# Equivalent Formulations of Reasonable Asymptotic Elasticity

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**Abstract**

We provide four equivalent formulations of the condition of Reasonable Asymptotic Elasticity of a utility function. These equivalent conditions are formulated in terms of the conjugate of the utility function. The notion of Reasonable Asymptotic Elasticity was introduced by Schachermayer who showed its relevance in utility maximization problems.

**JEL:** G11, G12, G13. **AMS:** 91B16, 91B28.

## 1 Introduction

In this paper we study a growth condition $\mathbf{G}(\phi)$ which plays a relevant role in many important topics in mathematical finance. The function $\phi : (0, +\infty) \to \mathbb{R}$ is assumed to be strictly convex and differentiable and is interpreted as the conjugate of the utility function $u$. Let:

$\mathbf{G}(\phi)$ : $\quad \forall\,[\lambda_0, \lambda_1] \subseteq (0, +\infty)$ there exist $\alpha > 0$, $\beta > 0$ such that:

$$\phi(\lambda y) \leq \alpha\phi(y) + \beta(y + 1), \ \forall y > 0, \ \forall \lambda \in [\lambda_0, \lambda_1].$$

It is clear that $\mathbf{G}(\phi)$ imposes limitations on the growth of $\phi$ for large $y$ and on the behavior of $\phi$ near the origin. In Section 2 we provide four equivalent formulations of this condition and we show their equivalence with the notion of Reasonable Asymptotic Elasticity of $u$, which was introduced by Schachermayer [8]. In Section 3 we provide a comparison with similar conditions already known in literature. To motivate our analysis let us first introduce the financial market model, the expected utility maximization criterion and its dual approach, useful for pricing in such a market.

**The financial model**

We will consider (see Bellini and Frittelli [1] for details) a semimartingale model of a financial market defined on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$, where $X = (X_t)_{t\in[0,T]}$ is the (discounted) $\mathbb{R}^d-$valued price process of $d$ tradeable assets and $K \triangleq \left\{ \int_0^T H_s \cdot dX_s \mid H \text{ is admissible} \right\}$ is the cone of all bounded from below claims (i.e. $\mathcal{F}_T-$measurable random variables) that are replicable, at zero initial cost, via admissible trading strategies $H = (H_t)_{t\in[0,T]}$ . With $M$ we will denote the set of all *pricing probability measures.* If $X$ is bounded (resp. locally bounded) then $M$ is the set of $P-$absolutely continuous *martingale* (resp. *local martingale*) probability measures for $X$. The absence of arbitrage is equivalent, in this market, to the existence of an equivalent probability in $M$. In general this market is incomplete, $M$ consists of infinite elements and the no arbitrage theory does not lead to a unique pricing rule. One possible solution to the valuation problem is to take into account the preferences of the investor, via the specification of an utility function $u$, which we assume to be strictly concave, increasing and finite valued on $\mathbb{R}$.

Let $x \in \mathbb{R}$ be the initial wealth. The expected utility maximization problem

$$\sup_{g\in D} E_P[u(x+g)] \tag{1}$$

in general does not admit an optimal solution if $D = K$ (see Schachermayer [8]). In the duality approach to this problem a crucial role is played by the (opposite of the) conjugate of $u$, defined by: $\Phi(y) \triangleq -u^*(y) = \sup_{x\in\mathbb{R}} \{u(x) - xy\}$ , $y > 0$, which is strictly convex and differentiable. Given $\Phi$ we define

$$M_\Phi \triangleq \left\{ Q \in M : E_P[\Phi(\frac{dQ}{dP})] < +\infty \right\},$$

which is the set of pricing measures having finite *generalized entropy* (see [1]).

It was first shown in Bellini and Frittelli [1] that if:

$$U(x) \triangleq \sup_{g\in K} E_P[u(x+g)] < u(+\infty) \tag{2}$$

then the fundamental duality relation

$$\sup_{g\in K} E_P[u(x+g)] = \min_{Q\in M} \min_{\lambda>0} \left\{ \lambda x + E_P\left[ \Phi\left( \lambda \frac{dQ}{dP} \right) \right] \right\} \tag{3}$$

holds true, without any further assumption on the utility function.

Now we briefly illustrate four cases where the growth condition $\mathbf{G}(\Phi)$ is relevant . This partially justify the analysis in this note.

(I) In order to have existence of the optimal solution in (1), with an appropriate selection of the domain $D \supset K$, further conditions on the utility function are needed. One determinant contribution in this direction was given by Schachermayer [8], who introduced, based on a previous paper by Kramkov and Schachermayer [5], the concept of Reasonable Asymptotic Elasticity and showed that this condition is sufficient for the optimal solution to exist. In [8] it is also shown that without this condition many other important features of the indirect utility function $U$ defined in (2) do not hold true any more.

(II) The dual problem

$$V(\lambda) \triangleq \inf \left\{ E_P \left[ \phi\left(\lambda \frac{dQ}{dP}\right) \mid Q \in M_\phi \right] \right\}, \tag{4}$$

where $\lambda > 0$, naturally arises from (3) and has been extensively studied in finance (for pricing purpose, for example, in Bellini & Frittelli [1] or Goll & Ruschendorf [3]) as well as in the theory of statistical inference (see for example Ruschendorf [7] or Liese & Vajda [6]). A necessary and sufficient condition for the existence of the optimal solution $Q_\lambda$ of the dual problem (4) (i.e. for the existence of $Q_\lambda \in M_\phi$ such that $V(\lambda) = E_P[\phi(\lambda \frac{dQ_\lambda}{dP})]$) is known, provided $\phi$ satisfies $\mathbf{G}(\phi)$ :

**Proposition 1** *(Frittelli [2], Corollary 1). Let $\phi$ satisfy $\mathbf{G}(\phi)$ and $\lambda > 0$. Then $Q_\lambda \in M_\phi$ is optimal for the dual problem (4) if and only if $E_{Q_\lambda}[\phi'(\lambda \frac{dQ_\lambda}{dP})] \leq E_Q[\phi'(\lambda \frac{dQ_\lambda}{dP})]$, for all $Q \in M_\phi$.*

For $\lambda = 1$ this result was proved under slightly different assumptions by Ruschendorf [7] and Liese & Vajda [6]. If $\lambda = 1$ the weaker condition $v(\phi)$ (to be introduced in Section 3.3) may replace $\mathbf{G}(\phi)$ in the above proposition (see Liese & Vajda [6], Theorem 8.10).

(III) Frittelli [2] proposed a clear financial interpretation for the class $M_\Phi$. Indeed, fix $Q \in M$ and consider the problem:

$$U_Q(x) \triangleq \sup\{E_P[u(x+g)] \mid g \in L^1(Q),\ E_Q[g] \leq 0,\ u^-(x+g) \in L^1(P)\}.$$

This is precisely the utility maximization problem one would face if $Q$ is selected as pricing measure.

**Proposition 2** *(Frittelli [2], Proposition 4) If $Q \in M$ and $\Phi = -u^*$ satisfies $\mathbf{G}(\Phi)$ then*

$$Q \in M_\Phi \Leftrightarrow U_Q(x) < u(+\infty) \text{ for all } x \in \mathbb{R}.$$

More explicitly this means that pricing by $Q \in M_\Phi$ guarantees that the investor can't reach her maximum possible utility, $u(+\infty)$, starting with an arbitrarily low initial wealth $x$. If $\mathbf{G}(\Phi)$ is satisfied then $M_\Phi$ is the class of pricing measures which makes the model free of this types of *utility based arbitrage opportunities.*

(IV) Kabanov and Stricker showed (Corollary 1.4 [4]) that if the condition $\mathfrak{a}(\phi)$ of Section 3.2 below is satisfied then the set $M_\phi$ is dense in $M$ for the variation topology. It can be checked that this important result is also true under the condition $\mathbf{G}(\phi)$, which is weaker than $\mathfrak{a}(\phi)$ (see section 3.2).

## 2 The growth condition

We will adopt the following
**Assumption (1)** $\phi : (0, +\infty) \to \mathbb{R}$ is strictly convex and differentiable.

Set: $\phi(y) = +\infty$ on $(-\infty, 0)$, $\phi(0) = \lim_{y \downarrow 0} \phi(y)$ and $\phi(+\infty) = \lim_{y \uparrow +\infty} \phi(y)$. Consider the following other conditions:

$\mathbb{G}(\phi)$ : There exist $\gamma > 0$, $\delta > 0$ s.t. : $y \left| \phi'(y) \right| \leq \gamma \phi(y) + \delta(y+1)$, $\forall y > 0$.

$\mathbf{G}^{+}(\phi)$ : $\forall [\lambda_0, \lambda_1] \subseteq (0, +\infty)$ there exist $\alpha > 0$, $\beta > 0$ such that:

$$\phi^{+}(\lambda y) \leq \alpha \phi^{+}(y) + \beta(y+1), \ \forall y > 0, \ \forall \lambda \in [\lambda_0, \lambda_1].$$

$\mathbb{G}^{+}(\phi)$ : There exist $\gamma > 0$, $\delta > 0$ s.t. : $y \left| \phi'(y) \right| \leq \gamma \phi^{+}(y) + \delta(y+1)$, $\forall y > 0$.

**Proposition 3** *Suppose that $\phi$ satisfies assumption* (1)*. Then the conditions $\mathbf{G}(\phi)$, $\mathbf{G}^{+}(\phi)$, $\mathbb{G}(\phi)$, $\mathbb{G}^{+}(\phi)$ are equivalent. Each condition holds true for $\phi$ if and only if it holds true for the positive affine transformation $(a\phi + k)$ of the function $\phi$, where $a > 0$, $k \in \mathbb{R}$. In condition $\mathbf{G}(\phi)$ (resp. $\mathbb{G}(\phi)$) we may assume $\alpha > 1$ (resp. $\gamma > 1$).*

Consider also the following
**Assumption (1′)** $\phi'(0^{+}) = -\infty$ and $\phi'(+\infty) = +\infty$.
**Assumption (2)** $u : \mathbb{R} \to \mathbb{R}$ is strictly concave increasing and differentiable.
**Assumption (2′)** $u'(-\infty) = +\infty$ and $u'(+\infty) = 0$.
Recall that the concave conjugate $u^{*} : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ of a concave function $u : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ and the convex conjugate $\phi^{*} : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ of a convex function $\phi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ are defined, respectively, by $u^{*}(y) \triangleq \inf_{x \in \mathbb{R}} \{xy - u(x)\}$ and $\phi^{*}(x) \triangleq \sup_{y \in \mathbb{R}} \{xy - \phi(y)\}$. With the notation " $\phi \triangleq -u^{*}$ " we will denote the function $\phi : (0, +\infty) \to \mathbb{R}$ defined on the positive semi axes by $\phi(y) = -u^{*}(y)$, $\forall y > 0$. From standard arguments of convex analysis it can be easily shown that if $u$ satisfies assumptions (2) and (2′) then the function $\phi = -u^{*}$ satisfies assumptions (1) and (1′), $\phi' = -(u')^{-1}$ and

$$u(x) = -\phi^{*}(-x) = \phi(u'(x)) + xu'(x), \ \forall x \in \mathbb{R}. \tag{5}$$

**Definition 4** *(Schachermayer [8] - Definition 1.5) The function $u$ satisfying assumptions* (2) *and* (2′) *has* Reasonable Asymptotic Elasticity (RAE(u)) *if*

$$AE_{-\infty}(u) \triangleq \liminf_{x \to -\infty} \frac{xu'(x)}{u(x)} > 1 \text{ and } AE_{+\infty}(u) \triangleq \limsup_{x \to +\infty} \frac{xu'(x)}{u(x)} < 1.$$

**Proposition 5** *If $u$ satisfies assumptions* (2) *and* (2′) *and if $\phi \triangleq -u^{*}$, then the condition $RAE(u)$ is equivalent to any one of the equivalent growth conditions $\mathbf{G}(\phi)$, $\mathbf{G}^{+}(\phi)$, $\mathbb{G}(\phi)$, $\mathbb{G}^{+}(\phi)$.*

# 3 Comparison with other growth conditions

In this section $\phi$ satisfies assumption (1).

## 3.1 Conditions of Schachermayer

Consider the two conditions introduced by Schachermayer [8] in Corollary 4.2:
$a(\phi)$ : $\forall [\lambda_0, \lambda_1] \subseteq (0, +\infty)$ there exists $C > 0$ such that:

$$\phi(\lambda y) \leq C\phi(y), \ \forall y > 0, \ \forall \lambda \in [\lambda_0, \lambda_1].$$

$b(\phi)$ : There exists $K > 0$ s.t. : $y \left|\phi'(y)\right| \leq K\phi(y)$, $\forall y > 0$.
Obviously we have:
(1a) $a(\phi) \Rightarrow \mathbf{G}(\phi)$.
(1b) $b(\phi) \Rightarrow \mathbb{G}(\phi)$.

**Lemma 6** *(Corollary 4.2 Schachermayer [8]) If $u$ satisfies assumptions* (2), (2′), $u(0) > 0$ *and if* $\phi \triangleq -u^*$ *then:*
*(2a) $RAE(u) \Rightarrow a(\phi)$*
*(2b) $RAE(u) \Rightarrow b(\phi)$.*

As shown in the counterexamples $a(\phi)$ and $b(\phi)$ are not equivalent and, if $\phi \triangleq -u^*$, neither $a(\phi)$ nor $b(\phi)$ is equivalent to $RAE(u)$. The reason is that the conditions $\mathbf{G}(\phi)$, $\mathbf{G}^+(\phi)$, $\mathbb{G}(\phi)$, $\mathbb{G}^+(\phi)$ and $RAE(u)$ are invariant with respect to positive affine transformations, but the conditions $a(\phi)$ and $b(\phi)$ do not have this property. Suppose that $u$ satisfies assumptions (2) and (2′) and that $RAE(u)$ holds true. When $u(0) < 0$ then $\inf \phi < 0$ and this implies that $a(\phi)$ and $b(\phi)$ do not hold true.

**Counterexamples:**

- $\mathbb{G}(\phi) \nRightarrow b(\phi)$ and $\mathbf{G}(\phi) \nRightarrow a(\phi)$. Let $u_k(x) = k - e^{-x}$, $\phi_k = -(u_k)^*$, $\phi_k(y) = y \ln y - y + k$. Then, for all $k \in \mathbb{R}$, $\phi_k$ satisfies assumptions (1) and (1′), $RAE(u_k)$, $\mathbf{G}(\phi_k)$ and $\mathbb{G}(\phi_k)$ are satisfied, but $b(\phi_k)$ and $a(\phi_k)$ aren't satisfied for $k < 1$.

- $a(\phi) \nRightarrow b(\phi)$. The function $\phi(y) \triangleq \frac{1}{y+1} - 2$ satisfies assumption (1) and $\mathbf{G}^+(\phi)$ (hence $\mathbf{G}(\phi)$, $\mathbb{G}(\phi)$ and $\mathbb{G}^+(\phi)$), but $b(\phi)$ isn't satisfied. Since $\frac{\phi(z)}{\phi(y)} \geq \frac{1}{2}$ for all $y \geq 0$ and $z \geq 0$, and since $\phi$ is negative and decreasing it follows that $\phi(\lambda y) \leq \phi(\lambda_0 y) \leq \frac{1}{2}\phi(y)$ for all $y > 0$ and all $\lambda \in [\lambda_0, \lambda_1]$, and $a(\phi)$ is satisfied.

## 3.2 Conditions of Kabanov & Stricker

The following two conditions are considered in Kabanov & Stricker [4]:
$\mathfrak{a}(\phi)$ : There exist increasing positive functions $r_1$ and $r_2$ such that:

$$\phi^+(\lambda y) \leq r_1(\lambda)\phi^+(y) + r_2(\lambda)(y+1), \ \forall y > 0, \ \forall \lambda > 0.$$

$\mathfrak{b}(\phi)$ : There exists $c > 0$ s.t. : $y|\phi'(y)| \leq c(\phi^+(y) + 1)$, $\forall y \geq 0$.
Obviously we have: $\mathfrak{a}(\phi) \Rightarrow \mathbf{G}(\phi)$; $\mathfrak{b}(\phi) \Rightarrow \mathbb{G}(\phi)$.

But, as shown in the counterexamples, $\mathbf{G}(\phi)$ doesn't imply $\mathfrak{a}(\phi)$ and $\mathbb{G}(\phi)$ doesn't imply $\mathfrak{b}(\phi)$.

Indeed $\mathbf{G}(\phi)$ and $\mathbb{G}(\phi)$ are weaker conditions. Consider the following: $\mathfrak{a}(\phi)$ implies (taking $y = 1$ and $\lambda_0 > 0$) that, for all $\lambda$ satisfying $0 < \lambda \leq \lambda_0$, $\phi^+(\lambda) \leq r_1(\lambda)\phi^+(1) + r_2(\lambda)(2) \leq r_1(\lambda_0)\phi^+(1) + 2r_2(\lambda_0)$. Therefore if $\phi$ satisfies $\mathfrak{a}(\phi)$ then $\phi^+$ (and so $\phi$) must be bounded in a right neighborhood of the origin, which is not necessarily the case for functions satisfying $\mathbf{G}(\phi)$ or $\mathbb{G}(\phi)$. If $\phi = -u^*$ then $u(+\infty) = \phi(0)$. Condition $\mathbf{G}(\phi)$ may be satisfied by functions $\phi = -u^*$, where the utility $u$ is unbounded from above, so that $\phi(0) = u(+\infty) = +\infty$, while $\mathfrak{a}(\phi)$ excludes a priory this type of functions.

**Counterexamples**

- $\mathbb{G}(\phi) \nRightarrow \mathfrak{b}(\phi)$. The function $\phi(y) = 2(1 - \sqrt{y})$ if $y \in (0, 1)$, and $\phi(y) = \frac{1}{2}(-y + \frac{1}{y})$ if $y \in [1, +\infty)$ satisfies assumption (1) and condition $\mathbb{G}^+(\phi)$, but doesn't satisfy $\mathfrak{b}(\phi)$.
- $\mathbf{G}(\phi) \nRightarrow \mathfrak{a}(\phi)$. The function $\phi(y) = -\ln(y)$ if $y \in (0, 1)$, and $\phi(y) = y^2 - 3y + 2$ if $y \in [1, +\infty)$ satisfies assumption (1) and $\mathbb{G}^+(\phi)$, but doesn't satisfy $\mathfrak{a}(\phi)$.

### 3.3 Condition of Liese and Vajda

The following condition was introduced by Liese and Vajda [6] in the framework of the minimization of $f-$divergences over a set of probability measures:
$v(\phi)$ : For any $\lambda > 1$ there exist $y_0 > 0$ and $c_1, c_2, c_3 \in \mathbb{R}$ such that:

$$\phi(\lambda y) \leq c_1\phi(y) + c_2 y + c_3, \ \forall y > y_0.$$

Obviously we have: $\mathbf{G}(\phi) \Rightarrow v(\phi)$. But, as shown below, $v(\phi)$ is weaker than $G(\phi)$. Indeed, the condition $v(\phi)$ requires that $\phi(y)$ behaves well for large $y$ while it doesn't impose any constraint on $\phi(y)$ for small $y$.

**Counterexample:** $v(\phi) \nRightarrow \mathbf{G}(\phi)$. The function $\phi(y) = e^{\frac{1}{y}}$ if $y \in (0, 1)$, and $\phi(y) = y^2 - (2 + e)y + 2e + 1$ if $y \geq 1$ satisfies assumptions (1) and (1$'$) and condition $v(\phi)$, but does not satisfy $\mathbb{G}(\phi)$ (and hence nor $\mathbf{G}(\phi)$ ). Indeed, for any $\gamma > 0$ and $\delta > 0$ there exists $0 < y_1 < 1$ sufficiently small such that $y\left|\phi'(y)\right| = \frac{1}{y}e^{\frac{1}{y}} > \gamma\phi(y) + \delta(y + 1) = \gamma e^{\frac{1}{y}} + \delta(y + 1)$ for any $y \in (0, y_1)$.

# 4 Proofs

Let $k \in \mathbb{R}$ and let $u_k : \mathbb{R} \to \mathbb{R}$ and $\phi_k : (0, +\infty) \to \mathbb{R}$ be defined as $u_k(x) \triangleq u(x) + k$, $\forall x \in \mathbb{R}$ ; $\phi_k(y) \triangleq \phi(y) + k$, $\forall y > 0$. It is immediate to notice that $u$ (resp. $\phi$) satisfies assumption (2) (resp. (1)) if and only if $u_k$ (resp. $\phi_k$) satisfies assumption (2) (resp. (1)). Since $(u_k)^* = u^* - k$, if $\phi \triangleq -u^*$ we have: $\phi_k = \phi + k = -u^* + k = -(u_k)^*$.

**Lemma 7** *If $\phi$ satisfies assumption* (1) *then:*

*(3a)* $\mathbf{G}(\phi) \Leftrightarrow \mathbf{G}^+(\phi)$ *and $\alpha$ can be chosen to be the same in* $\mathbf{G}(\phi)$ *and* $\mathbf{G}^+(\phi)$.

*(3b)* $\mathbb{G}(\phi) \Leftrightarrow \mathbb{G}^+(\phi)$ *and $\gamma$ can be chosen to be the same in* $\mathbb{G}(\phi)$ *and* $\mathbb{G}^+(\phi)$.

*(3c) In condition* $\mathbf{G}(\phi)$ *or* $\mathbb{G}(\phi)$ *we may assume $\alpha > 1$ or $\gamma > 1$.*

*(4a)* $\mathbf{G}(\phi) \Rightarrow \mathbb{G}(\phi)$.

*(5a)* $\mathbf{G}(\phi) \Leftrightarrow \mathbf{G}(\phi_k)$ *for all $k \in \mathbb{R}$.*

*(5b)* $\mathbb{G}(\phi) \Leftrightarrow \mathbb{G}(\phi_k)$ *for all $k \in \mathbb{R}$.*

*If $u$ satisfies assumption* (2) *then*

*(5c)* $RAE(u) \Leftrightarrow RAE(u_k)$ *for all $k \in \mathbb{R}$.*

**Proof.** (3a) and (3b). We assume that $\inf \phi < 0$, otherwise $\mathbb{G}^+(\phi) \equiv \mathbb{G}(\phi)$ and $\mathbf{G}^+(\phi) \equiv \mathbf{G}(\phi)$.

Since $(a+b)^+ \leq a^+ + b^+$ and $a \leq a^+$ it follows easily that $\mathbf{G}(\phi) \Rightarrow \mathbf{G}^+(\phi)$ and $\mathbb{G}(\phi) \Rightarrow \mathbb{G}^+(\phi)$. To show the converse note that a differentiable strictly convex function $\phi : (0, +\infty) \to \mathbb{R}$ is either bounded from below or is unbounded from below and decreasing. In the latter case, we now show that there exist $m > 0$ and $q \geq 0$ such that $\phi(y) + my + q \geq 0$ for all $y > 0$. Indeed, recall from Kramkov & Schachermayer- Lemma 6.1 [5], that for a differentiable strictly concave increasing function $f : (0, +\infty) \to \mathbb{R}$ satisfying $f(+\infty) = +\infty$ we have: $AE_{+\infty}(f) \leq 1$. Hence, setting $f = -\phi$, we get $AE_{+\infty}(\phi) = AE_{+\infty}(f) \leq 1$. Therefore there exists $x_0 \in (0, +\infty)$ such that $\phi(x_0) < 0$ and $x_0 \phi'(x_0) \geq \phi(x_0)$. From convexity it follows that, for any $y > 0$, $0 \leq \phi(y) - \phi(x_0) - (y - x_0)\, \phi'(x_0) = \phi(y) + my + q$ where $m = -\phi'(x_0) > 0$ and $q = x_0 \phi'(x_0) - \phi(x_0) \geq 0$.

$\mathbf{G}^+(\phi) \Rightarrow \mathbf{G}(\phi)$. From $\mathbf{G}^+(\phi)$ we deduce that, given $[\lambda_0, \lambda_1] \subset (0, +\infty)$, there exist $\alpha > 0$ and $\beta > 0$ such that for all $\lambda \in [\lambda_0, \lambda_1]$ and for all $y > 0$, $\phi^+(\lambda y) \leq \alpha \phi^+(y) + \beta(y + 1)$.

In the case where $-\infty < \inf \phi < 0$, it is easy to check that $\mathbf{G}(\phi)$ holds true with the same $\alpha$ as in $\mathbf{G}^+(\phi)$ and a suitable $\beta$.

Let $\inf \phi = -\infty$. For all $y > 0$ and $\lambda \in [\lambda_0, \lambda_1]$ such that $\phi(\lambda y) \leq 0$, we have: $\phi(\lambda y) \leq 0 \leq \alpha(\phi(y) + my + q) \leq \alpha \phi(y) + \widehat{\beta}(y+1)$ where $\widehat{\beta} = \max(\alpha m, \alpha q) > 0$. For all $y > 0$ and $\lambda \in [\lambda_0, \lambda_1]$ such that $\phi(\lambda y) \geq 0$ and $\phi(y) \leq 0$, from $\mathbf{G}^+(\phi)$ we have: $\phi(\lambda y) \leq \beta(y+1) \leq \beta(y+1) + \alpha(\phi(y) + my + q) \leq \alpha \phi(y) + \beta^*(y+1)$, where $\beta^* = \max(\beta + \alpha m, \beta + \alpha q) > \beta$ and $\beta^* > \widehat{\beta}$. Hence for all $\lambda \in [\lambda_0, \lambda_1]$ and for all $y > 0$, $\phi(\lambda y) \leq \alpha \phi(y) + \beta^*(y+1)$.

With similar arguments, it can be shown that $\mathbb{G}^+(\phi) \Rightarrow \mathbb{G}(\phi)$ and that the coefficient $\gamma$ can be chosen to be the same in $\mathbb{G}^+(\phi)$ and in $\mathbb{G}(\phi)$.

(3c) Note that, from (3a), $\mathbf{G}(\phi)$ implies $\mathbf{G}^+(\phi)$ and $\mathbf{G}^+(\phi)$ clearly implies: $\forall [\lambda_0, \lambda_1] \subseteq (0, +\infty)$ there exist $\alpha > 1, \beta > 0$ such that $\phi^+(\lambda y) \leq \alpha \phi^+(y) + \beta(y+1)$. From (3a) it then follows that $\mathbf{G}(\phi)$ holds with the same coefficient $\alpha > 1$. Similarly for condition $\mathbb{G}(\phi)$.

(4a) Assume that $\mathbf{G}(\phi)$ holds. Taking $[\lambda_0, \lambda_1] \subseteq (1, +\infty)$ or $[\lambda_0, \lambda_1] \subseteq (0, 1)$, from $\mathbf{G}(\phi)$ it follows respectively:

i) $\exists \lambda_* > 1, \alpha_1 > 1, \beta_1 > 0$ such that: $\phi(\lambda_* y) \leq \alpha_1 \phi(y) + \beta_1(y+1)$, $\forall y > 0$;

ii) $\exists \lambda^* \in (0, 1), \alpha_2 > 1, \beta_2 > 0$ s. t.: $\phi(\lambda^* y) \leq \alpha_2 \phi(y) + \beta_2(y+1)$, $\forall y > 0$.

From the convexity of $\phi$ we get $\forall y>0$ and $\forall \lambda>0$

$$\phi'(y)(\lambda y-y)\leq \phi(\lambda y)-\phi(y). \tag{6}$$

Since $\lambda_* y-y>0$ (resp. $\lambda^* y-y<0$), from i) (resp. ii)) and (6) we get $\forall y>0$

$$\begin{aligned}\phi'(y) &\leq \frac{\phi(\lambda_* y)-\phi(y)}{(\lambda_*-1)\,y}\leq \frac{1}{y}\left\{\frac{\alpha_1-1}{\lambda_*-1}\phi(y)+\frac{\beta_1}{\lambda_*-1}(y+1)\right\}\\ -\phi'(y) &\leq \frac{1}{y}\left\{\frac{\alpha_2-1}{1-\lambda^*}\phi(y)+\frac{\beta_2}{1-\lambda^*}(y+1)\right\}.\end{aligned}$$

Taking $\gamma=\max\left\{\frac{\alpha_1-1}{\lambda_*-1},\frac{\alpha_2-1}{1-\lambda^*}\right\}>0$ and $\delta=\max\left\{\frac{\beta_1}{\lambda_*-1},\frac{\beta_2}{1-\lambda^*}\right\}>0$ we deduce $\mathbb{G}(\phi)$.

(5a) We first show that if $\mathbf{G}(\phi_k)$ holds for some $k\in\mathbb{R}$ then also $\mathbf{G}(\phi)$ holds. We assume that $\forall\,[\lambda_0,\lambda_1]\subseteq(0,+\infty)$ there exist $\alpha>1$ and $\beta>0$ such that: $\phi_k(\lambda y)\leq\alpha\phi_k(y)+\beta(y+1)$ for all $y>0$ and for all $\lambda\in[\lambda_0,\lambda_1]$. Then: $\forall y>0$, $\forall\lambda\in[\lambda_0,\lambda_1]$ we get $\phi(\lambda y)\leq\alpha\phi(y)+\beta(y+1)+k(\alpha-1)\leq\alpha\phi(y)+\beta^*(y+1)$, where $\beta^*=\beta+k^+(\alpha-1)\geq\beta>0$.

Assume now that $\mathbf{G}(\phi)$ holds and let $k\in\mathbb{R}$. Let $\widetilde{\phi}=\phi_k$. Since $\phi=\widetilde{\phi}_{-k}$ we have that $\mathbf{G}(\widetilde{\phi}_{-k})$ holds by assumption. For what just proved then also $\mathbf{G}(\widetilde{\phi})$ holds, i.e. $\mathbf{G}(\phi_k)$ holds.

(5b) We skip proof of (5b), as analogous to that of (5a).

(5c) Note that $\frac{xu_k'(x)}{u_k(x)}=\frac{xu'(x)}{u(x)}\frac{u(x)}{u(x)+k}$. Since $u(-\infty)=-\infty$, $AE_{-\infty}(u_k)=AE_{-\infty}(u)$ for all $k\in\mathbb{R}$.

If $u(+\infty)=+\infty$ then $AE_{+\infty}(u_k)=AE_{+\infty}(u)$ for all $k\in\mathbb{R}$.

Let $u(+\infty)=0$. Then $u(x)<0$ for all $x\in\mathbb{R}$ and $\frac{xu'(x)}{u(x)}<0$ for all $x>0$. Therefore $AE_{+\infty}(u)=\limsup_{x\to+\infty}\frac{xu'(x)}{u(x)}\leq 0$. Moreover, if $k<0$, then $\frac{xu_k'(x)}{u_k(x)}=\frac{xu'(x)}{u(x)+k}<0$ for all $x>0$ and so $AE_{+\infty}(u_k)\leq 0$. If $k\geq 0$, then $\frac{xu_k'(x)}{u_k(x)}\leq\frac{xu'(x)}{u(x)}$ for all $x>0$ and so $AE_{+\infty}(u_k)\leq AE_{+\infty}(u)\leq 0$.

Finally, let $u(+\infty)<+\infty$, and set $k_0=-u(+\infty)$. Define $\widetilde{u}=u_{k_0}$. Then $\widetilde{u}(+\infty)=0$ and from above we get $AE_{+\infty}(\widetilde{u}_k)\leq 0$ for all $k\in\mathbb{R}$. Since $u_k=\widetilde{u}_{k-k_0}$, we have $AE_{+\infty}(u_k)\leq 0$ for all $k\in\mathbb{R}$. ∎

**Lemma 8** *If $u$ satisfies assumptions* (2) *and* (2′), *and if $\phi\triangleq-u^*$ then:*
*(6a)* $\mathbf{G}(\phi)\Rightarrow RAE(u)$.
*(6b)* $\mathbb{G}(\phi)\Rightarrow RAE(u)$.

**Proof.** As stated in Section 2, $\phi\triangleq-u^*$ satisfies assumptions (1) and (1′) and so $\phi$ is bounded from below and attains the minimum at a point $y_0\in(0,+\infty)$. Set $x_0\triangleq-\phi'(y_0)=0$, so that $u'(x_0)=y_0$. We need only to prove (6b), since then (6a) follows from (4a) and (6b).

The condition $\mathbb{G}(\phi)$ implies that there exist $\gamma>0$ and $\delta>0$ such that:

i) $0 \leq y\phi'(y) \leq \gamma\phi(y) + \delta(y+1)$, $\forall y \geq y_0 > 0$;
ii) $0 \leq -y\phi'(y) \leq \gamma\phi(y) + \delta(y+1)$, $\forall 0 < y < y_0$.

We show that $AE_{-\infty}(u) > 1$.

For $x < x_0 = 0$ we have $u'(x) > u'(x_0) = y_0$ and we may apply i) to $y = u'(x)$ and deduce, from (5) and $\phi'(u'(x)) = -x$,

$$u(x) = \phi(u'(x)) + xu'(x) \geq xu'(x)\left[1 - \frac{1}{\gamma}\right] - \frac{\delta}{\gamma}(u'(x) + 1).$$

For $x < x_0 = 0$ "small" enough, both $u(x)$ and $xu'(x)$ are negative and therefore: $0 \leq \lim_{x\to-\infty} \frac{u(x)}{xu'(x)} \leq 1 - \frac{1}{\gamma} - \frac{\delta}{\gamma}\lim_{x\to-\infty}\left\{\frac{1}{x} + \frac{1}{xu'(x)}\right\} = 1 - \frac{1}{\gamma} < 1$, since $\lim_{x\to-\infty} u'(x) = +\infty$. Hence $AE_{-\infty}(u) = \liminf_{x\to-\infty} \frac{xu'(x)}{u(x)} > 1$.

We show now that $AE_{+\infty}(u) < 1$. Recall that if $u(+\infty) < +\infty$ then $AE_{+\infty}(u) \leq 0$ (see proof of item (5c)). So we need only to consider the case: $u(+\infty) = +\infty$. Reminding that $u'(+\infty) = 0$ and proceeding as above, it is easy to check that $AE_{+\infty}(u) < 1$. ■

**Lemma 9** *If $\phi$ satisfies assumption* (1) *then:*
*(4b)* $\mathbb{G}(\phi) \Rightarrow \mathbf{G}(\phi)$.

**Proof.** We consider the four cases :
i) $\phi'(+\infty) = +\infty$ and $\phi'(0^+) = -\infty$ (i.e. assumption ($1'$) for $\phi$)
ii) $\phi'(+\infty) = +\infty$ and $\phi'(0^+) > -\infty$ (hence also $\phi(0^+) < +\infty$)
iii) $\phi'(+\infty) < +\infty$ and $\phi(0^+) = +\infty$ (hence also $\phi'(0^+) = -\infty$)
iv) $\phi'(+\infty) < +\infty$ and $\phi(0^+) < +\infty$.

i) From boundedness from below of $\phi$ and by (5a) and (5b) we may assume that $\phi > 0$. It can be checked that $u(x) \triangleq \inf_{y>0}\{xy + \phi(y)\}$ satisfies assumptions (2), ($2'$), $\phi \triangleq -u^*$ and $u(0) > 0$. From (6b) it follows that $RAE(u)$ holds. From (2a) we deduce that $a(\phi)$ holds and (1a) then shows that $\mathbf{G}(\phi)$ holds.

Since proofs of ii) and iii) are similar, we will show only iii).

iii) Pasting "ad hoc" some different functions, it can be proved that there exist $y_0 > 0$ and a function $\hat{\phi} : (0, +\infty) \to \mathbb{R}$ such that $\hat{\phi}$ satisfies assumption (1), $\hat{\phi}'(+\infty) = +\infty$, $\hat{\phi}(y) = \phi(y)$ for all $0 < y \leq y_0$ and $\mathbb{G}\left(\hat{\phi}\right)$ holds true. Note that $\hat{\phi}'(0^+) = \phi'(0^+) = -\infty$. Therefore, we may apply i) to the function $\hat{\phi}$ and deduce that $\mathbf{G}(\hat{\phi})$ and hence $\mathbf{G}^+(\hat{\phi})$ holds true: $\forall [\lambda_0, \lambda_1] \subseteq (0, +\infty)$ there exist $\alpha > 0$, $\beta > 0$ such that:

$$\hat{\phi}^+(\lambda y) \leq \alpha\hat{\phi}^+(y) + \beta(y+1), \ \forall y > 0, \ \forall \lambda \in [\lambda_0, \lambda_1]. \tag{7}$$

Set $y_1 = \frac{y_0}{\lambda_1} > 0$. Then for all $0 < y < y_1$ and $\lambda \in [\lambda_0, \lambda_1]$ we have: $\lambda y < y_0$ and $\hat{\phi}(\lambda y) = \phi(\lambda y)$. Hence for all $y < \min\{y_1, y_0\} \triangleq y_2 > 0$ and $\lambda \in [\lambda_0, \lambda_1]$ we have: $\hat{\phi}^+(\lambda y) = \phi^+(\lambda y)$ and $\hat{\phi}^+(y) = \phi^+(y)$ Therefore from (7) we get

$$\phi^+(\lambda y) \leq \alpha\phi^+(y) + \beta(y+1), \ \forall 0 < y < y_2, \ \forall \lambda \in [\lambda_0, \lambda_1]. \tag{8}$$

Since $\phi'(+\infty) < +\infty$, there exists $C > 0$ such that $|\phi'(\lambda y)| \leq C$ for all $y \geq y_2$ and $\lambda \in [\lambda_0, \lambda_1]$. From the convexity of $\phi$ we get, $\forall y > 0$ and $\forall \lambda > 0$,

$$\begin{aligned} \phi(\lambda y) &\leq \phi(y) + \phi'(\lambda y)(\lambda - 1)y \\ \phi^+(\lambda y) &\leq \phi^+(y) + [\phi'(\lambda y)(\lambda - 1)]^+ y \leq \phi^+(y) + |(\phi')(\lambda y)||(\lambda - 1)|y \end{aligned}$$

Therefore

$$\phi^+(\lambda y) \leq \phi^+(y) + C \max\{|\lambda_0 - 1|, |\lambda_1 - 1|\} y, \ \forall y \geq y_2, \ \forall \lambda \in [\lambda_0, \lambda_1]. \quad (9)$$

From (8) and (9) we deduce the thesis.

iv) In this case it is not difficult to show that $\mathbf{G}(\phi)$ always holds true, even without assumption $\mathbb{G}(\phi)$. We leave this proof for exercise. ■

**Proof of Proposition 3.** It follows from (4a), (4b), (3a) and (3b). The last two sentences follow from (5a), (5b) and (3c). ■

**Proof of Proposition 5.** If $u(0) > 0$ the proposition follows from (6b), (2b), (1b) and Proposition 3. From (5c) we deduce the thesis. ■

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