

Law Invariant Convex Risk Measures

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Abstract

As a generalization of a result by Kusuoka (2001), we provide the representation of law invariant convex risk measures. Very particular cases of law invariant coherent and convex risk measures are also studied.

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1 Introduction

Coherent risk measures have been introduced and deeply analyzed by Artzner et al. (1999) and Delbaen (2002). Convex risk measures have been introduced and studied by Föllmer and Schied (2002a), (2002b), (2002c), Frittelli and Rosazza (2002), (2004), and Heath (2000).

A smaller class of coherent risk measures was characterized by Kusuoka (2001). This author indeed imposed the further axiom of “law invariance”. Roughly speaking, a risk measure is law invariant if it assigns the same value to two risky positions having the same distribution with respect to the initial probability measure. Examples of law invariant coherent risk measures are provided by the class of insurance prices characterized by Wang et al. (1997).

The main result of this paper (see Theorem 7) is the representation of the wider class of law invariant convex risk measures. Moreover, thanks to the characterization of law invariant coherent and convex risk measures (the former by Kusuoka (2001), the latter proved in Section 2), we will study two degenerate

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classes of law invariant risk measures. Proposition 8 is a minor extension of a result of Castagnoli et al. (2004) and characterizes a degenerate class of coherent risk measures. Proposition 9 extends this result to the class of law invariant convex risk measures under a slightly stronger assumption.

A result similar to Theorem 7 has also been independently proved by Kunze (2003).

We will recall briefly the definitions and the representations of risk measures in the following setting.

Let T be a fixed future date and $\mathcal{X}$ denote the set of all financial positions whose riskiness has to be quantified.

In the furthering, we will assume that $\mathcal{X} = L^\infty(\Omega, \mathcal{F}, P)$, i.e. the space of all essentially bounded random variables defined on the general probability space $(\Omega, \mathcal{F}, P)$. We will endow $L^\infty \triangleq L^\infty(\Omega, \mathcal{F}, P)$ with the weak topology $\sigma(L^\infty, L^1)$, so that its topological dual space is precisely $L^1 \triangleq L^1(\Omega, \mathcal{F}, P)$. The set of density functions will be denoted with $\mathcal{Z} \triangleq \{x' \in L^1_+ : x'(1) = 1\}$ and any element $x' \in \mathcal{Z}$ will be identified with a probability measure $Q \ll P$ via the Radon-Nikodym derivative $\frac{dQ}{dP} \triangleq x'$, which will sometimes also be denoted by $\varphi_Q = \frac{dQ}{dP}$.

We remind that a *risk measure* is a functional $\rho : \mathcal{X} \rightarrow \mathbb{R}$ satisfying some “financially reasonable” axioms.

Axioms

(co) convexity:

$$\rho(\alpha X + (1 - \alpha) Y) \leq \alpha \rho(X) + (1 - \alpha) \rho(Y), \forall \alpha \in [0, 1], \forall X, Y \in \mathcal{X};$$

(sub) sublinearity:

$$(ph) \rho(aX) = a\rho(X), \forall a \geq 0, \forall X \in \mathcal{X} \text{ (positive homogeneity);}$$

$$(sa) \rho(X + Y) \leq \rho(X) + \rho(Y), \forall X, Y \in \mathcal{X} \text{ (subadditivity);}$$

(mon) monotonicity: $X \geq 0 \Rightarrow \rho(X) \leq \rho(0)$, $\forall X \in \mathcal{X}$;

(ti) translation invariance: $\rho(X + a) = \rho(X) - a$, $\forall a \in \mathbb{R}, \forall X \in \mathcal{X}$;

(lsc) lower semi-continuity:

$$\forall c \in \mathbb{R} \text{ the set } \{X \in \mathcal{X} : \rho(X) \leq c\} \text{ is } \sigma(L^\infty, L^1)\text{-closed};$$

(li) law invariance under P :

if $X, Y \in \mathcal{X}$ have the same distribution with respect to P , then

$$\rho(X) = \rho(Y).$$

The financial interpretation of axioms (co)-(lsc) may be easily find in the literature, for example in Artzner et al. (1999), in Föllmer and Schied (2002c) and in Frittelli and Rosazza (2002), (2004), among many others.

Law invariant risk measures allot the same riskiness to financial positions that are identically distributed with respect to the probability P given a priori.

We recall briefly the definitions and the representation of coherent and convex risk measures in the L^∞ -case.

Definition 1 (Artzner et al. (1999) and Delbaen (2002)) *A functional $\rho : \mathcal{X} \rightarrow \mathbb{R}$ is a coherent risk measure if it satisfies axioms (sub), (mon) and (ti).*

Theorem 2 (Theorem 3.2 Delbaen (2002)) *(On the representation of coherent risk measures).*

$\rho : L^\infty(\Omega, \mathcal{F}, P) \rightarrow \mathbb{R}$ is a coherent risk measure such that its acceptance set $\{X \in L^\infty : \rho(X) \leq 0\}$ is $\sigma(L^\infty, L^1)$ -closed if and only if there exists a closed convex set $\mathcal{P}$ of P -absolutely continuous probability measures such that

$$\rho(X) = \sup_{Q \in \mathcal{P}} E_Q[-X], \quad \forall X \in L^\infty. \quad (1)$$

In this paper we will adopt the following

Definition 3 (Frittelli and Rosazza (2002)) *A functional $\rho : \mathcal{X} \rightarrow \mathbb{R}$ is a convex risk measure if it satisfies axioms (co), (lsc) and $\rho(0) = 0$.*

Following the setting in Frittelli and Rosazza (2002), we have directly assumed in the definition of a convex risk measure the (lsc) axiom, while the two axioms (mon) and (ti) will be assumed whenever are useful. From Föllmer and Schied (2002c)- Theorem 4.26- it is known that for a risk measures ρ satisfying (co), (ti) and (mon), the following conditions are equivalent: a) ρ satisfies (lsc); b) the acceptance set is $\sigma(L^\infty, L^1)$ -closed; c) ρ has the Fatou property; d) ρ is continuous from above. So in the sequel any convex risk measure (in the sense of definition 3) satisfying (mon) and (ti) will also satisfy any of these "continuity" conditions.

Theorem 4 (Corollary 7 Frittelli and Rosazza (2002)) *(On the representation of convex risk measures).*

A) $\rho : L^\infty \rightarrow \mathbb{R}$ is a convex risk measure if and only if there exists a convex functional $G : L^1 \rightarrow \mathbb{R} \cup \{+\infty\}$ such that $\inf_{x' \in L^1} G(x') = 0$ and

$$\rho(X) = \sup_{x' \in \mathcal{H}} \{x'(-X) - G(x')\}, \quad \forall X \in L^\infty, \quad (2)$$

where $\mathcal{H} \triangleq \{x' \in L^1 : G(x') < +\infty\}$. G can always be represented as

$$G(x') = \sup_{X \in L^\infty} \{x'(-X) - \rho(X)\}.$$

B) $\rho : L^\infty \rightarrow \mathbb{R}$ is a convex risk measure satisfying (mon) and (ti) if and only if ρ admits the representation in (2) where $\mathcal{H} \subseteq \mathcal{Z}$, and therefore it can be written as

$$\rho(X) = \sup_{Q \in \mathcal{M}} \left\{ E_Q[-X] - G\left(\frac{dQ}{dP}\right) \right\}, \quad \forall X \in L^\infty, \quad (3)$$

where $\mathcal{M} \triangleq \left\{ Q \ll P : G\left(\frac{dQ}{dP}\right) < +\infty \right\}$.

While the previous representations hold true for general coherent and convex risk measures, Kusuoka (2001) characterized coherent risk measures satisfying the law invariance property, besides lower semi-continuity.

In the furthering, F_X will denote the cumulative distribution function of X with respect to P . For simplicity, we'll write " $X \sim Y$ " when $F_X \equiv F_Y$. Following the notations of Kusuoka (2001) we set:

$$Z_X(x) \triangleq \inf \{z \in \mathbb{R} : F_X(z) > x\}, \quad \forall x \in [0, 1], \quad \forall X \in L^\infty, \quad (4)$$

$$\hat{Z}_X(x) \triangleq \begin{cases} 0, & \text{if } x < 0 \\ Z_X(x), & \text{if } 0 \leq x < 1 \\ \lim_{x \rightarrow 1^-} Z_X(x), & \text{if } x \geq 1 \end{cases} \quad (5)$$

$$\rho_0(X) \triangleq \text{ess sup}(-X), \quad \forall X \in L^\infty, \quad (6)$$

$$\rho_\alpha(X) \triangleq \frac{1}{\alpha} \int_{1-\alpha}^1 Z_{-X}(x) dx, \quad \forall X \in L^\infty, \quad \forall \alpha \in (0, 1], \quad (7)$$

where ess sup stands for the essential supremum.

Proposition 5 (Proposition 22 Kusuoka (2001)) *For any $\alpha \in [0, 1]$, ρ_α is a law invariant coherent risk measure satisfying (lsc).*

Theorem 6 (Proposition 15 Kusuoka (2001)).

Let $(\Omega, \mathcal{F}, P)$ be a probability space with P atomless and $\rho : L^\infty \rightarrow \mathbb{R}$. The following are equivalent:

- i) ρ is a law invariant coherent risk measure satisfying (lsc);*
- ii) there exists a set $\mathfrak{M}_0$ of probability measures on $(0, 1]$ such that*

$$\rho(X) = \sup_{m \in \mathfrak{M}_0} \left\{ \int_{(0,1]} \rho_\alpha(X) m(d\alpha) \right\}, \quad \forall X \in L^\infty. \quad (8)$$

In Theorem 7 we will provide the characterization of the larger class of law invariant convex risk measures (with some further axioms). As we will see later, such a representation contains as a particular case the characterization of law invariant coherent risk measures established by Kusuoka (2001), but it is more general than the previous one since it is "corrected" with the penalty term G , that comes from the representation of convex risk measures.

Theorem 7 *Let $(\Omega, \mathcal{F}, P)$ be a probability space with P atomless and $\rho : L^\infty \rightarrow \mathbb{R}$.*

The following are equivalent:

- iii) ρ is a law invariant convex risk measure satisfying axioms (mon) and (ti);*
- iv) there exist a non empty convex set $\mathcal{M}$ of probability measures and a convex functional $G : L^1 \rightarrow \mathbb{R} \cup \{+\infty\}$ with $\inf_{Q \in \mathcal{M}} G(\varphi_Q) = 0$ such that*

$$\rho(X) = \sup_{Q \in \mathcal{M}} \left\{ \int_{(0,1]} \rho_\alpha(X) m_Q(d\alpha) - G(\varphi_Q) \right\}, \quad \forall X \in L^\infty, \quad (9)$$

where, for any $Q \in \mathcal{M}$, m_Q is a probability measure on $(0, 1]$ defined by $m_Q(d\alpha) \triangleq \alpha d\hat{Z}_{\varphi_Q}(1 - \alpha)$.

Note that the definition of law invariance depends on the probability measure P given a priori, hence it is not surprising that in the representation of law invariant convex risk measures the set $\mathcal{M}$ of generalized scenarios will be dependent on P . As shown in the next two propositions, under suitable hypotheses $\mathcal{M}$ collapses to the singleton $\{P\}$ and the associate risk measure simply reduces to the expectation under P . The next Proposition is a minor extension of a result of Castagnoli et al. (2004) for coherent risk measures and is obtained applying Theorem 6.

Proposition 8 *Let $(\Omega, \mathcal{F}, P)$ be a probability space with P atomless and let $\rho : L^\infty \rightarrow \mathbb{R}$ be a law invariant coherent risk measure satisfying (lsc).*

If there exists a random variable $Y \in L^\infty$ which is not P -a.s. constant and such that $\rho(Y) = -\rho(-Y)$, then

$$\rho(X) = E_P[-X], \quad \forall X \in L^\infty.$$

Thanks to Theorem 7, a similar result still holds true for the larger family of law invariant convex risk measures satisfying (mon) and (ti).

When ρ is coherent, the condition

$$\rho(Y) = -\rho(-Y) \tag{10}$$

is equivalent to

$$\rho(\gamma Y) = \gamma \rho(Y) \quad \text{for all } \gamma \in \mathbb{R}, \tag{11}$$

which means that ρ is linear on the subspace generated by Y . If ρ is a convex risk measure, the condition (10) is equivalent only to

$$\rho(\gamma Y) = \gamma \rho(Y) \quad \text{for all } |\gamma| \leq 1$$

(this can be checked applying Remark 13). Since, in the convex case, (10) does not any more imply (11), in the following Proposition we need to assume it explicitly.

Proposition 9 *Let P be an atomless probability measure and $\rho : L^\infty \rightarrow \mathbb{R}$ be a law invariant convex risk measure satisfying axioms (mon) and (ti). If there exists a random variable $Y \in L^\infty$ which is not P -a.s. constant and satisfies (11), then*

$$\rho(X) = E_P[-X], \quad \forall X \in L^\infty.$$

Noticing that the space $L^\infty(\Omega, \mathcal{F}, P)$ as well as the property of atomless probability measure depend on the probability P only through its null sets, from the above proposition we also deduce the following

Corollary 10 *Let P be an atomless probability measure and $\rho : L^\infty \rightarrow \mathbb{R}$ be a convex risk measure satisfying axioms (mon) and (ti). Suppose further that there exists $Q \sim P$ such that ρ is law invariant under Q .*

ρ is not linear on L^∞ if and only if it is not linear on each linear subspace generated by each random variable $Y \in L^\infty$ which is not P -a.s. constant.

Remark 11 *As seen in Theorem 7, any law invariant convex risk measure satisfying (mon) and (ti) can be represented by means of the family of “simpler” law invariant coherent risk measures $(\rho_\alpha)_{\alpha \in [0,1]}$. Moreover, the measures (ρ_α) admit, under appropriate assumptions, also the representation as Tail Conditional Expectation (TCE), Worst Conditional Expectation (WCE) and Average Value at Risk (AVaR). Precisely (see for example Artzner et al. (1999), and Föllmer and Schied (2002c)) define:*

$$TCE_\alpha(X) \triangleq E_P[-X | X \leq -VaR_\alpha(X)],$$

$$WCE_\alpha(X) \triangleq \sup_{A \in \mathcal{F}: P(A) > \alpha} E_P[-X | A],$$

$$AVaR_\alpha(X) \triangleq \frac{1}{\alpha} \int_0^\alpha VaR_\gamma(X) d\gamma,$$

While TCE_α is not coherent in general, WCE_α and $AVaR_\alpha$ are coherent risk measures.

We assume that the probability measure P is atomless.

Then (see Theorem 4.39 and Corollary 4.47 in Föllmer and Schied (2002c)) for any $\alpha \in (0, 1)$:

$$AVaR_\alpha(X) \equiv WCE_\alpha(X) \equiv \max_{\{Q \ll P: \frac{dQ}{dP} \leq \frac{1}{\alpha} \text{ } P\text{-a.s.}\}} E_Q[-X].$$

On the other hand, from Proposition 22 of Kusuoka (2001) and from Examples 4 and 13 of Delbaen (2000), it follows that for any $\alpha \in (0, 1)$

$$\rho_\alpha(X) = \max_{\{Q \ll P: \frac{dQ}{dP} \leq \frac{1}{\alpha} \text{ } P\text{-a.s.}\}} E_Q[-X].$$

Therefore, we see that

$$\rho_\alpha \equiv WCE_\alpha \equiv AVaR_\alpha.$$

In particular (see Delbaen (2000), among others), for any $X \in L^\infty$ with continuous cumulative distribution function F_X , all the risk measures above collapse to TCE_α .

2 Proofs

Theorem 7 will be deduced from the following result.

Theorem 12 Let $(\Omega, \mathcal{F}, P)$ be a probability space with P atomless and $\rho : L^\infty \rightarrow \mathbb{R}$.

The following are equivalent:

- v) ρ is a law invariant convex risk measure satisfying axioms (mon) and (ti);
- vi) there exist a non empty convex set $\mathcal{M}$ of probability measures and a convex functional $G : L^1 \rightarrow \mathbb{R} \cup \{+\infty\}$ with $\inf_{Q \in \mathcal{M}} G(\varphi_Q) = 0$ such that

$$\rho(X) = \sup_{Q \in \mathcal{M}} \left\{ \int_0^1 Z_{-X}(x) Z_{\varphi_Q}(x) dx - G(\varphi_Q) \right\}, \quad \forall X \in L^\infty. \quad (12)$$

We recall now some properties of convex risk measures useful for the proof of the previous result.

Remark 13 (Frittelli and Rosazza (2002)).

If $\rho : \mathcal{X} \rightarrow \mathbb{R}$ is a convex risk measure, then:

1)

$$\begin{aligned} \rho(\alpha X) &\leq \alpha \rho(X), \quad \forall \alpha \in [0, 1], \quad \forall X \in \mathcal{X}; \\ \rho(\alpha X) &\geq \alpha \rho(X), \quad \forall \alpha \in (-\infty, 0] \cup [1, +\infty), \quad \forall X \in \mathcal{X}. \end{aligned}$$

2) axiom (mon) is equivalent to strong monotonicity:

$$\text{if } X, Y \in \mathcal{X} : X \geq Y \Rightarrow \rho(X) \leq \rho(Y).$$

Proof of Theorem 12.

v) $\Rightarrow$ vi):

We recall (from Kusuoka (2001) and §3.12 in Williams (1991)) that we can assume without loss of generality $(\Omega, \mathcal{F}, P) \equiv ([0, 1], \mathcal{B}([0, 1]), \text{Lebesgue}|_{[0, 1]})$. Indeed, for any random variable X defined on a general probability space $(\Omega, \mathcal{F}, P)$ and with cumulative distribution function F_X it is possible to obtain a random variable $Z_X(\cdot)$ defined on $([0, 1], \mathcal{B}([0, 1]), \text{Lebesgue}|_{[0, 1]})$ and with the same cdf F_X . Moreover, since P is atomless and ρ is law invariant by hypothesis, it is sufficient to study the case where $(\Omega, \mathcal{F}, P)$ reduces to $([0, 1], \mathcal{B}([0, 1]), \text{Lebesgue}|_{[0, 1]})$.

Let $\mathfrak{P}$ be the set of all P -absolutely continuous probability measures defined on $(\Omega, \mathcal{F})$. Suppose that ρ is a law invariant convex risk measure satisfying axioms (mon) and (ti) and set $\pi(X) \triangleq \rho(-X)$. Then from Theorem 4 it follows that ρ can be represented as in (3), i.e.

$$\pi(X) = \rho(-X) = \sup_{Q \in \mathcal{M}} \{E_Q[X] - G(\varphi_Q)\}, \quad \forall X \in L^\infty, \quad (13)$$

where $\mathcal{M}$ is a subset of $\mathfrak{P}$ and

$$G(\varphi_Q) = \sup_{X \in L^\infty} \{E_P[\varphi_Q X] - \pi(X)\}, \quad \forall Q \in \mathcal{M}. \quad (14)$$

We will keep almost the same notations as in Kusuoka (2001) and we will modify his proof of the characterization of law invariant coherent risk measures (see Proposition 15 and Theorem 4 in [13]) in order to obtain a representation result for the more general case of convex risk measures.

Consider for each $n \geq 1$ the σ -algebra $\mathcal{F}_n \triangleq \sigma \{ \mathbf{1}_{[2^{-n}(k-1), 2^{-n}k)}; k = 1, \dots, 2^n \}$, and $\mathcal{X}_{\mathcal{F}} \triangleq \{X \in L^\infty : X \text{ is } \mathcal{F}_n\text{-measurable for some } n \geq 1\}$.

Fix now $X \in L^\infty$ and set $\tilde{X} \triangleq Z_X$. Then $\tilde{X} \sim X$ and $\tilde{X}$ is non decreasing and right-continuous in $\omega \in [0, 1)$. Furthermore, for any $n \in \mathbb{N}$ set $U_n \triangleq E_P \left[\tilde{X} \middle| \mathcal{F}_n \right]$ and

$$W_n(\omega) \triangleq \begin{cases} \text{ess. inf} \left(\tilde{X} \right), & \omega \in [0, 2^{-n}) \\ U_n(\omega - 2^{-n}), & \omega \in [2^{-n}, 1) \end{cases}.$$

It is well known that U_n converges P -a.s. and in L^1 to $\tilde{X}$ and that $\{U_n\}_{n \geq 1}$ is uniformly integrable. It is easy to check that $W_n \leq \tilde{X}$ for any $n \geq 1$ and that $W_n \uparrow \tilde{X}$, P -a.s.. Furthermore, W_n is $\mathcal{F}_n$ -measurable for any $n \geq 1$ (therefore it belongs to $\mathcal{X}_{\mathcal{F}}$) and $\{W_n\}_{n \geq 1}$ is uniformly integrable.

Since the proof of Lemma 10 of Kusuoka (2001) is not completely clear, we adapt it to the more general case of convex risk measures and prove it as follows.

For any $Q \in \mathcal{M}$ and $X \in L^\infty$

$$\begin{aligned} E_Q[X] - G(\varphi_Q) &\leq \sup \left\{ E_{\tilde{Q}}[X]; \tilde{Q} \in \mathfrak{P}, \varphi_{\tilde{Q}} \sim \varphi_Q \right\} - G(\varphi_Q) \\ &= \int_0^1 Z_X(x) Z_{\varphi_Q}(x) dx - G(\varphi_Q) \end{aligned} \quad (15)$$

$$= \int_0^1 Z_{\tilde{X}}(x) Z_{\varphi_Q}(x) dx - G(\varphi_Q), \quad (16)$$

where (15) is due to Proposition 14 of Kusuoka (2001) and (16) to law invariance of $Z_X(\cdot)$.

It is well known that $W_n \xrightarrow{P\text{-a.s.}} \tilde{X}$ implies that F_{W_n} converges weakly to $F_{\tilde{X}}$. Then it follows that $Z_{W_n}(\cdot)$ converges P -a.s. to $Z_{\tilde{X}}(\cdot)$, therefore $Z_{W_n}(\cdot) Z_{\varphi_Q}(\cdot)$ converges P -a.s. to $Z_{\tilde{X}}(\cdot) Z_{\varphi_Q}(\cdot)$. Moreover, with the same arguments of Kusuoka (2001), since $\{W_n\}_{n \geq 1}$ is uniformly integrable and $W_n \sim Z_{W_n}(\cdot)$, then also $\{Z_{W_n}(\cdot)\}_{n \geq 1}$ is uniformly integrable. $\left\{ Z_{W_n}(\cdot) Z_{\varphi_Q}(\cdot) \right\}_{n \geq 1}$ is still uniformly integrable. From the P -a.s. convergence of $Z_{W_n}(\cdot) Z_{\varphi_Q}(\cdot)$ to $Z_{\tilde{X}}(\cdot) Z_{\varphi_Q}(\cdot)$ as $n \rightarrow +\infty$, it follows that the above convergence is also in L^1 .

From the above arguments and from (16) it follows that

$$E_Q[X] - G(\varphi_Q) \leq \lim_{n \rightarrow +\infty} \int_0^1 Z_{W_n}(x) Z_{\varphi_Q}(x) dx - G(\varphi_Q) \quad (17)$$

$$= \lim_{n \rightarrow +\infty} \left\{ \sup_{\tilde{W}_n \in \mathcal{X}_{\mathcal{F}}: \tilde{W}_n \sim W_n} \left\{ E_Q[\tilde{W}_n] - G(\varphi_Q) \right\} \right\}, \quad (18)$$

where (17) holds true because of the L^1 -convergence of $Z_{W_n}(\cdot) Z_{\varphi_Q}(\cdot)$ to $Z_{\tilde{X}}(\cdot) Z_{\varphi_Q}(\cdot)$, (18) is due to Lemma 11 of Kusuoka (2001) (because $W_n \in \mathcal{X}_{\mathcal{F}}$) and to the fact that $G(\varphi_Q)$ depends neither on W_n nor on n , but only on φ_Q .

From (18) we obtain that for any $Q \in \mathcal{M}$ and $X \in L^\infty$

$$E_Q[X] - G(\varphi_Q) \leq \limsup_{n \rightarrow +\infty} \left\{ \sup_{\tilde{W}_n \in \mathcal{X}_{\mathcal{F}}: \tilde{W}_n \sim W_n} \pi(\tilde{W}_n) \right\} \quad (19)$$

$$= \limsup_{n \rightarrow +\infty} \pi(W_n) \quad (20)$$

$$\leq \pi(\tilde{X}) = \pi(X), \quad (21)$$

where inequality in (19) is due to (13), equalities in (20) and (21) to the law invariance of ρ (hence of π), and inequality in (21) follows from $W_n \leq \tilde{X}$ for any $n \geq 1$ and from Remark 13.

Till now we have shown that for any $Q \in \mathcal{M}$

$$E_Q[X] - G(\varphi_Q) \leq \int_0^1 Z_X(x) Z_{\varphi_Q}(x) dx - G(\varphi_Q) \leq \pi(X), \quad \forall X \in L^\infty.$$

Since $\pi(X) = \sup_{Q \in \mathcal{M}} \{E_Q[X] - G(\varphi_Q)\}$, it follows that

$$\pi(X) = \sup_{Q \in \mathcal{M}} \left\{ \int_0^1 Z_X(x) Z_{\varphi_Q}(x) dx - G(\varphi_Q) \right\}, \quad \forall X \in L^\infty,$$

hence the thesis follows immediately.

vi) $\Rightarrow$ v): Suppose that there exist a convex set $\mathcal{M}$ of probability measures and a convex functional $G: L^1 \rightarrow \mathbb{R} \cup \{+\infty\}$ with $\inf_{Q \in \mathcal{M}} G(\varphi_Q) = 0$ such that ρ has the form in (12).

Law invariance: if $X \sim Y$, then $Z_{-X}(x) \equiv Z_{-Y}(x)$ for any $x \in [0, 1]$. Hence $\rho(X) = \rho(Y)$.

Translation invariance: we recall from [9] that for any convex risk measure (ti) is equivalent to the property: $\rho(c) = -c$ for any $c \in \mathbb{R}$. Hence we need only to show this last property.

Note that for any $c \in \mathbb{R}$ the generalized inverse $Z_{-c}(\cdot)$ is identically equal to $(-c)$. From the representation in (12) we deduce therefore that for any $c \in \mathbb{R}$:

$$\begin{aligned} \rho(c) &= \sup_{Q \in \mathcal{M}} \left\{ -c \int_0^1 Z_{\varphi_Q}(x) dx - G(\varphi_Q) \right\} \\ &= \sup_{Q \in \mathcal{M}} \{ -cE[\varphi_Q] - G(\varphi_Q) \} = -c. \end{aligned} \quad (22)$$

Monotonicity: it is immediate to check that the generalized inverse $Z_{-X}(\cdot)$ is non positive for any $X \geq 0$ and that $Z_{\varphi_Q}(\cdot)$ is non negative for any $Q \in \mathcal{M}$.

Hence $\rho(X) \leq \sup_{Q \in \mathcal{M}} \{-G(\varphi_Q)\} = 0$. From the (ti) axiom just proved the thesis follows immediately.

It remains to show that ρ satisfies convexity and lower semi-continuity. We will prove these axioms with the help of Proposition 14 of Kusuoka (2001) which guarantees that for any $X \in L^\infty$ and for any $Q \in \mathcal{M}$:

$$\int_0^1 Z_{-X}(x) Z_{\varphi_Q}(x) dx = \sup \left\{ E_{\tilde{Q}}[-X]; \tilde{Q} \ll P : \varphi_{\tilde{Q}} \sim \varphi_Q \right\}. \quad (23)$$

Convexity: from identity (23) it follows that for any $X, Y \in L^\infty$ and for any $\alpha \in [0, 1]$

$$\begin{aligned} \rho(\alpha X + (1 - \alpha) Y) &= \sup_{Q \in \mathcal{M}} \left\{ \sup_{\tilde{Q} \ll P : \varphi_{\tilde{Q}} \sim \varphi_Q} \left\{ E_{\tilde{Q}}[-\alpha X - (1 - \alpha) Y] \right\} - G(\varphi_Q) \right\} \\ &\leq \alpha \rho(X) + (1 - \alpha) \rho(Y), \end{aligned}$$

where the inequality above comes from the sublinearity of supremum.

Lower semi-continuity: from equations (12) and (23), we deduce that ρ is the supremum of lower semi-continuous functionals, therefore ρ itself is lower semi-continuous. ■

Proof of Theorem 7. Thanks to Theorem 12, to get Theorem 7 we need only to prove that (12) can be written as (9) by taking $m_Q(d\alpha) \triangleq \alpha d\tilde{Z}_{\varphi_Q}(1 - \alpha)$. This can be done exactly as in the proof of Theorem 4 of Kusuoka (2001). ■

Remark 14 *Note that Theorem 7 contains, as a particular case, Theorem 6. Indeed, if ρ is a law invariant coherent risk measure satisfying (lsc), then $G(\varphi_Q) = 0$ for any $Q \in \mathcal{M}$. So, by setting $\mathfrak{M}_0 \triangleq \{m_Q : Q \in \mathcal{M}\}$, the representation in (9) can be rewritten as*

$$\rho(X) = \sup_{m \in \mathfrak{M}_0} \left\{ \int_{(0,1]} \rho_\alpha(X) m(d\alpha) \right\}, \quad \forall X \in L^\infty.$$

Proof of Proposition 8. As in the proof of Theorem 12, we may assume without loss of generality that $(\Omega, \mathcal{F}, P) \equiv ([0, 1], \mathcal{B}[0, 1], \text{Lebesgue}|_{[0,1]})$.

Suppose now that there exists a random variable $Y \in L^\infty$ which is not P -a.s. constant and such that $\rho(Y) = -\rho(-Y)$.

From Theorem 6, we know that there exists a set $\mathfrak{M}_0$ of probability measures on $(0, 1]$ such that

$$\rho(X) = \sup_{m \in \mathfrak{M}_0} \left\{ \int_{(0,1]} \rho_\alpha(X) m(d\alpha) \right\}, \quad \forall X \in L^\infty.$$

Hence

$$-\rho(-Y) = \inf_{m \in \mathfrak{M}_0} \left\{ \int_{(0,1]} -\rho_\alpha(-Y) m(d\alpha) \right\} \leq \inf_{m \in \mathfrak{M}_0} \left\{ \int_{(0,1]} \rho_\alpha(Y) m(d\alpha) \right\},$$

where the last inequality follows from sublinearity of any ρ_α . Furthermore, since by hypothesis $\rho(Y) = -\rho(-Y)$, it follows that

$$\sup_{m \in \mathfrak{M}_0} \left\{ \int_{(0,1]} \rho_\alpha(Y) m(d\alpha) \right\} = \rho(Y) \leq \inf_{m \in \mathfrak{M}_0} \left\{ \int_{(0,1]} \rho_\alpha(Y) m(d\alpha) \right\},$$

and therefore

$$\rho(Y) = \int_{(0,1]} \rho_\alpha(Y) m(d\alpha), \quad \forall m \in \mathfrak{M}_0.$$

Analogously, it can be checked that

$$\rho(-Y) = \int_{(0,1]} \rho_\alpha(-Y) m(d\alpha), \quad \forall m \in \mathfrak{M}_0.$$

Therefore, $0 = \rho(-Y) + \rho(Y) = \int_{(0,1]} [\rho_\alpha(Y) + \rho_\alpha(-Y)] m(d\alpha) = 0$ for any $m \in \mathfrak{M}_0$. Since $\rho_\alpha(Y) + \rho_\alpha(-Y) \geq 0$ for any $\alpha \in (0, 1]$, we deduce

$$\rho_\alpha(Y) + \rho_\alpha(-Y) = 0 \quad m - a.s., \quad \forall m \in \mathfrak{M}_0. \quad (24)$$

We claim that

$$\{\alpha \in (0, 1] : \rho_\alpha(Y) + \rho_\alpha(-Y) \neq 0\} = (0, 1). \quad (25)$$

From (24) and (25) it then follows that $m((0, 1)) = 0$ for any $m \in \mathfrak{M}_0$, hence $m(\{1\}) = 1$ for any $m \in \mathfrak{M}_0$. In other words, $\mathfrak{M}_0 = \{\delta_{\{1\}}\}$, where $\delta_{\{1\}}$ is the Dirac probability measure on $(0, 1]$, and we obtain, for any $X \in L^\infty$,

$$\begin{aligned} \rho(X) &= \sup_{m \in \mathfrak{M}_0} \left\{ \int_{(0,1]} \rho_\alpha(X) m(d\alpha) \right\} = \int_{(0,1]} \rho_\alpha(X) \delta_{\{1\}}(d\alpha) \\ &= \rho_1(X) = \int_0^1 Z_{-X}(x) dx = E_P[-X]. \end{aligned}$$

To prove (25) recall that for $\alpha \in (0, 1]$

$$\rho_\alpha(Y) + \rho_\alpha(-Y) = \frac{1}{\alpha} \int_{1-\alpha}^1 [Z_{-Y}(x) + Z_Y(x)] dx,$$

and define $\xi(\alpha) \triangleq \int_{1-\alpha}^1 [\hat{Z}_{-Y}(x) + \hat{Z}_Y(x)] dx$, for $\alpha \in [0, 1]$, where $\hat{Z}$ is the extension of Z , as in (5). It then suffices to show that $\xi(\alpha) > 0$ for any $\alpha \in (0, 1)$. Notice that $\xi(0) = 0$; $\xi(1) = E_P[-Y] + E_P[Y] = 0$; $\hat{Z}_{-Y}(\cdot) + \hat{Z}_Y(\cdot)$ is non decreasing and right-continuous; $\hat{Z}_{-Y}(0) + \hat{Z}_Y(0) = \text{ess. inf}(-Y) +$

$\text{ess. inf}(Y) < 0$, while $\hat{Z}_{-Y}(1) + \hat{Z}_Y(1) = \text{ess. sup}(-Y) + \text{ess. sup}(Y) > 0$. These facts imply that ξ is strictly increasing in a right neighborhood of 0 and strictly decreasing in a left neighborhood of 1 and that $\xi(\alpha) > 0$ for all $\alpha \in (0, 1)$. ■

Proof of Proposition 9. From Theorem 7, any law invariant convex risk measure $\rho : L^\infty \rightarrow \mathbb{R}$ satisfying axioms (mon) and (ti) can be represented as in equation (9). Remind that, for any $\alpha \in [0, 1]$, ρ_α are given in (6) and in (7). From the representation of ρ in (9) and from Remark 13, we get

$$-\tilde{\rho}(-X) \leq \rho(X) \leq \tilde{\rho}(X), \quad (26)$$

where

$$\tilde{\rho}(X) \triangleq \sup_{Q \in \mathcal{M}} \left\{ \int_{(0,1]} \rho_\alpha(X) m_Q(d\alpha) \right\}.$$

By taking $\mathfrak{M}_0 \triangleq \{m_Q \text{ probability measure on } (0, 1] : Q \in \mathcal{M}\}$, from Theorem 6 it follows that $\tilde{\rho}$ is a law invariant coherent risk measure satisfying lower semi-continuity.

From (11), we obtain that for any $\gamma > 0$

$$\rho(\pm Y) = \gamma \rho\left(\pm \frac{Y}{\gamma}\right) = \sup_{Q \in \mathcal{M}} \left\{ \int_{(0,1]} \rho_\alpha(\pm Y) m_Q(d\alpha) - \gamma G(\varphi_Q) \right\} \quad (27)$$

$$\rho(\pm Y) = -\gamma \rho\left(\mp \frac{Y}{\gamma}\right) = \inf_{Q \in \mathcal{M}} \left\{ \int_{(0,1]} -\rho_\alpha(\mp Y) m_Q(d\alpha) + \gamma G(\varphi_Q) \right\}. \quad (28)$$

From (27), we have that

$$\gamma G(\varphi_Q) \geq \int_{(0,1]} \rho_\alpha(\pm Y) m_Q(d\alpha) - \rho(\pm Y), \quad \forall Q \in \mathcal{M}, \quad \forall \gamma > 0,$$

then, since $\gamma > 0$ can be arbitrarily small,

$$\begin{aligned} \int_{(0,1]} \rho_\alpha(\pm Y) m_Q(d\alpha) - \rho(\pm Y) &\leq 0, \quad \forall Q \in \mathcal{M} \\ \sup_{Q \in \mathcal{M}} \left\{ \int_{(0,1]} \rho_\alpha(\pm Y) m_Q(d\alpha) \right\} &\leq \rho(\pm Y). \end{aligned} \quad (29)$$

As done above, from (28) we obtain that

$$\rho(\pm Y) \leq \inf_{Q \in \mathcal{M}} \left\{ \int_{(0,1]} \rho_\alpha(\pm Y) m_Q(d\alpha) \right\}. \quad (30)$$

From (29) and (30), it follows that

$$\inf_{Q \in \mathcal{M}} \left\{ \int_{(0,1]} \rho_\alpha(\pm Y) m_Q(d\alpha) \right\} = \rho(\pm Y) = \sup_{Q \in \mathcal{M}} \left\{ \int_{(0,1]} \rho_\alpha(\pm Y) m_Q(d\alpha) \right\}. \quad (31)$$

From (31) and the assumption $\rho(Y) = -\rho(-Y)$, it is immediate to check that

$$-\tilde{\rho}(-Y) = \rho(Y) = \tilde{\rho}(Y). \quad (32)$$

Since $\tilde{\rho}$ is a law invariant coherent risk measure satisfying (lsc) and (32), then from Proposition 8, it follows that $\tilde{\rho}(X) = E_P[-X]$ for any $X \in L^\infty$. From (26), the thesis follows immediately. ■

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