

Dynamic Convex Risk Measures

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New Risk Measures for the 21th Century, 2004

G. Szego ed., John Wiley & Sons, pp. 227-248.

(First version: April 2003)

Abstract

We provide a representation theorem for convex risk measures defined on $L^p(\Omega, \mathcal{F}, P)$ spaces, $1 \leq p \leq +\infty$, and we discuss the financial meaning of the convexity axiom.

We characterize those convex risk measures that are law invariant and show the link between convex risk measures and utility based prices in incomplete market models.

As a natural extension of the representation of convex risk measures, we introduce and study a class of dynamic convex risk measures.

Keywords: *Risk Measures, Convex Risk Measures, Dynamic Risk Measures, Entropic Risk Measures, Law Invariant Risk Measures, Coherent Risk Measures, g -expectations.*

1 Introduction

Risk measures were introduced in order to quantify the riskiness of financial positions and to provide a criterion to determine whether the risk was acceptable or not.

Well known static risk measures are Value at Risk (see Duffie & Pan [14]), coherent risk measures (see Artzner et al. [4], [5] and Delbaen [12], [13]), sub-linear (see Frittelli [20]) and convex risk measures (see Heath [24], Föllmer & Schied [15] and Frittelli & Rosazza [22]). Furthermore, a large part of literature is concerned with quantile-based alternatives to Value at Risk (see, among others, Acerbi & Tasche [1], Artzner et al. [5], Frey & Mc Neil [18] and Rockafellar & Uryasev [34]).

The basic idea of risk measures in a dynamic setting was already present in the papers of Cvitanic & Karatzas [11] and Wang [40]. Very recent approaches to this subject can be found in the papers by: Artzner et al. [6], where two different dynamic risk measures are proposed, Pflug & Ruszczynski [31] and Riedel [32].

After a brief introduction on the main static risk measures existing in literature we will discuss the class of “convex risk measures”. For a more exhaustive and “historical” presentation of risk measures see Szegö [38].

It is well known that, in spite of its easy computation and interpretation, the so called Value at Risk (see Duffie & Pan [14] for definition and details) has provoked several criticisms in literature (see Artzner et al. [4], [5] and Frey & Mc Neil [18], among others).

As an answer to the “drawbacks” of VaR and to the problem of “quantifying” the riskiness of financial positions, Artzner et al. [4], [5] introduced axiomatically the so called “coherent risk measures”. In particular, such risk measures satisfy four desirable axioms of “coherence”, i.e. monotonicity, subadditivity, positive homogeneity and translation invariance. Moreover, Artzner et al. [5] (in finite probability spaces) and later Delbaen [13] (in general spaces) proved a representation theorem for coherent risk measures.

The notion of a convex risk measure was firstly introduced by Heath [24] in finite probability spaces and later by Föllmer & Schied [15] and Frittelli & Rosazza [22] in more general spaces of random variables. Föllmer & Schied [15], [16] and, independently, Frittelli [20] and Frittelli & Rosazza [22] proved *representation theorems for sublinear and convex risk measures that generalize the one for coherent risk measures*. In section 1.2 we will discuss the financial interpretation of the axioms defining risk measures and we will explain why we consider reasonable to *weaken the coherence axioms*. In section 2 we will present the main features of convex risk measures defined on $L^p(\Omega, \mathcal{F}, P)$ spaces and we will characterize those convex risk measures that are law invariant.

Many interesting results on convex risk measures can be also found in the recent excellent book of Föllmer & Schied [17]. In particular, these authors proposed convex risk measures “dependent” on the preferences of the agents, by choosing the acceptance set according to the loss functions of investors.

In Section 3, we will show that the Dynamic Certainty Equivalent of a claim, introduced by Frittelli [19] in incomplete markets as a utility based pricing rule coherent with the no arbitrage principle, provides an example of a convex risk measure. In the case of the exponential utility function, the Dynamic Certainty Equivalent coincides with the buyer price of the claim and it generates the *entropic convex risk measure*.

Another example of convex risk measures, based on pricing techniques in incomplete markets, was pointed out by Carr et al. [9].

In a very recent paper, Rockafellar et al. [35] introduced a new class of risk measures, called “expectation-bounded risk measures”. As remarked in [35], neither the class of coherent risk measures is included in the class of expectation-bounded risk measures nor viceversa. Nevertheless, these classes have some elements in common, since the authors above showed that Worst Conditional Expectation and Conditional VaR are expectation-bounded, in addition to coherent, risk measures.

In section 4 we will suggest the *definition of a dynamic convex risk measure*. In particular, two “classes” of examples are presented: one comes from the dynamic version of the representation of convex risk measures, the other one from the theory of backward stochastic differential equations and the notion of g-expectation introduced by Peng [30].

1.1 Notations

Let T be a future date fixed in advance and let X be an ordered locally convex topological vector space which represents the “habitat” of all the financial positions whose riskiness we wish to quantify. To simplify the notations and without loss of generality, in the furthering we’ll always assume that the risk free rate is zero.

Assume that X is endowed with a topology τ for which X and its topological dual space X' , formed with all continuous linear functionals on X , form a dual system. Although most of the results hold true in an ordered locally convex T.V.S., in this paper we’ll assume for simplicity that:

$$\begin{aligned} X &= L^p(\Omega, \mathcal{F}, P), \quad 1 \leq p \leq +\infty, \\ X' &\subseteq L^1(\Omega, \mathcal{F}, P), \end{aligned}$$

where $(\Omega, \mathcal{F}, P)$ is a probability space. If the sample space Ω is finite (say, its cardinality is n), then $X = X' = \mathbb{R}^n$. Other examples are: $X = L^p(\Omega, \mathcal{F}, P)$ and $X' = L^q(\Omega, \mathcal{F}, P)$ where $p \in (1, +\infty)$, p and q are conjugate, and τ is the norm topology in $L^p(\Omega, \mathcal{F}, P)$; or $X = L^\infty(\Omega, \mathcal{F}, P)$ and $X' = L^1(\Omega, \mathcal{F}, P)$ and $\tau = \sigma(L^\infty, L^1)$. We denote with $\mathbf{1}$ the random variable P -a.s. equal to 1, with “ $\leq$ ” the natural preorder on the vector space X given by inequalities that hold P -a.s.

Let X'_+ be the set formed with all the positive continuous linear functionals on X , that is

$$X'_+ \triangleq \{x' \in X' \mid x'(x) \leq 0 \quad \forall x \in X : x \leq 0\},$$

and

$$\mathcal{Z} \triangleq \{x' \in X'_+ : x'(\mathbf{1}) = 1\}$$

be the set of all probability densities in X' . By the Radon Nikodym theorem, we may identify any probability density $x' \in \mathcal{Z}$ with its associated probability measure P' by setting $\frac{dP'}{dP} = x'$. Hence, $x'(\cdot)$ is nothing but the expected value $E_{P'}[\cdot]$, namely

$$x'(x) = E_P[x'x] = E_{P'}[x], \quad \text{if } x' \in \mathcal{Z}.$$

1.2 Axioms

A (*static*) *risk measure* is a functional $\rho : X \rightarrow \mathbb{R}$ satisfying some properties which seem to be “desirable” from a financial point of view. We present now a list of (non independent) axioms for ρ .

Axioms

- (a) convexity:
 $Epi(\rho) = \{(x, a) \in X \times \mathbb{R} : \rho(x) \leq a\}$ is convex in $X \times \mathbb{R}$;
- (b) lower semi-continuity:
the set $\{x \in X : \rho(x) \leq c\}$ is closed in X for all $c \in \mathbb{R}$;
- (c) *positivity: $x \geq 0 \Rightarrow \rho(x) \leq \rho(0)$, $\forall x \in X$;
- (c1) *monotonicity: $x \geq y \Rightarrow \rho(x) \leq \rho(y)$, $\forall x, y \in X$;
- (d) subadditivity: $\rho(x + y) \leq \rho(x) + \rho(y)$, $\forall x, y \in X$;
- (d1) positive homogeneity: $\rho(ax) = a\rho(x)$, $\forall a \geq 0, \forall x \in X$;
- (e) translation invariance: $\rho(x + a) = \rho(x) - a$, $\forall a \in \mathbb{R}, \forall x \in X$;
- (e1) constancy: $\rho(a) = -a \forall a \in \mathbb{R}$;
- (h) law invariance:
if $x, y \in X$ have the same distribution with respect to P then $\rho(x) = \rho(y)$
(this is the only axiom that effectively depends on the reference probability P).

Due to the financial interpretation of ρ (see the discussion of axioms (e) below), the axioms (c) and (c1) have the inequality sign opposite to what the name of the axioms would suggest (this is the reason why we added the symbol (*) to the denomination of the axioms).

We discuss now the financial interpretation of the above axioms.

Convexity (a) and Sublinearity (d) (d1). Recall that: ρ is convex if and only if $\rho(\alpha x + (1 - \alpha)y) \leq \alpha\rho(x) + (1 - \alpha)\rho(y)$, $\forall \alpha \in [0, 1], \forall x, y \in X$; ρ is sublinear if ρ satisfies both axioms (d) subadditivity and (d1) positive homogeneity. In the definition of a *convex risk measure* (see Definition 5) we require the convexity axiom but not necessarily the sublinearity axiom. In the following four items we show why we consider reasonable to weaken sublinearity with convexity (see also [22]).

(1) The convexity axiom clearly expresses the requirement that the *risk is not increased by the diversification* of the positions held in the portfolio.

(2) Convexity alone implies (see Remark 6) the following inequalities (if $\rho(0) = 0$):

- (a1) $\rho(\alpha x) \leq \alpha\rho(x)$, $\forall \alpha \in [0, 1], \forall x \in X$;
- (a2) $\rho(\alpha x) \geq \alpha\rho(x)$, $\forall \alpha \geq 1, \forall x \in X$.

Both conditions (a1) and (a2) are justified by *liquidity arguments*: Indeed, when α becomes large, the whole position (αx) is less liquid than α singular positions x , hence inequality (a2) seems reasonable. When α is small, the opposite inequality must hold for specular reasons.

While Artzner et al. [5] motivated the axiom (d1) of positive homogeneity because of liquidity arguments, we believe (see also [22]) that only properties (a1) and (a2) are required from this type of consideration.

(3) Some authors have argued that positive homogeneity is necessary to preserve the property that a *risk measure should be invariant with respect to the change of the currency*. In the discussion of the translation invariance axiom (e) below, we will see that this is not really the case (see also Remark 3.9 in [23]).

(4) If $\rho(0) = 0$ it can be easily checked (see also Lemma 7 (d)) that which ever two axioms, among convexity (a) subadditivity (d) positive homogeneity (d1), hold true, then the other one holds true as well.

Translation invariance (e) and Constancy (e1). (1) The axiom of translation invariance (e) allows for the *representation of $\rho(x)$ as a capital requirement*. It guarantees that $\rho(x)$ is the minimal amount of money to add to the initial position x to make it acceptable:

Lemma 1 $\rho : X \rightarrow \mathbb{R}$ satisfies the axiom of translation invariance (e) if and only if there exists a set $\mathcal{A} \subseteq X$ such that:

$$\rho(x) = \inf \{ \alpha \in \mathbb{R} \mid x + \alpha \in \mathcal{A} \}. \quad (1)$$

Proof. Indeed, if ρ is given by (1) then:

$$\begin{aligned} \rho(x + a) &= \inf \{ \alpha \in \mathbb{R} \mid x + a + \alpha \in \mathcal{A} \} = \inf \{ \beta - a \in \mathbb{R} \mid x + \beta \in \mathcal{A} \} \\ &= -a + \inf \{ \beta \in \mathbb{R} \mid x + \beta \in \mathcal{A} \} = -a + \rho(x). \end{aligned}$$

Viceversa, if ρ satisfies the axiom (e) then:

$$\begin{aligned} \rho(x) &= \inf \{ \alpha \in \mathbb{R} \mid \rho(x) \leq \alpha \} = \inf \{ \alpha \in \mathbb{R} \mid \rho(x + \alpha) \leq 0 \} \\ &= \inf \{ \alpha \in \mathbb{R} \mid x + \alpha \in \mathcal{A} \}, \end{aligned}$$

where $\mathcal{A} \triangleq \{x \in X \mid \rho(x) \leq 0\}$. ■

The set $\mathcal{A}$ is called the acceptance set associated with ρ . Hence $\rho(x)$ is positive for unacceptable positions x , while $\rho(x)$ is negative for acceptable positions.

(2) Note that in the statement of axiom (e) (and the same argument could be used also - and only - for the constancy axiom (e1)) it is required that both sums $x + a$ and $\rho(x) - a$ are well defined. This implies that x and $\rho(x)$ must be expressed in the same unit: the unit of the “constant” a . If the random variable x (or $x_{\$}$) represents a random amount expressed in \$, then also $\rho(x)$ (or $\rho_{\$}(x_{\$})$) will be a sure amount expressed in \$. Hence a risk measure satisfying the translation invariance axiom *does depend on the particular choice of the currency*. Therefore also the acceptance set associated to ρ (as well as the penalty function F that will be introduced later) will depend on it.

(3) Consider two currencies (to be concrete: dollar and pound) and let $\lambda > 0$ be the exchange rate: $1\$ = \lambda\mathcal{L}$. Let $A_{\mathcal{L}}$ be a subset of random variables which are expressed in $\mathcal{L}$. Then obviously, $x_{\mathcal{L}} \in A_{\mathcal{L}}$ if and only if $\lambda x_{\mathcal{L}} \in (\lambda A_{\mathcal{L}}) \triangleq \{y \mid \exists x_{\mathcal{L}} \in A_{\mathcal{L}} : y = \lambda x_{\mathcal{L}}\}$. Then the elements of the sets $\lambda A_{\mathcal{L}}$ and $A_{\mathcal{L}}$ are the

“same” random variables, denominated either in $\$$ or in $\mathcal{L}$. Hence if $\mathcal{A}_{\mathcal{L}}$ is the acceptance set associated with $\rho_{\mathcal{L}}$, then $\mathcal{A}_{\$} \triangleq \lambda \mathcal{A}_{\mathcal{L}}$ is the acceptance set associated with $\rho_{\$}$.

Remark 2 Let $\lambda > 0$ be the exchange rate: $1\$ = \lambda \mathcal{L}$, let $\rho : X \rightarrow \mathbb{R}$ satisfy the axiom of translation invariance (e), let $\rho_{\mathcal{L}}$ (resp. $\rho_{\$}$) be the risk measure ρ expressed in pound (resp. dollar) and let $\mathcal{A}_{\mathcal{L}}$ (resp. $\mathcal{A}_{\$}$) be the acceptance set associated with $\rho_{\mathcal{L}}$ (resp. $\rho_{\$}$). If $x_{\$} = \lambda x_{\mathcal{L}}$, then:

$$\mathcal{A}_{\$} = \lambda \mathcal{A}_{\mathcal{L}} \quad \text{iff} \quad \rho_{\$}(x_{\$}) = \lambda \rho_{\mathcal{L}}(x_{\mathcal{L}}),$$

which is the proper substitute of the positive homogeneity property.

Proof. If $\mathcal{A}_{\$} = \lambda \mathcal{A}_{\mathcal{L}}$ then

$$\begin{aligned} \rho_{\$}(x_{\$}) &\triangleq \inf \{ \alpha \$: x_{\$} + \alpha \$ \in \mathcal{A}_{\$} \} = \inf \{ \lambda \alpha \mathcal{L} : \lambda x_{\mathcal{L}} + \lambda \alpha \mathcal{L} \in \mathcal{A}_{\$} \} \\ &= \lambda \inf \left\{ \alpha \mathcal{L} : x_{\mathcal{L}} + \alpha \mathcal{L} \in \frac{\mathcal{A}_{\$}}{\lambda} \right\} = \lambda \inf \{ \alpha \mathcal{L} : x_{\mathcal{L}} + \alpha \mathcal{L} \in \mathcal{A}_{\mathcal{L}} \} = \lambda \rho_{\mathcal{L}}(x_{\mathcal{L}}). \end{aligned}$$

If $\rho_{\$}(x_{\$}) = \lambda \rho_{\mathcal{L}}(x_{\mathcal{L}})$ then:

$$\begin{aligned} \mathcal{A}_{\$} &\triangleq \{ x_{\$} \mid \rho_{\$}(x_{\$}) \leq 0 \} = \{ \lambda x_{\mathcal{L}} \mid \lambda \rho_{\mathcal{L}}(x_{\mathcal{L}}) \leq 0 \} \\ &= \lambda \{ x_{\mathcal{L}} \mid \rho_{\mathcal{L}}(x_{\mathcal{L}}) \leq 0 \} = \lambda \mathcal{A}_{\mathcal{L}}. \end{aligned}$$

■

(3) We will see in Lemma 7 that for a convex risk measure the translation invariance axiom (e) is equivalent to the, self evident, constancy axiom (e1).

***Positivity (c) and *Monotonicity (c1).** Consider the axiom:

$$(c^{\circ}) \quad x \leq 0 \Rightarrow \rho(x) \geq \rho(0).$$

Pay attention to the fact that there is no symmetry between the axioms (c°) and (c). It may be easily checked (see Remark 6 (iv)) that if ρ is convex then (c) implies (c°) but the converse implication is false, as shown by the simple counterexample $\rho(x) \triangleq |x|$, $x \in \mathbb{R}$.

The interpretations of the axiom (c) (as well as (c1)) follows immediately from the financial meaning of a risk measure: Suppose that $x \geq 0$, then the position x clearly is acceptable and so $\rho(x) \leq 0$. Note that $-\rho(x)$ is the maximum amount of money which we can withdraw from the position.

Lower semi-continuity (b). This axiom is technical and it is required essentially to achieve the adequate functional representation in Theorem 9.

Law invariance (h). In addition to the more “classical” axioms (a)-(e), law invariance is also recurrent in literature (see, for instance, Kusuoka [25] and Wang et al. [39]).

On the one hand, the financial motivation of law invariance is intuitive. Indeed, it is desirable to have risk measures which “allot the same riskiness” to financial positions that are identically distributed with respect to the probability P .

On the other hand, note that the definition of law invariance depends on the probability measure P given a priori, hence it is reasonable to expect that in the representations of law invariant coherent or convex risk measures the set $\mathcal{P}$ of generalized scenarios will be dependent on P .

1.3 Coherent risk measures

After having discussed the pros and cons of the axioms listed previously, we recall now the definition of coherent risk measures in general probability spaces.

Definition 3 (see Artzner et al. [4], [5] and Delbaen [13]) .

*A functional $\rho : X \rightarrow \mathbb{R}$ is a coherent risk measure if ρ satisfies axioms (c) *positivity, (d) subadditivity, (d1) positive homogeneity and (e) translation invariance.*

Starting from coherent risk measures, several quantile-based alternatives to VaR are proposed in literature: firstly, Tail Conditional Expectation (TCE) and Worst Conditional Expectation (WCE) (see Artzner et al. [5]); and later, Expected Shortfall (ES) (see Acerbi & Tasche [1]) and Conditional Value-at-Risk (CVaR) (see Rockafellar & Uryasev [34]). In particular, the main goal achieved by the above authors is to find out a coherent risk measure (except for TCE which is not coherent) which takes into account not only the quantile of the distribution of loss at a given level α , but also an information about the mean of the loss which could occur with a probability lower than α .

We recall (see Acerbi & Tasche [1] for details and proofs) that all the previous risk measures coincide for financial positions with continuous cumulative distribution function and that ES is easy to estimate (see Frey & Mc Neil [18] for the case of large portfolios of credit risks).

Coming back to general coherent risk measures, we recall from Delbaen [13] the following characterization of coherent risk measures satisfying a further continuity property. While this result holds true in general spaces, a similar (and earlier) result is shown by Artzner et al. [5] in the case of finite sample spaces.

Theorem 4 (Delbaen [13], Th. 3.2) .

Let $\rho : L^\infty(\Omega, \mathcal{F}, P) \rightarrow \mathbb{R}$ be a coherent risk measure.

There exists a closed convex set $\mathcal{P}$ of P -absolutely continuous probability measures such that

$$\rho(x) = \sup_{Q \in \mathcal{P}} E_Q[-x], \quad \forall x \in L^\infty \quad (2)$$

$$\Updownarrow$$

the acceptance set $\{x \in L^\infty : \rho(x) \leq 0\}$ is a $\sigma(L^\infty, L^1)$ -closed convex cone.

By the representation in (2), any coherent (and lower semi-continuous) risk measure ρ can be seen as the maximum expected loss taken over a set $\mathcal{P}$ of “generalized scenarios”.

A deeper discussion on coherent risk measures is beyond the main goal of this paper. A reader interested in ramifications of coherent risk measures can see the papers of Artzner [3] (about insurance prices), of Landsman & Sherris [26] and Wang et al. [39] (on distorted probabilities and insurance prices) and of Delbaen [12], [13] (on convex games).

2 Convex risk measures

As anticipated and motivated previously, convex risk measures were introduced as a generalization of coherent ones. They were firstly proposed by Heath [24] (there called “shareholder risk measures” or “weakly coherent risk measures”) in finite sample spaces and later in general probability spaces by Föllmer & Schied [15] and, independently, by Frittelli & Rosazza [22]. All above notions of “convex risk measure” are based on the convexity axiom. However, they differ from each other because of the different selection of the other axioms.

Definition 5 *A functional $\rho : X \rightarrow \mathbb{R}$ is a convex risk measure if ρ satisfies axioms: (a) convexity, (b) lower-semicontinuity and $\rho(0) = 0$.*

Remark 6 *A convex risk measure ρ satisfies:*

- (i) $\rho(\alpha x) \leq \alpha \rho(x)$, $\forall \alpha \in [0, 1]$, $\forall x \in X$;
- (ii) $\rho(\alpha x) \geq \alpha \rho(x)$, $\forall \alpha \in (-\infty, 0] \cup [1 + \infty)$, $\forall x \in X$;
- (iii) $\rho(x - y) \geq -\rho(-x) - \rho(y)$, $\forall x, y \in X$;
- (iv) if $(x \geq 0 \Rightarrow \rho(x) \leq 0)$ then $(x \leq 0 \Rightarrow \rho(x) \geq 0)$ (i.e. $(c) \Rightarrow (c^\circ)$).

Lemma 7 *For a convex risk measure, the following couples of axioms are equivalent:*

- (c) *positivity and (c1) *monotonicity;
- (d) subadditivity and (d1) positive homogeneity;
- (e) translation invariance and (e1) constancy.

Remark 8 *Let $\rho : X \rightarrow \mathbb{R}$ satisfy the translation invariance axiom (e). Then:*

- (i) *The l.s.c. axiom (b) is equivalent to: $\{x \in X : \rho(x) \leq 0\}$ is closed in X ;*
- (ii) *The following are equivalent:*
 - α ρ is convex (axiom (a));
 - β ρ is quasi convex (i.e.: $\{x \in X : \rho(x) \leq c\}$ is convex for all $c \in \mathbb{R}$);
 - γ $\{x \in X : \rho(x) \leq 0\}$ is convex;

See the Appendix for the proofs.

2.1 Representation of convex risk measures

In the following results (see Frittelli & Rosazza [22] Theorem 6 and Corollary 7), we provide the characterization of convex and sublinear risk measures. Remind that the set $\mathcal{Z}$ is formed with all probability densities in X' , hence any element $x' \in \mathcal{Z}$ may thus be identified with a probability measure Q via the Radon-Nikodym derivative, i.e. $\frac{dQ}{dP} \triangleq x'$.

Theorem 9 .

1) $\rho : X \rightarrow \mathbb{R}$ is a convex risk measure if and only if there exist a convex functional $F : X' \rightarrow \mathbb{R} \cup \{+\infty\}$, satisfying $\inf_{x' \in X'} F(x') = 0$, such that

$$\rho(x) = \sup_{x' \in \mathcal{P}} \{x'(-x) - F(x')\} < +\infty, \quad \forall x \in X, \quad (\text{Co})$$

where $\mathcal{P} = \{x' \in X' : F(x') < +\infty\}$ is the effective domain of F .

2) $\rho : X \rightarrow \mathbb{R}$ is a sublinear and lower semi-continuous risk measure (i.e. ρ satisfies axioms (b), (d) and (d1)) if and only if ρ is representable as in (Co) with $F \equiv 0$ on $\mathcal{P}$, i.e.

$$\rho(x) = \sup_{x' \in \mathcal{P}} \{x'(-x)\} < +\infty, \quad \forall x \in X. \quad (\text{S})$$

We wish to stress that the representation in (Co) (similar but more general than (2)) holds true with axioms that are much weaker than the coherence ones. Although only the convexity and lower semi-continuity axioms are necessary to represent ρ as in (Co), one might be interested in other properties. The following result shows how further axioms come into play in the representations (Co) and (S).

Corollary 10 *If $\rho : X \rightarrow \mathbb{R}$ is a convex risk measure, then:*

- i) ρ satisfies (c) *positivity iff in (Co) we have: $\mathcal{P} \subseteq X'_+$;
- ii) ρ satisfies (e) translation invariance iff in (Co) we have: $\mathcal{P} \subseteq \{x' \in X' : x'(\mathbf{1}) = 1\}$;
- iii) ρ satisfies (c) and (e) iff in (Co) we have: $\mathcal{P} \subseteq \mathcal{Z}$.

If ρ is sublinear and l.s.c. (i.e. ρ satisfies (b), (d) and (d1)), then:

- iv) ρ satisfies (c) *positivity iff in (S) we have: $\mathcal{P} \subseteq X'_+$;
- v) ρ satisfies (c) *positivity and (e) translation invariance iff in (S) we have: $\mathcal{P} \subseteq \mathcal{Z}$ (this is exactly the case of coherent risk measures).

The proofs of Theorem 9 and of Corollary 10 are in the appendix. From Corollary 10 iii) we deduce:

Corollary 11 *$\rho : X \rightarrow \mathbb{R}$ is a convex risk measure satisfying the axioms (c) *positivity and (e) translation invariance iff there exist a convex set of probability measures $\mathcal{P}$ and a convex functional $F : \mathcal{P} \rightarrow \mathbb{R} \cup \{+\infty\}$ satisfying $\inf \{F(Q) \mid Q \in \mathcal{P}\} = 0$ and*

$$\rho(x) = \sup_{Q \in \mathcal{P}} \{E_Q[-x] - F(Q)\} < +\infty, \quad \forall x \in X. \quad (3)$$

The representation in (3) has an easy financial interpretation. Indeed, ρ is the supremum over a set $\mathcal{P}$ of scenarios of the expected loss “corrected” with a “penalty” term F which depends on the scenarios. Moreover, while the set $\mathcal{P}$ of possible scenarios could be exogenously determined, for example by some regulatory institution or by the market itself, the functional F could be determined by the investors (by means of their preferences and utility functions). As we will see in details later (Section 3), the representation form in (3) underlines the link between convex risk measures, pricing rules and preferences of investors.

2.2 Law invariant convex risk measures

Representation results of risk measures so far recalled do not involve the law invariance axiom (h). A characterization of law invariant coherent risk measures satisfying lower semi-continuity is due to Kusuoka [25] (for L^∞ spaces). Indeed, he showed that any such risk measure ρ can be represented as the supremum of the expected value of (roughly speaking) a collection $\{\rho_\alpha\}_{\alpha \in [0,1]}$ of particular coherent risk measures over a set of probability measures, where the expected value is calculated with respect to the index α . Furthermore, he argued that for any $\alpha \in (0,1)$ the risk measure ρ_α coincides with the Worst Conditional Expectation WCE_α at the level α . We recall from Kusuoka [25] the following definitions and the representation of law invariant coherent risk measures.

In the furthering, F_x will denote the cumulative distribution function of x with respect to P and Z_x will represent the “generalized inverse” of the cumulative distribution function F_x :

$$Z_x(\xi) \triangleq \inf \{ \lambda \in \mathbb{R} : F_x(\lambda) > \xi \}, \quad \forall \xi \in [0,1], \quad \forall x \in L^\infty. \quad (4)$$

Set:

$$\rho_0(x) \triangleq \text{ess. sup}(-x), \quad \forall x \in L^\infty, \quad (5)$$

$$\rho_\alpha(x) \triangleq \frac{1}{\alpha} \int_{1-\alpha}^1 Z_{-x}(\xi) d\xi, \quad \forall x \in L^\infty, \quad \forall \alpha \in (0,1], \quad (6)$$

where ess. sup stands for the essential supremum.

Theorem 12 (Kusuoka [25], Prop. 15) *Let $(\Omega, \mathcal{F}, P)$ a probability space with P atomless and $\rho : L^\infty(\Omega, \mathcal{F}, P) \rightarrow \mathbb{R}$.*

ρ is a law invariant coherent risk measure satisfying the Fatou property

$$\Updownarrow$$

there exists a set M_0 of probability measures on $(0,1]$ such that

$$\rho(x) = \sup_{m \in M_0} \left\{ \int_{(0,1]} \rho_\alpha(x) m(d\alpha) \right\}, \quad \forall x \in L^\infty. \quad (7)$$

The following result generalizes the representation above to the larger class of law invariant convex risk measures satisfying some further axioms.

Theorem 13 (see [37], Cor. 34) *Let $(\Omega, \mathcal{F}, P)$ be a probability space with P atomless, $\rho : L^\infty(\Omega, \mathcal{F}, P) \rightarrow \mathbb{R}$ and let $L^\infty(\Omega, \mathcal{F}, P)$ be endowed with the $\sigma(L^\infty, L^1)$ -topology.*

*ρ is a law invariant convex risk measure satisfying axioms (c) *positivity and (e) translation invariance*

$\Updownarrow$

there exist a non empty convex set $\mathcal{P}$ of probability measures and a convex functional $F : \mathcal{P} \rightarrow \mathbb{R} \cup \{+\infty\}$ such that $\inf_{Q \in \mathcal{P}} F(Q) = 0$ and

$$\rho(x) = \sup_{Q \in \mathcal{P}} \left\{ \int_{(0,1]} \rho_\alpha(x) m_Q(d\alpha) - F(Q) \right\}, \quad \forall x \in L^\infty, \quad (8)$$

where, for any $Q \in \mathcal{P}$, m_Q is a probability measure on $(0, 1]$ properly defined.

Note that the proof of Theorem 13 (see [37]) shows that the correction term F in (8) is exactly the same which we would obtain by ignoring the law invariance axiom.

3 Indifferent prices and risk measures

In this section we discuss utility based risk measures and show the link between convex risk measures and pricing rules. The main references for this section are the papers by Bellini & Frittelli [8], Frittelli [19] and Frittelli & Rosazza [22]. We will review the so called “Dynamic Certainty Equivalent”, introduced in [19], which provides a pricing rule that takes into account the preferences of investors and the no arbitrage principle.

Let $[0, T]$ be the time interval and let $S = (S_t)_{t \in [0, T]}$ be an $\mathbb{R}^d$ -valued càd-làg semi-martingale on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, P)$, where we assume that the filtration satisfies the usual assumptions of right continuity and completeness. The process S represents the price process of d tradeable assets.

An $\mathbb{R}^d$ -valued predictable process $H = (H_t)_{t \in [0, T]}$ is called *admissible* if H is S -integrable and the stochastic process

$$\int_0^t H_s \cdot dS_s, \quad t \in [0, T],$$

is P -a.s. uniformly bounded from below. The integral $\int_0^T H_s \cdot dS_s$ is the time T financial gain from the admissible trading strategy H . The set

$$K \triangleq \left\{ \int_0^T H_s \cdot dS_s \mid H \text{ is admissible} \right\}$$

is the cone of bounded from below claims that are attainable, at zero initial cost, from trading in the d assets with admissible trading strategies. Define

$$M \triangleq \{Q \ll P : K \subseteq L^1(Q) \text{ and } E_Q[f] \leq 0 \forall f \in K\}.$$

The elements of the set M are called *separating measures*. If S is bounded (resp. locally bounded) then M is the set of probability measures Q that are absolutely continuous with respect to P and such that S is a Q -martingale (resp. local martingale). This justifies the use of separating measures as “pricing measures”. We assume the existence of an element in M that is equivalent to P .

Consider an investor whose preferences are described by an utility function $u : \mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty\}$ that is increasing on $\mathbb{R}$, differentiable and strictly concave on its effective domain.

Let $x \in L^\infty$ and define:

$$U(x) = \sup \left\{ Eu\left(x + \int_0^T H_s \cdot dS_s\right) \mid H \text{ is admissible} \right\} = \sup \{Eu(x + f) \mid f \in K\} \quad (9)$$

In the following definition we compare the maximum expected utility from the claim $x \in L^\infty$ with the maximum expected utility from the sure amount $w \in \mathbb{R}$.

Definition 14 (see [19]) *For any contingent claim $x \in L^\infty$, the Dynamic Certainty Equivalent $\mathbb{E}_u(x)$ of x is defined as the real number w which solves the following equation:*

$$U(w) = U(x), \quad x \in L^\infty, \quad w \in \mathbb{R}. \quad (10)$$

We then denote $w \triangleq \mathbb{E}_u(x)$.

Under appropriate assumptions, in [19] it was proved the existence and uniqueness of the solution of (10) and it was shown that the dynamic certainty equivalent is “coherent” with the no arbitrage principle and that it generalizes the “standard” certainty equivalent.

Clearly, the dynamic certainty equivalent $\mathbb{E}_u$ depends on the utility function u (hence, on the preferences of investors) and we will see that, for the exponential utility function u , $\rho(x) \triangleq -\mathbb{E}_u(x)$ will provide an example of a convex risk measure that, in general, is not sublinear; while for u of logarithmic or power HARA type, $\rho(x) \triangleq -\mathbb{E}_u(x)$ will provide examples of sublinear risk measures, in general not coherent.

The idea behind the computation of $\mathbb{E}_u(x)$ is the following (for a rigorous treatment one should consult [8] and [19]). The first step is to compute the maximum expected utility $U_Q(w)$ from the wealth $w \in \mathbb{R}$ when the “pricing”

measure Q is fixed:

$$\begin{aligned} U_Q(w) &\triangleq \sup \{Eu(w+f) \mid f \in L^1(Q), E_Q[f] \leq 0, u^-(w+f) \in L^1(P)\} \\ &= \min_{\lambda \in (0, +\infty)} \left\{ \lambda w - E_P \left[u^* \left(\lambda \frac{dQ}{dP} \right) \right] \right\} \\ &= E_P \left[u \left(I \left(\lambda^* \frac{dQ}{dP} \right) \right) \right], \end{aligned}$$

where u^* is the concave conjugate of u , $I = (u')^{-1}$ and λ^* is the unique solution of the equation

$$E_P \left[\frac{dQ}{dP} I \left(\lambda^* \frac{dQ}{dP} \right) \right] = w.$$

The second step is to show that, under suitable hypothesis,

$$U(x) = \inf_{Q \in M} U_Q(E_Q[x]), \quad x \in L^\infty. \quad (11)$$

Then from (10) and (11) the certainty equivalent $w \triangleq \mathbb{E}_u(x)$ is the solution of

$$\inf_{Q \in M} U_Q(w) = \inf_{Q \in M} U_Q(E_Q[x]).$$

Example 15 (Entropic convex risk measures, see [19] and [22]) .

Consider the exponential utility function $u(y) = -e^{-y}$. Then:

$$U_Q(w) = -e^{-w-H(Q,P)} = u(w + H(Q,P)), \quad (12)$$

where $H(Q,P)$ is the relative entropy defined by

$$H(Q,P) \triangleq \begin{cases} E_P \left[\frac{dQ}{dP} \ln \frac{dQ}{dP} \right], & \text{if } Q \ll P \\ +\infty, & \text{otherwise} \end{cases}, \quad (13)$$

and the Dynamic Certainty Equivalent is:

$$\mathbb{E}_u(x) = \inf_{Q \in M} \{E_Q[x] + H(Q,P)\} - \inf_{Q \in M} \{H(Q,P)\}, \quad \forall x \in L^\infty. \quad (14)$$

By taking

$$F(Q) \triangleq H(Q,P) - \inf_{Q' \in M} H(Q',P) \quad (15)$$

and

$$\rho(x) \triangleq -\mathbb{E}_u(x)$$

it follows, from (14), that

$$\rho(x) = \sup_{Q \in M} \{E_Q[-x] - F(Q)\}, \quad x \in L^\infty.$$

Hence, ρ is the entropic convex risk measure, where in the representation (3) we have: $\mathcal{P} = M$ and $\inf_{Q \in M} F(Q) = 0$.

From Corollary 10 it follows that ρ satisfies axioms (a) convexity, (b) lower semi-continuity, (c) *positivity, (e) translation invariance, but in general not (d) or (d1) (sublinearity). Hence, ρ is a convex risk measure that in general is not sublinear nor coherent.

The correction term F , which is defined through (15), (12) and (13), is strongly linked with the functional U_Q and so it depends on the preferences of the investors, while the set M of generalized scenarios is “independent” from them. From (12) we deduce (see also Proposition 3.1 [21]) that

$$\begin{aligned} H(Q, P) &= u^{-1}(U_Q(w)) - w \\ &= \sup \{ u^{-1}E[u(w + f)] \mid f \in L^\infty : E_Q[f] \leq 0 \} - w \\ &= \sup \{ u^{-1}E[u(f)] \mid f \in L^\infty : E_Q[f] \leq 0 \}, \end{aligned}$$

which explicitly shows the financial interpretation of the relative entropy and hence of the "penalty" function F .

Recall that the *buyer price* of the claim $x \in L^\infty$ is defined as the solution $p = p(x) \in \mathbb{R}$ of the equation:

$$U(x_0 - p + x) = U(x_0),$$

where $x_0 \in \mathbb{R}$ is the initial wealth and U is defined in (9). Due to the particular properties of the exponential utility function, it is easy to check that $p(x)$ is independent from the initial wealth x_0 and that the buyer price coincides with the Dynamic Certainty Equivalent $p(x) = \mathbb{E}_u(x)$, given in (14).

An example (strongly linked to the previous one) of the entropic convex risk measures can be found also in Musiela & Zariphopoulou [27], [28] defined via “indifference prices”.

A remarkable application of the entropic convex risk measures to problems arising from pricing of weather derivatives and insurance contracts can be found in a recent work of Barrieu & El Karoui [7].

Example 16 (Sublinear risk measures, see [20] and [22]) .

Let the utility function u be of logarithmic or power HARA type. Then (under appropriate hypothesis) the dynamic certainty equivalent is given by

$$\mathbb{E}_u(x) = \inf_{Q \in M} \{ G_u(Q) E_Q[x] \}, \quad \forall x \in L_+^\infty, \quad (16)$$

where the functional $G_u : M \rightarrow \mathbb{R}$ satisfies

$$\inf_{Q \in M} G_u(Q) = 1. \quad (17)$$

As in the case of the exponential utility function, the dependence of G_u from the utility u is given by means of the conjugate of the functional U_Q defined previously.

By taking $\mathcal{P} \triangleq \{ x' \in X'_+ : x' = \frac{dQ}{dP} G_u(Q), \text{ for } Q \in M \}$ and $\rho(x) \triangleq -\mathbb{E}(x)$, it follows that

$$\rho(x) = \sup_{Q \in M} \{ G_u(Q) E_Q[-x] \} = \sup_{x' \in \mathcal{P}} \{ x'(-x) \}. \quad (18)$$

Hence, ρ is a sublinear l.s.c monotone risk measure that in general is not coherent, since the translation invariance axiom is not satisfied. Moreover, ρ is composed by a part (given by the dilation factor G_u) which depends only on the preferences of the investor and another (given by the set M of generalized scenarios and their associated expected values $E_Q[\cdot]$) which depends on the market through the set M of all separating measures.

4 Dynamic risk measures

Risk measures so far discussed deal with the problem of quantifying *today* the riskiness of financial positions with maturity at a future date T . In this sense, such risk measures can be considered as “static”.

A further problem is of “monitoring” the riskiness of our financial positions at different times (ideally, all times) between today and the final date T . In order to solve this problem, we need to treat risk measures in a dynamic setting.

Cvitanic & Karatzas [11] are mainly interested in “measuring” the risk coming from the inability of an investor of hedging a contingent claim x with an initial endowment x_0 . In a complete market model, Cvitanic & Karatzas [11] proposed the following dynamic (in their meaning) risk measure:

$$\rho^{x_0}(x) = \sup_{\gamma \in \Gamma} \inf_{\pi \in A(x_0)} E_{Q_\gamma} \left[(x - \pi_T^{x_0})^+ \right],$$

where $\{Q_\gamma\}_{\gamma \in \Gamma}$ is a net of probability measures (or “generalized scenarios”) indexed by $\gamma \in \Gamma$, $A(x_0)$ stands for the class of all admissible portfolios with initial endowment x_0 and $\pi_T^{x_0}$ for the value of the portfolio $\pi \in A(x_0)$ at the final time T . Roughly speaking, therefore, the “dynamism” of the risk measure above is due to readjustments of portfolios π at any time between 0 and T and *not* to the valuation of the risk measure all along the interval $[0, T]$.

In finite sample spaces and in discrete time models, Wang [40] pointed out the need of introducing risk measures in a dynamic setting and proposed the class of “Likelihood-Based Risk Measures”, able to value financial positions with readjustments along the interval $[0, T]$ and which contains as a particular case a possible extension of the VaR approach in a multiperiod context.

Recently, in Artzner et al. [6] dynamic risk measures have been studied. In particular, their “starting point” is the representation given below in example 20 and one of their goals is to investigate when the “recursivity” property is satisfied and when this risk measure coincides with the one arising from backward proceedings.

Other recent approaches to dynamic risk measures can be found in Pflug & Ruszczynski [31] and Riedel [32].

Here, we propose *an axiomatic definition of dynamic (convex or not) risk measures*. Intuitively, it seems reasonable to look for a map $(\rho_t)_{t \in [0, T]}$ indexed by the interval of time $[0, T]$, where at any instant t (between today and the future date T) ρ_t is a random variable which represents the “riskiness” of our

financial position at time t , i.e. “conditionally” to the information available in t . Moreover, dynamic risk measures should fulfill some “boundary conditions” at times 0 and T . In particular, we impose that ρ_0 is a static risk measure (in the sense explained before) and that ρ_T is nothing but the opposite of the worth of the financial position.

Let $T > 0$ be a fixed time in the future, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, P)$ be a filtered probability space and let $L^p(\mathcal{F}_T) = L^p(\Omega, \mathcal{F}_T, P)$ for $p \geq 1$ denote the space of all real-valued, $\mathcal{F}_T$ -measurable and p -integrable random variables. Denote with $L^0(\Omega, \mathcal{F}_t, P)$ the space of all random variables defined on $(\Omega, \mathcal{F}_t, P)$.

Definition 17 (see [36], [37]) *We call a dynamic risk measure any map $(\rho_t)_{t \in [0, T]}$ such that:*

- a) $\rho_t : L^p(\mathcal{F}_T) \rightarrow L^0(\Omega, \mathcal{F}_t, P)$, for all $t \in [0, T]$;
- b) ρ_0 is a static risk measure;
- c) $\rho_T(x) = -x$ P -a.s., for all $x \in L^p(\mathcal{F}_T)$.

In analogy with the static case, we will list now some desirable properties for $(\rho_t)_{t \in [0, T]}$. Remind that, for any $t \in [0, T]$ and for any $x \in L^p(\mathcal{F}_T)$, $\rho_t(x)$ is a random variable. According to this, in the furthering for any fixed $t \in [0, T]$ any equality and inequality of ρ_t should be understood to be valid P -a.s..

Dynamic axioms for $(\rho_t)_{t \in [0, T]}$

- (A) convexity: $\forall t \in [0, T]$, ρ_t is convex P -a.s.;
- (C) *positivity: $x \geq 0 \Rightarrow \forall t \in [0, T]$, $\rho_t(x) \leq \rho_t(0)$ P -a.s.;
- (C1) *monotonicity: $x \geq y \Rightarrow \forall t \in [0, T]$, $\rho_t(x) \leq \rho_t(y)$ P -a.s.;
- (D) subadditivity:
 $\forall x, y \in L^p(\mathcal{F}_T)$, $\forall t \in [0, T]$, $\rho_t(x + y) \leq \rho_t(x) + \rho_t(y)$ P -a.s.;
- (D1) positive homogeneity:
 $\forall \alpha \geq 0$, $\forall x \in L^p(\mathcal{F}_T)$, $\forall t \in [0, T]$, $\rho_t(\alpha x) = \alpha \rho_t(x)$ P -a.s.;
- (E) translation invariance:
 $\forall t \in [0, T]$, $\forall a$ $\mathcal{F}_t$ -measurable in $L^p(\mathcal{F}_T)$, $\forall x \in L^p(\mathcal{F}_T)$,
 $\rho_t(x + a) = \rho_t(x) - a$ P -a.s.;
- (E1) constancy: $\forall c \in \mathbb{R}$, $\forall t \in [0, T]$, $\rho_t(c) = -c$ P -a.s..

Note that in axiom (E) a translation property for ρ_t is not imposed only for constant random variables a , but for any $\mathcal{F}_t$ -measurable ones. The motivation of this stronger axiom is that any $\mathcal{F}_t$ -measurable random variable is, roughly speaking, “known at time t ”, hence it behaves more or less as a constant.

We propose the following extension of the definitions of coherent and convex risk measures to a dynamic setting.

Definition 18 (see [37]) .

- 1) A dynamic risk measure $(\rho_t)_{t \in [0, T]}$ is called convex if it satisfies axiom (A) convexity and $\rho_t(0) = 0$.

2) A dynamic risk measure $(\rho_t)_{t \in [0, T]}$ is called coherent if it satisfies axioms (C) **positivity*, (D), (D1) *sublinearity* and (E) *translation invariance*.

3) $(\rho_t)_{t \in [0, T]}$ is said to be time-consistent if

$$\rho_0 [x \mathbf{1}_A] = \rho_0 [-\rho_t (x) \mathbf{1}_A], \quad \forall t \in [0, T], \quad \forall x \in L^p(\mathcal{F}_T), \quad \forall A \in \mathcal{F}_t. \quad (19)$$

Notice that the reason to take $-\rho_t (x)$ instead of $\rho_t (x)$ in (19) is a consequence of the financial interpretation of ρ_t .

We wish to stress that the time-consistency property above is nothing but the translation of the “filtration-consistency” of Coquet et al. [10] to the setting of risk measures. Take note that this definition does not coincide with the time-consistency as in Artzner et al. [6], but it is linked with their recursivity property.

Two families of examples of convex and coherent dynamic risk measures are here proposed: the former comes from the extension of representations in (Co) and in (2) in a dynamic setting (and it can be found also in the work of Artzner et al. [6]); the latter arises from backward stochastic differential equations and the so called “conditional g -expectation” introduced by Peng [30]. We wish to stress that it was this latter family to suggest to us our definition of a dynamic risk measure.

Example 19 (dynamic convex risk measure) .

Let $\mathcal{P}$ be a convex set of P -absolutely continuous probability measures defined on $(\Omega, \mathcal{F}_T)$. For any $t \in [0, T]$ let $F_t : \mathcal{P} \rightarrow \mathbb{R}$ be a convex functional such that $\inf_{Q \in \mathcal{P}} F_t(Q) = 0$. Set

$$\rho_t (x) \triangleq \text{ess. sup}_{Q \in \mathcal{P}} \{E_Q [-x | \mathcal{F}_t] - F_t(Q)\}, \quad \forall x \in L^p(\mathcal{F}_T), \quad \forall t \in [0, T]. \quad (20)$$

From the properties of conditional expectations and essential supremum, it follows that $(\rho_t)_{t \in [0, T]}$ is a dynamic convex risk measure. Moreover, ρ_t satisfies axioms (C) **positivity*, (E) *translation invariance* and (E1) *constancy*.

Example 20 (dynamic coherent risk measure) .

Let $\mathcal{P}$ be a convex set of P -absolutely continuous probability measures defined on $(\Omega, \mathcal{F}_T)$ and set

$$\rho_t (x) \triangleq \text{ess. sup}_{Q \in \mathcal{P}} E_Q [-x | \mathcal{F}_t], \quad \forall x \in L^p(\mathcal{F}_T), \quad \forall t \in [0, T].$$

Then $(\rho_t)_{t \in [0, T]}$ is a dynamic coherent risk measure.

See also Artzner et al. [6] for more details on this dynamic risk measure.

We present now another class of examples of dynamic risk measures arising from the notion of “conditional g -expectation” introduced by Peng [30]. We

present the definition of conditional g -expectation by referring to Peng [30] and to the more recent work of Coquet et al. [10] for more details.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $(B_t)_{t \geq 0}$ a standard d -dimensional Brownian motion and $(\mathcal{F}_t)_{t \geq 0}$ the augmented filtration associated to that one generated by $(B_t)_{t \geq 0}$.

Let $T > 0$ be a fixed horizon of time. Let $L^2_{\mathcal{F}}(T; \mathbb{R}^n)$ be the space of all $\mathbb{R}^n$ -valued, adapted processes $(v_t)_{t \in [0, T]}$ such that

$$E \left[\int_0^T \|v_t\|_n^2 dt \right] < +\infty,$$

where $\|\cdot\|_n$ stands for the Euclidean norm on $\mathbb{R}^n$.

Consider a function $g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ satisfying the usual assumptions prescribed in Coquet et al. [10] and Peng [30] in order to guarantee the existence and uniqueness of the solution of the backward stochastic differential equation below.

We recall (see Pardoux and Peng [29] -Theorem 4.1- and Peng [30]- Theorem 35.1- or Coquet et al. [10]) that for every terminal condition $x \in L^2(\mathcal{F}_T)$ the following backward stochastic differential equation (shortly, BSDE)

$$\begin{aligned} -dy_t &= g(t, y_t, z_t)dt - z_t dB_t, \quad 0 \leq t \leq T; \\ y_T &= x \end{aligned} \tag{21}$$

has a unique solution, i.e. there is a unique pair $(y_t, z_t)_{t \in [0, T]} \in L^2_{\mathcal{F}}(T; \mathbb{R}) \times L^2_{\mathcal{F}}(T; \mathbb{R}^d)$ which solves (21).

Thanks to the above result, Peng [30] defined the g -expectation and the conditional g -expectation as follows. The “usual” expectations correspond to the case when $g \equiv 0$.

Definition 21 (see Peng [30], def. 36.1 - def. 36.5) .

For every $x \in L^2(\mathcal{F}_T)$ and for every $t \in [0, T]$, the conditional g -expectation of x under $\mathcal{F}_t$ (denoted by $\mathcal{E}_g[x | \mathcal{F}_t]$) is defined by

$$\mathcal{E}_g[x | \mathcal{F}_t] \triangleq y_t, \tag{22}$$

where y_t is (the first component of) the solution of the BSDE (21) with terminal condition x .

In particular, for $t = 0$

$$\mathcal{E}_g[x] \triangleq y_0, \tag{23}$$

is called g -expectation.

Definition 22 Let $g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ and let $\rho : L^2(\mathcal{F}_T) \rightarrow L^2_{\mathcal{F}}(T; \mathbb{R})$ be defined as

$$\rho = (\rho_t)_{t \in [0, T]}, \quad \rho_t(x) \triangleq \mathcal{E}_g[-x | \mathcal{F}_t], \quad \forall x \in L^2(\mathcal{F}_T). \quad (24)$$

Proposition 23 1) If the functional g is convex in $(y, z) \in \mathbb{R} \times \mathbb{R}^d$, then $(\rho_t)_{t \in [0, T]}$ defined as in (24) is a dynamic convex risk measure. Moreover, $(\rho_t)_{t \in [0, T]}$ is time-consistent and satisfies axioms (C), (E) and (E1).

2) If the functional g is sublinear in $(y, z) \in \mathbb{R} \times \mathbb{R}^d$, then $(\rho_t)_{t \in [0, T]}$ defined as in (24) is a dynamic coherent risk measure. Moreover, it is time-consistent.

Notice that the above examples are in line with a recent work of Barrieu & El Karoui [7], where the authors show that the so called dynamic “market risk measures” solve a quadratic backward stochastic differential equation.

In Proposition 23 examples of dynamic risk measures are generated via conditional g -expectation with g satisfying some suitable assumptions.

Viceversa, as a simple rephrasing of Theorem 7.1 of Coquet et al. [10], in the next result (see also [37]) we show some sufficient conditions for a dynamic risk measure to come from a g -expectation. Recall (see [10]) that ρ_0 is $\mathcal{E}^\mu$ -dominated if $\forall x, y \in L^2(\mathcal{F}_T)$, $\rho_0(x + y) - \rho_0(x) \leq \mathcal{E}_{g_\mu}[-y]$ for some $g_\mu(t, y, z) = \mu |z|$ with $\mu > 0$,

Proposition 24 1) Let $d = 1$ and let $(\rho_t)_{t \in [0, T]}$ be a dynamic convex risk measure on $L^2(\mathcal{F}_T)$ satisfying (C1) *monotonicity and (E) translation invariance. If

- i) $(\rho_t)_{t \in [0, T]}$ is time-consistent;
- ii) ρ_0 is strictly monotone, i.e.: $x \geq y$ and $\rho_0(x) = \rho_0(y) \Leftrightarrow x = y$;
- iii) ρ_0 is $\mathcal{E}^\mu$ -dominated,

then there exists a unique $g : \Omega \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ (g is independent from y) satisfying the usual assumptions, such that $|g(t, z)| \leq \mu |z|$ and

$$\rho_0(x) = \mathcal{E}_g[-x] \quad \text{and} \quad \rho_t(x) = \mathcal{E}_g[-x | \mathcal{F}_t], \quad \forall x \in L^2(\mathcal{F}_T). \quad (25)$$

Moreover, if such a g is P -a.s. continuous in t for all $z \in \mathbb{R}$, then it is also convex in z .

2) Let $d = 1$ and let $(\rho_t)_{t \in [0, T]}$ be a dynamic coherent risk measure satisfying i), ii) and iii). Then there exists a unique $g : \Omega \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ (g is independent from y) satisfying the usual assumptions, such that $|g(t, z)| \leq \mu |z|$ and (25) holds true. Moreover, if such a g is P -a.s. continuous in t for all $z \in \mathbb{R}$, then it is also sublinear in z .

5 Appendix

Recall that the conjugate f^* and the biconjugate f^{**} of a convex function $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ are defined by

$$\begin{aligned} f^* & : X' \rightarrow \mathbb{R} \cup \{+\infty\}, \quad f^*(x') \triangleq \sup_{x \in X} \{x'(x) - f(x)\}, \\ f^{**} & : X \rightarrow \mathbb{R} \cup \{+\infty\}, \quad f^{**}(x) \triangleq \sup_{x' \in X'} \{x'(x) - f^*(x')\}. \end{aligned}$$

Theorem 25 (Rockafellar [33], Theorem 5) .

*If the function $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is convex and lower semi-continuous, then $f = f^{**}$, i.e.*

$$f(x) = \sup_{x' \in X'} \{x'(x) - f^*(x')\}, \quad \forall x \in X.$$

Proof of Theorem 9. .

1) If ρ is a convex risk measure, then $f(x) \triangleq \rho(-x)$ satisfies (a) convexity, (b) lower semi-continuity and $f(0) = 0$. From Theorem 25, it follows that

$$f(x) = \sup_{x' \in X'} \{x'(x) - f^*(x')\}, \quad \forall x \in X.$$

Take

$$F = f^*, \tag{26}$$

$$\mathcal{P} = \{x' \in X' : F(x') < +\infty\}. \tag{27}$$

Then we have shown that:

$$\rho(x) = f(-x) = \sup_{x' \in \mathcal{P}} \{x'(-x) - F(x')\}, \quad \forall x \in X, \tag{28}$$

and

$$0 = f(0) = \sup_{x' \in \mathcal{P}} \{x'(0) - F(x')\} = - \inf_{x' \in \mathcal{P}} \{F(x')\} = - \inf_{x' \in X'} \{F(x')\}.$$

From (28) we see that ρ displays the representation form in (Co).

The reverse implication is straightforward. Indeed, the representation of ρ given in (Co) implies the convexity of ρ , while $\mathcal{P} \subseteq X'$ implies the lower semi-continuity of ρ . Since $0 = \inf_{x' \in X'} F(x') = \inf_{x' \in \mathcal{P}} F(x')$, then $\rho(0) = 0$.

2) Since sublinearity implies convexity, it follows that ρ can be represented as in (Co), where $f(x) \triangleq \rho(-x)$ and F and $\mathcal{P}$ are given in (26) and (27). It remains to be shown that

$$F(x') = 0 \quad \text{if } x' \in \mathcal{P}.$$

Since $\inf_{x' \in X'} \{F(x')\} = 0$, we have: $0 \leq F(x') = f^*(x') \triangleq \sup_{x \in X} \{x'(x) - f(x)\}$ for any $x' \in X'$. Consider now $x' \in X'$ and $\bar{x} \in X$ such that $[x'(\bar{x}) - f(\bar{x})] > 0$. Then

$$F(x') \geq \sup_{\lambda > 0} \{x'(\lambda \bar{x}) - f(\lambda \bar{x})\} = \sup_{\lambda > 0} \{\lambda [x'(\bar{x}) - f(\bar{x})]\} = +\infty, \quad (29)$$

since f is positively homogeneous and by assumption $[x'(\bar{x}) - f(\bar{x})] > 0$.

It then follows that

$$F(x') < +\infty \Rightarrow x'(x) - f(x) \leq 0, \forall x \in X \Rightarrow F(x') \leq 0,$$

hence

$$F(x') < +\infty \Rightarrow F(x') = 0.$$

Conversely, the sublinearity of ρ follows immediately from the representation in (S). ■

Proof of Corollary 10. Let ρ be a convex risk measure. Then (Co) holds true with (27).

i) If ρ satisfies *positivity (c) then it is easy to check from (Co) and (27) that:

$$x'(x) \leq F(x'), \forall x' \in \mathcal{P} \text{ and } \forall x \in X : x \leq 0. \quad (30)$$

Since $F(x')$ is finite and does not depend on x , from (30) we deduce that

$$x'(x) \leq 0, \forall x' \in \mathcal{P} \text{ and } \forall x \in X : x \leq 0.$$

Therefore

$$x' \in \mathcal{P} \Rightarrow x' \in X'_+.$$

Conversely, we prove that the axiom of *positivity (c) is satisfied by ρ .

For any $x \in X$ such that $x \geq 0$ we have from (Co):

$$\rho(x) = \sup_{x' \in \mathcal{P}} \{x'(-x) - F(x')\} \leq \sup_{x' \in \mathcal{P}} \{-F(x')\} = \rho(0) = 0,$$

since $\mathcal{P} \subseteq X'_+$.

ii) From $\rho(0) = 0$ and axiom (e) (translation invariance) we get $\rho(a) = -a$ $\forall a \in \mathbb{R}$. From (Co) we deduce that

$$-a = \rho(a) = \sup_{x' \in \mathcal{P}} \{-ax'(\mathbf{1}) - F(x')\}, \forall a \in \mathbb{R}. \quad (31)$$

Hence

$$a[1 - x'(\mathbf{1})] \leq F(x'), \forall x' \in \mathcal{P}, \forall a \in \mathbb{R}.$$

Since a is arbitrary, then it is necessary that $x'(\mathbf{1}) = 1$ for any $x' \in \mathcal{P}$.

Conversely, if ρ is representable as in (Co) with $\mathcal{P} = \{x' \in X' : x'(\mathbf{1}) = 1\}$ then ρ is a convex risk measure satisfying

$$\rho(x + a) = \sup_{x' \in \mathcal{P}} \{x'(-x) - a - F(x')\} = \rho(x) - a, \forall x \in X, \forall a \in \mathbb{R}.$$

iii) is a consequence of items i) and ii).

iv) (resp. v)) follows from item 2) of Theorem 9 and from item i) (resp. iii)). ■

Proof of Remark 6.

(i) For any $\alpha \in [0, 1]$ and for any $x \in X$

$$\begin{aligned}\rho(\alpha x) &= \rho(\alpha x + (1 - \alpha)0) \\ &\leq \alpha\rho(x) + (1 - \alpha)\rho(0) = \alpha\rho(x).\end{aligned}$$

(ii) For any $x \in X$ and any $\alpha > 1$:

$$\rho(x) = \rho\left(\frac{1}{\alpha}\alpha x\right) \leq \frac{1}{\alpha}\rho(\alpha x),$$

since $\frac{1}{\alpha} \in (0, 1)$ and (i) holds true.

For any $x \in X$

$$0 = 2\rho(0) = 2\rho\left(\frac{1}{2}x + \frac{1}{2}(-x)\right) \leq \rho(x) + \rho(-x),$$

that is

$$\rho(x) \geq -\rho(-x). \quad (32)$$

From (32) and (i), it follows that for any $x \in X$ and any $\alpha \in [-1, 0)$

$$\rho(\alpha x) \geq -\rho(-\alpha x) \geq \alpha\rho(x).$$

The case of $\alpha < -1$ can be obtained as for $\alpha > 1$.

(iii) For any $x, y \in X$

$$2\rho\left(-\frac{1}{2}y\right) = 2\rho\left(-\frac{1}{2}x + \frac{1}{2}(x - y)\right) \leq \rho(-x) + \rho(x - y).$$

Thanks to the previous inequality and to (ii):

$$\rho(x - y) \geq -\rho(-x) + 2\rho\left(-\frac{1}{2}y\right) \geq -\rho(-x) - \rho(y).$$

(iv) If $x \leq 0$, then $-x \geq 0$ and, from (ii), $-\rho(x) \leq \rho(-x) \leq 0$. ■

Proof of Lemma 7.

(c) $\Leftrightarrow$ (c1):

The implication (c1) $\Rightarrow$ (c) is trivial.

Viceversa, suppose that ρ is a convex risk measure satisfying axiom (c). From Corollary 10- item i)- it follows that

$$\rho(x) = \sup_{x' \in \mathcal{P}} \{x'(-x) - F(x')\}, \quad \forall x \in X,$$

where the set $\mathcal{P}$ is contained in X'_+ .

Hence, for any $x, y \in X$ such that $x \geq y$:

$$\begin{aligned} \rho(x) - \rho(y) &= \sup_{x' \in \mathcal{P}} \{x'(-x) - F(x')\} - \sup_{x' \in \mathcal{P}} \{x'(-y) - F(x')\} \\ &\leq \sup_{x' \in \mathcal{P}} \{[x'(-x) - F(x')] - [x'(-y) - F(x')]\} \end{aligned} \quad (33)$$

$$= \sup_{x' \in \mathcal{P}} \{x'(-x + y)\} \leq 0, \quad (34)$$

where inequality (33) is due to the definition of the supremum and inequality (34) follows from $y - x \leq 0$ and $\mathcal{P} \subseteq X'_+$.

(d) $\Rightarrow$ (d1):

Let $\alpha \geq 0$ and remind that α can be written as $\alpha = [\alpha] + \text{mant}(\alpha)$, where $[\alpha] \in \mathbb{N}$ and $\text{mant}(\alpha) \in [0, 1)$.

From (d), we have, by induction on n , that:

$$\rho(nx) \leq n\rho(x), \quad \forall n \in \mathbb{N}. \quad (35)$$

From subadditivity, (35) and Remark 6 (i) we have, for any $\alpha \geq 0$ and any $x \in X$:

$$\begin{aligned} \rho(\alpha x) &= \rho([\alpha]x + \text{mant}(\alpha)x) \\ &\leq \rho([\alpha]x) + \rho(\text{mant}(\alpha)x) \\ &\leq [\alpha]\rho(x) + \text{mant}(\alpha)\rho(x) \\ &= \alpha\rho(x). \end{aligned}$$

From the inequality just established and from Remark 6 (ii), we have

$$\rho(\alpha x) = \alpha\rho(x), \quad \forall \alpha \geq 1. \quad (36)$$

For any $\alpha \in (0, 1)$, $\frac{1}{\alpha} > 1$ and then we get

$$\rho(x) = \rho\left(\frac{1}{\alpha}\alpha x\right) = \frac{1}{\alpha}\rho(\alpha x).$$

(d1) $\Rightarrow$ (d):

For any $x, y \in X$:

$$\begin{aligned}\rho(x+y) &= 2 \left[\frac{1}{2} \rho(x+y) \right] = 2\rho\left(\frac{1}{2}x + \frac{1}{2}y\right) && \text{(by (d1))} \\ &\leq 2\frac{1}{2}[\rho(x) + \rho(y)] = \rho(x) + \rho(y). && \text{(by (a))}\end{aligned}$$

(e) $\Leftrightarrow$ (e1):

The implication (e) $\Rightarrow$ (e1) is immediate.

Viceversa, suppose that axiom (e1) (constancy) is satisfied. In the proof of item ii) in Corollary 10 we used only the constancy axiom (see equation (31)). Hence we may analogously represent ρ as

$$\rho(x) = \sup_{x' \in \mathcal{P}} \{x'(-x) - F(x')\}, \quad \forall x \in X,$$

where $\mathcal{P} \subseteq \{x' \in X' : x'(1) = 1\}$.

Hence, for any $x \in X$ and any $\beta \in \mathbb{R}$:

$$\begin{aligned}\rho(x+\beta) &= \sup_{x' \in \mathcal{P}} \{x'(-x-\beta) - F(x')\} \\ &= \sup_{x' \in \mathcal{P}} \{x'(-x) - \beta - F(x')\} = \rho(x) - \beta.\end{aligned}$$

■

Proof of Remark 8.

Assume that ρ is a functional satisfying the translation invariance axiom. Then:

$$\begin{aligned}\mathcal{A}_c &\triangleq \{x \in X : \rho(x) \leq c\} = \{x \in X : \rho(x+c) \leq 0\} \\ &= \{y-c \in X : \rho(y) \leq 0\} = \{y \in X : \rho(y) \leq 0\} - c = \mathcal{A}_0 - c.\end{aligned} \tag{37}$$

(i) The thesis follows immediately from $\mathcal{A}_c = \mathcal{A}_0 - c$, since the sum of a closed set and a compact set is a closed set (see, for instance, Lemma 5.2. in [2]).

(ii) The implications $\alpha \Rightarrow \beta$ and $\beta \Rightarrow \gamma$ are trivial.

Suppose that ρ satisfies γ). We have to show that the set $\mathcal{C} \triangleq \text{Epi}(\rho) = \{(x, d) \in X \times \mathbb{R} : \rho(x) \leq d\}$ is a convex set in $X \times \mathbb{R}$.

For the translation invariance of ρ , $\mathcal{C}$ can be rewritten as

$$\mathcal{C} \triangleq \{(x, d) \in X \times \mathbb{R} : \rho(x+d) \leq 0\}. \tag{38}$$

Consider two arbitrary elements (y, a) and (z, b) in $\mathcal{C}$ and an arbitrary $\alpha \in [0, 1]$. Then, by (38), $y+a \in \mathcal{A}_0$ and $z+b \in \mathcal{A}_0$. We have to show that $\alpha(y, a) + (1-\alpha)(z, b) \in \mathcal{C}$. This follows from (38) and:

$$\rho(\alpha y + (1-\alpha)z + \alpha a + (1-\alpha)b) = \rho(\alpha(y+a) + (1-\alpha)(z+b)) \leq 0,$$

since the level set $\mathcal{A}_0$ is convex by hypothesis. ■

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