

The Minimal Entropy Martingale Measure and the Valuation Problem in Incomplete Markets

Marco Frittelli
Università degli Studi di Milano

MATHEMATICAL FINANCE* 2000,
Vol. 10/1 pp. 39-52.

Abstract

Let χ be a family of stochastic processes on a given filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathcal{T}}, P)$ with $\mathcal{T} \subseteq \mathbb{R}_+$. Under the assumption that the set $\mathcal{M}_e$ of equivalent martingale measures for χ is not empty, we give sufficient conditions for the existence of a unique equivalent martingale measure which minimizes the relative entropy, with respect to P , in the class of martingale measures. We then provide the characterization of the density of the Minimal Entropy Martingale measure, which suggests the equivalence between the maximization of expected exponential utility and the minimization of the relative entropy.

Mathematics Subject Classification: 60G42, 60G44, 90A09, 90A10.

Keywords: *Martingales, Equivalent Martingale Measures, Relative Entropy, Incomplete Markets, Asset Pricing.*

Acknowledgment: I am very grateful to an anonymous referee for his suggestions which helped me in improving the paper.

*Manuscript received March 1996; final version received February 1999.

1 Introduction

In this paper we will investigate the properties of the *Minimal Entropy Martingale* measure (MEM) in relation to the problem of financial asset valuation in incomplete markets.

Let $\mathcal{T} \subseteq [0, +\infty)$ represent the set of trading dates and let χ be a family of stochastic processes on a given filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathcal{T}}, P)$ that represents deflated price processes of the securities available for trading in the market. With $\mathcal{M}_e$ we denote the set of martingale probability measures that are equivalent to P . $\mathcal{M}_e$ represents the set of positive linear pricing functionals compatible with some sort of “no arbitrage condition” (see [3], [5], [10] and [15]). We assume throughout the paper that $\mathcal{M}_e \neq \emptyset$.

In complete markets every contingent claim C is replicable and its price $\pi(C)$ is uniquely determined by the unique martingale measure Q via the formula

$$\pi(C) = E_Q[\beta C]$$

where β here represents the discount factor.

However, in incomplete markets a claim is not, in general, perfectly replicable. The set $\mathcal{M}_e$ is not in general a singleton, and so we are typically left with the set of prices

$$\Pi = \{E_Q[\beta C] : Q \in \mathcal{M}_e\}$$

all compatible with the “no arbitrage condition”. So, one may invoke some other principle in order to select one specific martingale measure (which we call the *pricing measure*) in $\mathcal{M}_e$ and accordingly price contingent claims.

Several criteria have been proposed in literature (see for example [6], [7], [8], [9], [22], [23]) in order to find, among all martingale measures, the one which is the “closest” (in some sense to be specified) to P . Here we only note that even in the simple context of a finite state multiperiod market the *minimal variance measure* Q^{var} (see [6], [23]) and the *minimal martingale measure* Q^{min} (see [9]) may not be equivalent to P so that the corresponding pricing functional will not be positive (see Section 3.2).

Other criteria (see [4] and [17]) are based on the consideration that, in incomplete markets, asset valuation is no longer independent from the preference structure of investors so that asset pricing will again depend on the utility functions of investors.

In this paper we try to partially reconcile the above mentioned approaches: we show that the probability Q_0 that minimizes the relative entropy, with

respect to P , in the class $\mathcal{M}$ of P -absolutely continuous martingale measures can be a good selection for the pricing measure and can be legitimate from a utility maximization argument.

Csiszar [2] studied in detail the problem of the existence of the probability measure that minimizes the relative entropy on a set of probability measures defined by linear constraints and some of our results, as specified later, may also be derived from [2].

A review of the applications of the notion of relative entropy to economics and finance may be found in Stutzer ([25] and [26]) in the framework of a finite probability space. An introduction to this subject may also be found in Frittelli [12].

In [19], Miyahara independently developed an approach similar to ours and proved a result regarding the existence of the MEM for bounded processes which is the analogue of our Theorem 1. When the processes are solutions of a SDE he studied the relations between the MEM and the minimal martingale measure; in a more recent paper ([20]) he also analyzed the properties of the MEM in a jump-diffusion model.

In this paper we consider a general incomplete security market model, with possibly an infinite number of price processes defined on a general stochastic bases.

When the stochastic processes are bounded, we prove that the MEM exists and is unique (Th.1). We also show (Th.2) that, in contrast with the minimal and minimal variance measures, the MEM is always equivalent to P , even if the processes are unbounded.

We then provide the characterization of the density of the MEM (Th. 3 and 4) without any restrictions on the processes and, in a multiperiod model, we give (Th.5) a more explicit form of this density.

In section 3 we illustrate the financial interpretation of our approach and show the relationship between the minimization of the relative entropy and the maximization of the exponential utility function. More recent papers on this subject are Bellini-Frittelli [1], Frittelli [13], Grandts - Rheinlaender [14], Samperi [21].

2 The minimal entropy martingale measure

Let $\mathcal{T} \subseteq [0, +\infty)$, $0 \in \mathcal{T}$, be the time index set and consider a family χ of real adapted stochastic processes $X = (X_t)_{t \in \mathcal{T}}$ defined on a filtered probability

space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathcal{T}}, P)$. With $L^0(P)$ (resp. $L^\infty(P)$, $L^1(P)$) we denote the space of P -a.s. finite (resp. P -essentially bounded, P -integrable) random variables on $(\Omega, \mathcal{F})$. Set:

$$K = \text{Lin} \{H(X_t - X_s) : H \in L^\infty(\Omega, \mathcal{F}_s, P), s, t \in \mathcal{T} : s \leq t, X \in \chi\}.$$

A probability measure Q will be called a martingale measure if all processes $X \in \chi$ are $(\mathcal{F}_t)_{t \in \mathcal{T}}$ -martingales under Q . So, the sets $\mathcal{M}$ (resp. $\mathcal{M}_e$) of martingale measures that are absolutely continuous (resp. equivalent) w.r.to P may be defined as follows:

$$\mathcal{M} \triangleq \{Q \ll P : E_Q[k] = 0 \quad \forall k \in K\}, \quad \mathcal{M}_e \triangleq \{Q \in \mathcal{M} : Q \sim P\}.$$

Definition 1 Let Q be a probability measure on $(\Omega, \mathcal{F})$. The relative entropy $I(Q, P)$ of Q with respect to P is defined by

$$I(Q, P) = \begin{cases} E_P \left[\frac{dQ}{dP} \ln \left(\frac{dQ}{dP} \right) \right] & \text{if } Q \ll P \\ +\infty & \text{otherwise} \end{cases}.$$

If M is a set of probability measures on $(\Omega, \mathcal{F})$ we set:

$$I(M, P) = \inf \{I(Q, P) : Q \in M\}.$$

A probability measure $Q_0 \in \mathcal{M}$ will be called the Minimal Entropy Martingale measure (MEM) if it satisfies

$$I(Q_0, P) = \min_{Q \in \mathcal{M}} I(Q, P) = \min_{Q \in \mathcal{M}} E_P \left[\frac{dQ}{dP} \ln \left(\frac{dQ}{dP} \right) \right].$$

Set $\mathcal{M}^0 = \{Q \in \mathcal{M} : I(Q, P) < +\infty\}$ and $\mathcal{L} = \bigcap_{Q \in \mathcal{M}^0} L^1(Q)$.

For future use, we introduce the function $\phi(x) = x \ln x$, $x \in (0, +\infty)$, $\phi(0) = 0$, which allows us to write $I(Q, P) = E_P \left[\phi \left(\frac{dQ}{dP} \right) \right]$, for $Q \ll P$.

For any probability measure Q , it is well known that: $I(Q, P) \geq 0$, $I(Q, P) = 0$ if and only if $Q = P$, and the functional $Q \rightarrow I(Q, P)$ is strictly convex.

The notion of relative entropy will be used intuitively as a measure of the “distance” between two probability measures, even though the relative entropy is not a metric.

The next theorem provides a simple sufficient condition for the existence of the minimal entropy martingale measure.

Theorem 1 *If $I(\mathcal{M}, P) < +\infty$ and if the processes in χ are bounded then there exists a unique minimal entropy martingale measure.*

Proof. Uniqueness follows from the strict convexity of the relative entropy.

Let $\{Q_n\}_{n \geq 1} \in \mathcal{M}^0$ be such that $I(Q_n, P) \downarrow I(\mathcal{M}, P)$. Since

$$\sup_{n \geq 1} E_P[\phi(\frac{dQ_n}{dP})] \leq I(Q_1, P),$$

the de la Vallée-Poussin criterion guarantees that $\{\frac{dQ_n}{dP}\}_{n \geq 1}$ is uniformly integrable and Dunford-Pettis compactness theorem assures that we may find a subsequence, still denoted by $\{\frac{dQ_n}{dP}\}_{n \geq 1}$, that converges in the $\sigma(L^1(P), L^\infty(P))$ topology. Then, by the Hahn Banach Theorem, we may find a sequence $\{\tilde{Q}_n\}_{n \geq 1}$ of convex combinations of $\{Q_n\}_{n \geq 1}$, $\tilde{Q}_n \in \text{conv}(Q_n, Q_{n+1}, \dots)$, such that $\{\frac{d\tilde{Q}_n}{dP}\}$ is convergent, in the $L^1(P)$ -norm topology, to a density $\varphi_0 = \frac{dQ_0}{dP} \in L^1(P)$. Since $\tilde{Q}_n \in \mathcal{M}$ and the processes are bounded, we have

$$0 = E_{\tilde{Q}_n}[f] = E_P[f \frac{d\tilde{Q}_n}{dP}] \rightarrow E_P[f \frac{dQ_0}{dP}] = E_{Q_0}[f], \quad \forall f \in K \subseteq L^\infty(P),$$

and so $Q_0 \in \mathcal{M}$. From the convexity of the relative entropy we deduce $I(\tilde{Q}_n, P) \leq I(Q_n, P)$ and by Fatou Lemma we get:

$$I(Q_0, P) \leq \lim_{n \rightarrow +\infty} \inf I(\tilde{Q}_n, P) \leq \lim_{n \rightarrow +\infty} \inf I(Q_n, P) = I(\mathcal{M}, P). \blacksquare$$

Remark 1 *If the processes in χ are assumed to be only locally bounded, then Theorem 1 holds if martingale measures are replaced with local martingale measures.*

If the processes are unbounded the $\inf \{I(Q, P) : Q \in \mathcal{M}\}$ may not be attained by a martingale measure. This is shown in counterexample A) in the very simple context of a one period process defined on a countable state space.

One may easily check that if the processes in χ are bounded, then the set $\mathcal{M}$ is closed in variation. So Theorem 1 can also be derived from Csiszar's Theorem 2.1 [2]. The proof of Csiszar's Theorem is similar to the proof of the existence of projections in Hilbert space and it is based on a parallelogram identity that holds for the relative entropy and on the inequality

$|Q - P| \leq \sqrt{2I(Q, P)}$. The proof we gave of theorem 1, which relies on different arguments, can be generalized to functionals of the form

$$\delta(Q, P) = \begin{cases} E_P \left[\Phi \left(\frac{dQ}{dP} \right) \right] & \text{if } Q \ll P \\ +\infty & \text{otherwise} \end{cases},$$

where $\Phi : [0, +\infty) \rightarrow \mathbb{R}$, provided that: $\lim_{x \rightarrow +\infty} \frac{\Phi(x)}{x} = +\infty$, Φ is bounded below and $Q \rightarrow \delta(Q, P)$ is strictly convex.

We now show that, under the assumption $I(\mathcal{M}_e, P) < \infty$, the minimal entropy martingale measure, if it exists, is equivalent to P (see also the analogue result, Th. 2.2, in Csiszar [2]). This fact is true even for unbounded processes. The following Lemma will be needed in the next two theorems.

Lemma 1 *Let $Q_0, Q_1 \in \mathcal{M}^0$ and set $Q_x = xQ_1 + (1-x)Q_0$, $x \in [0, 1]$. Then $E_{Q_1} \left[\ln \left(\frac{dQ_0}{dP} \right) \right]$ is well defined and:*

$$\frac{d}{dx} I(Q_x, P)|_{x=0} = E_{Q_1} \left[\ln \left(\frac{dQ_0}{dP} \right) \right] - I(Q_0, P). \quad (1)$$

Proof. Set $I(x) = \phi \left(\frac{dQ_x}{dP} \right)$, $x \in [0, 1]$. Since ϕ is convex, it is easy to check that $\left(\frac{I(x) - I(0)}{x} \right)$ is not increasing as $x \downarrow 0$. From the assumptions $I(Q_1, P) < \infty$ and $I(Q_0, P) < \infty$ we deduce that $E_P[I(1) - I(0)] < +\infty$ and $\left(\frac{dQ_1}{dP} \ln \left(\frac{dQ_0}{dP} \right) \right)^+ \in L^1(P)$. By monotone convergence the following limit can be exchanged with the integration operator and the right derivative is given by:

$$\begin{aligned} \frac{d}{dx} I(Q_x, P)|_{x=0} &= \lim_{x \downarrow 0} E_P \left[\frac{I(x) - I(0)}{x} \right] = E_P [I'(0)] \\ &= E_P \left[\phi' \left(\frac{dQ_0}{dP} \right) \left(\frac{dQ_1}{dP} - \frac{dQ_0}{dP} \right) \right]. \end{aligned}$$

Equation (1) then follows from $\phi'(x) = \ln x + 1$. ■

Theorem 2 *Suppose that Q_0 is the minimal entropy martingale measure. If there exists a probability $Q_1 \in \mathcal{M}_e$ such that $I(Q_1, P) < +\infty$, then Q_0 is equivalent to P .*

Proof. To prove the theorem we simply employ the condition $\phi'(0) = -\infty$. Set $Q_x = xQ_1 + (1-x)Q_0 \in \mathcal{M}$, $x \in [0, 1]$. Then we must necessarily have $(\frac{d}{dx}I(Q_x, P) |_{x=0}) \geq 0$ (otherwise there would exist some $x > 0$ for which $I(Q_x, P) < I(Q_0, P)$). However, from equation (1), we see that this condition can't be satisfied if Q_0 is not equivalent to P , since then $P(\frac{dQ_0}{dP} = 0) > 0$ and $Q_1(\frac{dQ_0}{dP} = 0) > 0$. ■

Therefore, for bounded processes the sole hypothesis $I(\mathcal{M}_e, P) < \infty$ is sufficient to assure the uniqueness and existence of an *equivalent* minimal entropy martingale measure. Note that a sufficient condition for $I(\mathcal{M}_e, P)$ to be finite is that there exists a measure $Q \in \mathcal{M}_e$ such that $\frac{dQ}{dP} \in L^{1+\epsilon}(P)$ for some (arbitrarily small) $\epsilon > 0$.

We now provide the characterization of the MEM. Define:

$$C_0 = \{f \in \mathcal{L} : E_Q[f] \leq 0 \ \forall Q \in \mathcal{M}^0\}.$$

Theorem 3 Assume that $I(\mathcal{M}_e, P) < +\infty$. A probability Q_0 is the minimal entropy martingale measure if and only if:

- i) $Q_0 \in \mathcal{M}$,
- ii)

$$\frac{dQ_0}{dP} = c \exp(-f_0), \quad P - a.s., \quad (2)$$

where $f_0 \in L^1(Q_0)$, $E_{Q_0}[f_0] = 0$ and $c > 0$,

- iii) $f_0 \in C_0$.

Proof. We notice that, when Q_0 is given, condition ii) uniquely determines both f_0 and c .

Assume that i) and ii) are satisfied. Then $Q_0 \sim P$ and $I(Q_0, P) < +\infty$. For any given probability measure Q_1 such that $I(Q_1, P) < +\infty$ we have:

$$\begin{aligned} I(Q_1, P) &\geq I(Q_1, P) - I(Q_1, Q_0) \\ &= E_{Q_1}[\ln(\frac{dQ_0}{dP})] \end{aligned} \quad (3)$$

$$\begin{aligned} &= I(Q_0, P) + E_P \left[\left(\frac{dQ_1}{dP} - \frac{dQ_0}{dP} \right) \ln(\frac{dQ_0}{dP}) \right] \\ &= I(Q_0, P) - E_{Q_1}[f_0]. \end{aligned} \quad (4)$$

From iii) and $Q_1 \in \mathcal{M}^0$ we then get $I(Q_1, P) \geq I(Q_0, P)$. This inequality holds for all $Q_1 \in \mathcal{M}$ and so Q_0 is the MEM.

Conversely, suppose that Q_0 is the MEM. Then $Q_0 \sim P$. Define $f_0 = I(Q_0, P) - \ln(\frac{dQ_0}{dP})$, P -a.s.. Then i) and ii) hold, since $I(Q_0, P) = E_{Q_0}[\ln(\frac{dQ_0}{dP})] < +\infty$. Let $Q_1 \in \mathcal{M}^0$ and set $Q_x = xQ_1 + (1-x)Q_0$. Since Q_0 is the MEM, we necessarily have $(\frac{d}{dx}I(Q_x, P)|_{x=0}) \geq 0$. From Lemma 1 and equation (3)-(4) we then deduce:

$$\begin{aligned} 0 &\leq E_{Q_1}[\ln(\frac{dQ_0}{dP})] - I(Q_0, P) \\ &= -E_{Q_1}[f_0] \leq I(Q_1, P) - I(Q_0, P) < +\infty, \end{aligned}$$

which implies iii). ■

It is possible to give a different description of the set C_0 . Let $C = K - L_+^\infty(P)$ and denote with $\overline{C}^Q$ the closure of a set C in the $L^1(Q)$ norm topology.

Theorem 4

$$C_0 = \bigcap_{Q \in \mathcal{M}^0} \overline{C}^Q$$

Proof. If $Q \in \mathcal{M}$ then for all $f \in C$ we have $E_Q[f] \leq 0$, so it is clear that $\left(\bigcap_{Q \in \mathcal{M}^0} \overline{C}^Q\right) \subseteq C_0$. To show the converse it is sufficient to prove that $C_0 \subseteq \overline{C}^Q$, $\forall Q \in \mathcal{M}^0$. Let $Q \in \mathcal{M}^0$, $f_0 \in C_0$ and suppose by contradiction that $f_0 \notin \overline{C}^Q$. By the Hahn Banach Theorem there exists a continuous linear functional on $L^1(Q)$ that strictly separates f_0 from the closed convex cone $\overline{C}^Q$, i.e. there exists $\eta \in L^\infty(Q)$ such that:

$$0 = \sup_{f \in C} E_Q[\eta f] < E_Q[\eta f_0]. \quad (5)$$

Since $-1_{\{\eta < 0\}} \in C$, we deduce that $\eta \geq 0$ Q -a.s. We then may normalize η and define a probability $Q_1 \ll Q$ by setting: $\frac{dQ_1}{dQ} = \eta$. Since $K \subseteq C$ is a linear space, $\sup_{f \in C} E_Q[\eta f] = 0$ implies $E_{Q_1}[f] = 0$ for all $f \in K$, so that $Q_1 \in \mathcal{M}$. Since $\eta \in L^\infty(Q)$ and $I(Q, P) < +\infty$, we have:

$$I(Q_1, P) = E_Q[\eta \ln \eta] + E_P \left[\eta \frac{dQ}{dP} \ln \left(\frac{dQ}{dP} \right) \right] < +\infty.$$

Hence $Q_1 \in \mathcal{M}^0$ and, from equation (5), $E_{Q_1}[f_0] > 0$, which contradicts $f_0 \in C_0$. ■

The following result is a reformulation of Theorem 3.1 of Csiszar [2]. We immediately deduce it from Theorems 3 and 4 and from the inclusions $K \subseteq C \subseteq C_0 \subseteq \overline{C}^{Q_0}$. Notice that, when $E_{Q_0}[f_0] = 0$, $f_0 \in \overline{C}^{Q_0}$ if and only if $f_0 \in \overline{K}^{Q_0}$.

Corollary 1 *Assume that $I(\mathcal{M}_e, P) < +\infty$ and that Q_0 satisfies conditions i) and ii) of theorem 3. Then:*

- a) *If $f_0 \in K$ or if $f_0 \in C$ then Q_0 is the MEM.*
- b) *If Q_0 is the MEM, then $f_0 \in \overline{C}^{Q_0} \cap \mathcal{L}$ or equivalently $f_0 \in \overline{K}^{Q_0} \cap \mathcal{L}$.*

In counterexample **B)** we show that it is possible to find Q_0 satisfying both i) and ii) with $f_0 \in \overline{C}^{Q_0}$ but $f_0 \notin C_0$. In this case Q_0 can't be the MEM.

2.1 Multiperiod model

In this section we assume $\mathcal{T} = \{0, 1, \dots, T\}$, $\mathcal{F} = \mathcal{F}_T$ and that χ is a *finite* family of real adapted stochastic processes: $\chi = X = (X^1, \dots, X^d)$, $X^j = (X_t^j)_{t \in \mathcal{T}}$, $j = 1, \dots, d$. If we denote with $x \cdot y$ the scalar product in $\mathbb{R}^d$, under these assumptions we have:

$$K = \left\{ \sum_{t=1}^T H_t \cdot (X_t - X_{t-1}) : H_t \in L^\infty(\Omega, \mathcal{F}_{t-1}, P; \mathbb{R}^d) \right\}.$$

Define:

$$\begin{aligned} \hat{K} &\triangleq \left\{ \sum_{t=1}^T H_t \cdot (X_t - X_{t-1}) : H_t \in L^0(\Omega, \mathcal{F}_{t-1}, P; \mathbb{R}^d) \right\}, \\ K_0 &\triangleq \hat{K} \cap \mathcal{L}. \end{aligned}$$

We already know that the density of the MEM belongs to the exponential family. We now show that the exponent f_0 has a specific form, i.e. it belongs to $K_0 \subseteq C_0$, not only to C_0 .

Theorem 5 *In this multiperiod model, Theorem 3 holds true if we replace C_0 with K_0 .*

Proof. The sufficient condition of Theorem 3 follows from $K_0 \subseteq C_0$. To show the necessary condition of Theorem 3 suppose that $Q_0 \sim P$ is the MEM. Then we know that $f_0 \in \overline{K}^{Q_0} \cap \mathcal{L}$ (Corollary 1, b)). We claim that

$$\overline{K}^Q \subseteq \hat{K} \cap L^1(Q), \quad Q \in \mathcal{M}^0,$$

which implies the thesis, since then $\overline{K}^{Q_0} \cap \mathcal{L} \subseteq \hat{K} \cap \mathcal{L} = K_0$. The claim basically depends on the property that $\hat{K}$ is closed with respect to convergence in probability. Let $f_T \in \overline{K}^Q$ and let $k_T^n = \sum_{t=1}^T H_t^n \cdot (X_t - X_{t-1}) \in K$ be such that $k_T^n \rightarrow f_T$ in $L^1(Q)$, as $n \rightarrow +\infty$. Set $f_t \triangleq E_Q[f_T | \mathcal{F}_t]$, $k_t^n \triangleq E_Q[k_T^n | \mathcal{F}_t] = \sum_{i=1}^t H_i^n \cdot (X_i - X_{i-1})$, $\forall t = 1, \dots, T$. Then $k_t^n \rightarrow f_t$ in $L^0(P)$, and so also $H_t^n \cdot (X_t - X_{t-1}) = k_t^n - k_{t-1}^n \rightarrow f_t - f_{t-1}$ in $L^0(P)$. From Proposition 2 of Stricker [24] we deduce that there exists an $\mathbb{R}^d$ -valued $\mathcal{F}_{t-1}$ -measurable random vector H_t such that $f_t - f_{t-1} = H_t \cdot (X_t - X_{t-1})$ and hence there exists a $\mathbb{R}^d$ -valued predictable process $H = (H_t, t = 1, \dots, T)$ such that $f_T = \sum_{t=1}^T H_t \cdot (X_t - X_{t-1}) \in \hat{K}$. ■

2.2 Counterexamples

A) We pick this example from Frittelli-Lakner [11].

We consider a one period model with countable state space Ω and show that there exists an unbounded process X such that:

- $\mathcal{M}_e \neq \emptyset$
- $I(\mathcal{M}_e, P) < \infty$
- There is no MEM for $\{X\}$.

Let $\mathcal{T} = \{0, 1\}$, $\Omega = \mathbb{Z}$ (the set of integers), the σ -field $\mathcal{F}$ contains all subsets of Ω , and the filtration be given by $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_1 = \mathcal{F}$.

We define the probability measure P in the following way:

$$P(i) = P(-i) = \frac{P(0)}{i^2} \quad \text{for } i = 1, 2, \dots$$

where the value $P(0)$ is selected so that P becomes a probability measure (and so $0 < P(0) < 1$).

Let $\chi = \{X\}$ where $X_0 = 0$ and $X_1(i) = i$ for $i \in \mathbb{Z}$. Clearly X is not a P -martingale because $E_P[X_1^+] = E_P[X_1^-] = +\infty$. To see that $\mathcal{M}_e$ is not empty, let Q be defined by $Q(i) = Q(-i) = \frac{Q(0)}{i^3}$ for $i = 1, 2, \dots$ where the value $Q(0)$ is selected so that Q becomes a probability measure. Then Q is an equivalent martingale measure.

We define a sequence of probability measures $(Q_n)_{n \geq 1}$ in the following way:

$$Q_n(i) = \begin{cases} P(i) & \text{if } i = \pm 1, \dots, \pm n \\ 0 & \text{if } |i| > n \\ P(0) + \sum_{|j| > n} P(j) & \text{if } i = 0 \end{cases}$$

It is elementary to check that $(Q_n)_{n \geq 1} \subseteq \mathcal{M}$. However, one can see that

$$\lim_{n \rightarrow +\infty} I(Q_n, P) = 0.$$

Indeed $I(Q_n, P) = Q_n(0) \ln \frac{Q_n(0)}{P(0)} \rightarrow 0$, as $n \rightarrow +\infty$ because $\lim_{n \rightarrow +\infty} Q_n(0) = P(0)$. Since $I(\mathcal{M}, P) = 0$, but $P \notin \mathcal{M}$, the MEM does not exist.

B) We adapt an example in [2] to our context. Let Ω be the unit interval $(0, 1)$, $\mathcal{F}$ and Q_0 be the σ -algebra and measure of Lebesgue. We consider a one period model with $\mathcal{T} = \{0, 1\}$ and let $\mathcal{F}_0$ be the trivial σ -algebra and $\mathcal{F}_1 = \mathcal{F}$. Let $f_0 = -\frac{3}{4}1_{(0, \frac{1}{4})} + \frac{1}{4}1_{[\frac{1}{4}, 1)}$. Define $P \sim Q_0$ by setting $\frac{dP}{dQ_0} = \frac{e^{f_0}}{c}$, where $c = E_{Q_0}[e^{f_0}]$. The family χ consists of countably many bounded processes $\{X^n\}_{n \geq 1}$ defined by:

$$X_0^n = 0, \quad X_1^n = \begin{cases} -\frac{3}{4} - \frac{1}{4}\sqrt{n} & \text{if } 0 < x < \frac{1}{4n} \\ -\frac{3}{4} & \text{if } \frac{1}{4n} \leq x < \frac{1}{4} \\ \frac{1}{4} + \frac{1}{4}\frac{1}{\sqrt{n}} & \text{if } \frac{1}{4} \leq x < \frac{1}{2} \\ \frac{1}{4} & \text{if } \frac{1}{2} \leq x < 1 \end{cases}.$$

Note that the MEM exists, since the processes are bounded, and that K is the linear space spanned by $\{X_1^n\}_{n \geq 1}$.

Define Q_1 by setting $\frac{dQ_1}{dQ_0} = \begin{cases} \frac{1}{5\sqrt{x}} & \text{if } 0 < x < \frac{1}{4} \\ 0 & \text{if } \frac{1}{4} \leq x < \frac{3}{5} \\ 2 & \text{if } \frac{3}{5} \leq x < 1 \end{cases}.$

One may easily check that:

- $Q_0 \in \mathcal{M}_e$, $I(Q_0, P) < +\infty$, $\frac{dQ_0}{dP} = ce^{-f_0}$ and $E_{Q_0}[f_0] = 0$

- $X_1^n \rightarrow f_0$ in $L^1(Q_0)$
- $Q_1 \in \mathcal{M}$, $I(Q_1, P) < +\infty$ and $E_{Q_1}[f_0] = \frac{1}{20}$.

Then $f_0 \notin C_0$ and, by theorem 3, Q_0 is not the MEM, even though $f_0 \in \overline{K}^{Q_0} \subseteq \overline{C}^{Q_0}$.

3 Financial interpretation of the minimal entropy martingale measure.

In this section we will provide the financial justification of the entropy criterion previously described. The processes in χ are the discounted price processes of the available tradable assets and K is the space generated by elementary gains, i.e. the space of attainable claims, with zero initial cost, from elementary and bounded trading strategies. The cone $C = K - L_+^\infty$ is formed with all the claims that can be super replicated by such attainable claims with zero initial costs. The representation Theorem 3 states that the density of the MEM is proportional to $\exp(-f_0)$, where f_0 is a “claim” in $C_0 = \bigcap_{Q \in \mathcal{M}^0} \overline{C}^Q$. This suggests that the density of the MEM is proportional to the marginal utility of terminal wealth when we deal with *exponential utility* functions. This is shown in the next section under the assumption that the optimal solution of a utility maximization problem exists. We now show that the relative entropy admits a dual representation which provides a very clear and explicit financial interpretation of it.

Proposition 1 *If u_a is the exponential utility defined by*

$$u_a(x) = -\frac{1}{a}e^{-ax}, \quad a > 0, \quad (6)$$

then:

$$\frac{1}{a}I(Q, P) = \sup_{w \in L^\infty: E_Q[w] \leq 0} u_a^{-1}(E_P[u_a(w)]).$$

Proof. Set $u = u_1$ and let $\mathbb{E}(w) = u^{-1}(E_P[u(w)])$ be the certainty equivalent.

$$\begin{aligned} \sup_{w \in L^\infty: E_Q[w] \leq 0} \mathbb{E}(w) &= \sup_{w \in L^\infty: E_Q[w] = 0} \mathbb{E}(w) \\ &= \sup_{z \in L^\infty} \{\mathbb{E}(z) - E_Q[z]\}, \end{aligned}$$

since if $z \in L^\infty$ and $w = z - E_Q[z]$, then $E_Q[w] = 0$ and $\mathbb{E}(w) = \mathbb{E}(z) - E_Q[z]$. Hence:

$$\sup_{z \in L^\infty} \{\mathbb{E}(z) - E_Q[z]\} = \sup_{z \in L^\infty} \{E_Q[z] - \ln E_P[\exp(z)]\} = I(Q, P),$$

where the last equality follows from the functional form of the relative entropy (see for example [16], Theorem 1.4.4). When $a \neq 1$ the proof is similar. ■

We conclude that the relative entropy $I(Q, P)$ is the *supremum of the certainty equivalent* $\mathbb{E}(w)$ of all claims $w \in L^\infty$ having prices less than or equal to zero, when Q is selected as pricing functional.

3.1 Multiperiod model

We consider the multiperiod model described in section 2.1 and use the notations there introduced. We assume that X is *bounded* and we denote with $\mathcal{H}$ the vector space of predictable *bounded* $\mathbb{R}^d$ -valued stochastic processes $H = (H_t, t = 1, \dots, T)$. The space K of elementary gains and the space K_0 were introduced in section 2.1. Assume that the discount factor is non stochastic and, without loss of generality, set it equal to 1. The final wealth $W(x, H)$ from initial endowment $x \in \mathbb{R}$ and trading strategy $H \in \mathcal{H}$ is then given by:

$$W(x, H) = x + \sum_{t=1}^T H_t \cdot (X_t - X_{t-1}).$$

If u_a is the exponential utility function defined in equation (6), then a utility maximizer agent is led to solve the problem $\sup_{w \in K} E_P[u_a(x + w)]$ or, since the problem is independent from x , to solve:

$$\sup_{w \in K} E_P[u_a(w)] \text{ i.e. } \sup_{H \in \mathcal{H}} E_P[u_a(W(0, H))]. \quad (7)$$

If $I(\mathcal{M}_e, P) < +\infty$ then the MEM Q_0 exists, it is equivalent to P and, from Theorem 5:

$$\begin{aligned} \frac{dQ_0}{dP} &= c \exp(-f_0), \\ f_0 &= \sum_{t=1}^T H_t^* \cdot (X_t - X_{t-1}) \in K_0, \end{aligned}$$

$$E_P \left[(X_t^j - X_{t-1}^j) \exp \left(- \sum_{i=1}^T H_i^* \cdot (X_i - X_{i-1}) \right) | \mathcal{F}_t \right] = 0 \quad \forall j, \forall t. \quad (8)$$

If $f_0 \in K$ i.e. if $H_t^* \in L^\infty(\Omega, \mathcal{F}_{t-1}, P; \mathbb{R}^d)$ then from equations (8) we deduce that $H^a \triangleq \frac{H^*}{a}$ solves the optimality conditions of problem (7), i.e. it solves

$$E_P[(X_t^j - X_{t-1}^j) u'_a(W(0, H^a)) | \mathcal{F}_t] = 0 \quad \forall j, \forall t,$$

and H^a is an optimal solution of (7). In this case *the minimal entropy martingale measure Q_0 coincides with the martingale measure proportional to the marginal utility of terminal wealth:*

$$\frac{dQ_0}{dP} \approx u'_a(W(0, H^a)).$$

However, in general $f_0 \in K_0$ only, and problem (7) may not admit an optimal solution in K .

This suggests selecting a different space of attainable claims and accordingly reformulating the utility maximization problem.

Conditions that guarantee the existence of an optimal solution of such problems have been recently determined in a general setting by Kramkov and Schachermayer [18] for utility functions defined on $(0, +\infty)$.

3.2 The finite state case

The aim of this section is to compare, in some specific concrete contexts, the pricing characteristics of the MEM, the minimal variance and the minimal martingale measure. We will confine the analysis to a one period *finite state* model and consider one risky asset only.

Let $n \geq 2$ be the cardinality of Ω , $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^n$ and $\mathcal{T} = \{0, 1\}$. Let r be 1 plus the interest rate and $S = (S_0, S_1)$ be the price process of the risky asset. We assume that $S_0 = 1$ and that the random variable S_1 takes n different positive values $(a_1, \dots, a_n) = a$ with probability $P(S_1 = a_i) = p_i > 0$, $\forall i = 1, \dots, n$, $p_1 + \dots + p_n = 1$. The minimal entropy martingale measure $Q_0 = (\vec{q}_1, \dots, \vec{q}_n)$ is the solution of the problem:

$$\begin{aligned} \min_{q \in \mathbb{R}^n, q \geq 0} & \left(\sum_{i=1}^n q_i \ln \left(\frac{q_i}{p_i} \right) \right) \\ q \cdot \mathbf{1} &= 1 \\ q \cdot a &= r \end{aligned}$$

One easily checks that Q_0 is given by:

$$\vec{q}_i = \frac{p_i e^{\gamma a_i}}{\sum_{j=1}^n p_j e^{\gamma a_j}} = \frac{p_i e^{\gamma a_i}}{E[e^{\gamma a}]}, \quad i = 1, \dots, n, \quad (9)$$

where $\gamma \in \mathbb{R}$ is the unique real solution (that always exists under the assumption of no arbitrage opportunities) of the equation:

$$\sum_{i=1}^n p_i (a_i - r) e^{\gamma a_i} = 0. \quad (10)$$

The *entropy price* of a contingent claim $C = (c_1, \dots, c_n)$ is then given by:

$$\pi(C) = E_{Q_0} \left[\frac{1}{r} C \right] = \sum_{i=1}^n \frac{1}{r} \frac{p_i c_i e^{\gamma a_i}}{E[e^{\gamma a}]}.$$

Define: $\mu \triangleq E_P[a] = p \cdot a$, $\sigma^2 \triangleq Var_P[a] = \sum_{i=1}^n p_i a_i^2 - \mu^2$. It is easy to check that, in this setting, the minimal variance martingale measure Q^{var} and the minimal martingale measure $Q^{\min}$ coincide and are given by the explicit formula:

$$\frac{dq^{var}}{dp} = \frac{dq^{\min}}{dp} = \mathbf{1} + \frac{\mu - r}{\sigma^2} (\mu \mathbf{1} - a). \quad (11)$$

In order to compare (9) and (11) we consider a particular case.

Let $n = 3$, $a_1 = x + r + k$, $a_2 = x + r$, $a_3 = x + r - k$, $x, k \in \mathbb{R}$, $k > 0$, and suppose also that $p_2^2 = 4p_1 p_3$. Under the no arbitrage assumption ($\mathcal{M}_e \neq \emptyset \Leftrightarrow |x| < k$) the MEM is always equivalent to P and is given by (from 9 and 10):

$$\vec{q}_1 = \frac{(k - x)^2}{4k^2}, \quad \vec{q}_2 = \frac{k^2 - x^2}{2k^2}, \quad \vec{q}_3 = \frac{(k + x)^2}{4k^2}. \quad (12)$$

On the other hand, from (11) we deduce that the minimal variance martingale measure (or equivalently, the minimal martingale measure) is given by:

$$q_1^{var} = \frac{1 - \sqrt{p_3}}{k} [\sqrt{p_3} k - x], \quad q_2^{var} = 1 - q_1^{var} - q_3^{var}, \quad q_3^{var} = \frac{\sqrt{p_3}}{k} [(1 - \sqrt{p_3})k + x]. \quad (13)$$

It is evident that Q^{var} is not equivalent to P for $\frac{x}{k} \leq \sqrt{p_3} - 1$ and $\frac{x}{k} \geq \sqrt{p_3}$. As a (negative) consequence it may assign zero value (or even a negative price) to a non negative contingent claim, as for example an option on S .

From (12) and (13) we derive that:

$$\vec{q}_1 \geq q_1^{var}, \quad \vec{q}_2 \leq q_2^{var}, \quad \vec{q}_3 \geq q_3^{var}$$

and the equality holds if and only if $\frac{x}{k} = 2\sqrt{p_3} - 1$.

From this comparison one sees that, in order to keep Q_0 equivalent to P , the entropy approach shifts the mass from the middle state to the extreme states. In other words, the MEM assigns higher state prices to the extreme states than to the middle ones.

References

- [1] Bellini, F., and M. Frittelli (1998): "On the existence of minimax martingale measures", Working paper #2, Università di Milano - Bicocca, Italy. Submitted.
- [2] Csiszar, I. (1975): "I-divergence geometry of probability distributions and minimization problems," *Ann. Prob.*, 3, No. 1, 146-158.
- [3] Dalang, R.C., A. Morton, and W. Willinger (1990): "Equivalent martingale measures and no arbitrage in stochastic securities market models," *Stoc. & Stoc. Rep.*, 29, 185-201.
- [4] Davis, M.H.A. (1994): "A general option pricing formula," Preprint, Imperial College, London.
- [5] Delbaen, F., and W. Schachermayer (1994): "A general version of the fundamental theorem of asset pricing," *Mathematische. Ann.*, 300, 463-520.
- [6] Delbaen, F., and W. Schachermayer (1996): "The variance optimal martingale measure for continuous processes", *Bernoulli*, 9, 81-105.
- [7] Duffie, D., and H.R. Richardson (1991): "Mean-Variance Hedging in Continuous time," *Annals of Applied Probability*, 1, 1-15.

- [8] Föllmer, H., and D. Sondermann (1986): "Hedging of Non-Redundant Contingent Claims," in W. Hildenbrand and A. Mas Colell (eds), *Contributions to Mathematical Economics*.
- [9] Föllmer, H., and M. Schweizer (1991): "Hedging of Contingent Claims under Incomplete Information," in M.H.A. Davis and R.J. Elliott (eds), *Applied Stochastic Analysis*, Stochastics Monographs, Vol. 5, Gordon and Breach, London/New York.
- [10] Frittelli, M., and P. Lakner (1994): "Almost sure characterization of martingales," *Stoc. & Stoc. Rep.*, 49, 181-190.
- [11] Frittelli, M., and P. Lakner (1996): "Counterexamples for the existence of the minimal entropy martingale measure", Working paper #115, Brescia University, Italy.
- [12] Frittelli, M. (1995): "Minimal Entropy Criterion for Pricing in One Period Incomplete Markets," Working Paper #99, Dip. Met. Quant., University of Brescia, Italy.
- [13] Frittelli, M. (1998): "Introduction to a Theory of Value Coherent with the No-Arbitrage Principle", Working Paper # 1, Dip. Met. Quant. Università di Milano-Bicocca, Italy. To appear on *Finance and Stochastics*.
- [14] Grandts, T. Rheinlaender (1998): "On the minimal entropy martingale measure" Working Paper.
- [15] Harrison, J.M., and S.R. Pliska (1981): "Martingales and stochastic integrals in the theory of continuous trading," *Stoc. Proc. Appl.*, 11, 215-260.
- [16] Ihara, S. (1993): "Information Theory", World Scientific.
- [17] Karatzas I., Shreve S., Lehoczky J., Xu G., (1991): "Martingale and duality methods for utility maximization in an incomplete market", *SIAM J. Control and Optim.*, 29, 702-730.
- [18] Kramkov, D. and W. Schachermayer (1998): "The Asymptotic Elasticity of Utility Functions and Optimal Investment in Incomplete Markets", Working Paper, Wien University.

- [19] Miyahara, Y. (1996): "Canonical Martingale Measures of Incomplete Asset Market", *Probability Theory and Mathematical Statistics: Proceedings of the Seventh Japan-Russia Symposium*, Tokyo, World Scientific, 343-352.
- [20] Miyahara, Y. (1998): "Minimal Entropy Martingale Measures of Jump Type Processes in Incomplete Asset Markets", Working paper, Nagoya City University.
- [21] Samperi D. (1999): "On Entropy and mean-variance hedging" Preprint, Samperi Research.
- [22] Schweizer, M. (1995): "On the minimal martingale measure and the Föllmer-Schweizer decomposition," *Stoc. Anal. Appl.*, 13, 573-599.
- [23] Schweizer, M. (1995): "Variance-Optimal Hedging in Discrete Time", *Mathematics of Operation Research* 20, 1-32.
- [24] Stricker, C. (1990): "Arbitrage et lois de martingale," *Ann. Inst. H. Poincaré*, 26, 451-460.
- [25] Stutzer, M. (1992): "State prices and Gibbs States," Working paper, Dept. of Finance, Univ. of Minnesota.
- [26] Stutzer, M. (1996): "A Simple nonparametric Approach to Derivative Security Valuation", *Journal of Finance*, 5, 1633-1652.