

Some Remarks on Arbitrage and Preferences in Securities Market Models

Marco Frittelli

Università degli Studi di Firenze

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Abstract

We introduce the notion of a *Market Free Lunch* that depends on the preferences of all agents participating in the market. In semimartingale models of securities markets, we characterize No Arbitrage (NA) and No Free Lunch with Vanishing Risk (NFLVR) in terms of the Market Free Lunch and show that the difference between NA and NFLVR consists in the selection of the class of monotone, resp. monotone and continuous, utility functions that determines the absence of the Market Free Lunch.

We also provide a *direct* proof of the equivalence between the absence of a Market Free Lunch, with respect to monotone concave preferences, and the existence of an equivalent (local/sigma) martingale measure.

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JEL classification: G11, G12, G13 **AMS classification:** 60G42, 60G44

1 Introduction

In a seminal paper Kreps (1981) showed that, in ordered locally convex topological vector space, the conditions of Viability and No Free Lunch are equivalent to the existence of a strictly positive continuous linear extension functional. In a stochastic model of a securities market this extension operator is representable by an equivalent martingale measure (Harrison and Pliska (1981)). The Fundamental Theorem of Asset Pricing (Delbaen and Schachermayer (1994), (1998)) shows that the equivalence between No Free Lunch with Vanishing Risk (*NFLVR*) and the existence of such equivalent (local/sigma) martingale measure holds in great generality. In finite period markets (Dalang Morton and Willinger (1990)) the condition of *NFLVR* can be substituted by the simpler notion of No Arbitrage (*NA*). In Bellini and Frittelli (2002) is pointed out the

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equivalence between *NFLVR* and an appropriate notion of Viability, which depends, as in Kreps (1981), on a particular class of preferences. In Section 1.2.3 below we introduce the new concept of a Market Free Lunch (*MFL*), that also depends on the preferences of the agents acting in the market, and in Section 2 we compare the absence of a Market Free Lunch with *NA* and *NFLVR*. The results of this note, see Proposition 5 for the precise statements, can be summarized as follows: *There are No Market Free Lunch, with respect to monotone preferences, if and only if NA holds. There are No Market Free Lunch, with respect to monotone continuous preferences, if and only if NFLVR holds.* Furthermore, in Theorem 7 we formulate a new version of the Fundamental Theorem of Asset Pricing: *There exists an equivalent (local/sigma) martingale measure if and only if there are No Market Free Lunch with respect to monotone concave preferences.* This result is not based on any other version of the Fundamental Theorem of Asset Pricing.

1.1 The Market Model

We consider a general semimartingale model of a securities market, as presented by Delbaen and Schachermayer (1998) and we defer to this reference for unexplained notations and concepts.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, P)$ be a filtered probability space, where we assume that the filtration satisfies the usual assumptions of right continuity and completeness, let $\mathbb{P}$ be the class of probability measures equivalent to P and set $L^\infty = L^\infty(\Omega, \mathcal{F}, P)$ and $L^0 = L^0(\Omega, \mathcal{F}, P)$.

The $\mathbb{R}^d$ -valued càd-làg semimartingale $X = (X_t)_{t \in [0, T]}$ represents the price process of d tradeable assets.

An $\mathbb{R}^d$ -valued predictable X -integrable process $H = (H_t)_{t \in [0, T]}$ is called admissible if there exists a constant a such that, for all $t \in [0, T]$, $\int_0^t H_s \cdot dX_s \geq a$ P -a.s. Set

$$K \triangleq \left\{ \int_0^T H_s \cdot dX_s \mid H \text{ admissible} \right\}, \quad C \triangleq (K - L_+^0) \cap L^\infty.$$

K is the cone of all claims that are replicable, at zero initial cost, via admissible trading strategies. The set C is the cone of all claims in L^∞ that can be dominated by a replicable claim in K , hence is the cone of bounded super-replicable claims. It will be more convenient to work with the set C instead than with K , but the definitions of Arbitrage and of a *MFL* in equations (1) and (3) could be equivalently formulated with the set K (see the comment just before Definition 4).

The set M of separating measures is defined by:

$$M \triangleq \{Q \ll P : K \subseteq L^1(\Omega, \mathcal{F}, Q) \text{ and } E_Q[f] \leq 0 \forall f \in K\}.$$

It is well known that if X is bounded (resp. locally bounded) then

$$M = \{Q \ll P : X \text{ is a } (Q, (\mathcal{F}_t)_{t \in [0, T]}) \text{ martingale (resp. local martingale)}\},$$

i.e. M is the set of P -absolutely continuous *martingale* (resp. *local martingale*) measures. In general, for possibly unbounded X , M is the set of P -absolutely continuous probabilities such that the admissible stochastic integrals are supermartingales and if $M \cap \mathbb{P} \neq \emptyset$, then the set M_σ of absolutely continuous σ -martingale probabilities is not empty and M_σ is dense in M for the total variation topology (see Delbaen and Schachermayer (1998)).

1.2 Hedging and Arbitrage

Set $L_{++}^\infty = \{w \in L^\infty : P(w \geq 0) = 1 \text{ and } P(w > 0) > 0\}$. Let us interpret an element $w \in L_{++}^\infty$ as the time T payoff of a claim and assume, for simplicity, that the risk free interest rate is zero. If we short sell this claim today we receive a (strictly) positive amount of money. At time T we'll then have the obligation to pay back $-w$. But today we may also choose an admissible trading strategy, having zero (or negative!) initial cost, that might "hedge" the claim w . At time T our wealth will be $(f - w)$, where $f \in C$. The various notions of "Arbitrage" differ from each other for the precise meaning of the term "hedge", i.e. of the exact meaning of " f hedges w ", as shown in equations (1), (2), (3).

1.2.1 No Arbitrage

The notion of No Arbitrage is defined as follows (see for example Delbaen and Schachermayer (1998), Section 3):

$$K \cap L_+^0 = \{0\},$$

or equivalently

$$\mathbf{NA} : \quad C \cap L_+^\infty = \{0\}.$$

Therefore, the Arbitrage condition can be expressed as:

$$\mathbf{A} : \quad \exists w \in L_{++}^\infty \exists f \in C : (f - w) \geq 0 \text{ } P - a.s. \quad (1)$$

i.e. an Arbitrage opportunity requires a $P - a.s.$ hedge of the claim w .

Remark 1 (Finite Period Markets) Consider a Finite Period Market, where the filtration is formed by a finite number of σ -algebra $(\mathcal{F}_t)$, $t = 0, 1, \dots, T$. Then the classical notion of NA, as defined by Dalang Morton and Willinger (1990), is

$$\mathbf{NA} - \text{Finite Period Markets} : \quad K_0 \cap L_+^0 = \{0\},$$

where

$$K_0 = \left\{ \sum_{s=1}^T H_s \cdot (X_s - X_{s-1}) \mid H_s \in L^0(\Omega, \mathcal{F}_{s-1}, P) \right\}.$$

All elements of K are bounded from below, but no boundedness conditions are required on the elements of K_0 . In Finite Period Markets however, $\mathbf{NA}$ is equivalent to $\mathbf{NA} - \text{Finite Period Market}$. Indeed, each condition is equivalent to the existence of an equivalent martingale measure, as shown in the Fundamental Theorem of Asset Pricing (see, for example, Dalang Morton and Willinger (1990) and Delbaen and Schachermayer (1998), Lemma 3.1).

1.2.2 No Free Lunch with Vanishing Risk

The notion of No Free Lunch with Vanishing Risk was introduced by Delbaen and Schachermayer (1994) (1998) and is defined as follows:

$$\mathbf{NFLVR} : \quad \overline{C} \cap L_+^\infty = \{0\},$$

where $\overline{C}$ is the L^∞ -norm closure of C .

A Free Lunch with Vanishing Risk is equivalent to the existence of $w \in L_{++}^\infty$ and a sequence $f_n \in C$ such that $\|f_n - w\|_{L^\infty} \rightarrow 0$, i.e. a uniform approximation of the claim w . It can be expressed as follows:

$$\mathbf{FLVR} : \quad \exists w \in L_{++}^\infty : \sup_{f \in C} \left\{ \operatorname{ess\,inf}_\Omega (f - w) \right\} \geq 0. \quad (2)$$

The following is the celebrated Fundamental Theorem of Asset Pricing in semimartingale securities market models.

Theorem 2 (Delbaen and Schachermayer (1994), (1998))

$$\mathbf{NFLVR} \Leftrightarrow M \cap \mathbb{P} \neq \emptyset.$$

1.2.3 No Market Free Lunch

Let $\mathbb{U}$ be a certain class of utility functions $u : \mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty\}$ that will be determined later and let $D(u) = \{x \in \mathbb{R} : u(x) > -\infty\}$ be the effective domain of $u \in \mathbb{U}$. We assume that the preferences “ $\succeq$ ” of the agents in the market under consideration may be represented by expected utility:

$$f_1 \succeq f_2 \iff E_R[u(f_1)] \geq E_R[u(f_2)],$$

where $R \in \mathbb{P}$ and $u \in \mathbb{U}$.

We introduce the notion of a Free Lunch that depends on the preferences of the investors present in the market. We define a Market Free Lunch, with respect to $(\mathbb{U})$, as follows:

$$\mathbf{MFL}(\mathbb{U}) : \quad \exists w \in L_{++}^\infty : \forall P \in \mathbb{P} \, \forall u \in \mathbb{U} \, \sup_{f \in C} E_P[u(f - w)] \geq u(0). \quad (3)$$

This means that all agents in the market (represented by their beliefs $P \in \mathbb{P}$ and preferences $u \in \mathbb{U}$) regard the same claim w as a free lunch, since each investor may “hedge” w in a way which is coherent with his preferences and beliefs. We convene that the supremum in equations (3) and (4) is computed on the set $\{f \in C : E_P[u^-(f - w)] < +\infty\}$ and if this set is empty then the supremum is equal to $-\infty$.

Definition 3 *There are No Market Free Lunch w.r.to $(\mathbb{U})$ if:*

$$\mathbf{NMFL}(\mathbb{U}) : \quad \forall w \in L_{++}^\infty \, \exists P \in \mathbb{P} \, \exists u \in \mathbb{U} : \sup_{f \in C} E_P[u(f - w)] < u(0). \quad (4)$$

This definition clearly depends on the class $\mathbb{U}$ of utility functions under consideration. There are several possible choices of the class $\mathbb{U}$. It is also possible to further elaborate the definition of a $MFL(\mathbb{U})$: Requiring, for example, that the probability measures P in (3) belong only to a subset $\mathcal{P} \subset \mathbb{P}$.

Notice that the definition of a $MFL(\mathbb{U})$ is economically meaningful only if we restrict our attention to utility functions u that are non decreasing on $\mathbb{R}$. It can be checked that if $u : \mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty\}$ is non decreasing on $\mathbb{R}$ then $\sup_{f \in C} E_P[u(f - w)] = \sup_{f \in K} E_P[u(f - w)]$, which justifies the use of C in (3) and (4). The following families of utility functions will provide the characterization of NA , $NFLVR$ and $M \cap \mathbb{P} \neq \emptyset$.

Definition 4 .

- $\mathbb{U}_0 = \{u : \mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty\} \text{ such that } u \text{ is non decreasing on } \mathbb{R}\},$
- $\mathbb{U}_1 = \{u \in \mathbb{U}_0 : u \text{ is left continuous at } 0 \in \text{interior}(D(u))\},$
- $\mathbb{U}_2 = \{u \in \mathbb{U}_0 : u \text{ is finite valued and concave on } \mathbb{R}\}.$

Notice that the larger is $\mathbb{U}$, the weaker the condition of $NMFL(\mathbb{U})$ will be. Since $\mathbb{U}_2 \subset \mathbb{U}_1 \subset \mathbb{U}_0$ it is clear that $\mathbf{NMFL}(\mathbb{U}_2) \Rightarrow \mathbf{NMFL}(\mathbb{U}_1) \Rightarrow \mathbf{NMFL}(\mathbb{U}_0)$.

2 Characterization of NA , $NFLVR$ and $M \cap \mathbb{P} \neq \emptyset$

The results in this Section hold true in the general setting of the semimartingale securities market model described in Section 1.1.

Proposition 5 .

- (a) $\mathbf{NMFL}(\mathbb{U}_0) \Leftrightarrow \mathbf{NA}.$
- (b) $\mathbf{NMFL}(\mathbb{U}_1) \Leftrightarrow \mathbf{NFLVR}.$

Proof. (a) If $w \in C \cap L_{++}^\infty$ is an Arbitrage opportunity then $\sup_{f \in C} E_P[u(f - w)] \geq E_P[u(w - w)] = u(0)$.

Conversely, suppose that there exists a $\mathbf{MFL}(\mathbb{U}_0)$. Take $P \in \mathbb{P}$ and take $u \in \mathbb{U}_0$ defined by: $u(x) = -\infty$ if $x < 0$, $u(x) = 0$ if $x \geq 0$. Since $\exists w \in L_{++}^\infty : \sup_{f \in C} E_P[u(f - w)] \geq u(0)$, then there exists $f \in C$ such that $P(f - w \geq 0) = 1$.

(b) Suppose that (2) is satisfied and let $n \geq 1$. From the left continuity of u at 0 there exists $\delta_n > 0$ such that if $-\delta_n < x \leq 0$ then $u(x) > u(0) - \frac{1}{n}$. From (2) there exists $f_n \in C$ such that $\text{ess inf}(f_n - w) > -\delta_n$. Set $\tilde{f}_n \triangleq f_n - (f_n - w)^+ \in C$. Then $-\delta_n < (\tilde{f}_n - w) \leq 0$ $\mathbb{P} - a.s.$ Hence $u(\tilde{f}_n - w) > u(0) - \frac{1}{n}$, $\mathbb{P} - a.s.$ and $\sup_{f \in C} E_P[u(f - w)] \geq u(0)$.

Conversely, suppose that there exists a $\mathbf{MFL}(\mathbb{U}_1)$. Take $P \in \mathbb{P}$ and for all $n \geq 1$ take $u_n \in \mathbb{U}_1$ defined by: $u_n(x) = -\infty$ if $x \leq -\frac{1}{n}$, $u_n(x) = 0$ if $x > -\frac{1}{n}$. Since $\exists w \in L_{++}^\infty : \forall n \geq 1 \sup_{f \in C} E_P[u_n(f - w)] \geq u_n(0) = 0$ then $\forall n \geq 1$ there exists $f_n \in C$ such that $P(f_n - w \leq -\frac{1}{n}) = 0$, i.e. $\text{ess inf}(f_n - w) \geq -\frac{1}{n}$. ■

Proposition 5 points out that the difference, from an economic perspective, between NA and $NFLVR$ rests on the preferences of the agents in the market, i.e. on the selection of $\mathbb{U}_0$ or $\mathbb{U}_1$.

Remark 6 As already mentioned, the assumption that $u \in \mathbb{U}$ is non decreasing is necessary for the economic interpretation of the definition of a $MFL(\mathbb{U})$. However, the proof of Proposition 5 is valid also without this assumption. Hence, the following two statements are also formally true:
(a) $\mathbf{NMFL}(\mathbb{U}) \Leftrightarrow \mathbf{NA}$, if $\mathbb{U} = \{u : \mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty\}\}$,
(b) $\mathbf{NMFL}(\mathbb{U}) \Leftrightarrow \mathbf{NFLVR}$, if $\mathbb{U} = \{u : \mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty\} \mid u \text{ is left l.s.c. at } 0 \text{ (i.e.: } \lim_{x \rightarrow 0^-} u(x) \geq u(0))\}$.

The following Theorem shows in a direct way the equivalence between the existence of an equivalent separating measure and the absence of a Market Free Lunch with respect to $(\mathbb{U}_2)$. Notice that the proof of Theorem 7 does not rely on any previous version of the Fundamental Theorem of Asset Pricing (as for example Theorem 2), but is instead based on the relatively simple result stated below as Theorem 8 and proved in Bellini and Frittelli (2002), Theorem 2.1 and Corollary 2.2.

Theorem 7 (Fundamental Theorem of Asset Pricing)
 $\mathbf{NMFL}(\mathbb{U}_2) \Leftrightarrow M \cap \mathbb{P} \neq \emptyset$.

Note that the implications

$$\mathbf{NMFL}(\mathbb{U}_2) \Rightarrow \mathbf{NMFL}(\mathbb{U}_1) \Rightarrow \mathbf{NFLVR} \Rightarrow M \cap \mathbb{P} \neq \emptyset \Rightarrow \mathbf{NMFL}(\mathbb{U}_2)$$

follow immediately from: $\mathbb{U}_2 \subset \mathbb{U}_1$, Proposition 5 (b), Theorem 2 and the easy part of Theorem 7 (see (5)). However, in the above chain of implications, Theorem 2 plays a determinant non trivial role.

Theorem 8 Let $u : \mathbb{R} \rightarrow \mathbb{R}$ be non decreasing and concave on $\mathbb{R}$ (i.e. $u \in \mathbb{U}_2$), $P \in \mathbb{P}$ and G be a convex cone with $-L_+^\infty \subseteq G \subseteq L^\infty$. Suppose that $w \in L^\infty$ satisfies $\sup \{E_P[u(f+w)] \mid f \in G\} < u(+\infty)$. Then

$$N \triangleq \left\{ Q \ll P : \frac{dQ}{dP} \in G^0 \cap L^1(\Omega, \mathcal{F}, P) \right\} \neq \emptyset,$$

$$\sup \{E_P[u(f+w)] \mid f \in G\} = \min_{Q \in N} \min_{\lambda > 0} E_P[\lambda \frac{dQ}{dP} w - u^*(\lambda \frac{dQ}{dP})],$$

where u^* is the concave conjugate of u , $G^0 = \{\xi \in ba(\Omega, \mathcal{F}, P) : \xi(f) \leq 0 \ \forall f \in G\}$ is the polar of G and $ba(\Omega, \mathcal{F}, P)$ is the space of all bounded finitely additive measures on $(\Omega, \mathcal{F})$ absolutely continuous w.r.to P .

Proof of Theorem 7. It can be easily checked that

$$M = \{Q \ll P : E_Q[f] \leq 0 \ \forall f \in C\} = \left\{ Q \ll P : \frac{dQ}{dP} \in C^0 \cap L^1(\Omega, \mathcal{F}, P) \right\}.$$

$M \cap \mathbb{P} \neq \emptyset \Rightarrow \mathbf{NMFL}(\mathbb{U}_2)$. Let $w \in L_{++}^\infty$. If $Q \in M \cap \mathbb{P}$ and $u(x) = x$ then:

$$\sup_{f \in C} E_Q[u(f-w)] = \sup_{f \in C} (E_Q[f] - E_Q[w]) \leq -E_Q[w] < 0 = u(0). \quad (5)$$

NMFL($\mathbb{U}_2$) $\implies M \cap \mathbb{P} \neq \emptyset$. For all $w \in L_{++}^\infty$ there exist $u \in \mathbb{U}_2$ and $P \in \mathbb{P}$ such that:

$$\sup \{E_P[u(f - w)] \mid f \in C\} < u(0) \leq u(+\infty).$$

Then we may apply Theorem 8 to $u \in \mathbb{U}_2$, $G = C$ and $N = M$, and deduce that there exist $Q_w \in M$ and $\lambda_w > 0$ that attain the minimum in the dual problem:

$$u(0) > \sup_{f \in C} E_P[u(f - w)] = \min_{Q \in M} \min_{\lambda > 0} E_P[-\lambda \frac{dQ}{dP} w - u^*(\lambda \frac{dQ}{dP})] \quad (6)$$

$$= -\lambda_w E_{Q_w}[w] + E_P[-u^*(\lambda_w \frac{dQ_w}{dP})] \geq -\lambda_w E_{Q_w}[w] + u(0), \quad (7)$$

where the last inequality follows from $-u^*(y) \triangleq \sup_{x \in \mathbb{R}} \{u(x) - xy\} \geq u(0) \forall y \geq 0$. Taking $w = 1_A$, where $A \in \mathcal{F}$ satisfies $P(A) > 0$, we get from (6)-(7): $0 > -\lambda_{1_A} Q_{1_A}(A)$. To summarize: For each $A \in \mathcal{F} : P(A) > 0$ there exists $Q_{1_A} \in M : Q_{1_A}(A) > 0$. Then the existence of $Q \in M$ equivalent to P results from the application of the Halmos-Savage Lemma (see for example Frittelli and Lakner (1994)). ■

Remark 9 For the existence of $Q_{1_A} \in M : Q_{1_A}(A) > 0$ it would be sufficient to have in (6) an infimum over M , instead of a minimum. This suggests that it could be possible to weaken the assumptions on u . However, the condition that u is finite valued on $\mathbb{R}$ is necessary to guarantee that the dual optimization problem in (6) is over $C^0 \cap L^1(\Omega, \mathcal{F}, P)$ and not just over $C^0 \subset ba(\Omega, \mathcal{F}, P)$ (see Bellini and Frittelli (2002) for details), and so this assumption can't be suppressed.

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