

Optimal solutions to utility maximization and to the dual problem

Marco Frittelli
Università di Firenze

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Dept. Metodi Quantitativi S.E.A.,
Università Milano - Bicocca, Italy.
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Abstract

We consider increasing strictly concave utility functions that are finite valued on the whole real line and a continuous-time model of a financial incomplete market where asset price processes are semimartingales. We show that if the utility function has reasonable asymptotic elasticity, as defined by Schachermayer [18], then there exists an optimal solution to utility maximization from terminal wealth and to the dual problem of the minimization of generalized divergence distances. The proof is based on a direct application of the properties of the solution of the dual problem and on the dual characterization of the reasonable asymptotic elasticity condition.

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1 Introduction

Let $X = (X_t)_{t \in [0, T]}$ be an $\mathbb{R}^d$ -valued càd-làg semi-martingale on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, P)$, where we assume that the filtration satisfies the usual assumptions of right continuity and completeness. The process X represents the deflated price process of d tradeable assets.

An $\mathbb{R}^d$ -valued predictable process $H = (H_t)_{t \in [0, T]}$ is called *admissible* if H is X -integrable and the stochastic process

$$\int_0^t H_s \cdot dX_s, \quad t \in [0, T],$$

is P -a.s. uniformly bounded from below. $\int_0^T H_s \cdot dX_s$ is the time T deflated financial gain from the admissible trading strategy H . The set

$$K \triangleq \left\{ \int_0^T H_s \cdot dX_s \mid H \text{ is admissible} \right\}.$$

is the cone of bounded from below claims that are attainable, at zero initial cost, from trading in the d assets with admissible trading strategies.

We denote with $L^\infty = L^\infty(\Omega, \mathcal{F}, P; \mathbb{R})$ (resp. $L^0 = L^0(\Omega, \mathcal{F}, P; \mathbb{R})$, $L^1(P) = L^1(\Omega, \mathcal{F}, P; \mathbb{R})$) the space of P -essentially bounded (resp. P -a.s. finite, P -integrable) random variables on $(\Omega, \mathcal{F})$ having values in $\mathbb{R}$, with $L_+ = \{l \in L : l \geq 0\}$ the positive cone of a vector lattice $(L, \geq)$. Let $\mathbb{P}$ be the class of probability measures equivalent to P .

Without further mention, in the sequel we will always assume that the utility function $u : \mathbb{R} \rightarrow \mathbb{R}$ is increasing, differentiable, strictly concave, and satisfies

$$u'(-\infty) \triangleq \lim_{x \rightarrow -\infty} u'(x) = +\infty, \quad u'(+\infty) \triangleq \lim_{x \rightarrow +\infty} u'(x) = 0. \quad (1)$$

The optimization problem

$$\left\{ Eu\left(x + \int_0^T H_s \cdot dX_s\right) \mid H \text{ is admissible} \right\} = \{Eu(x + f) \mid f \in K\} \rightarrow \max!$$

in general doesn't admit an optimal solution in K (see Section 1.3 for a discussion on the literature on this subject). To find an optimal solution in the utility maximization problem we have to define a domain larger than K . First, let us introduce some more definitions.

$$M \triangleq \{Q \ll P : K \subseteq L^1(Q) \text{ and } E_Q[f] \leq 0 \ \forall f \in K\}.$$

The elements of the set M are called *separating measures*. The following representations of M are well known (see, for example, Delbaen-Schachermayer [4] or Bellini-Frittelli [1] Lemma 1.1) and justify the use of separating measures as “pricing measures”.

Lemma 1 $M = \{Q \ll P : X \text{ is a } Q \text{ martingale}\}$, if X is bounded.

$M = \{Q \ll P : X \text{ is a } Q \text{ local martingale}\}$, if X is locally bounded.

$$M = \left\{ Q \ll P : \int_0^\cdot H_s \cdot dX_s \text{ is a } Q \text{ supermartingale for each admissible } H \right\}.$$

Throughout the paper we assume that $\Psi : (0, +\infty) \rightarrow \mathbb{R}$ is a strictly convex differentiable function and set $\Psi(0) = \lim_{y \downarrow 0} \Psi(y)$. Then Ψ' is strictly increasing and continuous on $(0, +\infty)$. Define

$$M_\Psi \triangleq \left\{ Q \in M : \Psi^+\left(\frac{dQ}{dP}\right) \in L^1(P) \right\}.$$

M_Ψ is the set of probability measures $Q \in M$ having finite Ψ -divergence distance or “generalized entropy”.

Remark 1 Note that a strictly convex differentiable function $\Psi : (0, +\infty) \rightarrow \mathbb{R}$ is either bounded from below or is unbounded from below and decreasing. In the latter case, it can be easily shown that there exist $m > 0$ and $q \geq 0$ such that $\Psi(y) \geq -my - q$ for all $y > 0$. Therefore, $\Psi^-\left(\frac{dQ}{dP}\right) \in L^1(P)$ for all $Q \ll P$. Henceforth, $Q \in M_\Psi$ implies $\Psi\left(\frac{dQ}{dP}\right) \in L^1(P)$.

The conjugate $u^* : (0, +\infty) \rightarrow \mathbb{R}$ of the concave function u is defined by:

$$u^*(y) = \inf_{x \in \mathbb{R}} \{xy - u(x)\} = -\sup_{x \in \mathbb{R}} \{u(x) - xy\}.$$

Set

$$\Phi = -u^*.$$

Then $\Phi : (0, +\infty) \rightarrow \mathbb{R}$ is a strictly convex differentiable function, $\Phi(+\infty) = +\infty$, $\Phi(0^+) = u(+\infty)$ and, from (1), $\Phi'(0^+) = -\infty$, $\Phi'(+\infty) = +\infty$. Moreover, $\Phi(y) = y\Phi'(y) + u(-\Phi'(y))$ for $y \geq 0$, where the usual rule $0 \cdot \infty = 0$ is applied.

A relevant case is the exponential utility function $u(x) = 1 - e^{-x}$, where $\Phi(y) = 1 - y + y \ln y$, $M_\Phi = \{Q \in M : H(Q, P) < +\infty\}$ and $H(Q, P) = E[\frac{dQ}{dP} \ln(\frac{dQ}{dP})]$ is the relative entropy. Thus, if X is locally bounded and if $u(x) = 1 - e^{-x}$, then M_Φ is the set of local martingale measures having finite relative entropy.

Now we can formalize the *primal optimization problem* that we will analyze in this paper. Set

$$U(x) \triangleq \sup \{Eu(x + f) \mid f \in K_\Phi, u^-(x + f) \in L^1(P)\}, \quad (2)$$

where

$$K_\Phi \triangleq \left\{ f \in \bigcap_{Q \in M_\Phi} L^1(Q) \mid \sup_{Q \in M_\Phi} E_Q[f] \leq 0 \right\}$$

(the economic motivation for the selection of K_Φ will be given in Section 1.1). We say that f_x is optimal to (2) if $U(x) = Eu(x + f_x) > -\infty$ and $f_x \in K_\Phi$. Define also:

$$U_Q(x) \triangleq \sup \{Eu(x + f) \mid f \in L^1(Q), E_Q[f] \leq 0, u^-(x + f) \in L^1(P)\}, \quad (3)$$

Note that if $Q \in M_\Phi$ then $U(x) \leq U_Q(x)$ and if $M_\Phi = \{Q\}$ then $U(x) = U_Q(x)$.

To solve the primal problem we'll use the characterization of the solution of the following *dual optimization problem*. For $\lambda > 0$ set

$$V(\lambda) \triangleq \inf \left\{ E\Phi\left(\lambda \frac{dQ}{dP}\right) \mid Q \in M, \Phi^+\left(\lambda \frac{dQ}{dP}\right) \in L^1(P) \right\}. \quad (4)$$

We say that Q_λ is optimal for $V(\lambda)$ if $V(\lambda) = E\Phi\left(\lambda \frac{dQ_\lambda}{dP}\right) < +\infty$ and $Q_\lambda \in M$. When $u : \mathbb{R} \rightarrow \mathbb{R}$ it is well known (see [1] and [9]) that for each $\lambda > 0$ there exists an optimal solution for $V(\lambda)$.

Recall the following condition introduced by Schachermayer.

Definition 1 (*Definition 1.5, Schachermayer [18]*) *The utility function u has reasonable asymptotic elasticity (RAE(u)) if*

$$AE_{+\infty}(u) \triangleq \limsup_{x \rightarrow +\infty} \frac{xu'(x)}{u(x)} < 1 \text{ and } AE_{-\infty}(u) \triangleq \liminf_{x \rightarrow -\infty} \frac{xu'(x)}{u(x)} > 1.$$

As shown by Schachermayer, the assumption $RAE(u)$ is rather weak and economically “reasonable”. We will discuss this condition in Section 1.2.

The main result of the paper is the following Theorem. Note that we do not require a priori that $M \neq \emptyset$.

Theorem 1 *If u has reasonable asymptotic elasticity and if*

$$\sup \{Eu(x + f) | f \in K\} < u(+\infty), \text{ for some } x \in \mathbb{R},$$

then $M_\Phi \neq \emptyset$ and, for all $x \in \mathbb{R}$,

i) There exists an optimal solution $Q_x \in M_\Phi$ of the problem

$$U_{Q_x}(x) = \min_{Q \in M_\Phi} U_Q(x);$$

ii) Q_x is the optimal solution of the problem

$$V(\lambda_x) = E[\Phi(\lambda_x \frac{dQ_x}{dP})] = \min_{Q \in M_\Phi} E[\Phi(\lambda_x \frac{dQ}{dP})],$$

where λ_x is the unique solution of the equation

$$E[\frac{dQ_x}{dP} \Phi'(\lambda_x \frac{dQ_x}{dP})] = -x;$$

iii) There exists an optimal solution $f_x \in K_\Phi$ to (2):

$$U(x) = Eu(x + f_x) = \max \{Eu(x + f) | f \in K_\Phi, Eu^-(x + f) < +\infty\},$$

f_x is given by

$$f_x = -x - \Phi'(\lambda_x \frac{dQ_x}{dP}),$$

and satisfies:

$$u'(x + f_x) = \lambda_x \frac{dQ_x}{dP};$$

iv) The following duality relation holds true

$$U(x) = \lambda_x x + V(\lambda_x);$$

v)

$$\sup \{Eu(x + f) | f \in K\} = U(x) = U_{Q_x}(x);$$

vi) If $M_\Phi \cap \mathbb{P} \neq \emptyset$ then $Q_x \sim P$ and

$$K_\Phi = C_\Phi \triangleq \bigcap_{Q \in M_\Phi} \overline{(K - L_+^0) \cap L^\infty}^Q = \bigcap_{Q \in M_\Phi} \overline{K - L_+^1(Q)}^Q,$$

where $\overline{C}^Q$ denotes the closure of a set C in the norm topology of $L^1(Q)$;

vii) If $u(+\infty) = +\infty$ then $M_\Phi \cap \mathbb{P} \neq \emptyset$.

The proof of Theorem 1 is based on three distinct facts: the existence of a minimax measure, the characterization of the measure with minimal Φ -divergence distance, the equivalent formulations of the assumption of $RAE(u)$.

The conditions that guarantee the existence of minimax measures are taken from Bellini-Frittelli [1] (see Theorem 2 below).

The measures having minimal Φ -divergence distance were studied in details by Csiszar [2], Ruschendorf [17], Liese-Vajda [16] and were applied in the financial context by Bellini-Frittelli [1] and Goll-Ruschendorf [9]. The basic result on this subject is presented in Theorem 4. A novelty in the characterization of these minimal distances is represented by Theorem 3, in conjunction with the above mentioned Theorem 4.

1.1 The set K_Φ

We motivate the selection of the set K_Φ , or C_Φ , in the primal optimization problem. Set

$$C \triangleq (K - L_+^0) \cap L^\infty,$$

and note that C is a convex cone satisfying $L_-^\infty \subseteq C \subseteq L^\infty$, and it consists of all essentially bounded random variables that are dominated by some element of K ; in other words, C is the cone of bounded claims that can be super-replicated by attainable claims at zero initial cost. Lemma 3) will also show that $\sup \{Eu(x + f) \mid f \in K\} = \sup \{Eu(x + f) \mid f \in C\}$.

- a) (Complete market). Suppose that $M = \{Q\}$ and $Q \sim P$, i.e. assume that the market is “free of arbitrage opportunities” and is complete. Under the assumption $\sup \{Eu(x + f) \mid f \in K\} < u(+\infty)$, from Theorem 1 we deduce $M_\Phi = \{Q\}$ and

$$\{f \in L^1(Q) : E_Q[f] \leq 0\} = K_\Phi = C_\Phi = \overline{C}^Q.$$

Hence the budget constraint set $\{f \in L^1(Q) : E_Q[f] \leq 0\}$ that one would naturally select from economic considerations is precisely the set K_Φ .

b) Recall that the super replication price $\hat{f}$ of $f \in L^0$ is defined as:

$$\begin{aligned}\hat{f} &\triangleq \inf \{x \in \mathbb{R} \mid \exists g \in K \text{ s.t. } x + g \geq f \text{ } P - a.s.\} \\ &= \inf \{x \in \mathbb{R} \mid f - x \in (K - L_+^0)\},\end{aligned}$$

and assume that $M \cap \mathbb{P} \neq \emptyset$.

It was shown by Delbaen and Schachermayer, Theorem 5.10 [4], that if $f \in L^0$ is bounded from below, then $\hat{f} = \sup_{Q \in M} E_Q[f]$.

From Kabanov-Stricker [12] and the growth condition $G(\Phi)$ it follows that if $f \in L^0$ is bounded from below then

$$\sup_{Q \in M_\Phi} E_Q[f] = \sup_{Q \in M} E_Q[f].$$

Therefore, *those elements $f \in K_\Phi$ that are bounded from below have super-replication price at most equal to 0:*

$$\hat{f} = \sup_{Q \in M_\Phi} E_Q[f] \leq 0.$$

c) Not all separating measures $Q \in M$ are compatible with the preference structure of the investor.

Suppose that $RAE(u)$ holds true and that $Q \in M$. By Proposition 4 (d) we deduce that $Q \notin M_\Phi$ if and only if

$$\forall x \in \mathbb{R} \ U_Q(x) = \sup \{Eu(x + f) \mid f \in L^1(Q) : E_Q[f] \leq 0\} = u(+\infty).$$

Let $Q \notin M_\Phi$ and suppose that Q is taken as the pricing measure. Then, even with an arbitrarily large debt ($x \downarrow -\infty$), the investor might become arbitrarily close to his supremum utility $u(+\infty)$, by investing in claims $f \in L^1(Q)$ having at most zero cost, i.e. $E_Q[f] \leq 0$. The separating measure Q that allows for this type of “utility based arbitrage opportunity” should be considered incompatible with the preferences of the investor, represented by his utility function. Hence, only those separating measures $Q \in M_\Phi$ are taken in the definition of K_Φ .

- d) Suppose that $M_\Phi \cap \mathbb{P} \neq \emptyset$. Then, by Theorem 1 vi), $K_\Phi = C_\Phi$. The cone C_Φ is the closure of C in the topology defined as the upper bound of all $L^1(Q)$ norm topology, for all $Q \in M_\Phi$.

The set $\overline{K - L_+^1(Q)}^Q$ is the cone of Q -integrable claims that can be approximated, in the $L^1(Q)$ norm topology, by elements in $K - L_+^1(Q) = (K - L_+^0) \cap L^1(Q)$, i.e. by Q -integrable claims that can be dominated by claims attainable with zero initial wealth via admissible trading strategies. Since only those $Q \in M_\Phi$ are compatible with the utility function u , we end up with $C_\Phi = \bigcap_{Q \in M_\Phi} \overline{K - L_+^1(Q)}^Q$. Note that if there exists $Q \in M_\Phi$ such that $\frac{dP}{dQ} \in L^\infty(P)$, then $C_\Phi \subseteq \overline{C}^P$.

1.2 The equivalent formulations of the RAE(u) condition.

Recall that it is assumed that $\Psi : (0, +\infty) \rightarrow \mathbb{R}$ is a strictly convex, differentiable function.

Condition $A(\Psi)$

For each compact interval $[\lambda_0, \lambda_1]$ contained in $(0, +\infty)$ there exist constants $\alpha > 0$ and $\beta > 0$ such that:

$$\Psi^+(\lambda y) \leq \alpha \Psi^+(y) + \beta(y + 1), \text{ for } y > 0 \text{ and } \lambda \in [\lambda_0, \lambda_1]. \quad (5)$$

Condition $B(\Psi)$

There exist constants $\alpha > 0$ and $\beta > 0$ such that:

$$y|\Psi'(y)| \leq \alpha \Psi^+(y) + \beta(y + 1), \text{ for } y > 0, \quad (6)$$

(if $y = 0$ and $\Psi'(0) = \infty$, the usual convention $0 \cdot \infty = 0$ is applied).

Consider also the conditions $\mathbb{A}(\Psi)$ and $\mathbb{B}(\Psi)$ which are obtained from $A(\Psi)$ and $B(\Psi)$ replacing, in equations (5) and (6), Ψ^+ with Ψ .

The next proposition was proved in Frittelli-Rosazza [8], where more details can also be found.

Proposition 1 (a) *The conditions $A(\Psi)$, $B(\Psi)$, $\mathbb{A}(\Psi)$, $\mathbb{B}(\Psi)$ are equivalent.*

(b) *If $u : \mathbb{R} \rightarrow \mathbb{R}$ is increasing, differentiable, strictly concave and satisfies $u'(-\infty) = +\infty$, $u'(+\infty) = 0$ and if $\Phi = -u^*$ then $\text{RAE}(u)$ is equivalent to any of the equivalent conditions $A(\Psi)$, $B(\Psi)$, $\mathbb{A}(\Psi)$, $\mathbb{B}(\Psi)$.*

Definition 2 *We say that a strictly convex differentiable function $\Psi : (0, +\infty) \rightarrow \mathbb{R}$ satisfies the growth condition $G(\Psi)$ if any one of the equivalent conditions $A(\Psi)$, $B(\Psi)$, $\mathbb{A}(\Psi)$, $\mathbb{B}(\Psi)$ is satisfied.*

Conditions similar to $\mathbb{A}(\Psi)$ and $\mathbb{B}(\Psi)$ appear in Schachermayer [18] Corollary 4.2. But Schachermayer conditions are not equivalent to each other and are not equivalent to $RAE(u)$. The reason is that the conditions $A(\Phi)$, $\mathbb{A}(\Phi)$, $B(\Phi)$, $\mathbb{B}(\Phi)$ and $RAE(u)$ are invariant with respect to positive affine transformations, but the Schachermayer conditions do not have this property.

In Kabanov-Stricker [13] the following two assumptions are considered (condition $a(\Psi)$ was introduced in Kabanov-Stricker [12]):

$a(\Psi)$ There exist increasing positive functions r_1 and r_2 such that:

$$\Psi^+(\lambda y) \leq r_1(\lambda)\Psi^+(y) + r_2(\lambda)(y + 1), \quad \forall y > 0, \quad \forall \lambda > 0.$$

$b(\Psi)$ There exists a constant $c > 0$ such that:

$$y|\Psi'(y)| \leq c(\Psi^+(y) + 1), \quad \forall y \geq 0.$$

Obviously, $a(\Psi) \Rightarrow A(\Psi)$ and $b(\Psi) \Rightarrow B(\Psi)$. But $A(\Psi)$ doesn't imply $a(\Psi)$ and $B(\Psi)$ doesn't imply $b(\Psi)$. Indeed $A(\Psi)$ and $B(\Psi)$ are weaker conditions. To point out the main difference between $a(\Psi)$ and $A(\Psi)$ consider the following: $a(\Psi)$ implies (taking $y = 1$ and $\lambda_0 > 0$) that, for all λ satisfying $0 < \lambda \leq \lambda_0$,

$$\Psi^+(\lambda) \leq r_1(\lambda)\Psi^+(1) + r_2(\lambda)(2) \leq r_1(\lambda_0)\Psi^+(1) + 2r_2(\lambda_0).$$

Therefore if Ψ satisfies $a(\Psi)$ then Ψ^+ , and so Ψ , must be bounded in a right neighborhood of the origin, which is not necessarily the case for functions satisfying $A(\Psi)$ or $B(\Psi)$. Condition $A(\Phi)$ may be satisfied by functions $\Phi = -u^*$, where the utility u is unbounded from above, so that $\Phi(0) = u(+\infty) = +\infty$, while $a(\Phi)$ excludes a priori this type of functions.

1.3 On existing literature

Utility maximization in the context of incomplete financial markets received increasing interest in recent years. A wide review of this area of research, and a comprehensive and updated list of references can be found in Schachermayer

[19]. In the following we mention only those papers, closely related to the subject of the present work, that are not based on one specific utility function.

Duality methods from convex analysis were introduced in the study of incomplete markets by He and Pearson [10], [11] and Karatzas et Al. [14]. In these papers the utility maximization problem and its dual problem were solved, both in the discrete time, finite probability space, and in a continuous time diffusion setting.

For the general case of semimartingale asset price processes, Bellini and Frittelli [1] showed that, if the utility function is finite valued on $\mathbb{R}$ then the duality relation holds true and the dual problem admits always a solution in the set of separating measures. Kramkov and Schachermayer [15] considered utility functions finite valued only on $(0, +\infty)$ and proved the existence of the optimal solution to the primal utility maximization problem, under the condition that the asymptotic elasticity at $+\infty$ is strictly less than 1.

Schachermayer [18] considered utility functions finite valued on $\mathbb{R}$ and showed that the duality relation holds true, and that both the primal and the dual problems admit optimal solutions, under the assumption that u has reasonable asymptotic elasticity. The main difference with Theorem 1 is that Schachermayer takes the closure in the $L^1(P)$ - norm topology of a set generated by K , while we take the intersection of all the sets $\overline{C}^Q$, where $\overline{C}^Q$ is the closure of C in $L^1(Q)$, and Q varies in M_Φ . The proof in Schachermayer is based on the construction of a sequence of optimal solutions to problems having utility functions finite valued only on $(-n, +\infty)$, and on the verification that this sequence converges, as $n \rightarrow +\infty$, to the optimal solution of the original problem. Instead, our proof is based on a direct application of the properties of the solution to the dual problem. Our proof is much shorter than the one by Schachermayer: This is mainly due to the different selection of the domain of the utility maximization problem, which considerably simplify the problem.

The relationship between minimax measures and measures with minimal Φ -divergence distance was pointed out in Bellini-Frittelli [1] and is the main subject of the paper by Goll and Ruschendorf [9]. In the latter paper, the characterization of the measures with minimal Φ -divergence distance is provided in great generality. Goll and Ruschendorf [9] also characterized optimal utility based portfolios in terms of minimal Φ -divergence distance measures, for utility functions finite valued only on $(\alpha, +\infty)$, $\alpha > -\infty$.

In Frittelli [7], the duality results are used to compute pricing functionals π_u coherent with the no arbitrage principle. These functionals need not to be linear and, depending on which utility u is chosen, admit representations of the forms

$$\pi_u(w) = \inf_{Q \in M} F_u(Q) E_Q[w], \text{ where } F_u : M \rightarrow [1, +\infty),$$

or

$$\pi_u(w) = \inf_{Q \in M} \{E_Q[w] + H(Q, P)\} - \inf_{Q \in M} H(Q, P), \quad \text{if } u(x) = -e^{-x}.$$

After having essentially finished the paper we received a copy of the paper by Kabanov and Stricker [13], which extends some of the results in the paper by Delbaen et Al. [3] concerning the exponential utility function, and where some results similar to the ones here presented are independently obtained.

2 Proof and independent results

Without further mentioning, we will identify a P -absolute continuous probability measure Q (Q_i) with its Radon Nikodym density $\varphi = \frac{dQ}{dP}$ ($\varphi_i = \frac{dQ_i}{dP}$) in $L^1(P)$.

2.1 On minimax measures

In Lemma 2 and Theorem 2 we reformulate in a form convenient for our purpose, Corollary 2.1 and Corollary 2.2 of Bellini-Frittelli [1]. Define:

$$U_Q^*(x) = \sup \{Eu(x + f) \mid f \in L^\infty, E_Q[f] \leq 0\}.$$

ad note that $U_Q^*(x) \leq U_Q(x) \leq u(+\infty)$.

Lemma 2 *If $Q \ll P$, $x \in \mathbb{R}$ and $U_Q^*(x) < u(+\infty)$ then*

$$U_Q^*(x) = \min_{\lambda > 0} \{\lambda x + E\Phi(\lambda\varphi)\}. \quad (7)$$

The measure that satisfies the thesis of Theorem 2 is called the *minimax measure*.

Theorem 2 *If $x \in \mathbb{R}$ satisfies $\sup \{Eu(x + f) | f \in C\} < u(+\infty)$, then $M \neq \emptyset$ and there exists $Q_x \in M$ that satisfies*

$$\sup \{Eu(x + f) | f \in C\} = U_{Q_x}^*(x) = \min_{Q \in M} U_Q^*(x).$$

If in addition $u(+\infty) = +\infty$ then $Q_x \in M \cap \mathbb{P}$.

Lemma 3

$$\begin{aligned} M &= \{Q \ll P : E_Q[f] \leq 0 \ \forall f \in C\}, \\ \sup \{Eu(x + f) | f \in K\} &= \sup \{Eu(x + f) | f \in C\}. \end{aligned}$$

Proof. Let $k \in K$ and $k_n \triangleq \min(k, n) = k - (k - k_n) \in (K - L_+^0) \cap L^\infty = C$. Then $k_n \uparrow k$, $P - a.s.$, and both statments of Lemma 3 can be easily proved using the monotone convergence Theorem. ■

2.2 Consequences of the growth condition $G(\Psi)$

Proposition 2 *Suppose that $G(\Psi)$ holds true and let $\varphi = \frac{dQ}{dP}$.*

(a) $\Psi^+(\lambda\varphi) \in L^1(P)$ for some $\lambda > 0$ if and only if $\Psi^+(\lambda\varphi) \in L^1(P) \ \forall \lambda > 0$. Hence, for all $\lambda > 0$ we have: $M_{\Psi_\lambda} = M_\Psi$, where $\Psi_\lambda(y) \triangleq \Psi(\lambda y)$.

(b) If $\Psi^+(\varphi) \in L^1(P)$ then, $\forall \lambda > 0$, $\Psi(\lambda\varphi) \in L^1(P)$ and $(\varphi\Psi'(\lambda\varphi)) \in L^1(P)$.

(c) If $\Psi^+(\varphi) \in L^1(P)$ and if $\Psi'(0^+) = -\infty$, $\Psi'(+\infty) = +\infty$ then the function $F(\lambda) \triangleq E[\varphi\Psi'(\lambda\varphi)]$ defines a bijection between $(0, +\infty)$ and $(-\infty, +\infty)$.

Proof. (a) Suppose that $\Psi^+(\lambda^*\varphi) \in L^1(P)$ for some $\lambda^* > 0$ and let $\lambda > 0$. Then, from $A(\Psi)$, we get: $\Psi^+(\lambda\varphi) \leq \alpha\Psi^+(\varphi) + \beta(\varphi + 1) = \alpha\Psi^+(\frac{1}{\lambda^*}\lambda^*\varphi) + \beta(\varphi + 1) \leq \alpha_1\Psi^+(\lambda^*\varphi) + \beta_1(\lambda^*\varphi + 1) + \beta(\varphi + 1)$.

(b) From (a) and Remark 1 we deduce $\Psi(\lambda\varphi) \in L^1(P)$. Applying first condition $B(\Psi)$ and then condition $A(\Psi)$ we deduce:

$$\lambda\varphi|\Psi'(\lambda\varphi)| \leq \alpha\Psi^+(\lambda\varphi) + \beta(\lambda\varphi + 1) \leq \alpha_1\Psi^+(\varphi) + \beta_1(\varphi + 1) + \beta(\lambda\varphi + 1). \quad (8)$$

(c) From (b) we deduce that F is finite valued on $(0, +\infty)$. From (8), the continuity of Ψ' on $(0, +\infty)$ and Lebesgue Theorem we deduce that F is continuous at each $\lambda > 0$. Note that Ψ' is strictly increasing from $\Psi'(0^+) = -\infty$ to $\Psi'(+\infty) = +\infty$ and F is strictly increasing on $(0, +\infty)$. Since $\lambda \rightarrow \varphi\Psi'(\lambda\varphi)$ is increasing, by the monotone convergence Theorem we deduce $\lim_{\lambda \rightarrow 0} F(\lambda) =$

$E[\lim_{\lambda \rightarrow 0} \varphi \Psi'(\lambda \varphi)] = -\infty$, $\lim_{\lambda \rightarrow +\infty} F(\lambda) = E[\lim_{\lambda \rightarrow +\infty} \varphi \Psi'(\lambda \varphi)] = +\infty$, and so item (c) follows. ■

If $\Psi(y) = y \ln y$, the following theorem was proved in Frittelli [6]. For arbitrary functions Ψ we need condition $A(\Psi)$.

Theorem 3 *If Ψ satisfies $G(\Psi)$ and if $M_\Psi \cap \mathbb{P} \neq \emptyset$ then*

$$C_\Psi = \bigcap_{Q \in M_\Psi} \overline{K - L_+^1(Q)}^Q = K_\Psi.$$

Proof. For all $Q \in M$ we have: $K \subseteq L^1(Q)$, $C \subseteq (K - L_+^0) \cap L^1(Q) = K - L_+^1(Q)$ and if $f \in \overline{K - L_+^1(Q)}^Q$ then $E_Q[f] \leq 0$. Therefore,

$$C_\Psi \triangleq \bigcap_{Q \in M_\Psi} \overline{C}^Q \subseteq \bigcap_{Q \in M_\Psi} \overline{K - L_+^1(Q)}^Q \subseteq K_\Psi.$$

To show that $K_\Psi \subseteq C_\Psi$ let $h \in K_\Psi$ and suppose by contradiction that $h \notin C_\Psi$. Then $\exists Q_0 \in M_\Psi : h \notin \overline{C}^{Q_0}$. Let $Q^* \in M_\Psi \cap \mathbb{P}$ and define $Q = \frac{Q_0 + Q^*}{2}$. Then $Q \in M_\Psi \cap \mathbb{P}$, since M_Ψ is convex. Since $\|\cdot\|_{L^1(Q_0)} \leq 2\|\cdot\|_{L^1(Q)}$, $h \notin \overline{C}^Q$.

By the Hahn Banach Theorem there exists a continuous linear functional on $L^1(Q)$ that strictly separates h from the closed convex cone $\overline{C}^Q$, i.e. there exists $\eta \in L^\infty$ such that:

$$\sup_{f \in C} E_Q[\eta f] \leq 0 < E_Q[\eta h]. \quad (9)$$

Since $-1_{\{\eta < 0\}} \in C$, we deduce that $\eta \geq 0$ Q -a.s. We then may normalize η and define a probability $Q_1 \ll Q$ by setting: $\frac{dQ_1}{dQ} = \eta$. From equation (9) we then get $Q_1 \in M$ and $E_{Q_1}[h] > 0$. Define $Q_\lambda = \lambda Q + (1 - \lambda)Q_1$, $\lambda \in [0, 1]$. Then, $\forall \lambda \in (0, 1]$, $Q_\lambda \in M \cap \mathbb{P}$ and, since $E_{Q_\lambda}[h] = \lambda E_Q[h] + (1 - \lambda)E_{Q_1}[h]$ there exists $\lambda_0 \in (0, 1]$ such that $E_{Q_{\lambda_0}}[h] > 0$ (note that $E_Q[h]$ is finite since $h \in K_\Psi$). Since $\frac{dQ_1}{dQ}$ is bounded, there exists some $\lambda_1 \in \mathbb{R}_+$ such that $0 < \lambda_0 \leq \frac{dQ_{\lambda_0}}{dQ} = \lambda_0 + (1 - \lambda_0)\frac{dQ_1}{dQ} \leq \lambda_1$, P -a.s. Since $Q \in M_\Psi$, condition $A(\Psi)$ guarantees that

$$\Psi^+\left(\frac{dQ_{\lambda_0}}{dP}\right) = \Psi^+\left(\frac{dQ_{\lambda_0}}{dQ} \frac{dQ}{dP}\right) \leq \alpha \Psi^+\left(\frac{dQ}{dP}\right) + \beta \left(\frac{dQ}{dP} + 1\right) \in L^1(P).$$

Therefore $Q_{\lambda_0} \in M_\Psi$, which contradicts $h \in K_\Psi$, since $E_{Q_{\lambda_0}}[h] > 0$. ■

2.3 Optimal solution to the dual problem

The following theorem provides a necessary and sufficient condition for the existence of the projection of P on M with respect to the Ψ -divergence distance. Replacing, in Theorem 4, the hypothesis that Ψ satisfies assumption $G(\Psi)$ with the hypothesis that $\Psi'(\varphi_1) \in L^1(Q_1)$, Theorem 4 is essentially Theorem 5 in Rüschendorf [17]. The proof we present here is the generalization of the proof of Theorem 2.3 in Frittelli [6] and is based on assumption $B(\Psi)$. Theorem 4 and its Corollary 1 are crucial for the proof of Theorem 1. As shown in Theorem 3, in the following theorem we may replace K_Ψ with C_Ψ , whenever $M_\Psi \cap \mathbb{P} \neq \emptyset$.

Theorem 4 *If Ψ satisfies $G(\Psi)$ then $Q_1 \in M_\Psi$ is optimal for*

$$\inf \left\{ E\Psi\left(\frac{dQ}{dP}\right) \mid Q \in M_\Psi \right\} \quad (10)$$

if and only if $f_1 \triangleq E[\varphi_1 \Psi'(\varphi_1)] - \Psi'(\varphi_1) \in K_\Psi$, where $\varphi_1 = \frac{dQ_1}{dP}$.

Proof. Suppose that Q_1 is optimal for (10) and let $Q_0 \in M_\Psi$. Set $\varphi_0 = \frac{dQ_0}{dP}$, $\xi_x = x\varphi_0 + (1-x)\varphi_1$, $x \in [0, 1]$. We necessarily have $\left(\frac{d}{dx}E[\Psi(\xi_x)]\right)|_{x=0} \geq 0$, otherwise there would exist $x > 0$ such that $E[\Psi(\xi_x)] < E[\Psi(\xi_0)] = E[\Psi(\varphi_1)]$, which contradicts the optimality of Q_1 .

From the convexity of Ψ we get

$$\varphi_0 \Psi'(\varphi_1) \leq \varphi_1 \Psi'(\varphi_1) + \Psi(\varphi_0) - \Psi(\varphi_1) \quad P - a.s. \quad (11)$$

From Remark 1 we know that $\Psi(\varphi_0) \in L^1(P)$, $\Psi(\varphi_1) \in L^1(P)$ and, from Proposition 2 (b),

$$\varphi_1 \Psi'(\varphi_1) \in L^1(P). \quad (12)$$

Hence, from (11),

$$(\varphi_0 \Psi'(\varphi_1))^+ \in L^1(P). \quad (13)$$

Set $H(x) = \Psi(\xi_x)$, $x \in [0, 1]$. Since Ψ is convex, it is easy to check that H is convex and that $\left(\frac{H(x)-H(0)}{x}\right)$ is increasing. The assumptions $\Psi(\varphi_0) \in L^1(P)$ and $\Psi(\varphi_1) \in L^1(P)$ imply $E[H(1) - H(0)] < +\infty$. Then, by monotone convergence, the following limit can be exchanged with the integration operator, and the right derivative is given by:

$$0 \leq \frac{d}{dx}E[\Psi(\xi_x)]|_{x=0} = \lim_{x \downarrow 0} E \left[\frac{H(x) - H(0)}{x} \right] = E[H'(0)] = E[\Psi'(\varphi_1)(\varphi_0 - \varphi_1)]. \quad (14)$$

From (12) (13) and (14) we deduce $(\varphi_0 \Psi'(\varphi_1)) \in L^1(P)$, $\Psi'(\varphi_1) \in L^1(Q_0)$ and $f_1 \in L^1(Q_0)$. From equation (14) we get: $0 \leq E_{Q_0}[\Psi'(\varphi_1)] - E_{Q_1}[\Psi'(\varphi_1)] = -E_{Q_0}[f_1]$. This holds for all $Q_0 \in M_\Psi$ and so $f_1 \in K_\Psi$.

Conversely, let $Q_1 \in M_\Psi$ and $f_1 \in K_\Psi$. For any given probability measure $Q_0 \in M_\Psi$ we get, from the convexity of Ψ ,

$$\Psi(\varphi_0) \geq \Psi(\varphi_1) + (\varphi_0 - \varphi_1)\Psi'(\varphi_1), \quad P - a.s.$$

Therefore:

$$\begin{aligned} E[\Psi(\varphi_0)] &\geq E[\Psi(\varphi_1)] + E_{Q_0}[\Psi'(\varphi_1)] - E_{Q_1}[\Psi'(\varphi_1)] \\ &= E[\Psi(\varphi_1)] - E_{Q_0}[f_1] \geq E[\Psi(\varphi_1)]. \end{aligned}$$

■

From Theorem 4 and Proposition 2 (a) we deduce the following

Corollary 1 *If Ψ satisfies $G(\Psi)$ and if $\lambda > 0$ then Q_λ is optimal for*

$$\inf \left\{ E\Psi\left(\lambda \frac{dQ}{dP}\right) \mid Q \in M, \Psi^+\left(\lambda \frac{dQ}{dP}\right) \in L^1(P) \right\} \quad (15)$$

if and only if $f_\lambda \triangleq E[\varphi_\lambda \Psi'(\lambda \varphi_\lambda)] - \Psi'(\lambda \varphi_\lambda) \in K_\Psi$, where $\varphi_\lambda = \frac{dQ_\lambda}{dP}$.

Proof. Fix $\lambda > 0$ and define $\Psi_\lambda : (0, +\infty) \rightarrow \mathbb{R}$ by $\Psi_\lambda(y) \triangleq \Psi(\lambda y)$. Then Ψ_λ is convex and differentiable, $M_\Psi = M_{\Psi_\lambda}$ by Proposition 2 (a), $K_\Psi = K_{\Psi_\lambda}$, and Ψ_λ satisfies condition $B(\Psi_\lambda)$. Note that $Q_\lambda \in M_\Psi$ is optimal for (15) iff $Q_\lambda \in M_\Psi = M_{\Psi_\lambda}$ attains the infimum in $\inf \left\{ E\Psi_\lambda\left(\frac{dQ}{dP}\right) \mid Q \in M, \Psi_\lambda^+\left(\frac{dQ}{dP}\right) \in L^1(P) \right\}$. Theorem 4, applied to the function Ψ_λ , then assures that the last statement is equivalent to $f_1 \triangleq E[\varphi_\lambda \Psi'_\lambda(\varphi_\lambda)] - \Psi'_\lambda(\varphi_\lambda) \in K_{\Psi_\lambda} = K_\Psi$, and the thesis follows, since K_Ψ is a cone. ■

2.4 Consequences of the $RAE(u)$ condition

Under different assumptions, item (b) in the next proposition appears several times in literature (see Lemma 4.1 by Goll-Ruschendorf [9] for a similar result). In the present form, the proposition is based on the condition $G(\Phi)$.

Proposition 3 *Suppose that $\Phi = -u^*$ satisfies $G(\Phi)$ (or equivalently that $RAE(u)$ holds true), $\varphi = \frac{dQ}{dP}$ satisfies $\Phi^+(\varphi) \in L^1(P)$ and $x \in \mathbb{R}$. Then:*

- (a) for all $\lambda > 0$, $u(-\Phi'(\lambda\varphi)) \in L^1(P)$,
(b)

$$U_Q(x) = \min_{\lambda > 0} \{\lambda x + E\Phi(\lambda\varphi)\} \quad (16)$$

$$= \lambda_{x,\varphi}x + E\Phi(\lambda_{x,\varphi}\varphi) \quad (17)$$

$$= Eu(-\Phi'(\lambda_{x,\varphi}\varphi)) \quad (18)$$

$$= Eu(x + f) < +\infty,$$

where $\lambda_{x,\varphi} > 0$ is the unique solution of the equation

$$E[\varphi\Phi'(\lambda\varphi)] = -x, \quad (19)$$

and $f \triangleq -x - \Phi'(\lambda_{x,\varphi}\varphi) \in L^1(Q)$ satisfies $E_Q[f] = 0$;

(c) $U_Q(x) = U_Q^*(x)$

(d) we may replace in Theorem 2 the set M with the set M_Φ .

Proof. (a) Recall that

$$\Phi(y) = y\Phi'(y) + u(-\Phi'(y)), \quad y \geq 0. \quad (20)$$

Letting $y = \lambda\varphi$, (a) follows from Proposition 2 (b).

(b) From equation (1), we deduce that $\Phi'(0) = -\infty$ to $\Phi'(+\infty) = +\infty$ and so we know from Proposition 2 (c) that there exists a unique solution $\lambda_{x,\varphi} > 0$ of the equation (19). From the definition of Φ we have:

$$\begin{aligned} u(x) &\leq yx + \Phi(y), \quad \forall x \in \mathbb{R}, \quad \forall y \geq 0, \\ u(g) &\leq \lambda\varphi g + \Phi(\lambda\varphi), \quad P - a.s., \quad \forall g \in L^0(P), \quad \forall \lambda \geq 0. \end{aligned} \quad (21)$$

Note that: $U_Q(x) \triangleq \sup \{Eu(g) \mid g \in L^1(Q), E_Q[g] \leq x, u^-(g) \in L^1(P)\}$. Let $g \in L^0(P)$ satisfy $\varphi g \in L^1(P)$ and $E\varphi g \leq x$. Proposition 2 (b) and item (a) above guarantee that all the integrability requirements in the following inequalities are satisfied. From (21) we deduce:

$$\begin{aligned} Eu(g) &\leq \inf_{\lambda > 0} E[\lambda\varphi g + \Phi(\lambda\varphi)] \\ &\leq E[\lambda_{x,\varphi}\varphi g + \Phi(\lambda_{x,\varphi}\varphi)] \leq \lambda_{x,\varphi}x + E[\Phi(\lambda_{x,\varphi}\varphi)] \\ &= \lambda_{x,\varphi}x + E[\lambda_{x,\varphi}\varphi\Phi'(\lambda_{x,\varphi}\varphi) + u(-\Phi'(\lambda_{x,\varphi}\varphi))] \\ &= \lambda_{x,\varphi}x - \lambda_{x,\varphi}x + Eu(-\Phi'(\lambda_{x,\varphi}\varphi)) = Eu(-\Phi'(\lambda_{x,\varphi}\varphi)), \end{aligned}$$

where the first equality is due to (20) and the second one to (19). This shows that $U_Q(x) \leq Eu(-\Phi'(\lambda_{x,\varphi}\varphi))$. However, we may choose $g = -\Phi'(\lambda_{x,\varphi}\varphi)$ since in this case, from (a) and equation (19), we have: $u(g) \in L^1(P)$ and $E\varphi g = x$. Hence $U_Q(x) = Eu(g) = Eu(-\Phi'(\lambda_{x,\varphi}\varphi))$ and (b) follows.

(c) If $U_Q^*(x) < u(+\infty)$ then (c) follows from (7) and (16). If $U_Q^*(x) = u(+\infty)$ then (c) follows from $U_Q^*(x) \leq U_Q(x) \leq u(+\infty)$.

(d) Let $\Phi_\lambda(y) \triangleq \Phi(\lambda y)$. If $U_Q^*(x) < u(+\infty)$ and if $\bar{\lambda} > 0$ attains the minimum in equation (7), then $E[\Phi(\bar{\lambda}\varphi)] < +\infty$ and so $Q \in M_{\Phi_{\bar{\lambda}}}$. From Proposition 2 (a) we deduce $M_{\Phi_{\bar{\lambda}}} = M_\Phi$ and the thesis. ■

Proposition 4 *Suppose that $\Phi = -u^*$ satisfies $G(\Phi)$ (or equivalently that $RAE(u)$ holds true) and let $\varphi = \frac{dQ}{dP}$. Then: (a) $\Phi^+(\varphi) \in L^1(P)$ implies $U_Q(x) < u(+\infty) \forall x \in \mathbb{R}$; (b) $U_Q(x) < u(+\infty)$ for some $x \in \mathbb{R}$ implies $\Phi^+(\varphi) \in L^1(P)$; (c) $U_Q(x) < u(+\infty)$ for some $x \in \mathbb{R}$ implies $U_Q(x) < u(+\infty) \forall x \in \mathbb{R}$; (d) if $Q \in M$ then: $Q \in M_\Phi$ if and only if $\exists x \in \mathbb{R} : U_Q(x) < u(+\infty)$; (e) $\sup\{Eu(\bar{x} + f) \mid f \in K\} < u(+\infty)$ for some $\bar{x} \in \mathbb{R}$ implies $\sup\{Eu(x + f) \mid f \in K\} < u(+\infty) \forall x \in \mathbb{R}$.*

Proof. (a) If $\Phi^+(\varphi) \in L^1(P)$ then, from the last sentence in the proof of Proposition 3 (b), $U_Q(x) = Eu(g)$, where $u(g) \in L^1(P)$ and $E_Q g = x$. Since u is strictly increasing on $\mathbb{R}$, we necessarily have: $Eu(g) < u(+\infty)$. (b) Since $U_Q^*(x) \leq U_Q(x) < u(+\infty)$ then, by Lemma 2, $U_Q^*(x) = \lambda_x x + E\Phi(\lambda_x \varphi) < u(+\infty)$, for some optimal $\lambda_x > 0$. Hence $\Phi^+(\lambda_x \varphi) \in L^1(P)$ and, by Proposition 2 (a), $\Phi^+(\varphi) \in L^1(P)$. Items (c) and (d) follow immediately from (a) and (b). (e) From Lemma 3, Theorem 2, Proposition 3 (c) and (d) we deduce $U_{Q_{\bar{x}}}(\bar{x}) < u(+\infty)$ and $Q_{\bar{x}} \in M_\Phi$. Hence, from (c), $U_{Q_{\bar{x}}}(x) < u(+\infty) \forall x \in \mathbb{R}$ and (e) follows, since for each $x \in \mathbb{R}$ we have $\sup\{Eu(x + f) \mid f \in K\} \leq U_{Q_{\bar{x}}}(x)$. ■

2.5 Proof of Theorem 1

By Proposition 1 (b) we may assume that $\Phi = -u^*$ satisfies $G(\Phi)$.

i) From Proposition 4 (e) we may assume that $\sup\{Eu(x + f) \mid f \in K\} < u(+\infty) \forall x \in \mathbb{R}$. From Lemma 3

$$\sup\{Eu(x + f) \mid f \in C\} = \sup\{Eu(x + f) \mid f \in K\} < u(+\infty). \quad (22)$$

Therefore, we may apply Theorem 2 and Proposition 3 (d) and deduce that there exists $Q_x \in M_\Phi$ that satisfies:

$$\sup \{Eu(x+f) | f \in C\} = U_{Q_x}^*(x) = \min_{Q \in M_\Phi} U_Q^*(x) < u(+\infty). \quad (23)$$

From Proposition 3 (c), we have

$$U_Q^*(x) = U_Q(x) \quad \forall Q \in M_\Phi. \quad (24)$$

Equations (23) and (24) imply item i).

ii) We apply Proposition 3 to Q_x . Let $\varphi_x = \frac{dQ_x}{dP}$. From equations (19) and (17) there exists $\lambda_x \triangleq \lambda_{x, \varphi_x} > 0$ which satisfies:

$$E[\varphi_x \Phi'(\lambda_x \varphi_x)] = -x, \quad (25)$$

$$U_{Q_x}(x) = \lambda_x x + E\Phi(\lambda_x \varphi_x). \quad (26)$$

From item i) and equations (16) and (26) we also deduce that

$$\begin{aligned} U_{Q_x}(x) &= \min_{Q \in M_\Phi} U_Q(x) = \min_{Q \in M_\Phi} \min_{\lambda > 0} \{\lambda x + E\Phi(\lambda \varphi)\} \\ &\leq \min_{Q \in M_\Phi} \{\lambda_x x + E\Phi(\lambda_x \varphi)\} \leq \lambda_x x + E\Phi(\lambda_x \varphi_x) = U_{Q_x}(x). \end{aligned}$$

Therefore

$$U_{Q_x}(x) = \min_{Q \in M_\Phi} \{\lambda_x x + E\Phi(\lambda_x \varphi)\} = \lambda_x x + \min_{Q \in M_\Phi} E\Phi(\lambda_x \varphi), \quad (27)$$

and from equations (26) and (27) we deduce that $Q_x \in M_\Phi$ is optimal for $V(\lambda_x)$. Note that we'll continue to denote with Q_x (or φ_x) the optimal solution Q_{λ_x} (or φ_{λ_x}) of $V(\lambda_x)$.

iii) Since $Q_x \in M_\Phi$ is optimal for $V(\lambda_x)$, Corollary 1 implies that $f_{\lambda_x} \triangleq f_x = E[\varphi_x \Phi'(\lambda_x \varphi_x)] - \Phi'(\lambda_x \varphi_x) \in K_\Phi$. From (25) we get $f_x = -x - \Phi'(\lambda_x \varphi_x)$ and $u'(x + f_x) = \lambda_x \varphi_x$, since $u' = (-\phi')^{-1}$. From Proposition 3 (a) we know that $u(x + f_x) \in L^1(P)$. Since $Q_x \in M_\Phi$, the definitions of $U(x)$ and $U_{Q_x}(x)$ imply

$$U(x) \leq U_{Q_x}(x) = E[u(x + f_x)],$$

where the equality follows from equations (16)-(18). Since $f_x \in K_\Phi$, we get

$$E[u(x + f_x)] \leq U(x).$$

Therefore

$$U(x) = E[u(x + f_x)] = U_{Q_x}(x). \quad (28)$$

- iv) The duality relation follows from equations (26), (28) and item ii).
- v) It follows from equations (22), (23), (24) and (28).
- vi) The equalities follow from Theorem 3. Recall that $u'(+\infty) = 0$ and so $\Phi'(0) = -\infty$. If $Q(\varphi_x = 0) > 0$ then $\Phi'(\lambda_x \varphi_x) \notin L^1(Q)$. Since Q_x is optimal for $V(\lambda_x)$, from Corollary 1 we have that $Q \in M_\Phi$ implies $\Phi'(\lambda_x \varphi_x) \in L^1(Q)$ and therefore we deduce that $Q(\varphi_x = 0) = 0$ for all $Q \in M_\Phi$. The assumption $M_\Phi \cap \mathbb{P} \neq \emptyset$ implies $P(\varphi_x = 0) = 0$ and $Q_x \sim P$.
- vii) It follows from Theorem 2. ■

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