we have:

$$H(Q_1, P) = E_Q[\eta \ln \eta] + E_P \left[\eta \frac{dQ}{dP} \ln \left(\frac{dQ}{dP} \right) \right] < +\infty.$$

Hence $Q_1 \in \mathcal{M}_0$ and, from equation (11), $E_{Q_1}[h] > 0$, which contradicts $h \in \tilde{C}$. ■

Proof. of Theorem 2.

Suppose that Q_0 is the optimal solution to the MEP($\mathcal{M}_0$) and let $Q_1 \in \mathcal{M}_0$. As in the proof of proposition 8, we deduce that $\ln(\frac{dQ_0}{dP}) \in L^1(Q_1)$ and $f_0 \in L^1(Q_1)$. From (7) and (10) we get

$$0 \leq E_{Q_1}[\ln(\frac{dQ_0}{dP})] - H(Q_0, P) = -E_{Q_1}[f_0],$$

which implies $f_0 \in \tilde{C}$. The thesis follows from Lemma 9.

Conversely, let $f_0 \in \tilde{C}$. For any given probability measure Q_1 such that $H(Q_1, P) < +\infty$ we have:

$$\begin{aligned} H(Q_1, P) &\geq H(Q_1, P) - H(Q_1, Q_0) = E_{Q_1}[\ln(\frac{dQ_0}{dP})] \\ &= H(Q_0, P) - E_{Q_1}[f_0]. \end{aligned}$$

By Lemma 9, $f_0 \in \tilde{C}$ and therefore $E_{Q_1}[f_0] \leq 0$ for all $Q_1 \in \mathcal{M}_0$. Hence $H(Q_1, P) \geq H(Q_0, P)$ for all $Q_1 \in \mathcal{M}_0$. ■

Corollary 3 is an immediate consequence of Theorem 2 and the following

Lemma 10 *If $Q_0 \in \mathcal{M}_0 \cap \mathbb{P}$ and if f_0 is given in (4) then $f_0 \in L^0(P)$ and it satisfies condition (A).*

If $f_0 \in L^0(P)$ satisfies condition (A) and if Q_0 is given in (5) then $Q_0 \in \mathcal{M}_0 \cap \mathbb{P}$.

Proof. Suppose that $Q_0 \in \mathcal{M}_0 \cap \mathbb{P}$. Then $e^{-f_0} \in L^1(P)$ and, since $H(Q_0, P) = E_{Q_0}[\ln(\frac{dQ_0}{dP})] < +\infty$, we get $(e^{-f_0} f_0) \in L^1(P)$ and $E[e^{-f_0} f_0] = 0$. Since $Q_0 \in \mathcal{M}_0$, we have $E[e^{-f_0} f] = \frac{1}{E[e^{-f_0}]} E_{Q_0}[f] \leq 0$ for all $f \in C$. Conversely, if $f_0 \in L^0(P)$ satisfies condition (A) then $Q_0 \sim P$, $e^{-f_0} f_0 \in L^1(P)$ and

$$\forall f \in C \quad E_{Q_0}[f] \leq 0 = E_{Q_0}[f_0].$$

Hence $H(Q_0, P) < +\infty$ and $Q_0 \in \mathcal{M}_0 \cap \mathbb{P}$. ■

Proof. of Theorem 4

(a) Since Q_0 is the optimal solution of the EMP($\mathcal{M}_0$), from Corollary 3 we know that there exists $f_0 \in \widehat{C}$ satisfying condition (A). Note that condition (A) implies that $e^{-f_0} f_0 \in L^1(P)$ and $(e^{-f_0} f)^+ \in L^1(P)$ for all $f \in C$. Since $\widehat{C} \subseteq \overline{C}^{Q_0}$, from (A) it follows also that $E[e^{-f_0} f] \leq 0$ for all $f \in \widehat{C}$. Since u is concave and $\widehat{C}$ is convex, we get for all $f \in \widehat{C}$

$$\begin{aligned} E[u(f) - u(f_0)] &\leq E[u'(f_0)(f - f_0)] \\ &= E[e^{-f_0}(f - f_0)] \leq 0, \end{aligned}$$

and so, for all $f \in \widehat{C}$, $E[u(f)] \leq E[u(f_0)]$ (not excluding $E[u(f)] = -\infty$ for some $f \in \widehat{C}$).

(b) Suppose that $f_0 \in \widehat{C}$ is optimal for the UMP($\widehat{C}$) and so $e^{-f_0} \in L^1(P)$. Since $\widehat{C}$ is a convex cone, $f_0 + \alpha f \in \widehat{C}$ for all $f \in C$ and $\alpha \geq 0$. Define, for each $\alpha > 0$ and $f \in \widehat{C}$, the random variable

$$h(\alpha, f) = \frac{u(f_0 + \alpha f) - u(f_0)}{\alpha}$$

and note that $h(\alpha, f)$ is, for each fixed f , P -a.s. non decreasing as $\alpha \downarrow 0$. Since Q_0 satisfies condition (B), for all $f \in C$ there exists $\alpha > 0$ such that $E[h(\alpha, f)] > -\infty$. Therefore from the monotone convergence theorem and the optimality of f_0 we get:

$$E[u'(f_0)f] = E[\lim_{\alpha \downarrow 0} h(\alpha, f)] = \lim_{\alpha \downarrow 0} E[h(\alpha, f)] \leq 0 \quad \forall f \in C.$$

Hence:

$$E[e^{-f_0} f] \leq 0 \quad \forall f \in C. \quad (12)$$

Since C is positively homogeneous, $(-f_0)$ is also a feasible direction from f_0 , because $f_0(1 - \alpha) = f_0 + \alpha(-f_0) \in \widehat{C}$ for $\alpha \in [0, 1]$ and $E[h(1, -f_0)] = -1 - E[u(f_0)] > -\infty$ by assumption. Therefore $E[u'(f_0)(-f_0)] \leq 0$ and so $E[e^{-f_0} f_0] \geq 0$. From equation (12) we deduce that $Q_0 \in \mathcal{M}_0$ and, since $\widehat{C} \subseteq \overline{C}^{Q_0}$, $E[e^{-f_0} f_0] \leq 0$. Then $E[e^{-f_0} f_0] = 0$ and f_0 satisfies condition (A). The thesis follows from Corollary 3. ■

Counterexample 1

Let $\Omega = \mathbb{Z}$ (the set of integers), the σ -field $\mathcal{F}$ contain all subsets of Ω , and define the probability measure P in the following way:

$$P(i) = P(-i) = \frac{P(0)}{i^2} \quad \text{for } i = 1, 2, \dots$$

where the value $P(0)$ is selected so that P becomes a probability measure (and so $0 < P(0) < 1$).

Consider the random variable X defined by $X(i) = i$ for $i \in \mathbb{Z}$. Define $C = \text{Lin}\{X\} - L_+^\infty(P)$. Then

$$\mathcal{M} = \{Q \ll P : X \in L^1(Q) \text{ and } E_Q[X] = 0\}.$$

We show the following:

- i) $\mathcal{M}_0 \cap \mathbb{P} \neq \emptyset$ and $P \notin \mathcal{M}_0$,
- ii) $\inf_{Q \in \mathcal{M}_0} I(Q, P) = 0$,
- iii) $E[u(f)] \leq E[u(0)] = -1$ for all $f \in \overline{C}^{L^0}$.

From i) and ii) it follows that there is no optimal solution to the $\text{EMP}(\mathcal{M}_0)$ and from iii) it follows that $f_0 = 0$ is the optimal solution to the $\text{UMP}(\overline{C}^{L^0})$ and, since $f_0 \in C \subseteq \widehat{C} \subseteq \overline{C}^{L^0}$, to both the $\text{UMP}(C)$ and the $\text{UMP}(\widehat{C})$.

i) Let Q be defined by $Q(i) = Q(-i) = \frac{Q(0)}{i^3}$ for $i = 1, 2, \dots$ where the value $Q(0)$ is selected so that Q becomes a probability measure. Then $Q \in \mathcal{M}_0 \cap \mathbb{P}$. From $E_P[X^+] = E_P[X^-] = +\infty$ we deduce $X \notin L^1(P)$ and $P \notin \mathcal{M}_0$.

ii) We define a sequence of probability measures $(Q_n)_{n \geq 1}$ in the following way:

$$Q_n(i) = \begin{cases} P(i) & \text{if } i = \pm 1, \dots, \pm n \\ 0 & \text{if } |i| > n \\ P(0) + \sum_{|j| > n} P(j) & \text{if } i = 0 \end{cases}$$

It is elementary to check that $(Q_n)_{n \geq 1} \subseteq \mathcal{M}_0$. Since $\lim_{n \rightarrow +\infty} Q_n(0) = P(0)$, we get $I(Q_n, P) = Q_n(0) \ln \frac{Q_n(0)}{P(0)} \rightarrow 0$, as $n \rightarrow +\infty$.

iii) Set

$$\mathcal{A} \triangleq \{f \in L^0(P) : f = \alpha X - h, \alpha \in \mathbb{R}, h \in L_+^0(P)\}.$$

Since $E[e^{-\alpha X}] = +\infty \forall \alpha \neq 0$ we get $E[u(f)] \leq E[u(0)] = -1$ for all $f \in \mathcal{A}$. One can easily check that $\mathcal{A}$ is closed with respect to P -a.s. convergence, hence we have $\overline{C}^{L^0} \subseteq \mathcal{A}$ (indeed $\overline{C}^{L^0} = \mathcal{A}$) and iii) follows.

Counterexample 2

Let X be a random variable on $(\Omega, \mathcal{F}, P)$ such that $E_P[X] = 0$ and $E_P[e^{-\alpha X}] = +\infty \forall \alpha > 0$. Define $C = \text{Lin}\{X\} - L_+^\infty(P)$ and note that P doesn't satisfy condition (B), $P \in \mathcal{M}_0$, $H(P, P) = 0$. Hence P is the optimal solution of the EMP($\mathcal{M}_0$) and, by equation (4), $f_0 = 0$ P -a.s. is the optimal solution of the UMP($\widehat{C}$).

2.1 The polarity between $\widehat{C}$ and $\mathcal{M}_0$.

The aim of this section is to define a topology τ for which the closure of C is equal to $\widehat{C}$ and the polar of $\widehat{C}$ is $co\{\mathcal{M}_0\}$. To simplify the notation in this section we will identify probability measures Q , absolutely continuous with respect to P , with their Radon Nikodym derivatives $\frac{dQ}{dP} \in L^1(P)$. Recall assumptions (2) and (3) and define the linear spaces

$$L = \bigcap_{Q \in \mathcal{M}_0} L^1(Q) \text{ and } L' = \text{Lin}\{\mathcal{M}_0\} \subseteq L^1(P).$$

Notice that $C \subseteq L$ and $L^\infty(P) \subseteq L$. For all $z \in L$ and $z' \in L'$, we have that $(zz') \in L^1(P)$ and the bilinear form: $\langle z, z' \rangle = E[zz']$ is well defined. Then (L, L') defines a dual system and we denote with τ the weakest topology on L such that all linear functionals in L' are continuous. Endowed with

the τ topology L is a locally convex topological vector space where the set of continuous linear forms on L is precisely L' . We could also select any other topology compatible with the dual system (L, L') , since our results depend only on the property that the topological dual of L is L' . Another possible choice for τ is the weakest topology on L which is stronger than each $L^1(Q)$ norm topology for all $Q \in \mathcal{M}_0$, i.e. the upper bound of all such norm topologies.

Define:

$$\overline{C} = \overline{C}^\tau \text{ (the closure of } C \text{ with respect to the } \tau \text{ topology).}$$

Note that $\overline{C}$ is a convex cone and $\overline{C} \subseteq L \subseteq L^1(Q)$ for all $Q \in \mathcal{M}_0$. The polar of $\overline{C}$ with respect to the τ topology is given by

$$\overline{C}^0 \triangleq \{z' \in L' : E[zz'] \leq 0 \text{ for all } z \in \overline{C}\} \subseteq L_+^1(P),$$

since $L_-^\infty \subseteq C$. Define:

$$\overline{C}_1^0 \triangleq \overline{C}^0 \cap \{\text{unit ball of } L^1(P)\}.$$

We need the following simple result, which may perhaps be known.

Lemma 11 *If $H(Q_1, P) < +\infty$, $Q_2 \ll P$ and $Q_\beta = (1 + \beta)Q_1 - \beta Q_2$, then $H(Q_\beta, P) < +\infty$ for all $\beta \geq 0$ such that Q_β is a probability measure.*

Proof. The cases $\beta = 0$ and $Q_1 = Q_2$ are trivial. Let $\beta > 0$ and $\varphi_1 = \frac{dQ_1}{dP}$, $\varphi_2 = \frac{dQ_2}{dP}$, $\varphi_\beta = \frac{dQ_\beta}{dP}$. If $Q_\beta \ll P$, we necessarily must have $Q_2 \ll Q_1$. Let $\chi = \frac{dQ_2}{dQ_1}$. Then $\varphi_2 = \chi\varphi_1 \geq 0$ P -a.s. and $P(\chi \geq 0, \varphi_1 > 0) = 1$. Set $G = \{\chi > 1\} \cap \{\varphi_1 > 0\}$ and note that, since $E_{Q_1}[\chi] = 1$ we have $P(G) > 0$. Then: $\varphi_\beta = \varphi_1(1 + \beta - \beta\chi) \geq 0$, P -a.s., if and only if $\beta \leq \inf_G \frac{1}{\chi-1} = \frac{1}{x-1}$, where $x \triangleq \sup_G \chi > 1$. If $x = +\infty$ we have the trivial case $\beta = 0$. Suppose that $x < +\infty$, $0 < \beta \leq \frac{1}{x-1}$ and define $\varphi_x = \varphi_1 \frac{x-\chi}{x-1} \geq 0$ P -a.s. Then $E[\varphi_x] = 1$ and since, $H(Q_1, P) < +\infty$, it can be easily checked that $H(Q_x, P) < +\infty$. Note that $\varphi_x = \varphi_\beta$ for $\beta = \frac{1}{x-1}$ and each φ_β is a convex combination of φ_x and φ_1 :

$$\varphi_\beta = \alpha\varphi_x + (1 - \alpha)\varphi_1 \text{ if } 0 < \alpha = \beta(x - 1) \leq 1.$$

By the convexity of $H(\cdot, P)$ we deduce $H(Q_\beta, P) < +\infty$. ■

Proposition 12 $\mathcal{M}_0 = \overline{C}_1^0$.

Proof. First note that if $z \in \overline{C}$ then $E_Q[z] \leq 0$ for all $Q \in \mathcal{M}_0$. Since $\mathcal{M}_0$ is convex, L' admits the following representation

$$L' = \{z' \in L^1(P) : z' = \alpha z'_1 - \beta z'_2, \alpha, \beta \geq 0, z'_1, z'_2 \in \mathcal{M}_0\},$$

and therefore

$$\overline{C}_1^0 = \left\{ Q \ll P : Q = (1 + \beta)Q_1 - \beta Q_2, \beta \geq 0, Q_1, Q_2 \in \mathcal{M}_0 \text{ and } \forall z \in \overline{C} \ E_Q[z] \leq 0 \right\}.$$

The thesis then follows from Lemma 11, since one only needs to check that $H(Q, P) < +\infty$ if $Q \in \overline{C}_1^0$. ■

Proof. of Theorem 6

From Proposition 12 we deduce that $co\{\mathcal{M}_0\} = \overline{C}^0$ and hence the bipolar $\overline{C}^{00}$ of $\overline{C}$ is given by:

$$\begin{aligned} \overline{C}^{00} &\triangleq \{z \in L : E[zz'] \leq 0 \text{ for all } z' \in \overline{C}^0\} \\ &= \{z \in L : E_Q[z] \leq 0 \text{ for all } Q \in \mathcal{M}_0\} \\ &= \tilde{C} = \hat{C}, \end{aligned}$$

by Lemma 9. From the bipolar theorem we deduce that $\overline{C} = \overline{C}^{00} = \hat{C}$. From $co\{\mathcal{M}_0\} = \overline{C}^0$ we then get $(co\{\mathcal{M}_0\})^0 = \hat{C}$ and $(\hat{C})^0 = co\{\mathcal{M}_0\}$. ■

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