



Collective arbitrage and the value of cooperation

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Abstract

We introduce the notions of *collective arbitrage* and of *collective superreplication* in a discrete-time setting where agents invest in their markets and are allowed to cooperate through exchanges. We accordingly establish versions of the fundamental theorem of asset pricing and of the pricing–hedging duality. A reduction of the price interval of the contingent claims can be obtained by applying the collective superreplication price.

Keywords Arbitrage · Superreplication · Fundamental theorem of asset pricing · Cooperation · Fairness · Segmented markets

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1 Introduction

The theory developed in this paper aims at expanding the classical arbitrage pricing theory to a setting where N agents invest in stochastic security markets and are allowed to cooperate through suitable exchanges. More precisely, we suppose that each agent is allowed to invest in a subset of the available assets $(X^1, \dots, X^J)$, for a given $J \in \mathbb{N}$, and in a common riskless asset. Note that we do not exclude that this subset coincides with the full set $(X^1, \dots, X^J)$. The novel notions of collective arbitrage and collective superreplication are based on the possibility that the N agents may additionally enter in a zero-sum risk exchange mechanism, where no money is injected or taken out of the overall system. Cooperation and the multidimensional aspect are the key features of collective arbitrage and collective superreplication. In this

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setting, agents not only may invest in their respective market, but may additionally cooperate to improve their positions by taking advantage of the risk exchanges. In the case of one single agent, the theory reduces to the classical one. As we mention in the Sect. 1.3, there is an extensive literature in recent years on variations around the concept of one-agent no arbitrage or no free lunch. Departing from this stream of literature, the main aim of this paper is to understand the consequences of cooperation between several agents in relation to arbitrage and superreplication.

The theory we are about to discuss offers valuable insights into the broader context of arbitrage theory, with particular relevance in the context of market segmentation. Market segmentation within financial institutions entails the division of the global market into specialised submarkets or trading desks, each concentrating on a particular financial product or asset class. This segmentation can be influenced by various factors, including (i) product specialisation, such as stocks, bonds, derivatives or commodities; (ii) geographic focus, targeting specific regions or countries; and (iii) risk tolerance, as different desks may specialise in high-risk or low-risk asset classes. For additional motivations regarding market segmentation, we refer also to Carassus [8]. The integration of market segmentation with a risk exchange mechanism introduces novel challenges to the no-arbitrage framework. While each individual desk must adhere to traditional no-arbitrage principles, the overall system must also preclude the emergence of collective arbitrage opportunities.

Before moving into the details of our new setup, we briefly summarise the classical one-agent situation. Let a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, P)$, with filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathcal{T}}$, $\mathcal{T} = \{1, \dots, T\}$, be given, and denote by $X = (X^1, \dots, X^J)$ the J adapted stochastic processes representing the (discounted) prices of J securities. The set of admissible trading strategies consists of predictable processes (see the precise definition after Example 2.2) and is denoted by $\mathcal{H}$, and we let K be the set of time- T stochastic integrals of $H \in \mathcal{H}$ with respect to X . The set K represents all the possible terminal time- T payoffs available in the market from admissible trading strategies and having zero initial cost.

An arbitrage opportunity is an admissible trading strategy $H \in \mathcal{H}$ having zero initial cost and producing a nonnegative final payoff $k \in K$ which is strictly positive with positive probability. Equivalently, we have no arbitrage in this setting if the only nonnegative element in K is P -a.s. equal to 0, that is, $K \cap L_+^0(\Omega, \mathcal{F}, P) = \{0\}$. We now present the main concepts that we introduce in this paper, without giving full details. The precise setup as well as rigorous assumptions and formulations of the results are stated in the subsequent sections.

1.1 Collective arbitrage

Since each agent $i = 1, \dots, N$ is allowed to invest only in a subset of the available assets $(X^1, \dots, X^J)$, we define, similarly to the notion of the set K , the market K_i of agent i , that is, the space of all the possible time- T payoffs that agent i can obtain by using admissible trading strategies in his/her allowed investments and having zero initial cost.

We consider the set of all zero-sum risk exchanges

$$\mathcal{Y}_0 = \left\{ Y \in (L^0(\Omega, \mathcal{F}, P))^N : \sum_{i=1}^N Y^i = 0 \text{ } P\text{-a.s.} \right\}. \quad (1.1)$$

We call $\mathcal{Y}$ the set of possible/allowed exchanges, which we assume to satisfy $\mathcal{Y} \subseteq \mathcal{Y}_0$ and $0 \in \mathcal{Y}$. We stress that even if the overall sum is P -a.s. equal to 0, each component Y^i of a vector $Y \in \mathcal{Y}$ is in general a random variable. If Y^i is positive on some event, agent i receives on that event from the collection of the other agents the amount Y^i of capital. If Y^i is negative on some event, agent i provides on that event to the collection of the other agents the amount $|Y^i|$ of capital. So $Y \in \mathcal{Y}$ represents the capital that the agents may exchange among themselves with the requirement that the overall amount distributed is equal to zero. To implement a risk exchange $Y \in \mathcal{Y}$, the agents agree at the initial time by a binding agreement to exchange at the terminal time T the capital amounts implied by Y .

A *collective arbitrage* (CA) consists of a vector $(k^1, \dots, k^N)$, where $k^i \in K_i$ for each i , and a vector $Y = (Y^1, \dots, Y^N) \in \mathcal{Y}$ such that

$$k^i + Y^i \geq 0 \quad P\text{-a.s. for all } i \in \{1, \dots, N\}$$

and

$$P[k^j + Y^j > 0] > 0 \quad \text{for at least one } j \in \{1, \dots, N\}.$$

One may immediately notice that if $N = 1$, then $Y \in \mathcal{Y}$ must be equal to 0 and thus a collective arbitrage reduces to a classical arbitrage. However, for $N \geq 2$, in a collective arbitrage, agents are entangled by the vector $Y \in \mathcal{Y}$ of exchanges; this additional possible cooperation may create a collective arbitrage even if there is no arbitrage for each single agent. Observe that even if some agents – in some states of the world – lend some amounts to other agents, in a collective arbitrage, the condition $k^i + Y^i \geq 0$ holds for all i (for all agents), i.e., no agent incurs any loss.

We study the implications of the assumption of no collective arbitrage with respect to the set $\mathcal{Y}$, which we denote in short by $\text{NCA}(\mathcal{Y})$, or by NCA if $\mathcal{Y}$ is not essential. We also write NA_i for the no-arbitrage condition (in the classical sense) for agent i in the market K_i , and NA for the no-arbitrage condition (in the classical sense) in the global market K , where it is possible to invest in all the assets. In Item 6) of Sect. 3.1, we show that when considering the possibility that one agent invests in the global market, NA and NCA are equivalent notions. Thus the difference between NA and NCA can be fully appreciated by considering segmented markets.

As demonstrated in Sect. 3.1, the implications

$$\text{NA} \quad \implies \quad \text{NCA}(\mathcal{Y}) \quad \implies \quad \text{NA}_i, i = 1, \dots, N$$

essentially always hold, but none of the converse implications hold true in general. In Sect. 6.2, we establish that

$$\text{NA} \quad \iff \quad \text{NCA}(\mathcal{Y}_0),$$

i.e., NA and $\text{NCA}(\mathcal{Y})$ are equivalent notions for the “largest” choice $\mathcal{Y} = \mathcal{Y}_0$. Conversely, the weakest condition $\text{NA}_i, \forall i$, is equivalent to $\text{NCA}(\mathcal{Y})$ for the “smallest” choice $\mathcal{Y} = \mathcal{Y}_0 \cap (L^0(\Omega, \mathcal{F}_0, P))^N$. The latter space actually consists of zero-sum deterministic vectors, when $\mathcal{F}_0$ is the trivial σ -algebra. However, for general sets $\mathcal{Y}$, the notions of $\text{NCA}(\mathcal{Y})$ give rise to new concepts.

1.1.1 Economic motivation of the assumption of $\text{NCA}(\mathcal{Y})$

Various conceivable situations are covered by the collective arbitrage framework; the decision to cooperate and engage in an exchange might be based for example on the policies of a mother institution forcing sub-units to cooperate (e.g. several trading desks of a financial company), or on the regulatory policies of a supervising authority forcing banks in a network to cross-subsidise, or on taxation, or on rational behaviour from the individual agents’ points of view (as explained below).

We stress that similarly to the case of classical arbitrage, efficient markets with risk exchanges should not allow the possibility of collective arbitrage, and one of the main objectives of our paper is to study and characterise markets with $\text{NCA}(\mathcal{Y})$. We can distinguish two cases of collective arbitrage, the second being the more relevant one.

1) There exists at least one agent who has an individual (classical) arbitrage opportunity in her market. Together with zero strategies of the other agents, this would also be a collective arbitrage, which should obviously be ruled out in an efficient market.

2) More interestingly, no agent has an individual arbitrage opportunity, but there exists a “true” collective arbitrage where agents can achieve arbitrages by using the exchange opportunities. In this situation, it is not possible that one agent gives away capital to the other agents in *every* scenario; but every agent truly profits from the cooperation in the sense that she receives some positive amount from the other agents in at least one scenario. The example below addresses this case. In any case, also these collective arbitrages would be exploited by the agents, leading to the assumption that efficient markets should not allow collective arbitrage possibilities, just as in the classical arbitrage case.

Example 1.1 In the framework of case 2) above, we show how a collective arbitrage can be implemented as a rational choice for each agent. In the simple setting described in Sect. 7.1 below, two agents can obtain in their respective (complete) markets at *zero initial cost* the terminal-time payoffs

$$k^1 = \begin{cases} -5 & \text{on } A, \\ +5 & \text{on } A^c, \end{cases} \quad k^2 = \begin{cases} +10 & \text{on } A, \\ -2 & \text{on } A^c, \end{cases}$$

for a measurable event A with $0 < P[A] < 1$. Notice that $E_{Q^1}[k^1] = E_{Q^2}[k^2] = 0$ for the unique equivalent martingale measure Q^i in the market K_i . Clearly there are monotone preferences (and values of $P[A]$) for which k^1 is preferable to k^2 , and others for which the opposite happens. Now consider the exchanges (Y^1, Y^2) , satisfying $Y^1 + Y^2 = 0$, given by

$$Y^1 = \begin{cases} +5 & \text{on } A, \\ -2 & \text{on } A^c, \end{cases} \quad Y^2 = \begin{cases} -5 & \text{on } A, \\ +2 & \text{on } A^c, \end{cases}$$

which generate the collective arbitrage

$$k^1 + Y^1 = \begin{cases} 0 & \text{on } A, \\ 3 & \text{on } A^c, \end{cases} \quad k^2 + Y^2 = \begin{cases} 5 & \text{on } A, \\ 0 & \text{on } A^c. \end{cases}$$

With this exchange, agent 2 gives a part (5\$) of his profit to agent 1 in state A and receives 2\$ on the complement of A . There are monotone preferences for which *both agents are better off after the exchanges than before*. Take for example preferences represented by expected utility, where the utility functions belong to the class $\mathbb{U} := \{u : \mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty\} : u \text{ is increasing on } \mathbb{R} \text{ and } u(x) = -\infty \text{ on } (-\infty, 0)\}$. Thus a collective arbitrage could be viewed by all agents as an advantageous opportunity. Our assumption of NCA rules out such possibilities.

To understand how a collective arbitrage is implemented, consider in the above example the two agents which agree at the initial time to exchange at the terminal time precisely the Y^1 and Y^2 above. Then each of the agents freely decides her investment strategy – based on her own personal information – which generates some final payoff at the terminal time. If the agents select the investment strategies having the final payoffs k^1 and k^2 above (and this would be a rational choice as explained before), this would determine a collective arbitrage. Our assumption of NCA implies that generating k^1 and k^2 and establishing such an agreement is not feasible. Any other agreement to exchange money which could give rise to a CA is similarly impossible.

One could ask why the exchange (Y^1, Y^2) is feasible, i.e., where does each agent i find the capital $|Y^i|$, for $Y^i \leq 0$, that he needs to *provide* to the other agents? And which is, for each agent, its cost? Observe that in a collective arbitrage, $k^i + Y^i \geq 0$ for each i . Thus $k^i \geq |Y^i| = -Y^i$ on the set $\{Y^i \leq 0\}$. Each exchange Y^i , on the set where it is negative, can be financed by agent i *at zero initial cost* by investing in his/her market K_i . In the example above, agent 2 can generate at least 5\$ in the event A (on A , the exchange Y^2 is negative so that agent 2 has to provide $|Y^2| = 5\$$ to agent 1), while agent 1 can generate at least 2\$ in the event A^c . Since each agent i knows that he/she can finance the exchange Y^i (when negative) at zero initial cost, it is reasonable for each agent to enter at time zero in the agreement (Y^1, Y^2) .

Remark 1.2 Classically, terminal payoffs with zero initial cost from dynamic admissible trading strategies are represented by time- T stochastic integrals in the primal market, that is, by elements $k \in K$. Often this setup is extended to admit also semi-static strategies, which lead to terminal payoffs of the form

$$k + \sum_{m=1}^M \alpha^m (\phi_T^m - \phi_0^m),$$

where $\alpha^m \in \mathbb{R}$, $(\phi_T^m)_{m=1, \dots, M}$ are the terminal payoffs of some options traded in the market and $(\phi_0^m)_{m=1, \dots, M} \in \mathbb{R}$ their initial known costs. The approach taken in the present paper is different. We do not *extend* the market, but rather we entangle several agents in the market by allowing them to exchange among themselves the amounts represented by $Y \in \mathcal{Y}$.

Because $\mathcal{Y} \subseteq \mathcal{Y}_0$, it is only the total amount $\sum_{i=1}^N Y^i$ exchanged by the agents that is required to be equal to 0. The novel characteristic of this approach is the multidimensional notion of a collective arbitrage, a feature absent in the literature on arbitrage theory. We shall see in Sect. 4.3 that an endogenous vector $(\hat{Q}^1, \dots, \hat{Q}^N)$ of pricing measures arises a posteriori from the pricing–hedging duality and assigns zero price to each component of the optimal exchanges $\hat{Y}$, i.e., $E_{\hat{Q}^i}[\hat{Y}^i] = 0$ for each i .

1.1.2 On the collective FTAP

We analyse the conditions under which a new type of fundamental theorem of asset pricing (FTAP) holds, that we label collective FTAP (CFTAP). Differently from the classical version, the CFTAP depends of course on the properties of the set $\mathcal{Y}$ of exchanges, and so we provide in Sects. 3 and 6 several versions of such a theorem. On the technical side, in the classical case, the NA condition implies that the set $K - L_+^0(\Omega, \mathcal{F}, P)$ is closed in probability; see Delbaen and Schachermayer [11, Theorem 6.9.2]. This property is crucial to prove the FTAP and the dual representation of the superreplication price. Analogously, in our collective setting, we need to show the closedness in probability of the set $K^{\mathcal{Y}}$ defined in (3.5) below. We show this under some specific assumptions on the set $\mathcal{Y}$ and under the assumption of $\text{NCA}(\mathcal{Y})$. This technical part of the paper, in which some mathematical techniques are developed and others are adapted from the classical theory, is deferred to Sect. 6. The key novel feature in the CFTAP is that equivalent martingale measures have to be replaced by vectors $(Q^1, \dots, Q^N)$ of equivalent martingale measures, one for each agent and her corresponding market, fulfilling in addition the polarity property

$$\sum_{i=1}^N E_{Q^i}[Y^i] \leq 0 \quad \text{for all integrable } Y \in \mathcal{Y}; \quad (1.2)$$

see Theorems 3.12, 6.11 and 6.12 for the precise formulations. The findings of this paper take particularly tractable, yet informative and meaningful, forms in a finite probability space setup (see Sect. 5 and the examples in Sect. 7 in the very simple settings of a one- and a two-period model). The fact that the agents are allowed to cooperate and the assumption of $\text{NCA}(\mathcal{Y})$ has several consequences also for the pricing of contingent claims. This is particularly evident for the collective superreplication of N contingent claims (see Example 7.2.3), a notion which turns out to be well defined under $\text{NCA}(\mathcal{Y})$.

1.2 Collective superreplication

In Sect. 4, we consider the problem of N agents each superreplicating a contingent claim g^i , $i = 1, \dots, N$, which is an $\mathcal{F}$ -measurable random variable. We set $g = (g^1, \dots, g^N)$. As an immediate extension of the classical superreplication price, we first introduce the overall superreplication price

$$\rho_+^N(g) := \inf \left\{ \sum_{i=1}^N m^i : \exists k_i \in K_i, m \in \mathbb{R}^N \text{ with } m^i + k^i \geq g^i, \forall i \right\}.$$

If we use the notation $\rho_{i,+}(g^i)$ for the classical superreplication of the single claim g^i , we may easily recognise that

$$\rho_+^N(g) = \sum_{i=1}^N \rho_{i,+}(g^i).$$

In the spirit of systemic risk measures with random allocations in Biagini et al. [4], we introduce the collective superreplication of the N claims $g = (g^1, \dots, g^N)$ as

$$\rho_+^{\mathcal{Y}}(g) := \inf \left\{ \sum_{i=1}^N m^i : \exists k_i \in K_i, m \in \mathbb{R}^N, Y \in \mathcal{Y} \text{ with } m^i + k^i + Y^i \geq g^i, \forall i \right\},$$

and show that under $\text{NCA}(\mathcal{Y})$, the definition is well posed (see Lemma 4.4). The functionals $\rho_+^{\mathcal{Y}}(g)$ and $\rho_+^N(g)$ both represent the minimal total amount needed to superreplicate simultaneously all claims $(g^1, \dots, g^N)$. For the collective superreplication price $\rho_+^{\mathcal{Y}}(g)$, we allow an additional exchange among the agents as described by $\mathcal{Y}$. As $0 \in \mathcal{Y}$, we clearly have $\rho_+^{\mathcal{Y}} \leq \rho_+^N$. Thus collective superreplication is less expensive than classical superreplication: cooperation helps to reduce the cost of superreplication. The amount $(\rho_+^N(g) - \rho_+^{\mathcal{Y}}(g)) \geq 0$ is the *value of cooperation* with respect to g . To illustrate the advantage of adopting the collective superreplication price, consider the following situation. Suppose that each desk i (investing in a given market i) of a large bank is asked to superreplicate a claim g^i and to price that operation. Given this task, the bank charges to each of the clients the amount $\rho_{i,+}(g^i) := \inf\{m \in \mathbb{R} : \exists k^i \in K_i \text{ such that } m + k^i \geq g^i\}$ for each claim g^i , resulting in charging a total price $\sum_{i=1}^N \rho_{i,+}(g^i) = \rho_+^N(g)$ to superreplicate all the claims $(g^1, \dots, g^N) = g$. However, the N desks could cooperate via the exchanges $Y \in \mathcal{Y}$, and the *effective* total cost for the bank to superreplicate all the claims is only $\rho_+^{\mathcal{Y}}(g) \leq \rho_+^N(g)$. The positive difference $\rho_+^N(g) - \rho_+^{\mathcal{Y}}(g)$ is the *gain* obtained by the bank by selling g at the price $\rho_+^N(g)$ and spending only $\rho_+^{\mathcal{Y}}(g)$ to superreplicate it. See Example 7.2.3 for a simple example where we have $\rho_+^{\mathcal{Y}}(g) < \rho_+^N(g)$. Collective superreplication can also be considered as a theoretical tool to show how much the system of the N agents can lose without cooperation.

Under the $\text{NCA}(\mathcal{Y})$ assumption and using the closedness of the set $K^{\mathcal{Y}}$ defined in (3.5) below, we prove in Sect. 4.2 the version of the pricing–hedging duality

$$\rho_+^{\mathcal{Y}}(g) = \sup_{Q \in \mathcal{M}^{\mathcal{Y}}} \sum_{i=1}^N E_{Q^i}[g^i], \quad (1.3)$$

where $\mathcal{M}^{\mathcal{Y}}$ is the set of vectors of martingale measures satisfying the polarity condition (1.2). In the dual formulation of $\rho_+^{\mathcal{Y}}(g)$ in (1.3), we thus consider the specific subset $\mathcal{M}^{\mathcal{Y}}$ of the set of (vectors of) martingale measures. Hence our approach fits in the general trend in the literature: adding constraints implies using a smaller set of martingale measures, which in turns implies a reduction of the no-arbitrage price interval of the contingent claims.

When problem (1.3) admits an optimum $\widehat{Q} = (\widehat{Q}^1, \dots, \widehat{Q}^N) \in \mathcal{M}^{\mathcal{Y}}$, which clearly depends on $\mathcal{Y}$, we derive in Sect. 4.3 the formula

$$\rho_+^{\mathcal{Y}}(g) = \sum_{i=1}^N \inf\{m \in \mathbb{R} : \exists k^i \in K_i, Y^i \in L^1(\Omega, \mathcal{F}_T^i, \widehat{Q}^i) \text{ with } E_{\widehat{Q}^i}[Y^i] = 0 \\ \text{such that } m + k^i + Y^i \geq g^i\}. \quad (1.4)$$

Note that in (1.4), $(Y^1, \dots, Y^N)$ is not required to belong to $\mathcal{Y}$, but every Y^i must have zero cost under each component of the endogenously determined pricing vector $\widehat{Q}$. Thus $\rho_+^{\mathcal{Y}}(g)$ is the sum of the *individual superreplication prices* of each claim g^i under the assumption that each agent is using the pricing functional assigned by $\widehat{Q}^i$, so that both k^i and Y^i have zero value under $\widehat{Q}^i$.

In the spirit of Biagini et al. [5] and Biagini et al. [3], this fairness aspect as well as the connection with Bühlmann's notion of a risk exchange equilibrium in Bühlmann [7] is discussed in Sect. 4.3.

1.3 Relation with classical arbitrage

The concept of arbitrage in financial markets finds its roots in the seminal work of de Finetti [10]; see Maggis [22] for a historical excursion on this topic. However, it was not until the mid-1970s, with the groundbreaking works of Black and Scholes [6], Merton [23] and Ross [26], that arbitrage pricing theory became a cornerstone of financial mathematics. The late 1970s witnessed the establishment of a mathematical foundation of this theory through the contributions of Kreps [21], Harrison and Kreps [16] and Harrison and Pliska [17].

A pivotal moment arrived with the highly influential paper by Dalang et al. [9]. This work presented the fundamental theorem of asset pricing in discrete finite time in its current, widely recognised form. Subsequently, Schachermayer [27], Rogers [25], Kabanov and Kramkov [19], Jacod and Shiryaev [18] each developed independent proofs of this theorem. A comprehensive overview of this topic in discrete time can be found in the book by Föllmer and Schied [14, Chaps. 1 and 5].

Over the past few decades, a vast literature has emerged surrounding variations of the one-agent no-arbitrage (or no-free-lunch) concept in more general settings and in particular in continuous time. A detailed survey of this subject can be found in the book by Delbaen and Schachermayer [11, Chaps. 1, 5 and 6] and references therein. It is important to mention also the significant research conducted in the past decade on robust versions of these concepts.

Our approach offers novel insights that complement and extend the existing literature. We now present a comparison with the very general notion of arbitrage developed by Kreps [21]. An arbitrage for [21] is a *single* non-zero element in the vector space of marketed bundles M , contained in a locally convex topological vector lattice $\mathcal{X}$, having nonpositive price and belonging to the positive cone $\mathcal{X}_+$ associated to the lattice structure of $\mathcal{X}$. Due to its generality, this definition can encompass a wide range of possibilities. However, to the best of our knowledge, both this definition and the one of a free lunch have traditionally been interpreted as a single element (within

the abstract space M) constituting an arbitrage. The novelty of our approach lies in the concept of a collective arbitrage, which possesses the intrinsic multidimensional feature of being a collection of “arbitrage opportunities” in the precise sense already defined. The fact that M has a product structure $M = K^1 \times \cdots \times K^N$ and that the positive cone $\mathcal{K}_+$ has the particular structure $(L_+^0 - \mathcal{Y})$, $\mathcal{Y}$ being the particular set of exchanges, extends the ansatz of Kreps to a new setting.

We emphasise that our results are not simply a consequence of existing methods, but necessitate a dedicated technical effort, as elaborated upon throughout the paper, in particular in Sect. 6.

The paper is organised as follows. In Sect. 2, we introduce the setting for the financial market model. The definition and properties of collective arbitrage are collected in Sect. 3.1, while various versions of the collective FTAP are proved in Sect. 3.2, under the assumption that the set $K^{\mathcal{Y}}$ is closed in probability. In Sect. 4, we develop the theory of collective superreplication, prove the pricing–hedging duality (Sect. 4.2) and show the fairness property (Sect. 4.3) embedded in the collective superreplication scheme. The case of a finite probability space is covered in Sect. 5. In Sect. 6, we prove under different assumptions that the set $K^{\mathcal{Y}}$ is closed in probability and thus prove several versions of the CFTAP. Most of the examples are collected in the final Sect. 7.

The theory presented here has been extended in several directions. Frittelli [15] establishes a connection between a collective free lunch and the fundamental theorem of asset pricing in a continuous-time setting. Furthermore, Doldi et al. [12] investigate the concepts of collective replicability and collective market completeness. Collective dynamic risk measures are examined in Doldi et al. [13]. Finally, ongoing research efforts are focused on further generalisations, including collective optimal transport and robust collective arbitrage.

2 The setting

Let $\mathcal{T} = \{0, \dots, T\}$ be a finite set of discrete times and consider a given filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathcal{T}}$ and $\mathcal{F} = \mathcal{F}_T$. Except when explicitly stated otherwise, we also assume that $\mathcal{F}_0$ is trivial, i.e. that $P[A] = 0$ or $P[A] = 1$ for all $A \in \mathcal{F}_0$. We say that a probability measure Q on $(\Omega, \mathcal{F})$ belongs to $\mathcal{P}_e$ if $Q \approx P$, or to $\mathcal{P}_{ac}$ if $Q \ll P$, respectively. Unless differently stated, all inequalities between random variables are meant to hold P -a.s..

Remark 2.1 In this paper, we choose to work in a discrete-time setting in order to convey our main messages about collective arbitrage, collective superreplication and the role of cooperation, focusing more on the conceptual aspects of the theory. Most of the concepts we introduce also fit well in a more general framework (as continuous time or robust finance). However, the techniques need to be carefully adapted. This is the subject of a paper in preparation.

Let us consider N agents and denote their set by $I = \{1, \dots, N\}$. We assume the existence of a zero-interest-rate riskless asset in which all agents can invest. Additionally, the (global) securities market consists of J assets, for a given integer $J \geq 1$, with

discounted price processes $X^j = (X_t^j)_{t \in \mathcal{T}}$, $j = 1, \dots, J$. We assume that agent i can invest in the riskless asset and in the assets X^j , $j \in (i)$, where (i) stands for a given subset of $\{1, \dots, J\}$. We set $d_i = \#(i)$. We assume without loss of generality that $\bigcup_i (i) = \{1, \dots, J\}$ as we may ignore assets that cannot be used by any agent.

Example 2.2 Let $N = 3$ and $J = 4$, $(1) = \{1, 2\}$, $(2) = \{2\}$, $(3) = \{1, 3, 4\}$. Thus in this example, agent 1 may invest in assets (X^1, X^2) , agent 2 in asset X^2 and agent 3 in assets (X^1, X^3, X^4) .

We denote by

$$\mathbb{F}^i = (\mathcal{F}_t^i)_{t \in \mathcal{T}} \subseteq \mathbb{F}$$

the filtration representing the information available to agent i . We assume that all processes X^j with $j \in (i)$ are adapted to the filtration $\mathbb{F}^i$.

A stochastic process $H = (H_t)_{t \in \mathcal{T}}$ is called an *admissible trading strategy for agent i* if it is d_i -dimensional and predictable with respect to $\mathbb{F}^i$. The space of admissible trading strategies for agent i is denoted by $\mathcal{H}^i$. If $H \in \mathcal{H}^i$, we set

$$(H \cdot X)_T := \sum_{h \in (i)} (H^h \cdot X^h)_T,$$

where $(H^h \cdot X^h)_T := \sum_{t=1}^T H_t^h (X_t^h - X_{t-1}^h)$ denotes the stochastic integral of H^h with respect to the asset X^h , $h \in (i)$, and we write

$$K_i := \{(H \cdot X)_T : H \in \mathcal{H}^i\} \subseteq L^0(\Omega, \mathcal{F}_T^i, P). \quad (2.1)$$

The set of martingale measures with bounded densities for the assets in (i) is defined by

$$M_i := \left\{ Q \in \mathcal{P}_{ac} : \frac{dQ}{dP} \in L^\infty(\Omega, \mathcal{F}_T^i, P) \text{ and } X^j \text{ is a } (Q, \mathbb{F}^i)\text{-martingale for all } j \in (i) \right\}. \quad (2.2)$$

The existence of an element in M_i which is also equivalent to P characterises the no-arbitrage condition for agent i (defined below), as recalled in Theorem 2.6.

Remark 2.3 By setting $\frac{d\hat{P}}{dP} := \frac{c}{1 + \sum_{j,t} |X_t^j|}$ for some positive normalising constant c , we see that $\hat{P}$ is a probability equivalent to P such that the price process X is integrable under $\hat{P}$. The notions of no arbitrage in Definition 2.5, of no collective arbitrage in Definition 3.1, of superreplication price in Definitions 4.1 and 4.6 depend on the underlying probability only through its nullsets, i.e., they are invariant with respect to an equivalent change of measure. Thus we directly assume that P satisfies

$$X_t^j \in L^1(\Omega, \mathcal{F}_t^i, P) \quad \text{for each } t \in \mathcal{T} \text{ and } j \in (i), i = 1, \dots, N.$$

We recall from [11, Sect. 6.11] or [14, Theorem 5.14] that M_i can also be written as

$$M_i = \left\{ Q \in \mathcal{P}_{ac} : \frac{dQ}{dP} \in L^\infty(\Omega, \mathcal{F}_T^i, P) \text{ and } E_Q[k] = 0, \forall k \in K_i \cap L^1(\Omega, \mathcal{F}_T^i, P) \right\}. \quad (2.3)$$

Notation 2.4 We set $L^0(\Omega, \mathbf{F}_T, P) = L^0(\Omega, \mathcal{F}_T^1, P) \times \cdots \times L^0(\Omega, \mathcal{F}_T^N, P)$ and similarly for $L^1(\Omega, \mathbf{F}_T, P)$ and $L^\infty(\Omega, \mathbf{F}_T, P)$. For a vector $\mathbf{P} := (P^1, \dots, P^N)$ of probability measures, we also set

$$L^0(\Omega, \mathbf{F}_T, \mathbf{P}) := L^0(\Omega, \mathcal{F}_T^1, P^1) \times \cdots \times L^0(\Omega, \mathcal{F}_T^N, P^N)$$

and similarly for $L^1(\Omega, \mathbf{F}_T, \mathbf{P})$ and $L^\infty(\Omega, \mathbf{F}_T, \mathbf{P})$.

2.1 Classical no arbitrage and superhedging duality

In the classical setting, we have the following definition and characterisation of absence of arbitrage for agent i and for the global market.

Definition 2.5 The *classical no-arbitrage condition for agent i* is

$$\text{NA}_i : K_i \cap L_+^0(\Omega, \mathcal{F}_T^i, P) = \{0\}.$$

The classical no-global-arbitrage condition is defined as follows. Let K be the set of time- T stochastic integrals, with $\mathbb{F}$ -predictable integrands $H \in \mathcal{H}$, with respect to all the assets $\{X^1, \dots, X^J\}$. Let M be the set of martingale measures with bounded densities for the whole market $\{X^1, \dots, X^J\}$ with respect to the filtration $\mathbb{F}$. We say that there is *no global arbitrage*, denoted with NA, if there is no classical arbitrage in the whole market $\{X^1, \dots, X^J\}$, i.e.,

$$\text{NA} : K \cap L_+^0(\Omega, \mathcal{F}_T, P) = \{0\}.$$

We can now state the celebrated fundamental theorem of asset pricing (FTAP) by Dalang, Morton and Willinger.

Theorem 2.6 In our discrete-time setting, we have

$$\text{NA}_i \iff M_i \cap \mathcal{P}_e \neq \emptyset, \quad \text{and} \quad \text{NA} \iff M \cap \mathcal{P}_e \neq \emptyset.$$

Definition 2.7 We define the *classical superreplication price for agent i* and for a random variable $f \in L^0(\Omega, \mathcal{F}_T^i, P)$ by

$$\rho_{i,+}(f) := \inf\{m \in \mathbb{R} : \exists k \in K_i \text{ such that } m + k \geq f\}.$$

The following superhedging duality holds as in the classical case; so we omit its proof. Under NA_i , we have

$$\rho_{i,+}(f) = \sup_{Q \in M_i} E_Q[f] \quad \text{for all } f \in L^1(\Omega, \mathcal{F}_T^i, P).$$

3 Collective arbitrage

3.1 Definitions and properties

After recalling the classical setup and definitions, we now introduce some novel concepts in the multi-agent setting of Sect. 2.

Our notions of collective arbitrage and collective superreplication are based on the possibility that the N agents may enter in risk exchange mechanisms in a possibly scenario-dependent way. We model such exchanges as (possibly random) N -dimensional vectors Y belonging to a set

$$\mathcal{Y} \subseteq L^0(\Omega, \mathbf{F}_T, P) \quad \text{with } 0 \in \mathcal{Y},$$

using Notation 2.4 and assuming in particular that Y^i is $\mathcal{F}_T^i$ -measurable for every component i of such vectors. These risk exchanges are accounted at the terminal time T and measured in cash units. We also assume $0 \in \mathcal{Y}$, allowing the agents to not enter in any exchange at all. In general, to talk about risk exchange, we consider zero-sum exchanges with $\sum_{i=1}^N Y^i = 0$ P -a.s.. However, this requirement is not needed for some technical results; so we explicitly make this assumption later.

Each agent i follows an investment strategy $H_i \in \mathcal{H}^i$ in his own market (i) , with zero initial cost and terminal value $k^i = (H_i \cdot X)_T = \sum_{h \in (i)} (H_i^h \cdot X^h)_T \in K_i$. The agents also enter in a risk exchange corresponding to a vector $Y \in \mathcal{Y}$. This procedure leads to the terminal-time value $(H_i \cdot X)_T + Y^i$ for agent i . A *collective arbitrage* is given by a collection of strategies $(H_i)_{i \in I}$ and a risk exchange $Y \in \mathcal{Y}$ such that each component $(H_i \cdot X)_T + Y^i$ is nonnegative and there exists at least one $j \in \{1, \dots, N\}$ for which $P[(H_j \cdot X)_T + Y^j > 0] > 0$. Now we can introduce the formal definition.

Definition 3.1 We say that *no collective arbitrage* for $\mathcal{Y}$ ($\text{NCA}(\mathcal{Y})$) holds if

$$\left(\bigtimes_{i=1}^N K_i + \mathcal{Y} \right) \cap L_+^0(\Omega, \mathbf{F}_T, P) = \{0\}, \quad (3.1)$$

where $\bigtimes_{i=1}^N K_i$ denotes the Cartesian product of the sets K_i defined in (2.1).

By selecting the zero vector in $\bigtimes_{i=1}^N K_i$, we infer from (3.1) that necessarily the condition $\mathcal{Y} \cap L_+^0(\Omega, \mathbf{F}_T, P) = \{0\}$ must hold, which rules out the possibility of collective arbitrage through elements in $\mathcal{Y}$ only.

Proposition 3.2 *The following conditions are equivalent:*

$$\text{NCA}(\mathcal{Y}), \quad (3.2)$$

$$K^{\mathcal{Y}} \cap L_+^0(\Omega, \mathbf{F}_T, P) = \{0\}, \quad (3.3)$$

$$C^{\mathcal{Y}} \cap L_+^1(\Omega, \mathbf{F}_T, P) = \{0\}, \quad (3.4)$$

where

$$\begin{aligned} K^{\mathcal{Y}} &:= \bigtimes_{i=1}^N (K_i - L_+^0(\Omega, \mathcal{F}_T^i, P)) + \mathcal{Y}, \\ C^{\mathcal{Y}} &:= K^{\mathcal{Y}} \cap L^1(\Omega, \mathbf{F}_T, P). \end{aligned} \quad (3.5)$$

Proof “(3.2) $\Rightarrow$ (3.3)”: Take $(k^i - \ell^i + Y^i)_i \in K^{\mathcal{Y}}$ with $k^i - \ell^i + Y^i \geq 0$ for all i . Then $k^i + Y^i \geq \ell^i \geq 0$ for all i , and (3.2) yields $k^i + Y^i = 0$ for all i , thus $\ell^i = 0$. Hence we have $k^i - \ell^i + Y^i = 0$ for all i .

“(3.3) $\Rightarrow$ (3.4)”: Follows from $C^{\mathcal{Y}} \subseteq K^{\mathcal{Y}}$ and $L_+^1(\Omega, \mathbf{F}_T, P) \subseteq L_+^0(\Omega, \mathbf{F}_T, P)$.

“(3.4) $\Rightarrow$ (3.2)”: Take $k \in \bigtimes_{i=1}^N K_i$ and $Y \in \mathcal{Y}$ and suppose $f^i := k^i + Y^i \geq 0$ for all i . Set $f_m^i := \min\{f^i, m\}$ for some $m \in \mathbb{N}$. Then we have $0 \leq f_m^i \leq m$ and $f_m^i \in L_+^1(\Omega, \mathcal{F}_T^i, P)$, and we can write $f_m^i = f^i - (f^i - f_m^i) = k^i + Y^i - \ell^i$, where $\ell^i := f^i - f_m^i$ is in $L_+^0(\Omega, \mathcal{F}_T^i, P)$. Thus

$$(f_m^1, \dots, f_m^N) \in \left(\bigtimes_{i=1}^N (K_i - L_+^0(\Omega, \mathcal{F}_T^i, P)) + \mathcal{Y} \right) \cap L^1(\Omega, \mathbf{F}_T, P) = C^{\mathcal{Y}}.$$

Condition (3.4) then implies $f_m^i = 0$ for all i . Thus $k^i + Y^i = 0$ for all i . $\square$

Define

$$\mathcal{Y}_0 := \left\{ Y \in L^0(\Omega, \mathbf{F}_T, P) : \sum_{i=1}^N Y^i = 0 \right\}. \quad (3.6)$$

When $\mathcal{F}_T^i = \mathcal{F}_T$ for all i , then $L^0(\Omega, \mathbf{F}_T, P) = (L^0(\Omega, \mathcal{F}_T, P))^N$ so that (1.1) and (3.6) coincide. If $\mathcal{Y} \subseteq \mathcal{Y}_0$, the exchange is a zero-sum game; in other words, the exchange is self-financed in the sense that no money is injected or taken out of the system. We next collect some simple considerations regarding $\text{NCA}(\mathcal{Y})$ and no arbitrage.

1) When $\mathcal{Y} = \mathcal{Y}_0$ and $N = 1$, $\text{NCA}(\mathcal{Y})$ coincides with the classical notion of NA_1 , namely no classical arbitrage for the market (1) of the single agent in Definition 2.5.

2) When $\mathcal{Y} = \{0\}$, $N = 1$, $\mathbb{F}^1 = \mathbb{F}$ and we consider one single market consisting of all assets $\{X^1, \dots, X^J\}$, then $\text{NCA}(\mathcal{Y})$ reads $K \cap L_+^0(\Omega, \mathcal{F}_T, P) = \{0\}$, i.e., it coincides with the classical notion of NA for the whole market $\{X^1, \dots, X^J\}$.

3) From (3.1), we see that in general, for any set $\mathcal{Y} \subseteq L^0(\Omega, \mathbf{F}_T, P)$ with $0 \in \mathcal{Y}$,

$$\text{NCA}(\mathcal{Y}) \implies \text{NA}_i, \forall i. \quad (3.7)$$

More precisely, taking $Y = 0$ in (3.1), we get

$$\begin{aligned} \text{NCA}(\mathcal{Y}) &\implies K_i \cap L_+^0(\Omega, \mathcal{F}_T^i, P) = \{0\}, \forall i \\ &\iff \text{NA}_i, \forall i \\ &\iff M_i \cap \mathcal{P}_e \neq \emptyset, \forall i. \end{aligned}$$

4) When $\mathcal{Y} \subseteq \{Y \in L^0(\Omega, \mathbf{F}_T, P) : \sum_{i=1}^N Y^i \leq 0\}$, then

$$\text{NA} \implies \text{NCA}(\mathcal{Y}), \quad (3.8)$$

where NA denotes classical no arbitrage in the global market K . Indeed, we prove the implication “CA $\Rightarrow$ A”. Suppose there exist vectors $k \in \times_{i=1}^N K_i$ and $Y \in \mathcal{Y}$ such that each component $k^i + Y^i$ is nonnegative and at least one $j \in \{1, \dots, N\}$ for which $P[k^j + Y^j > 0] > 0$. Then summing up all the components shows that $\sum_{i=1}^N k^i$ is $\mathcal{F}_T$ -measurable, $\sum_{i=1}^N k^i \geq -\sum_{i=1}^N Y^i \geq 0$ and $P[\sum_{i=1}^N k^i > 0] > 0$. Then $\sum_{i=1}^N k^i \in \sum_{i=1}^N K_i$ is an arbitrage in the whole market.

5) The converse implication in (3.7) is false in general, as shown in Examples 7.1.2 and 7.2.2. Also, the converse implication in (3.8) does not hold in general as shown in Examples 7.1.1, 7.1.3, 7.1.4, 7.2.1 and 7.2.3. The features of these examples are presented in Sect. 7, Table 1. In Sects. 6.1 and 6.2, we provide sufficient conditions ensuring that these converse implications do hold true. However, the interesting cases are (i) when there exists a classical arbitrage in the global market, but $\text{NCA}(\mathcal{Y})$ holds true (i.e., the converse of (3.8) does *not* hold); (ii) when an arbitrage cannot be realised by each single agent, say due to constraints on the investments of the agents, but a collective arbitrage can be realised through the exchanges $\mathcal{Y}$ (i.e., the converse of (3.7) does *not* hold).

6) Suppose that $0 \in \mathcal{Y} \subseteq \{Y \in L^0(\Omega, \mathbf{F}_T, P) : \sum_{i=1}^N Y^i \leq 0\}$ and that the market of some agent j coincides with the global market K . Then $\text{NA} \Leftrightarrow \text{NA}_j$ so that from (3.8) and (3.7), we deduce that $\text{NA} \Leftrightarrow \text{NCA}(\mathcal{Y})$.

3.2 Collective fundamental theorem of asset pricing (FTAP)

In the classical case, the NA_i condition implies that the set $K_i - L_+^0(\Omega, \mathcal{F}_T^i, P)$ is closed in $L^0(\Omega, \mathcal{F}_T^i, P)$; see Delbaen and Schachermayer [11, Theorem 6.9.2]. This property is crucial for proving the FTAP and the dual representation of the superreplication price. Analogously, in our collective setting, we would like to show the closedness of the set $K^{\mathcal{Y}}$ appearing in the definition of $\text{NCA}(\mathcal{Y})$ in (3.3). Some closedness properties of the set of semistatic trading strategies have been recently investigated in Nutz et al. [24]. However, an example of non-closedness is given in Acciaio et al. [1]; so clearly we expect this to be a delicate point. But in the present theory, developed in Sect. 6, we have two additional facts that lead to such a closedness: some specific assumptions on the set $\mathcal{Y}$ and the assumption of $\text{NCA}(\mathcal{Y})$.

We now consider the polarity with respect to the dual system

$$(L^1(\Omega, \mathbf{F}_T, P), L^\infty(\Omega, \mathbf{F}_T, P)).$$

Define for $C^{\mathcal{Y}}$ introduced in (3.5) the sets

$$\begin{aligned}(C^{\mathcal{Y}})^0 &:= \left\{ z \in L^\infty(\Omega, \mathbf{F}_T, P) : \sum_{i=1}^N E[z^i f^i] \leq 0, \forall f \in C^{\mathcal{Y}} \right\}, \\ (C^{\mathcal{Y}})^{00} &:= \left\{ f \in L^1(\Omega, \mathbf{F}_T, P) : \sum_{i=1}^N E[z^i f^i] \leq 0, \forall z \in (C^{\mathcal{Y}})^0 \right\}, \\ (C^{\mathcal{Y}})_1^0 &:= \left\{ (Q^1, \dots, Q^N) \in (\mathcal{P}_{ac})^N : \left(\frac{dQ^1}{dP}, \dots, \frac{dQ^N}{dP} \right) \in (C^{\mathcal{Y}})^0 \right\}.\end{aligned}\quad (3.9)$$

Remark 3.3 Note that $(C^{\mathcal{Y}})^0 \subseteq L_+^\infty(\Omega, \mathbf{F}_T, P)$ since using $Y = 0 \in \mathcal{Y}$ and $k = 0 \in \times_{i=1}^N K_i$ in the definition of $K^{\mathcal{Y}}$, we have $-L_+^1(\Omega, \mathbf{F}_T, P) \subseteq C^{\mathcal{Y}}$.

We now define the vector space

$$\mathbb{R}_0^N := \left\{ x \in \mathbb{R}^N : \sum_{i=1}^N x^i = 0 \right\}$$

and observe that it can be equivalently expressed as:

$$\mathbb{R}_0^N = \text{span}\{(e^i - e^j) : i, j \in \{1, \dots, N\}\}, \quad (3.10)$$

where $(e^i)_i$ is the canonical basis of $\mathbb{R}^N$. We often require that

$$\mathcal{Y} \subseteq L^0(\Omega, \mathbf{F}_T, P) \text{ is a convex cone which contains } \mathbb{R}_0^N, \quad (3.11)$$

which also implies $\mathcal{Y} + \mathbb{R}_0^N = \mathcal{Y}$. Under this condition, every agent is allowed to exchange with any other agent any constant (i.e., deterministic) amount as long as the total amount exchanged by all agents is equal to 0. Equivalently, each pair of agents is allowed to exchange between them any given deterministic amount. Even if condition (3.11) is not necessary for the development of our theory, this assumption simplifies the presentation of our results.

Remark 3.4 If (3.11) is fulfilled, then for all $z \in (C^{\mathcal{Y}})^0$, we have $E[z^i] = E[z^j]$ for all i, j . Indeed, taking $0 \in \times_{i=1}^N (K_i - L_+^0(\Omega, \mathcal{F}_T^i, P))$ gives from the definition (3.9) of the polar set $(C^{\mathcal{Y}})^0$ that $\sum_{i=1}^N E[z^i Y^i] \leq 0$ for all $Y \in \mathcal{Y} \cap L^1(\Omega, \mathbf{F}_T, P)$. By (3.10), we can take $Y = \lambda(e^i - e^j) \in \mathbb{R}_0^N$ for an arbitrary $\lambda \in \mathbb{R}$, which immediately yields the desired equalities. Thus when $z \in (C^{\mathcal{Y}})^0$, $z \neq 0$, and we consider the polarity condition $\sum_{i=1}^N E[z^i Y^i] \leq 0$ for all $Y \in \mathcal{Y} \cap L^1(\Omega, \mathbf{F}_T, P)$, it is possible to normalise it by dividing by the same amount $E[z^1] = \dots = E[z^N]$ to obtain that $\sum_{i=1}^N E_{Q^i}[Y^i] \leq 0$ for all $Y \in \mathcal{Y} \cap L^1(\Omega, \mathbf{F}_T, P)$, with $Q \in (C^{\mathcal{Y}})_1^0$.

Definition 3.5 Recall the notation (2.2) for martingale measures and introduce

$$\mathcal{M}^{\mathcal{Y}} = \left\{ (Q^1, \dots, Q^N) \in M_1 \times \dots \times M_N : \sum_{i=1}^N E_{Q^i}[Y^i] \leq 0 \right. \\ \left. \text{for all } Y \in \mathcal{Y} \cap L^1(\Omega, \mathbf{F}_T, P) \right\}.$$

In the definition of $\mathcal{M}^{\mathcal{Y}}$, the term $\sum_{i=1}^N E_{Q^i}[Y^i]$ is well posed and finite since each component Y^i is required to be integrable under P and each Q^i has a bounded density with respect to P . In the classical no-arbitrage theory in discrete time, the no-arbitrage condition is characterised by equivalent martingale measures (see e.g. Delbaen and Schachermayer [11, Theorem 6.1.1] or Föllmer and Schied [14, Chap. 5]). In continuous time, these have to be replaced by separating measures, i.e., equivalent local/ σ -martingale measures. As the framework in this paper is in discrete time, we directly work with martingale measures.

Example 3.6 We present a grouping example. Consider a partition $\alpha = (\alpha_1, \dots, \alpha_m)$ of $\{1, \dots, N\}$ into m groups α_h , $h = 1, \dots, m$, for $1 \leq m \leq N$, and define

$$\mathcal{Y}_\alpha := \left\{ Y \in L^0(\Omega, \mathbf{F}_T, P) : \sum_{i \in \alpha_h} Y^i = 0 \text{ for each } h = \{1, \dots, m\} \right\}. \quad (3.12)$$

Observe that for the partition $\{1, \dots, N\}$ consisting of a single group, the set in (3.12) coincides with the set in (3.6). We can attribute to $\mathcal{Y}_\alpha$ an interpretation similar to the one given for $\mathcal{Y}_0$ in (3.6), with the only difference that now the exchanges in each single group α_h must sum up to 0.

Remark 3.7 In some of the following results, we assume $\mathcal{Y} \subseteq L^1(\Omega, \mathbf{F}_T, P)$. This is the case for example when $\mathcal{Y} \subseteq (L^0(\Omega, \mathcal{G}_T, P))^N$ for a σ -algebra $\mathcal{G}_T$ that is *finitely generated* and contained in $\mathcal{F}_T^i$ for all i . This is also the case developed in Sect. 5 where the σ -algebras in $\mathbb{F}$ are generated by a finite number of atoms of Ω .

The following result gives conditions so that $\mathcal{M}^{\mathcal{Y}}$ is the normalised polar cone of $C^{\mathcal{Y}}$.

Lemma 3.8 We have $(C^{\mathcal{Y}})_1^0 \subseteq \mathcal{M}^{\mathcal{Y}}$. If $\mathcal{Y} \subseteq L^1(\Omega, \mathbf{F}_T, P)$, then

$$(C^{\mathcal{Y}})_1^0 = \mathcal{M}^{\mathcal{Y}}.$$

Proof For the first inclusion, take $(Q^1, \dots, Q^N) \in (C^{\mathcal{Y}})_1^0$ and define the vector of densities $z := (\frac{dQ^1}{dP}, \dots, \frac{dQ^N}{dP}) \in (C^{\mathcal{Y}})^0$. Then

$$\sum_{i=1}^N E[z^i f^i] \leq 0 \quad \text{for all } f \in C^{\mathcal{Y}}. \quad (3.13)$$

From (3.13), by taking the zero element in $\times_{i=1}^N (K_i - L_+^0(\Omega, \mathcal{F}_T^i, P))$, we immediately get $\sum_{i=1}^N E[z^i Y^i] \leq 0$ for all $Y \in \mathcal{Y} \cap L^1(\Omega, \mathbf{F}_T, P)$. Furthermore, as $0 \in \mathcal{Y}$ and $0 \in K^i$ for every $i = 1, \dots, N$, we also get

$$E[z^i k^i] \leq 0 \quad \text{for all } k^i \in K_i \cap L^1(\Omega, \mathcal{F}_T^i, P), \quad \text{for all } i. \quad (3.14)$$

Since K_i is a vector space, we deduce from (3.14) that $E[z^i k^i] = 0$ for every $k^i \in K_i \cap L^1(\Omega, \mathcal{F}_T^i, P)$ and all i . Thus by (2.3), $Q^i \in M_i$ for all i . This proves that $(Q^1, \dots, Q^N)$ belongs to $\mathcal{M}^{\mathcal{Y}}$.

To prove $\mathcal{M}^{\mathcal{Y}} \subseteq (C^{\mathcal{Y}})_1^0$ under the additional assumption, consider an element $(Q^1, \dots, Q^N) \in \mathcal{M}^{\mathcal{Y}}$ and let $f \in C^{\mathcal{Y}}$. Then $f^i = k^i - \ell^i + Y^i \in L^1(\Omega, \mathcal{F}_T^i, P)$. We claim that $E_{Q^i}[k^i - \ell^i + Y^i] \leq E_{Q^i}[Y^i]$ for all i . This will imply the result since then $\sum_{i=1}^N E_{Q^i}[f^i] \leq \sum_{i=1}^N E_{Q^i}[Y^i] \leq 0$ as $\mathcal{Y} \subseteq L^1(\Omega, \mathbf{F}_T, P)$. To show the claim, observe that $k^i - \ell^i + Y^i \in L^1(\Omega, \mathcal{F}_T^i, P)$ and $\ell^i \geq 0$ imply $(k^i + Y^i)^- \in L^1(\Omega, \mathcal{F}_T^i, P)$. Then $k^i = (k^i + Y^i)^+ - (k^i + Y^i)^- - Y^i \geq -(k^i + Y^i)^- - Y^i$, so that

$$(k^i)^- \leq (-(k^i + Y^i)^- - Y^i)^- \in L^1(\Omega, \mathcal{F}_T^i, P)$$

and so $(k^i)^- \in L^1(\Omega, \mathcal{F}_T^i, Q^i)$. Now k^i is the time- T value of a stochastic integral, and [14, Theorem 5.14] then implies that $k^i \in L^1(\Omega, \mathcal{F}_T^i, Q^i)$ and $E_{Q^i}[k^i] = 0$. But from $k^i - \ell^i + Y^i \in L^1(\Omega, \mathcal{F}_T^i, Q^i)$, we conclude $\ell^i \in L^1(\Omega, \mathcal{F}_T^i, Q^i)$. Finally, we can write $E_{Q^i}[k^i - \ell^i + Y^i] \leq E_{Q^i}[k^i] + E_{Q^i}[Y^i] = E_{Q^i}[Y^i]$. $\square$

Proposition 3.9 *Suppose $\mathcal{Y}$ is a convex cone and $K^{\mathcal{Y}}$ is closed in probability. Then $\text{NCA}(\mathcal{Y})$ implies the existence of $z \in (C^{\mathcal{Y}})_1^0$ with $z^i > 0$ for all i . If additionally $\mathcal{Y}$ fulfils condition (3.11), then $\text{NCA}(\mathcal{Y})$ implies $(C^{\mathcal{Y}})_1^0 \cap \mathcal{P}_e^N \neq \emptyset$ and $\mathcal{M}^{\mathcal{Y}} \cap \mathcal{P}_e^N \neq \emptyset$.*

Proof The proof is based on the multidimensional version of the Kreps–Yan theorem in Theorem A.3. Indeed, the convex cone $G := C^{\mathcal{Y}} = K^{\mathcal{Y}} \cap L^1(\Omega, \mathbf{F}_T, P)$ is closed in $L^1(\Omega, \mathbf{F}_T, P)$ and from (3.4), we see that the assumptions on G in Theorem A.3 are satisfied. Thus there exists $z \in L^\infty(\Omega, \mathbf{F}_T, P)$ satisfying $z^i > 0$ for all i and (3.13), i.e., $z \in (C^{\mathcal{Y}})_1^0$. By Remark 3.4, $E[z^i] = E[z^j]$ for all i, j , and introducing $Q^i \in \mathcal{P}_{ac}$ by $\frac{dQ^i}{dP} := \frac{z^i}{E[z^i]}$, we have $(Q^1, \dots, Q^N) \in (C^{\mathcal{Y}})_1^0$. Since $z^i > 0$ for all i , we also have $Q^i \in \mathcal{P}_e$ so that $(Q^1, \dots, Q^N) \in (C^{\mathcal{Y}})_1^0 \cap \mathcal{P}_e^N$, and therefore the latter set is not empty. The last claim follows from the inclusion $(C^{\mathcal{Y}})_1^0 \subseteq \mathcal{M}^{\mathcal{Y}}$ proved in Lemma 3.8. $\square$

Example 3.10 In Example 7.1.3, we show that if $\mathcal{Y}$ does not fulfil (3.11), it is possible that $\text{NCA}(\mathcal{Y})$ holds true, NA_i holds true for all i and there exists a positive element in $(C^{\mathcal{Y}})_1^0$, but $\mathcal{M}^{\mathcal{Y}} = (C^{\mathcal{Y}})_1^0 = \emptyset$.

Proposition 3.11 (i) *If there exists $z \in (C^{\mathcal{Y}})_1^0$ with $z^i > 0$ for all $i = 1, \dots, N$, then $\text{NCA}(\mathcal{Y})$ holds true. In particular, $(C^{\mathcal{Y}})_1^0 \cap \mathcal{P}_e^N \neq \emptyset$ implies $\text{NCA}(\mathcal{Y})$.*

(ii) *Suppose that $\mathcal{Y} \subseteq L^1(\Omega, \mathbf{F}_T, P)$. If $\mathcal{M}^{\mathcal{Y}} \cap \mathcal{P}_e^N \neq \emptyset$, then $\text{NCA}(\mathcal{Y})$ holds true.*

Proof (i) Observe that if $f \in C^{\mathcal{Y}}$, then $\sum_{i=1}^N E[z^i f^i] \leq 0$, and if $f^i \geq 0$ for all i , this then implies $\sum_{i=1}^N E[z^i f^i] = 0$ so that $E[z^i f^i] = 0$ for all i . Given that $z^i > 0$ for all i , then $f^i = 0$ for all i , and $C^{\mathcal{Y}} \cap L_+^1(\Omega, \mathbf{F}_T, P) = \{0\}$.

(ii) This is a direct consequence of (i) and of the equality $(C^{\mathcal{Y}})_1^0 = \mathcal{M}^{\mathcal{Y}}$ proved in Lemma 3.8 under the condition $\mathcal{Y} \subseteq L^1(\Omega, \mathbf{F}_T, P)$. $\square$

As a consequence of the previous two results, we obtain the collective FTAP.

Theorem 3.12 *Suppose that $\mathcal{Y}$ is a convex cone and that $K^{\mathcal{Y}}$, given in (3.5), is closed in probability. Then*

$$\text{NCA}(\mathcal{Y}) \iff \text{there exists } z \in (C^{\mathcal{Y}})^0 \text{ with } z^i > 0 \text{ for all } i.$$

If additionally $\mathcal{Y}$ contains $\mathbb{R}_0^N$, then

$$\text{NCA}(\mathcal{Y}) \iff (C^{\mathcal{Y}})_1^0 \cap \mathcal{P}_e^N \neq \emptyset,$$

and if we also assume that $\mathcal{Y} \subseteq L^1(\Omega, \mathbf{F}_T, P)$, then

$$\text{NCA}(\mathcal{Y}) \iff \mathcal{M}^{\mathcal{Y}} \cap \mathcal{P}_e^N \neq \emptyset.$$

In Sect. 6, we provide conditions on the convex cone $\mathcal{Y}$ and on the market model ensuring that the condition $\text{NCA}(\mathcal{Y})$ implies that the set $K^{\mathcal{Y}}$ is closed in probability.

4 Collective superreplication price

In this section, we work in the multiperiod setting outlined in Sect. 2.

4.1 Definitions and properties

We now add to the market N claims

$$(g^1, \dots, g^N) \in L^0(\Omega, \mathbf{F}_T, P)$$

and we are interested in the prices $(p^1, \dots, p^N) \in \mathbb{R}^N$ for the claims $(g^1, \dots, g^N)$ which do not allow collective arbitrage in this extended market.

For this, we introduce in the following definition two novel concepts, namely those of superreplication of all claims and superreplication of all claims with exchanges, and introduce a notation for superreplication within each market and in the full market, respectively. These concepts identify in turns two superreplication functionals.

Definition 4.1 Let $g = (g^1, \dots, g^N) \in L^0(\Omega, \mathbf{F}_T, P)$.

- *Superreplication of all N claims:*

$$\rho_+^N(g) := \inf \left\{ \sum_{i=1}^N m^i : \exists k \in \bigtimes_{i=1}^N K_i, m \in \mathbb{R}^N \text{ with } m^i + k^i \geq g^i, \forall i \right\}.$$

- *Superreplication of all claims with exchanges $\mathcal{Y}$:*

$$\rho_+^{\mathcal{Y}}(g) := \inf \left\{ \sum_{i=1}^N m^i : \exists k \in \bigtimes_{i=1}^N K_i, m \in \mathbb{R}^N, Y \in \mathcal{Y} \text{ with } m^i + k^i + Y^i \geq g^i, \forall i \right\}.$$

- *(Classical) superreplication within each market:*

$$\rho_{i,+}(g^i) := \inf \{ m^i : \exists k^i \in K_i, m^i \in \mathbb{R} \text{ with } m^i + k^i \geq g^i \}. \quad (4.1)$$

- *(Classical) superreplication for the full market K (Definition 2.5) of a contingent claim $G \in L^0(\Omega, \mathcal{F}, P)$:*

$$\rho_{\text{full},+}(G) := \inf \{ m : \exists k \in K, m \in \mathbb{R} \text{ with } m + k \geq G \}.$$

The functionals $\rho_+^{\mathcal{Y}}(g)$ and $\rho_+^N(g)$ both represent the minimal amount needed to superreplicate simultaneously all claims $(g^1, \dots, g^N)$. We stress that for the collective superreplication price $\rho_+^{\mathcal{Y}}(g)$, we allow an additional exchange among the agents as described by $\mathcal{Y}$. We say that the value $p \in \mathbb{R}$ is a *selling price* for the claims $(g^1, \dots, g^N) =: g$ if there exists a vector $(p^1, \dots, p^N) \in \mathbb{R}^N$, where p^i is a selling price for g^i , such that $\sum_{i=1}^N p^i = p$.

Proposition 4.2 *If $p \in \mathbb{R}$ is a selling price for g satisfying $p > \rho_+^{\mathcal{Y}}(g)$, then*

$$\begin{aligned} \exists k \in \bigtimes_{i=1}^N K_i, Y \in \mathcal{Y} \text{ with } k^i + Y^i + (p^i - g^i) \geq 0, \forall i, \\ \text{and } P[k^j + Y^j + (p^j - g^j) > 0] > 0 \text{ for some } j \end{aligned} \quad (4.2)$$

holds, i.e., there exists a collective arbitrage in the extended market. Thus $\rho_+^{\mathcal{Y}}(g)$ is the maximum selling price compatible with $\text{NCA}(\mathcal{Y})$ in the extended market.

Proof Suppose that $p > \rho_+^{\mathcal{Y}}(g)$. Then there exist $\widehat{m} \in \mathbb{R}^N$, $\widehat{Y} \in \mathcal{Y}$ and $\widehat{k} \in \bigtimes_{i=1}^N K_i$ with $\sum_{i=1}^N \widehat{m}^i < p$ and $\widehat{m}^i + \widehat{k}^i + \widehat{Y}^i \geq g^i$ for all i . Set $p^i := \widehat{m}^i + \frac{p - \sum_{i=1}^N \widehat{m}^i}{N} > \widehat{m}^i$ for all i . Then $\sum_{i=1}^N p^i = p$.

Suppose we sell g^i at prices p^i , so that p is a selling price for g , and we use the strategy given by $(\widehat{m}, \widehat{k}, \widehat{Y})$. Then for each i , the value at time $t = 0$ is $p^i - \widehat{m}^i > 0$, and its time- T value is $-g^i + \widehat{m}^i + \widehat{k}^i + \widehat{Y}^i \geq 0$. By investing $p^i - \widehat{m}^i$ at time 0 in the bond, we have constructed a strategy in the extended market that at the initial time has value 0 and at the final time T has the value

$$0 < (p^i - \widehat{m}^i) + (-g^i + \widehat{m}^i + \widehat{k}^i + \widehat{Y}^i) = (p^i - g^i) + \widehat{k}^i + \widehat{Y}^i$$

for all i , i.e., (4.2) is fulfilled. $\square$

Proposition 4.3 *We have*

$$\rho_+^N(g) = \sum_{i=1}^N \rho_{i,+}(g^i). \quad (4.3)$$

Proof (4.3) is easily verified if $\rho_{i,+}(g^i) < \infty$ for each i , i.e., if the sets over which infima are taken in defining $\rho_{i,+}(g^i)$ are nonempty. If instead $\rho_{i,+}(g^i) = +\infty$ for some i , then $\rho_+^N(g) = +\infty$ since the set over which the infimum is taken in defining $\rho_+^N(g)$ is empty as well, and so (4.3) is still satisfied under the convention $+\infty - \infty = +\infty$. $\square$

We now turn to some simple properties of $\rho_+^{\mathcal{Y}}$. In general, $\rho_+^{\mathcal{Y}}$ may take the values $\pm\infty$, and in Example 7.1.4, we provide an example where there exists a global arbitrage, $\text{NCA}(\mathcal{Y})$ holds true, $\mathcal{Y}$ does not fulfil (3.11) and $\rho_+^{\mathcal{Y}}(g) = -\infty$ for all g .

We already mentioned in Sect. 1.2 that cooperation helps to reduce the cost of superreplication, i.e., $\rho_+^{\mathcal{Y}} \leq \rho_+^N$. We thus define $(\rho_+^N(g) - \rho_+^{\mathcal{Y}}(g)) \geq 0$ as the (selling) *value of cooperation* with respect to g . As both ρ_+^N and $\rho_+^{\mathcal{Y}}$ are positively homogeneous, we may also define a quantitative characteristic of the whole market, which we call the *market cooperation value*, as

$$V^{\mathcal{Y}} := \sup_{g \in L^\infty(\Omega, \mathbf{F}_T, P): \|g\|_\infty \leq 1} (\rho_+^N(g) - \rho_+^{\mathcal{Y}}(g)).$$

Lemma 4.4 *Suppose that $\mathcal{Y}$ fulfils (3.11) and $\text{NCA}(\mathcal{Y})$ holds true. Then:*

- 1) $\rho_+^{\mathcal{Y}}(0) = 0$.
- 2) $\rho_+^{\mathcal{Y}}$ is cash additive, i.e., $\rho_+^{\mathcal{Y}}(g + c) = \rho_+^{\mathcal{Y}}(g) + \sum_{i=1}^N c^i$ for all $c \in \mathbb{R}^N$.
- 3) $\rho_+^{\mathcal{Y}}$ is increasing with respect to the partial order of $L^0(\Omega, \mathbf{F}_T, P)$.
- 4) $\rho_+^{\mathcal{Y}} : L^\infty(\Omega, \mathbf{F}_T, P) \rightarrow \mathbb{R}$ is Lipschitz-continuous.
- 5) $\rho_+^{\mathcal{Y}}$ is positively homogeneous and $V^{\mathcal{Y}}$ takes values in $[0, 2N]$.
- 6) $\rho_+^{\mathcal{Y}} = \rho_+^{\mathcal{Y} + \mathbb{R}_0^N}$.

Items 2), 3) and 6) hold even without assuming $\text{NCA}(\mathcal{Y})$ and condition (3.11).

Proof We only prove 1), as the remaining items are standard. By definition, $\rho_+^{\mathcal{Y}}(0) \leq 0$. Suppose by contradiction that $\rho_+^{\mathcal{Y}}(0) < 0$. Then by the definition of $\rho_+^{\mathcal{Y}}(0)$, there exist $m \in \mathbb{R}^N$, $k \in \times_{i=1}^N K_i$ and $Y \in \mathcal{Y}$ with $m^i + k^i + Y^i \geq 0$ for all i and $\sum_{i=1}^N m^i = -\varepsilon$ for some $\varepsilon > 0$. Hence $k^i + (Y^i + m^i + \frac{\varepsilon}{N}) \geq \frac{\varepsilon}{N}$. As the vector with components $(m^i + \frac{\varepsilon}{N})$ is in $\mathbb{R}_0^N$, we deduce that there exist $k \in \times_{i=1}^N K_i$ and $\hat{Y} \in \mathcal{Y}$ with $k^i + \hat{Y}^i \geq \frac{\varepsilon}{N}$ for all i . This is a collective arbitrage. $\square$

Under the specific assumptions on the market in Proposition 4.5 below, the collective superreplication price $\rho_+^{\mathcal{Y}_0}(g)$ of the N claims $(g^1, \dots, g^N) = g$ with the exchange $\mathcal{Y} = \mathcal{Y}_0$ is equal to the classical superreplication price of the single claim $\sum_{i=1}^N g^i$ for a representative agent who is allowed to invest in the whole market.

Proposition 4.5 *If $\mathbb{R}^i = \mathbb{R}^j$ for all i, j and $\mathcal{Y} = \mathcal{Y}_0$ as given in (3.6), then*

$$\rho_+^{\mathcal{Y}_0}(g) = \rho_{\text{full},+}(G) \quad \text{for } G = \sum_{i=1}^N g^i.$$

Proof The inequality $\rho_+^{\mathcal{Y}_0}(g) \geq \rho_{\text{full},+}(G)$ follows by observing that for any $m \in \mathbb{R}^N$ satisfying the constraints for $\rho_+^{\mathcal{Y}_0}(g)$, $\sum_{i=1}^N m^i$ satisfies the constraints appearing in $\rho_{\text{full},+}(G)$ simply by summation. For the converse, take G as given and $m \in \mathbb{R}$, $k \in K$ with $m + k \geq G$. Then $m + k + L = G$ for some $L \in L_+^0(\Omega, \mathcal{F}, P)$, and setting $\tilde{k}^i := \frac{1}{N}k$ (which still belong to K) and $\tilde{L}^i = \frac{1}{N}L$, we have that $\sum_{i=1}^N (\tilde{m}^i + \tilde{k}^i + \tilde{L}^i)$ equals $\sum_{i=1}^N g^i$ for any $\tilde{m} \in \mathbb{R}^N$ with $\sum_{i=1}^N \tilde{m}^i = m$. Set additionally $\tilde{Y} := -(\tilde{m} + \tilde{k} + \tilde{L}) + g$. Then $\tilde{Y} \in \mathcal{Y}_0$ by direct verification. Thus $\tilde{m} + \tilde{k} + \tilde{Y} \geq g$ (componentwise). Moreover, $\tilde{k} + \tilde{Y} \in (K)^N + \mathcal{Y}_0 = \times_{i=1}^N K_i + \mathcal{Y}_0$, where the last is due to Lemma A.2. We can conclude that $\tilde{k} + \tilde{Y} = k + Y$ for some $k \in \times_{i=1}^N K_i$ and $Y \in \mathcal{Y}_0$. So $\tilde{m} + k + Y \geq g$, and $\rho_+^{\mathcal{Y}_0}(g) \leq \rho_{\text{full},+}(G)$ follows. $\square$

Set

$$\mathcal{R}^{\mathcal{Y}}(g) := \left\{ m \in \mathbb{R}^N : \exists k \in \times_{i=1}^N K_i, Y \in \mathcal{Y} \text{ with } m^i + k^i + Y^i \geq g^i, \forall i \right\}$$

so that $\rho_+^{\mathcal{Y}}(g) = \inf\{\sum_{i=1}^N m^i : m \in \mathcal{R}^{\mathcal{Y}}(g)\}$. We say that $\rho_+^{\mathcal{Y}}(g)$ is *attained by* $\hat{m}$ if there exists $\hat{m} \in \mathcal{R}^{\mathcal{Y}}(g)$ with $\rho_+^{\mathcal{Y}}(g) = \sum_{i=1}^N \hat{m}^i$. In this case, we say that $\hat{m} \in \mathbb{R}^N$, $\hat{Y} \in \mathcal{Y}$ and $\hat{k} \in \times_{i=1}^N K_i$ are *optimisers of* $\rho_+^{\mathcal{Y}}(g)$ if $\hat{m}^i + \hat{k}^i + \hat{Y}^i \geq g^i$ for all i .

4.1.1 Superreplication of any one claim

Differently from the one-dimensional case, if one has many contingent claims, say $(g^1, \dots, g^N) \in L^0(\Omega, \mathbf{F}_T, P)$, to superreplicate, there are many possible definitions to consider. For example, in the following definitions, $\pi_+^N(g)$ is an immediate extension to the multidimensional setting of the classical one-dimensional notion of superreplication.

Definition 4.6 Let $g = (g^1, \dots, g^N) \in L^0(\Omega, \mathbf{F}_T, P)$. We set

$$\begin{aligned} \pi_+^N(g) &:= \inf \left\{ m \in \mathbb{R} : \exists k \in \times_{i=1}^N K_i \text{ with } m + k^i \geq g^i, \forall i \right\}, \\ \pi_+^{\mathcal{Y}}(g) &:= \inf \left\{ m \in \mathbb{R} : \exists k \in \times_{i=1}^N K_i, Y \in \mathcal{Y} \text{ with } m + k^i + Y^i \geq g^i, \forall i \right\}. \end{aligned}$$

Notice that $\pi_+^N(g)$ and $\pi_+^{\mathcal{Y}}(g)$ both represent the minimal amount needed to superreplicate *any one* claim among $g^1, \dots, g^N$, but they differ as no cooperation is allowed for π_+^N . Both $\pi_+^N(g)$ and $\pi_+^{\mathcal{Y}}(g)$ guarantee the possibility to superreplicate just one claim among $g^1, \dots, g^N$ (so not necessarily all the contingent claims $g^1, \dots, g^N$), no matter which one is chosen.

For example, consider a multinational corporation having several nationally based subsidiary companies with each one allowed to invest only in the national market and each one facing a risk g^i . The corporation could be interested in knowing the minimal amount that is required to *save any one* of the subsidiary companies. This would

amount to computing $\pi_+^N(g)$, or $\pi_+^{\mathcal{Y}}(g)$ if cooperation is allowed. One can think of a principal/head of a corporation who wants to make sure to have an amount m of money such that at the terminal time, she can cover one option of her choice in a basket $(g_1, \dots, g_N)$. Each agent i in the system invests in their own market K^i as if they could use the amount m to superreplicate g^i .

Remark 4.7 One might also define all intermediate possibilities of superreplication costs, namely the cost to replicate $n \leq N$ contingent claims out of $(g^1, \dots, g^N)$. But in this paper, we only discuss the two extreme cases represented by $\pi_+^{\mathcal{Y}}$ and $\rho_+^{\mathcal{Y}}$ (or π_+^N and ρ_+^N).

Proposition 4.8 Recall from (4.1) the definition of $\rho_{i,+}(f)$ for $f \in L^0(\Omega, \mathcal{F}_T^i, P)$. We have

$$\pi_+^N(g) = \max_i \rho_{i,+}(g^i). \quad (4.4)$$

Proof From the definitions, it immediately follows that $\pi_+^N(g) \leq \max_i \rho_{i,+}(g^i)$. By contradiction, suppose that $\pi_+^N(g) < \max_i \rho_{i,+}(g^i) = \rho_{j,+}(g^j)$ for some j . By the definition of $\pi_+^N(g)$, there exists $m < \rho_{j,+}(g^j)$ such that $m + k^i \geq g^i$ for all i . Thus m satisfies the constraints in $\rho_{j,+}(g^j)$ and thus $\rho_{j,+}(g^j) \leq m$. $\square$

We now show that under (3.11), the relation between $\rho_+^{\mathcal{Y}}(g)$ and $\pi_+^{\mathcal{Y}}$ is very simple. In the remaining part of the paper, we thus mainly focus our analysis on $\rho_+^{\mathcal{Y}}(g)$.

Proposition 4.9 Suppose that $\mathcal{Y}$ fulfils (3.11). Then

$$\rho_+^{\mathcal{Y}}(g) = N\pi_+^{\mathcal{Y}}(g). \quad (4.5)$$

Moreover, $m_* \in \mathbb{R}$, $Y_* \in \mathcal{Y}$ and $k_* \in \times_{i=1}^N K_i$ are optimisers of $\pi_+^{\mathcal{Y}}(g)$ if and only if $\hat{m} \in \mathbb{R}^N$, $\hat{Y} \in \mathcal{Y}$, $\hat{k} \in \times_{i=1}^N K_i$ are optimisers of $\rho_+^{\mathcal{Y}}(g)$ with the following conditions: $\hat{m}^i = Np^i$ for any $p \in \mathbb{R}^N$ satisfying $\sum_{i=1}^N p^i = \pi_+^{\mathcal{Y}}(g)$, $\hat{k} = k_*$, $\hat{Y}^i = Y_*^i + m_* - \hat{m}^i$.

Proof Let $m \in \mathbb{R}^N$ and set $\bar{m} := \sum_{i=1}^N m^i$; then the vector with components $m^i - \frac{\bar{m}}{N}$ is in $\mathbb{R}_0^N$. If $Y \in \mathcal{Y}$, then by (3.11), the vector $\hat{Y}$ with components $\hat{Y}^i = Y^i + m^i - \frac{\bar{m}}{N}$

is in $\mathcal{Y}$ and

$$\begin{aligned}
 \rho_+^{\mathcal{Y}}(g) &= \inf \left\{ \sum_{i=1}^N m^i : \exists k \in \bigtimes_{i=1}^N K_i, Y \in \mathcal{Y} \text{ with } m^i + k^i + Y^i \geq g^i, \forall i \right\} \\
 &= \inf \left\{ \sum_{i=1}^N m^i : \exists k \in \bigtimes_{i=1}^N K_i, Y \in \mathcal{Y} \text{ with} \right. \\
 &\quad \left. \frac{\bar{m}}{N} + k^i + \left(Y^i + m^i - \frac{\bar{m}}{N} \right) \geq g^i, \forall i \right\} \\
 &= \inf \left\{ \bar{m} : \exists k \in \bigtimes_{i=1}^N K_i, \hat{Y} \in \mathcal{Y} \text{ with } \frac{\bar{m}}{N} + k^i + \hat{Y}^i \geq g^i, \forall i \right\} \\
 &= N\pi_+^{\mathcal{Y}}(g).
 \end{aligned}$$

The remaining statements are obvious computations. $\square$

Remark 4.10 We showed in Proposition 4.9 that $\pi_+^{\mathcal{Y}} = \frac{1}{N}\rho_+^{\mathcal{Y}}$ which in particular illustrates the symmetrisation allowed by including $\mathbb{R}_0^N$ in $\mathcal{Y}$ in (3.11). Instead, from Proposition 4.8, we have $\pi_+^N(g) = \max_i \rho_{i,+}(g^i)$, and from (4.3), we see $\pi_+^N \geq \frac{1}{N}\rho_+^N$ so that π_+^N could be strictly greater than $\frac{1}{N}\rho_+^N$. In Example 7.1.1, we show that it is possible to profit from cooperation, i.e., $\pi_+^{\mathcal{Y}}(g) < \pi_+^N(g)$.

4.2 Pricing-hedging duality

Assumption 4.11 $\mathcal{Y} \subseteq L^0(\Omega, \mathbf{F}_T, P)$ is a convex cone, $\text{NCA}(\mathcal{Y})$ holds true and $K^{\mathcal{Y}}$ is closed in $L^0(\Omega, \mathbf{F}_T, P)$.

This assumption implies that $C^{\mathcal{Y}}$ is closed in $L^1(\Omega, \mathbf{F}_T, P)$. We stress that under appropriate conditions on $\mathcal{Y}$ and on the market model (see Sect. 6), the assumption that $K^{\mathcal{Y}}$ is closed in $L^0(\Omega, \mathbf{F}_T, P)$ is redundant.

As in the classical case and using the same argument as in Remark 2.3, we assume $(g^1, \dots, g^N) \in L^1(\Omega, \mathbf{F}_T, P)$ and recall that $X_t^j \in L^1(\Omega, \mathcal{F}_t^i, P)$ for each $t \in \mathcal{T}$, each i and $j \in (i)$. We can now state the pricing-hedging duality for $\pi_+^{\mathcal{Y}}$ and $\rho_+^{\mathcal{Y}}$.

Theorem 4.12 Suppose Assumption 4.11 holds true. For any $g \in L^1(\Omega, \mathbf{F}_T, P)$, we have

$$\pi_+^{\mathcal{Y}}(g) = \sup_{z \in (C^{\mathcal{Y}})^0 \setminus \{0\}} \frac{\sum_{i=1}^N E[z^i g^i]}{\sum_{i=1}^N E[z^i]}, \quad (4.6)$$

$$\rho_+^{\mathcal{Y}}(g) = \inf \left\{ \sum_{i=1}^N m^i : m \in \mathbb{R}^N, \sum_{i=1}^N m^i E[z^i] \geq \sum_{i=1}^N E[z^i g^i], \forall z \in (C^{\mathcal{Y}})^0 \right\}. \quad (4.7)$$

If additionally $\mathcal{Y}$ fulfils (3.11), then

$$\pi_+^{\mathcal{Y}}(g) = \sup_{Q \in (C^{\mathcal{Y}})_1^0} \frac{1}{N} \sum_{i=1}^N E_{Q^i}[g^i], \quad (4.8)$$

$$\rho_+^{\mathcal{Y}}(g) = \sup_{Q \in (C^{\mathcal{Y}})_1^0} \sum_{i=1}^N E_{Q^i}[g^i], \quad (4.9)$$

and if $\pi_+^{\mathcal{Y}}(g) < \infty$ (respectively $\rho_+^{\mathcal{Y}}(g) < \infty$), then $\pi_+^{\mathcal{Y}}(g)$ (respectively $\rho_+^{\mathcal{Y}}(g)$) is attained by some $m_* \in \mathbb{R}$ (respectively by some $\widehat{m} \in \mathbb{R}^N$). If we additionally have $\mathcal{Y} \subseteq L^1(\Omega, \mathbf{F}_T, P)$, then we can replace $(C^{\mathcal{Y}})_1^0$ with $\mathcal{M}^{\mathcal{Y}}$ in (4.8) and (4.9).

Proof First observe that by Proposition 3.9 and $\text{NCA}(\mathcal{Y})$, there exists a non-zero element in $(C^{\mathcal{Y}})^0$. Moreover, there exist $k^i \in K_i$ with $m + k^i + Y^i \geq g^i$ if and only if $g^i - m - Y^i \in (K_i - L_+^0(\Omega, \mathcal{F}_T^i, P))$. Thus taking the vector $\mathbf{1} \in \mathbb{R}^N$ with all components equal to 1, there exists $k \in \times_{i=1}^N K_i$ and $Y \in \mathcal{Y}$ with $m + k^i + Y^i \geq g^i$ for all i if and only if there exists $Y \in \mathcal{Y}$ with $g - m\mathbf{1} - Y \in \times_{i=1}^N (K_i - L_+^0(\Omega, \mathcal{F}_T^i, P))$, if and only if $g - m\mathbf{1} \in (\times_{i=1}^N (K_i - L_+^0(\Omega, \mathcal{F}_T^i, P)) + \mathcal{Y}) = K^{\mathcal{Y}}$. Since $g \in L^1(\Omega, \mathbf{F}_T, P)$, we obtain

$$\pi_+^{\mathcal{Y}}(g) = \inf\{m \in \mathbb{R} : g - m\mathbf{1} \in C^{\mathcal{Y}}\}.$$

We now apply a standard argument. Since the convex cone $K^{\mathcal{Y}}$ is closed in probability by Assumption 4.11, $C^{\mathcal{Y}} = K^{\mathcal{Y}} \cap L^1(\Omega, \mathbf{F}_T, P)$ is norm-closed in $L^1(\Omega, \mathbf{F}_T, P)$, and so $C^{\mathcal{Y}}$ is a convex cone which is closed in the topology $\sigma(L^1(\Omega, \mathbf{F}_T, P), L^\infty(\Omega, \mathbf{F}_T, P))$. The bipolar theorem implies that $(C^{\mathcal{Y}})^{00} = C^{\mathcal{Y}}$. More specifically, we use for example Aliprantis and Border [2, Theorem 5.103.2, in conjunction with footnote 8 on page 215, for the notation and characterisation of polars and bipolars of cones]. Here, we use the Köthe one-sided polar or one-sided polar in [2]. Thus we get

$$\begin{aligned} \pi_+^{\mathcal{Y}}(g) &= \inf\{m \in \mathbb{R} : g - m\mathbf{1} \in C^{\mathcal{Y}}\} \\ &= \inf\left\{m \in \mathbb{R} : \sum_{i=1}^N E[z^i(g^i - m)] \leq 0, \forall z \in (C^{\mathcal{Y}})^0\right\} \\ &= \inf\left\{m \in \mathbb{R} : m \geq \frac{\sum_{i=1}^N E[z^i g^i]}{\sum_{i=1}^N E[z^i]}, \forall z \in (C^{\mathcal{Y}})^0 \setminus \{0\}\right\} \\ &= \sup_{z \in (C^{\mathcal{Y}})^0 \setminus \{0\}} \frac{\sum_{i=1}^N E[z^i g^i]}{\sum_{i=1}^N E[z^i]}, \end{aligned}$$

which shows (4.6). Using a similar argument, one can directly prove (4.7). If $\mathcal{Y}$ fulfils (3.11), then $E[z^i] = E[z^j]$ for all i, j and all $z \in (C^{\mathcal{Y}})^0$. Then using

$\frac{dQ^i}{dP} := \frac{z^i}{E[z^i]}$, (4.8) follows from (4.6) and (4.9) follows from (4.8) and (4.5) (alternatively, (4.9) follows directly from (4.7)). Thanks to Proposition 4.9, we have to prove the existence of the optimiser only for $\pi_+^{\mathcal{Y}}(g)$. Define m_* as the right-hand side of (4.8). By the assumption and the condition NCA($\mathcal{Y}$), we have $(C^{\mathcal{Y}})_1^0 \neq \emptyset$ and $m_* \in \mathbb{R}$. Then by the above argument, $g - m_* \mathbf{1} \in C^{\mathcal{Y}}$, thus $g - m_* \mathbf{1} = k_* - \ell_* + Y_*$ for some appropriate choices $k_* \in \times_{i=1}^N K_i$, $\ell_* \in L_+^0(\Omega, \mathbf{F}_T, P)$ and $Y_* \in \mathcal{Y}$. Thus $m_* + k_*^i + Y_*^i \geq g^i$ for all i , showing that m_* , k_* and Y_* are optimisers for $\pi_+^{\mathcal{Y}}(g)$. The last assertion follows from the fact that $(C^{\mathcal{Y}})_1^0 = \mathcal{M}^{\mathcal{Y}}$, as proved in Lemma 3.8. $\square$

In Example 7.1.4, $(C^{\mathcal{Y}})_1^0 = \emptyset$, and so we then use the representation in (4.7) to compute $\rho_+^{\mathcal{Y}}(g)$.

4.3 Collective and individual superreplication costs

We let $Q = (Q^1, \dots, Q^N)$, $Q^i \in \mathcal{P}_{ac}$, and recall the notation 2.4 for $L^1(\Omega, \mathbf{F}_T, \mathbf{Q})$. We set

$$\begin{aligned} A_{Q^i} &:= \{Y^i \in L^1(\Omega, \mathcal{F}_T^i, Q^i) : E_{Q^i}[Y^i] = 0\}, \\ A_Q &:= A_{Q^1} \times \dots \times A_{Q^N}, \\ \rho_{A_Q}(g) &:= \inf \left\{ \sum_{i=1}^N m^i : \exists k \in \times_{i=1}^N K_i, Y \in A_Q \text{ with } m^i + k^i + Y^i \geq g^i, \forall i \right\}, \\ \rho_{Q^i}(g^i) &:= \inf \{m \in \mathbb{R} : \exists k^i \in K_i, Y^i \in A_{Q^i} \text{ with } m + k^i + Y^i \geq g^i\} \\ &= \inf \{E_{Q^i}[Y^i] : Y^i \in L^1(\Omega, \mathcal{F}_T^i, Q^i) \text{ such that} \\ &\quad \exists k^i \in K_i \text{ with } k^i + Y^i \geq g^i\}. \end{aligned} \quad (4.10)$$

Equation (4.10) is obtained by removing the constant m and incorporating it in the exchange variables and by replacing $m + Y^i$ with Y^i .

Observe that in (4.10), the Y^i are no longer constrained to have zero expectation. For a fixed agent i and a fixed probability Q^i , $\rho_{Q^i}(g^i)$ is thus the smallest *cost* needed to superreplicate the claim g^i , *independently* from any other agent, i.e., the *individual superreplication cost*. Observe that the infimum is taken with respect to all possible $Y^i \in L^1(\Omega, \mathcal{F}_T^i, Q^i)$ and without any reference to other agents.

Under Assumption 4.11 and if $\rho_+^{\mathcal{Y}}(g) < +\infty$, we know from Theorem 4.12 that the pricing–hedging duality holds true and $\rho_+^{\mathcal{Y}}(g)$ is attained, so that there exists an optimiser $(\widehat{m}, \widehat{k}, \widehat{Y}) \in \mathbb{R}^N \times \times_{i=1}^N K_i \times \mathcal{Y}$ of $\rho_+^{\mathcal{Y}}(g)$.

Proposition 4.13 *Suppose Assumption 4.11 holds and $\mathcal{Y}$ fulfils (3.11). Take any vector $g = (g^1, \dots, g^N) \in L^1(\Omega, \mathbf{F}_T, P)$. Suppose that $\rho_+^{\mathcal{Y}}(g) < +\infty$, that there exists*

$\widehat{Q} \in M^{\mathcal{Y}}$ satisfying

$$\rho_+^{\mathcal{Y}}(g) = \sup_{Q \in M^{\mathcal{Y}}} \sum_{i=1}^N E_{Q^i}[g^i] = \sum_{i=1}^N E_{\widehat{Q}^i}[g^i],$$

and there exists an optimiser $(\widehat{m}, \widehat{k}, \widehat{Y})$ of $\rho_+^{\mathcal{Y}}(g)$ with $\widehat{Y} \in \mathcal{Y} \cap L^1(\Omega, \mathbf{F}_T, \widehat{\mathbf{Q}})$. Then:

1) Any optimiser $(\widehat{m}, \widehat{Y}) \in \mathbb{R}^N \times \mathcal{Y} \cap L^1(\Omega, \mathbf{F}_T, \widehat{\mathbf{Q}})$ of $\rho_+^{\mathcal{Y}}(g)$ satisfies

$$E_{\widehat{Q}^i}[g^i] - \widehat{m}^i = E_{\widehat{Q}^i}[\widehat{Y}^i] \quad \text{for all } i, \quad (4.11)$$

$$\sum_{i=1}^N E_{\widehat{Q}^i}[\widehat{Y}^i] = 0. \quad (4.12)$$

2) Among the optimisers of $\rho_+^{\mathcal{Y}}(g)$, we can always find one optimiser $(\tilde{m}, \tilde{Y})$ with $\tilde{m} \in \mathbb{R}^N$ and $\tilde{Y} \in \mathcal{Y} \cap L^1(\Omega, \mathbf{F}_T, \widehat{\mathbf{Q}})$ such that

$$E_{\widehat{Q}^i}[g^i] - \tilde{m}^i = E_{\widehat{Q}^i}[\tilde{Y}^i] = 0 \quad \text{for all } i.$$

3) We have

$$\begin{aligned} \rho_+^{\mathcal{Y}}(g) &= \sum_{i=1}^N \inf\{m \in \mathbb{R} : \exists k^i \in K_i, Y^i \in L^1(\Omega, \mathcal{F}_T^i, \widehat{Q}^i) \text{ with } E_{\widehat{Q}^i}[Y^i] = 0 \\ &\quad \text{such that } m + k^i + Y^i \geq g^i\} \\ &= \sum_{i=1}^N \rho_{\widehat{Q}^i}(g^i) = \rho_{A_{\widehat{Q}}}(g). \end{aligned}$$

4) Any optimiser $(\tilde{m}, \tilde{Y}) \in \mathbb{R}^N \times \mathcal{Y}$ of $\rho_+^{\mathcal{Y}}(g)$ in 2) is an optimiser of $\rho_{A_{\widehat{Q}}}(g)$. In particular, $\tilde{m}^i$ is an optimiser for $\rho_{\widehat{Q}^i}(g^i)$ and $\tilde{m}^i = \rho_{\widehat{Q}^i}(g^i) = E_{\widehat{Q}^i}[g^i]$.

Proof 1) From the assumption, we have

$$\rho_+^{\mathcal{Y}}(g) = \sum_{i=1}^N E_{\widehat{Q}^i}[g^i] = \sum_{i=1}^N \widehat{m}^i$$

with $\widehat{m}^i + \widehat{k}^i + \widehat{Y}^i \geq g^i$ and $(\widehat{m}, \widehat{k}, \widehat{Y}) \in \mathbb{R}^N \times K_i \times \mathcal{Y} \cap L^1(\Omega, \mathbf{F}_T, \widehat{\mathbf{Q}})$. Consequently, we have $\widehat{k}^i \geq g^i - \widehat{Y}^i - \widehat{m}^i \in L^1(\Omega, \mathcal{F}_T^i, \widehat{Q}^i)$, which implies by Föllmer and Schied [14, Theorem 5.14] that $\widehat{k}^i$ belongs to $L^1(\Omega, \mathcal{F}_T^i, \widehat{Q}^i)$ and $E_{\widehat{Q}^i}[\widehat{k}^i] = 0$. By taking expectations, we then get

$$E_{\widehat{Q}^i}[g^i] - \widehat{m}^i \leq E_{\widehat{Q}^i}[\widehat{Y}^i] + E_{\widehat{Q}^i}[\widehat{k}^i] = E_{\widehat{Q}^i}[\widehat{Y}^i] \quad \text{for all } i$$

so that

$$0 = \rho_+^{\mathcal{Y}}(g) - \rho_+^{\mathcal{Y}}(g) = \sum_{i=1}^N E_{\widehat{Q}^i}[g^i] - \sum_{i=1}^N \widehat{m}^i \leq \sum_{i=1}^N E_{\widehat{Q}^i}[\widehat{Y}^i] \leq 0,$$

where the last inequality follows from $\widehat{Q} \in M^{\mathcal{Y}}$.

2) Observe that for any $x \in \mathbb{R}_0^N$, $(\widehat{m}, \widehat{Y}) \in \mathbb{R}^N \times \mathcal{Y} \cap L^1(\Omega, \mathbf{F}_T, \widehat{\mathbf{Q}})$ is an optimiser for $\rho_+^{\mathcal{Y}}(g)$ if and only if $(\tilde{m} = \widehat{m} + x, \tilde{Y} = \widehat{Y} - x) \in \mathbb{R}^N \times \mathcal{Y} \cap L^1(\Omega, \mathbf{F}_T, \widehat{\mathbf{Q}})$ is an optimiser for $\rho_+^{\mathcal{Y}}(g)$, and obviously $\sum_{i=1}^N \widehat{m}^i = \sum_{i=1}^N \tilde{m}^i$. Set $x^i := E_{\widehat{Q}^i}[\widehat{Y}^i]$ for each i so that $x \in \mathbb{R}_0^N$ by (4.12). Then $E_{\widehat{Q}^i}[\tilde{Y}^i] = 0$ for all i and $(\tilde{m}, \tilde{Y})$ is an optimiser for $\rho_+^{\mathcal{Y}}(g)$ and thus satisfies (4.11), i.e., $E_{\widehat{Q}^i}[g^i] - \tilde{m}^i = E_{\widehat{Q}^i}[\tilde{Y}^i] = 0$ for all i .

3) Observe that

$$\begin{aligned} \rho_{A_{\widehat{Q}}}(g) &= \inf \left\{ \sum_{i=1}^N m^i : \exists k \in \bigtimes_{i=1}^N K_i, Y \in A_{\widehat{Q}} \text{ with } m^i + k^i + Y^i \geq g^i, \forall i \right\} \\ &= \sum_{i=1}^N \inf \{ m^i \in \mathbb{R} : \exists k^i \in K_i, Y^i \in A_{\widehat{Q}^i} \text{ with } m^i + k^i + Y^i \geq g^i \} \\ &= \sum_{i=1}^N \rho_{\widehat{Q}^i}(g^i). \end{aligned} \quad (4.13)$$

Thus we need to show $\rho_{A_{\widehat{Q}}}(g) = \rho_+^{\mathcal{Y}}(g)$. From 2), we know that among the optimisers for $\rho_+^{\mathcal{Y}}(g)$, there exists $(\tilde{m}, \tilde{Y}) \in \mathbb{R}^N \times \mathcal{Y} \cap L^1(\Omega, \mathbf{F}_T, \widehat{\mathbf{Q}})$ such that

$$(\tilde{m}, \tilde{Y}) \in \left\{ (m, Y) \in \mathbb{R}^N \times A_{\widehat{Q}} : \exists k \in \bigtimes_{i=1}^N K_i \text{ with } m^i + k^i + Y^i \geq g^i, \forall i \right\}$$

and thus

$$\rho_{A_{\widehat{Q}}}(g) \leq \rho_+^{\mathcal{Y}}(g). \quad (4.14)$$

Now consider $\rho_{A_{\widehat{Q}}}(g)$ in (4.13). From the inequality $m^i + k^i + Y^i \geq g^i$ for all i in (4.13) with $k^i \in K_i$, $Y^i \in A_{\widehat{Q}^i}$, using $E_{\widehat{Q}^i}[Y^i] = 0$ and [14, Theorem 5.14] yields $m^i \geq E_{\widehat{Q}^i}[g^i]$ for all i and thus $\sum_{i=1}^N m^i \geq \sum_{i=1}^N E_{\widehat{Q}^i}[g^i]$, which implies

$$\rho_{A_{\widehat{Q}}}(g) \geq \sum_{i=1}^N E_{\widehat{Q}^i}[g^i] = \rho_+^{\mathcal{Y}}(g). \quad (4.15)$$

From (4.14) and (4.15), we deduce

$$\rho_+^{\mathcal{Y}}(g) \geq \rho_{A_{\widehat{Q}}}(g) \geq \sum_{i=1}^N E_{\widehat{Q}^i}[g^i] = \rho_+^{\mathcal{Y}}(g),$$

which proves 3).

4) For the optimiser $(\tilde{m}, \tilde{Y}) \in \mathbb{R}^N \times (\mathcal{Y} \cap L^1(\Omega, \mathbf{F}_T, \hat{\mathbf{Q}}))$ in 2), we obtain that $E_{\hat{Q}^i}[\tilde{Y}^i] = 0$ for all i and

$$\rho_{A_{\hat{Q}}}(g) = \sum_{i=1}^N E_{\hat{Q}^i}[g^i] = \rho_+^{\mathcal{Y}}(g) = \sum_{i=1}^N \tilde{m}^i.$$

Thus $(\tilde{m}, \tilde{Y})$ is also an optimiser for $\rho_{A_{\hat{Q}}}(g)$, and so $\tilde{m}^i$ is an optimiser for $\rho_{\hat{Q}^i}(g^i)$. $\square$

4.3.1 Economic interpretation and comparison with Bühlmann's risk exchange equilibrium

In this subsection, the assumptions of Proposition 4.13 hold true and we adopt the same notations. We already observed that cooperation reduces the cost for the system of N agents, i.e., $\rho_+^{\mathcal{Y}}(g) \leq \rho_+^N(g)$. Furthermore, from Proposition 4.13, we obtain the following conclusions.

1) Given $\rho_+^{\mathcal{Y}}(g)$, we define the *cost allocation for agent i* as

$$\rho^i(g) := E_{\hat{Q}^i}[g^i] = \tilde{m}^i,$$

where $\tilde{m}^i$ is given in Proposition 4.13, 2). From $\rho_+^{\mathcal{Y}}(g) = \sum_{i=1}^N E_{\hat{Q}^i}[g^i]$ and Proposition 4.13, 4), we see that the cost allocation $\rho^i(g)$ for agent i is equal to the superreplication cost $\rho_{\hat{Q}^i}(g^i)$ associated to the pricing measure $\hat{Q}^i$ that agent i would compute independently from all other agents $j \neq i$. Moreover, as $\hat{Q} \in \mathcal{M}^{\mathcal{Y}} \subseteq M_1 \times \dots \times M_N$, we have

$$\rho^i(g) := E_{\hat{Q}^i}[g^i] \leq \sup_{Q^i \in M_i} E_{Q^i}[g^i] := \rho_{i,+}(g^i),$$

so that the cost allocation $\rho^i(g)$ for agent i is smaller than the individual superreplication cost $\rho_{i,+}(g^i)$.

2) $\rho_+^{\mathcal{Y}}(g) = \sum_{i=1}^N \rho_{\hat{Q}^i}(g^i)$ is the sum of the individual superreplication prices of each claim g^i , when the price is assigned by $(\hat{Q}^1, \dots, \hat{Q}^N)$.

We now describe the connection of the notion of collective superreplication with Bühlmann's [7] risk sharing equilibrium, which provides an additional rationale to our approach. Bühlmann's reformulation of a risk sharing equilibrium in this context would consist of a pair of vectors

$$(\hat{Q}, \hat{Y}) \in \mathcal{M}^{\mathcal{Y}} \times (L^1(\Omega, \mathcal{F}_T^1, \hat{Q}^1) \times \dots \times L^1(\Omega, \mathcal{F}_T^N, \hat{Q}^N))$$

such that

a) $\hat{Y}^i$ is an optimiser of

$$\begin{aligned} \rho_{\hat{Q}^i}(g^i) &= \inf\{E_{\hat{Q}^i}[Y^i] : Y^i \in L^1(\Omega, \mathcal{F}_T^i, \hat{Q}^i) \text{ such that} \\ &\quad \exists k^i \in K_i \text{ with } k^i + Y^i \geq g^i\} \end{aligned}$$

given in (4.10);

$$\text{b) } \sum_{i=1}^N (\hat{Y}^i - E_{\hat{Q}^i}[\hat{Y}^i]) = 0.$$

The economic rationale is clear: in a), each agent optimises his own initial cost $\rho_{\hat{Q}^i}(g^i)$ assigned via $\hat{Q}^i$, while the exchanges take place at the terminal time and satisfy the clearing condition in b) that the total amount exchanged, $\sum_{i=1}^N \hat{Y}^i$, must be equal to the initial cost $\sum_{i=1}^N E_{\hat{Q}^i}[\hat{Y}^i]$.

Take now

$$\mathcal{Y} = \left\{ Y \in L^1(\Omega, \mathbf{F}_T, P) : \sum_{i=1}^N Y^i = 0 \right\}.$$

From Proposition 4.13, 4), we may easily deduce

Proposition 4.14 *Under the assumptions of Proposition 4.13, the pair $(\hat{Q}, \hat{Y})$ appearing there satisfies the Bühlmann-type conditions a) and b).*

Remark 4.15 We consider N probability measures $(\hat{Q}^1, \dots, \hat{Q}^N)$ instead of just one as in [7] because of the generality of the set $\mathcal{Y}$. If $\mathcal{Y}$ is the set

$$\mathcal{Y} = \left\{ Y \in (L^1(\Omega, \mathcal{F}_T, P))^N : \sum_{i=1}^N Y^i = 0 \text{ and } \mathcal{F}_T^i = \mathcal{F}_T, \forall i \right\},$$

then $\hat{Q}^1, \dots, \hat{Q}^N$ all coincide on $\mathcal{F}_T$, giving one single probability as in [7].

4.4 Collective subreplication price

Similarly to the superreplication case, but not symmetrically, we define the collective subreplication prices as follows.

Definition 4.16 Consider a vector $g = (g^1, \dots, g^N)$ in $L^0(\Omega, \mathbf{F}_T, P)$. We set

$$\begin{aligned} \rho_-^{\mathcal{Y}}(g) &:= \sup \left\{ \sum_{i=1}^N m^i : \exists k \in \bigtimes_{i=1}^N K_i, m \in \mathbb{R}^N, Y \in \mathcal{Y} \right. \\ &\quad \left. \text{with } m^i + k^i - Y^i \leq g^i, \forall i \right\}, \\ \pi_-^{\mathcal{Y}}(g) &:= \sup \left\{ m \in \mathbb{R} : \exists k \in \bigtimes_{i=1}^N K_i, Y \in \mathcal{Y} \text{ with } m + k^i - Y^i \leq g^i, \forall i \right\}. \end{aligned}$$

Observe that differently from the definition of superreplication, we subtract the contribution Y^i of the vector Y in the definition of subreplication. We clearly have

$$\rho_-^{\mathcal{Y}}(g) = -\rho_+^{\mathcal{Y}}(-g),$$

$$\pi_-^{\mathcal{Y}}(g) = -\pi_+^{\mathcal{Y}}(-g),$$

$$\rho_-^{\mathcal{Y}}(g) \geq \rho_-^N(g) := \sup \left\{ \sum_{i=1}^N m^i : \exists k \in \bigtimes_{i=1}^N K_i, m \in \mathbb{R}^N \text{ with } m^i + k^i \leq g^i, \forall i \right\},$$

$$\pi_-^{\mathcal{Y}}(g) \geq \pi_-^N(g) := \sup \{ m \in \mathbb{R} : \exists k^i \in K_i \text{ with } m + k^i \leq g^i, \forall i \},$$

$$\rho_-^N(g) = -\rho_+^N(-g) = \sum_{i=1}^N -\rho_{i,+}(-g^i),$$

$$\pi_-^N(g) = -\pi_+^N(-g) = \min_i (-\rho_{i,+}(-g^i)),$$

where we used Proposition 4.8. The (total) *value of cooperation* with respect to g is then

$$(\rho_+^N(g) - \rho_+^{\mathcal{Y}}(g)) + (\rho_-^{\mathcal{Y}}(g) - \rho_-^N(g)) \geq 0.$$

If $\mathcal{Y}$ fulfils (3.11), NCA($\mathcal{Y}$) holds true and $K^{\mathcal{Y}}$ is closed in $L^0(\Omega, \mathbf{F}_T, P)$, we deduce from Theorem 4.12 and from $\rho_-^{\mathcal{Y}}(g) = -\rho_+^{\mathcal{Y}}(-g)$ and $\pi_-^{\mathcal{Y}}(g) = -\pi_+^{\mathcal{Y}}(-g)$ that

$$\rho_-^{\mathcal{Y}}(g) = \inf_{Q \in (C^{\mathcal{Y}})_1^0} \sum_{i=1}^N E_{Q^i}[g^i] \quad \text{and} \quad \pi_-^{\mathcal{Y}}(g) = \inf_{Q \in (C^{\mathcal{Y}})_1^0} \frac{1}{N} \sum_{i=1}^N E_{Q^i}[g^i]$$

for any $g \in L^1(\Omega, \mathbf{F}_T, P)$.

5 Finite-dimensional multiperiod markets

In this section, we work in the multiperiod setting outlined in Sect. 2. We assume that $L^0(\Omega, \mathcal{F}, P)$ has **finite dimension**, or equivalently that the σ -algebra $\mathcal{F}$ is generated by a partition consisting of a finite number of atoms. Then any vector space contained in $(L^0(\Omega, \mathcal{F}, P))^N$ is a finite-dimensional closed vector space that can also be considered as a finitely generated convex cone. The positive orthant $L_+^0(\Omega, \mathbf{F}_T, P)$ is clearly a finitely generated convex cone. It is also evident that the sum of two (or a finite number of) finitely generated convex cones is again a convex cone that is finitely generated (for completeness, we prove these simple facts in Appendix A.1). By [2, Corollary 5.25], finitely generated convex cones in a topological metric space are closed. We thus conclude that if $\mathbb{K} \subseteq (L^0(\Omega, \mathcal{F}, P))^N$ is a vector space and at the same time $\mathcal{Y} \subseteq (L^0(\Omega, \mathcal{F}, P))^N$ is a finitely generated convex cone (as in the case of $\mathcal{Y}$ being a vector space, which is a frequent assumption in this paper), then

$$\mathbb{K} - L_+^0(\Omega, \mathbf{F}_T, P) + \mathcal{Y}$$

is a (finitely generated) *closed* convex cone in $(L^0(\Omega, \mathcal{F}, P))^N$. Recall the definitions of

$$\mathbb{R}_0^N := \left\{ x \in \mathbb{R}^N : \sum_{i=1}^N x^i = 0 \right\},$$

$$\rho_+^{\mathcal{Y}}(g) := \inf \left\{ \sum_{i=1}^N m^i : \exists k \in \bigtimes_{i=1}^N K_i, m \in \mathbb{R}^N, Y \in \mathcal{Y} \right.$$

$$\left. \text{with } m^i + k^i + Y^i \geq g^i, \forall i \right\}.$$

Theorem 5.1 Suppose $\mathcal{Y} \subseteq L^0(\Omega, \mathbf{F}_T, P)$ is a finitely generated convex cone containing $\mathbb{R}_0^N$. Then

$$\text{NCA}(\mathcal{Y}) \quad \text{if and only if} \quad \mathcal{M}^{\mathcal{Y}} \cap \mathcal{P}_e^N \neq \emptyset. \quad (5.1)$$

Assume that $\text{NCA}(\mathcal{Y})$ holds true. Then the superreplication price $\rho_+^{\mathcal{Y}}$ is a cash-additive, increasing, continuous functional $\rho_+^{\mathcal{Y}} : L^0(\Omega, \mathbf{F}_T, P) \rightarrow \mathbb{R}$ and we have:

1) The pricing–hedging duality holds, i.e.,

$$\rho_+^{\mathcal{Y}}(g) = \max_{Q \in \mathcal{M}^{\mathcal{Y}}} \sum_{i=1}^N E_{Q^i}[g^i] = \sum_{i=1}^N E_{\widehat{Q}^i}[g^i] \quad \text{with } \widehat{Q} \in \mathcal{M}^{\mathcal{Y}}.$$

2) There exists an optimiser $(\widehat{m}, \widehat{k}, \widehat{Y}) \in \mathbb{R}^N \times \bigtimes_{i=1}^N K_i \times \mathcal{Y}$ of $\rho_+^{\mathcal{Y}}(g)$ which satisfies

$$E_{\widehat{Q}^i}[g^i] = \widehat{m}^i \quad \text{and} \quad E_{\widehat{Q}^i}[\widehat{Y}^i] = 0 \quad \text{for all } i.$$

$$3) \quad \rho_+^{\mathcal{Y}}(g) = \sum_{i=1}^N \rho_{\widehat{Q}^i}(g^i) = \sum_{i=1}^N \widehat{m}^i,$$

where $\widehat{m}^i$ is for each i an optimiser of

$$\rho_{\widehat{Q}^i}(g^i) = \inf \{ m \in \mathbb{R} : \exists k^i \in K_i, Y^i \in L^1(\Omega, \mathcal{F}_T^i, \widehat{Q}^i) \text{ with } E_{\widehat{Q}^i}[Y^i] = 0$$

$$\text{such that } m + k^i + Y^i \geq g^i \}.$$

Proof The assumption that $L^0(\Omega, \mathcal{F}, P)$ has finite dimension and the discussion preceding the result imply that $K^{\mathcal{Y}} = \bigtimes_{i=1}^N K_i - L_+^0(\Omega, \mathbf{F}_T, P) + \mathcal{Y}$ is a closed convex cone. Moreover, the assumption that $\mathcal{Y} \subseteq (L^1(\Omega, \mathcal{F}, P))^N$ is always fulfilled as $\mathcal{Y} \subseteq (L^0(\Omega, \mathcal{F}, P))^N = (L^1(\Omega, \mathcal{F}, P))^N$. Thus we may apply Theorem 3.12 that proves the collective FTAP in (5.1). Lemma 4.4 provides the properties of $\rho_+^{\mathcal{Y}}$ and in

particular shows that it is finite-valued; see Lemma 4.4, 4). From (4.9), we deduce the pricing–hedging duality in which $(C^{\mathcal{Y}})_1^0$ can be replaced by $\mathcal{M}^{\mathcal{Y}}$ by Lemma 3.8. The existence of the dual maximiser $\widehat{Q} \in \mathcal{M}^{\mathcal{Y}}$ in 1) is a consequence of the compactness of $\mathcal{M}^{\mathcal{Y}}$ due to the finite-dimensional setup, which proves 1). The remaining assertions follow directly from Proposition 4.13. $\square$

6 Collective FTAP and the closedness of the set $K^{\mathcal{Y}}$ in multiperiod markets

In multiperiod markets, we provide sufficient conditions for (i) the equivalence of $\text{NCA}(\mathcal{Y})$ and NA_i for all i (Sect. 6.1), and (ii) the equivalence of $\text{NCA}(\mathcal{Y})$ and NA in the global market (Sect. 6.2). In Sect. 6.3, we show a general version of the collective FTAP when $\text{NCA}(\mathcal{Y})$ is not (in general) equivalent to any one of the classical notions of no arbitrage.

In the sequel, for technical reasons, we assume that the set $\mathcal{Y}$ is **closed in probability**. Observe that the sets in (3.6) and in (3.12) as well as $\mathcal{Y}_0 \cap L^0(\Omega, \mathbf{F}_T, P)$ satisfy this requirement.

6.1 When $\text{NCA}(\mathcal{Y})$ is equivalent to NA_i for all i

We adopt the same multiperiod setting as in Sect. 2. The main result in this section is Theorem 6.5, which collects five equivalent formulations of $\text{NCA}(\mathcal{Y})$.

Theorem 6.1 *If $\mathcal{Y} \subseteq L^0(\Omega, \mathbf{F}_0, P)$, $0 \in \mathcal{Y}$ and $\mathcal{Y} \subseteq \mathcal{Y}_0$, then*

$$\text{NCA}(\mathcal{Y}) \iff \text{NA}_i, \forall i \iff M_i \cap \mathcal{P}_e \neq \emptyset, \forall i, \quad (6.1)$$

and the first equivalence holds true even if for each i , the initial σ -algebra $\mathcal{F}_0^i$ is not trivial.

Proof The second equivalence in (6.1) is the classical FTAP by Dalang et al. [9]; see Theorem 2.6. For the first equivalence, observe that “ $\text{NCA}(\mathcal{Y}) \Rightarrow \text{NA}_i, \forall i$ ” was shown in (3.7). To prove “ $\text{NA}_i, \forall i \Rightarrow \text{NCA}(\mathcal{Y})$ ”, we show that a collective arbitrage implies the existence of a classical arbitrage for some agent. Suppose that $(\times_{i=1}^N K_i + \mathcal{Y}) \cap L_+^0(\Omega, \mathbf{F}_T, P) \neq \{0\}$ and let $k \in \times_{i=1}^N K_i$ and $Y \in \mathcal{Y}$ satisfy $k^i + Y^i \geq 0$ for all i and $P[k^j + Y^j > 0] > 0$ for some j . Then there are two possibilities:

Either $Y^i = 0$ for all i , in which case we have $k^i \geq 0$ for all i and $P[k^j > 0] > 0$, so that k^j is a classical arbitrage for agent j ;

or, since $\sum_{i=1}^N Y^i = 0$, there exists $n \in \{1, \dots, N\}$ such that $P[Y^n < 0] > 0$. In this case, take $A := \{Y^n < 0\} \in \mathcal{F}_0^n$. Thus $(k^n + Y^n)1_A \geq 0$, so that $1_A k^n \geq -1_A Y^n \geq 0$ and $P[1_A k^n > 0] > 0$. But then $1_A k^n = \sum_{h \in (n)} (1_A H^h \cdot X^h)_T$ is in K_n since $1_A H^h$ is predictable due to $A \in \mathcal{F}_0^n$, and so $1_A k^n$ is an arbitrage for agent n . $\square$

Remark 6.2 Differently from the rest of the paper, Theorems 6.3 and 6.9 do not assume that $\mathbf{F}_0$ is trivial, as they will be applied in the multiperiod setting of Theorem 6.11 for the time step $\{t, t + 1\}$, where $\mathbf{F}_t$ is not trivial. One may recognise from its proof that Theorem 6.3 holds true also if we replace time 0 by any time $t \in \{0, \dots, T - 1\}$ and the condition $\mathcal{Y} \subseteq L^0(\Omega, \mathbf{F}_0, P)$ with $\mathcal{Y} \subseteq L^0(\Omega, \mathbf{F}_t, P)$. Of course, in this case, all relevant notions need to be defined and computed from time t to T .

Theorem 6.3 *We do not assume that the initial σ -algebra $\mathcal{F}_0^i$ is trivial for any i . Suppose that $\mathcal{Y} \subseteq L^0(\Omega, \mathbf{F}_0, P)$ is a convex cone closed in probability and that $\mathcal{Y} \subseteq \mathcal{Y}_0$. Then $\text{NCA}(\mathcal{Y})$ implies that $\times_{i=1}^N K_i + \mathcal{Y}$ and $K^{\mathcal{Y}}$ are closed in probability.*

Proof We first show that $K^{\mathcal{Y}}$ is closed in probability. To that end, take a sequence $W_n = k_n + Y_n - \ell_n \in K^{\mathcal{Y}}$ converging a.s. to W_∞ . Now $\text{NCA}(\mathcal{Y})$ implies NA_i for every $i = 1, \dots, N$ as in (3.7), and by Lemma 6.4 applied componentwise, we infer that $\inf_{n \in \mathbb{N}} Y_n^i > -\infty$ a.s. for every $i = 1, \dots, N$. Since we assume $\mathcal{Y} \subseteq \mathcal{Y}_0$, this in turns implies $\sup_{n \in \mathbb{N}} \sum_{j=1}^N |Y_n^j| < \infty$ a.s. An iteration of Karatzas and Schachermayer [20, Theorem 3.1] yields a subsequence $(Y_{n_k})_{k \in \mathbb{N}} \subseteq \mathcal{Y}$ and $Y_\infty \in L^0(\Omega, \mathbf{F}_0, P)$ such that $\lim_{H \rightarrow \infty} \frac{1}{H} \sum_{h=1}^H Y_{n_h} = Y_\infty$ a.s., and $Y_\infty \in L^0(\Omega, \mathbf{F}_0, P)$ by a.s. boundedness of the original sequence $(Y_n)_{n \in \mathbb{N}}$. Since $\mathcal{Y}$ is a convex cone which is closed in probability, we get $Y_\infty \in \mathcal{Y}$. Now we see that $\frac{1}{H} \sum_{h=1}^H W_{n_h} \rightarrow W_\infty$ as $H \rightarrow \infty$, and

$$\frac{1}{H} \sum_{h=1}^H W_{n_h} - \frac{1}{H} \sum_{h=1}^H Y_{n_h} = \frac{1}{H} \sum_{h=1}^H k_{n_h} - \frac{1}{H} \sum_{h=1}^H \ell_{n_h} \in \times_{i=1}^N K_i - L_+^0(\Omega, \mathbf{F}_T, P).$$

Since the left-hand side converges in probability, so does the right-hand side. Moreover, as argued above, NA_i holds for every $i = 1, \dots, N$, so that

$$\times_{i=1}^N K_i - L_+^0(\Omega, \mathbf{F}_T, P) = \times_{i=1}^N (K_i - L_+^0(\Omega, \mathbf{F}_T^i, P))$$

is the product of sets closed in probability and therefore closed in probability itself. We conclude that $W_\infty - Y_\infty \in \times_{i=1}^N K_i - L_+^0(\Omega, \mathbf{F}_T, P)$, which implies $W_\infty \in K^{\mathcal{Y}}$ since $Y_\infty \in \mathcal{Y}$ as previously discussed. This shows that $K^{\mathcal{Y}}$ is closed in probability.

Now we show that $\times_{i=1}^N K_i + \mathcal{Y}$ is closed in probability. Note that $\times_{i=1}^N K_i + \mathcal{Y}$ can be obtained from $K^{\mathcal{Y}}$ by replacing in $K^{\mathcal{Y}}$ the cone $L_+^0(\Omega, \mathbf{F}_0, P)$ with $\{0\}$. Now recall that the set $\times_{i=1}^N K_i$ is closed in probability (even without assuming NA_i for every $i = 1, \dots, N$). Thus going through the previous argument and replacing there the elements of $L_+^0(\Omega, \mathbf{F}_0, P)$ with 0, one can similarly conclude that $\times_{i=1}^N K_i + \mathcal{Y}$ is closed in probability. $\square$

Lemma 6.4 *We do not assume that the initial σ -algebra $\mathcal{F}_0$ is trivial. Suppose K satisfies NA . Suppose also that $(k_n)_{n \in \mathbb{N}} \subseteq K$, $(Z_n)_{n \in \mathbb{N}} \subseteq L^0(\Omega, \mathcal{F}_0, P)$ and $(W_n)_{n \in \mathbb{N}} \subseteq L^0(\Omega, \mathcal{F}_T, P)$ are sequences with $k_n + Z_n \geq W_n$ a.s. for every n and $\lim_{n \rightarrow \infty} W_n = W_\infty$ a.s. for some $W_\infty \in L^0(\Omega, \mathcal{F}_T, P)$. Then $\inf_{n \in \mathbb{N}} Z_n > -\infty$ a.s.*

Proof Define $\ell_n := k_n + Z_n - W_n \in L_+^0(\Omega, \mathcal{F}_T, P)$. Then $k_n + Z_n - \ell_n = W_n$. Suppose by contradiction that

$$A := \left\{ \inf_{n \in \mathbb{N}} Z_n = -\infty \right\} \in \mathcal{F}_0 \text{ satisfies } P[A] > 0.$$

Observe that $\liminf_{n \rightarrow \infty} Z_n = -\infty$ on A . As a consequence, by Delbaen and Schachermayer [11, Proposition 6.3.4 (i)], there exists a sequence $(\tau_n)_{n \in \mathbb{N}}$ of $\mathcal{F}_0$ -measurable random variables taking values in $\mathbb{N}$ such that $\lim_{n \rightarrow \infty} Z_{\tau_n} 1_A = -\infty 1_A$ a.s.. Define

$$H_n := \frac{1}{1 + |Z_n|} 1_A \in L^0(\Omega, \mathcal{F}_0, P).$$

Then $\lim_{n \rightarrow \infty} H_{\tau_n} Z_{\tau_n} = -1_A$. Moreover, $\lim_{n \rightarrow \infty} W_{\tau_n} = W_\infty$ a.s. so that $\lim_{n \rightarrow \infty} H_{\tau_n} W_{\tau_n} = 0$ a.s.. Observe now that $H_n k_n \in K$ for every n since H_n is $\mathcal{F}_0$ -measurable, and that

$$H_{\tau_n} k_{\tau_n} = \lim_{M \rightarrow \infty} \sum_{m=0}^M 1_{\{\tau_n=m\}} H_m k_m.$$

On the right-hand side, each term of the summation still belongs to K since $1_{\{\tau_n=m\}}$ belongs to $L^0(\Omega, \mathcal{F}_0, P)$, and so does the limit since K is closed by [11, Proposition 6.8.1]. Hence also $H_{\tau_n} k_{\tau_n} \in K$ for every n , and $H_{\tau_n} \ell_{\tau_n} \in L_+^0(\Omega, \mathcal{F}_T, P)$ can be similarly verified. Now we have

$$H_{\tau_n} k_{\tau_n} - H_{\tau_n} \ell_{\tau_n} = H_{\tau_n} W_{\tau_n} - H_{\tau_n} Z_{\tau_n} \longrightarrow -(-1_A) = 1_A \quad \text{as } n \rightarrow \infty,$$

which implies $1_A \in K - L_+^0(\Omega, \mathcal{F}_T, P)$ since $K - L_+^0(\Omega, \mathcal{F}_T, P)$ is closed by NA (see [11, Theorem 6.9.2]). This contradicts NA. Hence $P[A] = 0$. $\square$

Theorem 6.5 Suppose $\mathcal{Y} \subseteq L^0(\Omega, \mathbf{F}_0, P)$ fulfils (3.11) and $\mathcal{Y} \subseteq \mathcal{Y}_0$, so that $\mathcal{Y} = \mathbb{R}_0^N$ and in particular $\mathcal{Y}$ is closed in probability. Then the following are equivalent:

- 1) $\text{NCA}(\mathcal{Y})$.
- 2) $\mathcal{M}^{\mathcal{Y}} \cap \mathcal{P}_e^N \neq \emptyset$.
- 3) $M_i \cap \mathcal{P}_e \neq \emptyset, \forall i$.
- 4) For each i , there exists an equivalent martingale measure for all processes X^j for all $j \in (i)$.
- 5) $\text{NA}_i, \forall i$.

Proof Under $\text{NCA}(\mathcal{Y})$, we know by Theorem 6.3 that $K^{\mathcal{Y}}$ is closed in probability; so by Proposition 3.9, $\text{NCA}(\mathcal{Y})$ implies $\mathcal{M}^{\mathcal{Y}} \cap \mathcal{P}_e^N \neq \emptyset$. The implications “2) $\Rightarrow$ 3)” and “3) $\Rightarrow$ 4)” are trivial, and “4) $\Rightarrow$ 5)” follows from the easy part of the classical Dalang–Morton–Willinger FTAP (Theorem 2.6), which is true even if $\frac{dQ^i}{dP}$ is not bounded. Finally, the implication “5) $\Rightarrow$ 1)” was proved in Theorem 6.1. $\square$

In Examples 7.1.1 and 7.2.1, the equivalence between $\text{NCA}(\mathcal{Y})$ and $\text{NA}_i, \forall i$, holds true, while in Examples 7.1.2 and 7.2.2, we show that when $\mathcal{Y}$ is not contained in $L^0(\Omega, \mathbf{F}_0, P)$, it is possible that $\text{NA}_i, \forall i$, holds true, but there exists a collective arbitrage.

6.2 When $\text{NCA}(\mathcal{Y})$ is equivalent to no global arbitrage NA

We adopt the same multiperiod setting as in Sect. 2. **Throughout this subsection**, the key assumption is a common filtration for all agents, i.e., $\mathbb{F}^i = \mathbb{F}^j = \mathbb{F}$ for all i, j . Thus $L^0(\Omega, \mathcal{F}_T, P) = (L^0(\Omega, \mathcal{F}, P))^N$. This fact and the selection $\mathcal{Y} = \mathcal{Y}_0$ so that we allow to exchange *all* $\mathcal{F}_T$ -measurable random variables with the sum of their components equal to 0 imply that $\text{NCA}(\mathcal{Y})$ is equivalent to no global arbitrage NA. The intuition for this is that $\mathcal{Y}$ is so large that it contains exchanges of any stochastic integrals; so an agent trying to achieve a collective arbitrage from an arbitrage can add to her portfolio stochastic integrals involving stocks not in her market by exchanges with the other agents at no cost. The equivalence between NCA and NA can also be reformulated by saying that a collective arbitrage for the N agents is equivalent to a (classical) arbitrage for a representative agent who is allowed to invest in the global market. We stress, however, that this equivalence holds only under the assumptions of Proposition 6.6 below. The fact that one cannot use a representative agent formulation in general (see Example 6.7) is an indication that our results do not trivialise to the classical case.

Proposition 6.6 *Suppose that $\mathbb{F}^i = \mathbb{F}^j = \mathbb{F}$ for all $i, j = 1, \dots, N$. Then for $\mathcal{Y} = \mathcal{Y}_0$ defined in (3.6), we have:*

- 1) $\text{NCA}(\mathcal{Y}_0)$ and NA are equivalent (see Definition 2.5 for the latter).
- 2) Under either $\text{NCA}(\mathcal{Y}_0)$ or NA, $K^{\mathcal{Y}_0}$ is closed in probability.

Proof 1) In the current setup, $\mathcal{Y}_0 \subseteq (L^0(\Omega, \mathcal{F}, P))^N$ and we have “NA $\Rightarrow$ $\text{NCA}(\mathcal{Y}_0)$ ” by (3.8); so we only need to prove the converse. To this end, by Lemma A.2, we have

$$K^{\mathcal{Y}_0} = \bigtimes_{i=1}^N K + \mathcal{Y}_0 - (L_+^0(\Omega, \mathcal{F}, P))^N.$$

In particular, by the definition of $\text{NCA}(\mathcal{Y}_0)$ and picking $Y = 0 \in \mathcal{Y}_0$, we deduce the equality $(K - L_+^0(\Omega, \mathcal{F}, P)) \cap L_+^0(\Omega, \mathcal{F}, P) = \{0\}$, which is NA.

2) Since NA holds, $K - L_+^0(\Omega, \mathcal{F}, P)$ is closed in probability (see e.g. [11, Theorem 6.9.2], noticing that we assume $\mathcal{F} = \mathcal{F}_T$). Let now

$$\mathcal{S} : (L^0(\Omega, \mathcal{F}, P))^N \rightarrow L^0(\Omega, \mathcal{F}, P)$$

be the map defined by $\mathcal{S}(W) = \sum_{i=1}^N W^i$, continuous for convergence in probability. It is then enough to show that

$$K^{\mathcal{Y}_0} = \mathcal{S}^{-1}(K - L_+^0(\Omega, \mathcal{F}, P)).$$

The inclusion $K^{\mathcal{Y}_0} \subseteq \mathcal{S}^{-1}(K - L_+^0(\Omega, \mathcal{F}, P))$ is directly checked. Indeed, an element in $K^{\mathcal{Y}_0}$ is of the form $k + Y - \ell$. Then

$$\mathcal{S}(k + Y - \ell) = \sum_{i=1}^N k^i + 0 - \sum_{i=1}^N \ell^i \in K - L_+^0(\Omega, \mathcal{F}, P).$$

It remains to prove the converse inclusion. Select $Z \in \mathcal{S}^{-1}(K - L_+^0(\Omega, \mathcal{F}, P))$. Then $\sum_{j=1}^N Z^j = k - \ell$ for some $k \in K$ and $\ell \in L_+^0(\Omega, \mathcal{F}, P)$. By Lemma A.1, $\mathcal{S}(\times_{i=1}^N K_i) = K$ and we write

$$k = \mathcal{S}(\widehat{k}) \quad \text{for some } \widehat{k} \in \bigtimes_{i=1}^N K_i.$$

Consider now $W = Z - \widehat{k}$ and note that $\mathcal{S}(W) = k - \ell - k = -\ell \in -L_+^0(\Omega, \mathcal{F}, P)$. We write

$$\begin{pmatrix} W^1 \\ \vdots \\ W^N \end{pmatrix} = \begin{pmatrix} W^1 - \mathcal{S}(W) \\ W^2 \\ \vdots \\ W^N \end{pmatrix} + \begin{pmatrix} \mathcal{S}(W) \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Observe that the first term on the right-hand side, call it $\widehat{Y}$, satisfies $\widehat{Y} \in \mathcal{Y}_0$ and the second term, call it $-\widehat{\ell}$, belongs to $-(L_+^0(\Omega, \mathcal{F}, P))^N$ since $\mathcal{S}(W) = -\ell$. Thus $Z = W + \widehat{k}$ and $W + \widehat{k} = \widehat{k} - \widehat{\ell} + \widehat{Y} \in K^{\mathcal{Y}_0}$, which concludes the proof. $\square$

We now present an example to illustrate the relevance of the condition $\mathcal{Y}_0 = \mathcal{Y}$ for the equivalence of $\text{NCA}(\mathcal{Y})$ and NA.

Example 6.7 Take $N = J = 3$, $(i) = \{i\}$ for $i = 1, 2, 3$, a finite Ω and one single filtration for all assets, i.e., $\mathcal{F}_t^i = \mathcal{F}_t^j = \mathcal{F}_t$ for all t and all $i, j \in \{1, 2, 3\}$. If

$$\mathcal{Y} = \mathcal{Y}_0 = \left\{ Y \in (L^0(\Omega, \mathcal{F}_T, P))^3 : \sum_{i=1}^3 Y^i = 0 \right\},$$

we obtain from Proposition 6.6 that $\text{NCA}(\mathcal{Y})$ is equivalent to

$$\begin{aligned} K^{\mathcal{Y}} \cap (L_+^0(\Omega, \mathcal{F}_T, P))^3 = \{0\} & \iff K^3 \cap (L_+^0(\Omega, \mathcal{F}_T, P))^3 = \{0\} \\ & \iff \left(\bigcap_{i=1}^3 M_i \right) \cap \mathcal{P}_e \neq \emptyset. \end{aligned}$$

Now take a different $\mathcal{Y} := \{Y \in (L^0(\Omega, \mathcal{F}_T, P))^3 : Y^1 = 0 \text{ and } Y^2 + Y^3 = 0\}$ such that $\mathcal{Y} \neq \mathcal{Y}_0$. One can easily verify that

$$\mathcal{M}^{\mathcal{Y}} = \{(Q^1, Q, Q) : Q^1 \in M_1 \text{ and } Q \in M_2 \cap M_3\}.$$

We may select X^1, X^2, X^3 in such a way that (a) $(\bigcap_{i=1}^3 M_i) \cap \mathcal{P}_e = \emptyset$ so that there exists a global arbitrage in the market in the classical sense; (b) $\mathcal{M}^{\mathcal{Y}} \cap (\mathcal{P}_e)^3 \neq \emptyset$ so that by Proposition 3.11 and finiteness of Ω , $\text{NCA}(\mathcal{Y})$ holds true. The existence of an element in $\mathcal{M}^{\mathcal{Y}} \cap (\mathcal{P}_e)^3$ only guarantees the existence of an equivalent martingale measure for X_1 and another for both processes X_2, X_3 , but not necessarily an equivalent martingale measure simultaneously for all the processes X^1, X^2, X^3 .

6.3 A general setting with $\mathcal{Y} \subseteq (L^0(\Omega, \mathcal{F}, P))^N$

We adopt the same multiperiod setting as in Sect. 2. Throughout this subsection, the key assumption is a common filtration for all agents, i.e., $\mathbb{F}^i = \mathbb{F}^j = \mathbb{F}$ for every i, j . In comparison to the setting in Sects. 6.1 and 6.2, we do not impose any measurability constraints on $\mathcal{Y}$ (except for $\mathcal{Y} \subseteq (L^0(\Omega, \mathcal{F}, P))^N$ which is a very general requirement) nor any structural geometric properties (as $\mathcal{Y} = \mathcal{Y}_0$).

As clarified in Sect. 6.3.2 below, the multiperiod results are based on the one-period model. We thus first present the one-period market in Sect. 6.3.1, which contains most of the technical details relative to the closedness of $K^{\mathcal{Y}}$.

In some results, we need not only that $\mathcal{Y}$ is closed in probability, but also that $\mathcal{Y} - (L_+^0(\Omega, \mathcal{F}, P))^N$ is closed in probability. The following auxiliary result shows that this latter condition is automatic as soon as $\mathcal{Y} \subseteq \mathcal{Y}_0$ is closed in probability.

Lemma 6.8 *Suppose that $\mathcal{Y} \subseteq \mathcal{Y}_0$ is a convex cone closed in probability. Then the convex cone $\mathcal{Y} - (L_+^0(\Omega, \mathcal{F}, P))^N$ is closed in probability.*

Proof Take a sequence $(Y_n - \ell_n)_{n \in \mathbb{N}} \subseteq \mathcal{Y} - (L_+^0(\Omega, \mathcal{F}, P))^N$ converging in probability (without loss of generality, almost surely) to $Z \in (L^0(\Omega, \mathcal{F}, P))^N$. Take also the event $A := \{\sup_{n \in \mathbb{N}} |Y_n| = \infty\}$. If $P[A] > 0$, then for some index k , $A_k := \{\inf_{n \in \mathbb{N}} Y_n^k = -\infty\}$ satisfies $P[A_k] > 0$. Consequently, we have

$$Z^k 1_{A_k} = \lim_{n \rightarrow \infty} (Y_n^k 1_{A_k} - \ell_n^k 1_{A_k}) \leq \liminf_{n \rightarrow \infty} Y_n^k 1_{A_k} = -\infty 1_{A_k},$$

which is a contradiction. Thus $P[A] = 0$ and $(Y_n)_{n \in \mathbb{N}}$ is a.s. bounded. So is then $(\ell_n)_{n \in \mathbb{N}}$ since $(Y_n - \ell_n)_{n \in \mathbb{N}}$ converges. By an iteration of Karatzas and Schachermayer [20, Theorem 3.1], we can extract subsequences $(Y_{n_k})_{k \in \mathbb{N}}$, $(\ell_{n_k})_{k \in \mathbb{N}}$ whose Cesàro means converge almost surely to Y and ℓ , say, respectively. Since $\mathcal{Y}$ and $(L_+^0(\Omega, \mathcal{F}, P))^N$ are convex and closed, the Cesàro means and their limits belong to the respective sets, i.e., $Y \in \mathcal{Y}$, $\ell \in (L_+^0(\Omega, \mathcal{F}, P))^N$. Since $(Y_{n_k} - \ell_{n_k})_{k \in \mathbb{N}}$ converges almost surely to Z , also its Cesàro means converge to Z (see Williams [28, Lemma 12.6]). Thus $Z = Y - \ell \in \mathcal{Y} - (L_+^0(\Omega, \mathcal{F}, P))^N$. $\square$

6.3.1 The one-period market with general $\mathcal{Y} \subseteq (L^0(\Omega, \mathcal{F}, P))^N$

In this section, we consider a one-period market $\mathcal{T} = \{0, 1\}$ and thus the elements in K_i are just one-period stochastic integrals.

Theorem 6.9 *Suppose that $\mathcal{T} = \{0, 1\}$, $\mathcal{F}_0^i = \mathcal{F}_0^j = \mathcal{F}_0$ for all $i, j \in \{1, \dots, N\}$ and that $\mathcal{F}_0$ is not necessarily trivial. Assume that $\mathcal{Y} \subseteq (L^0(\Omega, \mathcal{F}, P))^N$ is a vector space satisfying*

- 1) $\mathcal{Y}$ and $(\mathcal{Y} - (L_+^0(\Omega, \mathcal{F}, P))^N)$ are closed in probability;
- 2) for every $Z \in L^0(\Omega, \mathcal{F}_0, P)$ and $Y \in \mathcal{Y}$, we have $ZY \in \mathcal{Y}$.

Then $\times_{i=1}^N K_i + \mathcal{Y}$ is closed in probability. If furthermore NCA($\mathcal{Y}$) holds, then also $K^{\mathcal{Y}} = \times_{i=1}^N K_i - (L_+^0(\Omega, \mathcal{F}, P))^N + \mathcal{Y}$ is closed in probability.

Lemma 6.10 *Define*

$$\mathcal{H}_{\mathcal{Y}} := \left\{ (H_1, \dots, H_N) \in \bigtimes_{i=1}^N \mathcal{H}_i : ((H_i \cdot X)_T)_{i=1, \dots, N} \in \mathcal{Y} \right\},$$

$$\mathcal{H}_{\mathcal{Y}}^{\perp} := \left\{ (H_1, \dots, H_N) \in \bigtimes_{i=1}^N \mathcal{H}_i : \sum_{i=1}^N H_i H_i^* = 0, \forall (H_1^*, \dots, H_N^*) \in \mathcal{H}_{\mathcal{Y}} \right\}, \quad (6.2)$$

i.e., $\mathcal{H}_{\mathcal{Y}}$ is the (vector) space of vectors of strategies with each component admissible for the corresponding agent, replicating elements in $\mathcal{Y}$, and $\mathcal{H}_{\mathcal{Y}}^{\perp}$ is its a.s. orthogonal complement in $\bigtimes_{i=1}^N \mathcal{H}_i$. Note that in (6.2), both H_i, H_i^* are d_i -dimensional and $H_i H_i^*$ stands for the ω -wise inner product. Under the assumptions of Theorem 6.9, $\mathcal{H}_{\mathcal{Y}}$ and $\mathcal{H}_{\mathcal{Y}}^{\perp}$ are closed in probability, we have

$$\bigtimes_{i=1}^N \mathcal{H}_i = \mathcal{H}_{\mathcal{Y}} \oplus \mathcal{H}_{\mathcal{Y}}^{\perp},$$

and in particular,

$$(H_1, \dots, H_N) \in \mathcal{H}_{\mathcal{Y}}^{\perp}, ((H_i \cdot X)_T)_{i=1, \dots, N} \in \mathcal{Y} \implies (H_1, \dots, H_N) = 0, \quad (6.3)$$

$$(H_1, \dots, H_N) \in \mathcal{H}_{\mathcal{Y}}^{\perp}, Z \in L^0(\Omega, \mathcal{F}_0, P) \implies Z(H_1, \dots, H_N) \in \mathcal{H}_{\mathcal{Y}}^{\perp}. \quad (6.4)$$

Proof We follow Schachermayer [27, Lemma 2.4]. It is elementary to check that $\mathcal{H}_{\mathcal{Y}} \cap \mathcal{H}_{\mathcal{Y}}^{\perp} = \{0\}$ because if $(H_1, \dots, H_N)$ belongs to that intersection, we have $\sum_{i=1}^N H_i H_i = 0$ which immediately gives $H_1 = \dots = H_N = 0$. Observe first that $\mathcal{H}_{\mathcal{Y}}$ is closed in probability; indeed, if we take a sequence $(H_1(n), \dots, H_N(n))_{n \in \mathbb{N}}$ in $\mathcal{H}_{\mathcal{Y}}$ converging in probability to $(H_1, \dots, H_N)$, up to passing to a subsequence, the convergence is a.s. and we have for $n \rightarrow \infty$ that

$$Y_n := \left((H_1(n) \cdot X)_T, \dots, (H_N(n) \cdot X)_T \right) \longrightarrow ((H_1 \cdot X)_T, \dots, (H_N \cdot X)_T) \quad \text{a.s.}$$

Since $\mathcal{Y}$ is closed in probability $((H_1 \cdot X)_T, \dots, (H_N \cdot X)_T) \in \mathcal{Y}$.

We also observe that $\mathcal{L} := (\bigtimes_{i=1}^N \mathcal{H}_i) \cap (\bigtimes_{i=1}^N (L^2(\Omega, \mathcal{F}, P))^{d_i})$ is a Hilbert space, as it is a closed subspace of a Hilbert space. Furthermore, $\mathcal{H}_{\mathcal{Y}}^2 := \mathcal{H}_{\mathcal{Y}} \cap \mathcal{L}$ is a closed vector subspace of $\mathcal{L}$ by closedness in probability of $\mathcal{H}_{\mathcal{Y}}$. Call $\mathcal{H}_{\mathcal{Y}}^{2, \perp}$ its orthogonal complement in $\mathcal{L}$ and denote by $\pi, \pi^{\perp}$ the corresponding orthogonal projections. We also stress that whenever $(H_1, \dots, H_N) \in \bigtimes_{i=1}^N \mathcal{H}_i$, we have for $|(H_1, \dots, H_N)| := \sum_{i=1}^N |H_i|$ (the absolute value of a vector stands for the sum of the absolute values of its components) that $\frac{1}{1 + |(H_1, \dots, H_N)|} (H_1, \dots, H_N) \in \mathcal{L}$. This is a consequence of assuming a common initial σ -algebra $\mathcal{F}_0$. We now define for

$$(H_1, \dots, H_N) \in \times_{i=1}^N \mathcal{H}_i$$

$$\text{Proj}((H_1, \dots, H_N) | \mathcal{H}_{\mathcal{Y}})$$

$$:= (1 + |(H_1, \dots, H_N)|) \pi \left(\frac{1}{1 + |(H_1, \dots, H_N)|} (H_1, \dots, H_N) \right),$$

$$\text{Proj}((H_1, \dots, H_N) | \mathcal{H}_{\mathcal{Y}}^\perp)$$

$$:= (1 + |(H_1, \dots, H_N)|) \pi^\perp \left(\frac{1}{1 + |(H_1, \dots, H_N)|} (H_1, \dots, H_N) \right).$$

It is immediate to verify that

$$\text{Proj}((H_1, \dots, H_N) | \mathcal{H}_{\mathcal{Y}}) + \text{Proj}((H_1, \dots, H_N) | \mathcal{H}_{\mathcal{Y}}^\perp) = (H_1, \dots, H_N).$$

Thus we are only left to verify that

$$\text{Proj}((H_1, \dots, H_N) | \mathcal{H}_{\mathcal{Y}}^\perp) \in \mathcal{H}_{\mathcal{Y}}^\perp$$

and that the same holds with $\mathcal{H}_{\mathcal{Y}}$ mutatis mutandis. We verify the first claim as the second follows similarly. We heavily rely on Assumption 2) in Theorem 6.9. We first show that $\mathcal{H}_{\mathcal{Y}}^{2,\perp} \subseteq \mathcal{H}_{\mathcal{Y}}^\perp$. Take $(H_1, \dots, H_N) \in \mathcal{H}_{\mathcal{Y}}^{2,\perp}$ and $(H_1^*, \dots, H_N^*) \in \mathcal{H}_{\mathcal{Y}}$. If $A := \{\sum_{i=1}^N H_i H_i^* > 0\}$ has positive probability, observing that Assumption 2) in Theorem 6.9 gives $1_A \frac{1}{1 + |(H_1^*, \dots, H_N^*)|} (H_1^*, \dots, H_N^*) \in \mathcal{H}_{\mathcal{Y}} \cap \times_{i=1}^N (L^2(\Omega, \mathcal{F}, P))^{d_i}$, we get

$$E_P \left[1_A \frac{1}{1 + |(H_1^*, \dots, H_N^*)|} \sum_{i=1}^N H_i H_i^* \right] > 0,$$

which is a contradiction. Arguing similarly with $\{\sum_{i=1}^N H_i H_i^* < 0\}$ yields that a.s., $\sum_{i=1}^N H_i H_i^* = 0$ and hence the desired inclusion. To conclude, if $(H_1, \dots, H_N)$ is in $\mathcal{H}_{\mathcal{Y}}$ and $Z \in L^0(\Omega, \mathcal{F}_0, P)$, then $Z(H_1, \dots, H_N) \in \mathcal{H}_{\mathcal{Y}}$ by Assumption 2) in Theorem 6.9, and the same property holds for $\mathcal{H}_{\mathcal{Y}}^\perp$ as a consequence. Equation (6.3) follows by observing that $(H_1, \dots, H_N)$ then belongs to $\mathcal{H}_{\mathcal{Y}} \cap \mathcal{H}_{\mathcal{Y}}^\perp$. $\square$

Proof of Theorem 6.9 We start by proving closedness in probability of the set $K^{\mathcal{Y}}$. Observe that the assumption NCA($\mathcal{Y}$) is only used in the very last step of the proof. Let $W_n = k_n + Y_n - \ell_n$, $n \in \mathbb{N}$, be a sequence in $K^{\mathcal{Y}}$ converging in probability (without loss of generality, almost surely) to some $W_\infty \in (L^0(\Omega, \mathcal{F}, P))^N$. Suppose that $k_n^i = (H_i(n) \cdot X)_T$ for $H_i(n) \in \mathcal{H}_i$. Set

$$(\hat{H}_1(n), \dots, \hat{H}_N(n)) := \text{Proj}((H_1, \dots, H_N) | \mathcal{H}_{\mathcal{Y}}^\perp).$$

Then $k_n^i = (\hat{H}_i(n) \cdot X)_T + \hat{Y}_n^i = \hat{k}_n^i + \hat{Y}_n^i$, where $\hat{Y}_n := (\hat{Y}_n^1, \dots, \hat{Y}_n^N) \in \mathcal{Y}$ by Lemma 6.10. Hence $W_n = \hat{k}_n + (Y_n + \hat{Y}_n) - \ell_n = \hat{k}_n + \bar{Y}_n - \ell_n$ and

$$W_n - \hat{k}_n = \bar{Y}_n - \ell_n. \quad (6.5)$$

Define now

$$A := \left\{ \liminf_{n \rightarrow \infty} \sum_{i=1}^N |\hat{H}_i(n)| = \infty \right\} \in \mathcal{F}_0.$$

Suppose first that $P[A] = 0$. In this case, we can apply an argument already used in the classical proof of the FTAP. By applying Föllmer and Schied [14, Lemma 1.64] to the sequence $(\eta_n)_{n \in \mathbb{N}} := (\hat{H}_1(n), \dots, \hat{H}_N(n))_{n \in \mathbb{N}}$, one can extract a strictly increasing sequence $(\tau_n)_{n \in \mathbb{N}}$ of $\mathcal{F}_0$ -measurable integer-valued random variables and $\eta \in \times_{i=1}^N (L^0(\Omega, \mathcal{F}_0, P))^{d_i}$ such that

$$\lim_{n \rightarrow \infty} (\hat{H}_1(\tau_n), \dots, \hat{H}_N(\tau_n)) = \eta \quad P\text{-a.s..}$$

One now verifies that

$$(\hat{H}_1(\tau_n), \dots, \hat{H}_N(\tau_n)) \in \mathcal{H}_{\mathcal{Y}}^\perp, \quad \eta \in \mathcal{H}_{\mathcal{Y}}^\perp, \quad \bar{Y}_{\tau_n} \in \mathcal{Y}, \quad \ell_{\tau_n} \in (L_+^0(\Omega, \mathcal{F}, P))^N.$$

To this end, observe that

$$(\hat{H}_1(\tau_n), \dots, \hat{H}_N(\tau_n)) = \lim_{K \rightarrow \infty} \sum_{k=0}^K 1_{\{\tau_n=k\}} (\hat{H}_1(k), \dots, \hat{H}_N(k)) \quad P\text{-a.s..} \quad (6.6)$$

Since $1_{\{\tau_n=k\}} \in L^0(\Omega, \mathcal{F}_0, P)$ and $(\hat{H}_1(k), \dots, \hat{H}_N(k)) \in \mathcal{H}_{\mathcal{Y}}^\perp$, we use (6.4) to get that $1_{\{\tau_n=k\}} (\hat{H}_1(k), \dots, \hat{H}_N(k)) \in \mathcal{H}_{\mathcal{Y}}^\perp$. Since $\mathcal{H}_{\mathcal{Y}}^\perp$ is a vector space, also the sums on the right-hand side of (6.6) belong to $\mathcal{H}_{\mathcal{Y}}^\perp$, and since the latter is closed under almost sure convergence, we also get that $(\hat{H}_1(\tau_n), \dots, \hat{H}_N(\tau_n))$ as the limit over K belongs to $\mathcal{H}_{\mathcal{Y}}^\perp$. Since then η is the a.s. limit of a sequence in $\mathcal{H}_{\mathcal{Y}}^\perp$ by definition, it also belongs to $\mathcal{H}_{\mathcal{Y}}^\perp$. A similar argument yields also $\ell_{\tau_n} \in (L_+^0(\Omega, \mathcal{F}, P))^N$ and

$$\bar{Y}_{\tau_n} = \lim_{K \rightarrow \infty} \sum_{k=0}^K 1_{\{\tau_n=k\}} \bar{Y}_k \in \mathcal{Y}.$$

From $W_{\tau_n} \rightarrow W_\infty$ as $n \rightarrow \infty$ and using (6.5), we then get

$$\begin{aligned} & W_\infty - ((\eta_1 \cdot X)_T, \dots, (\eta_N \cdot X)_T) \\ &= \lim_{n \rightarrow \infty} \left(W_{\tau_n} - ((\hat{H}_1(\tau_n) \cdot X)_T, \dots, (\hat{H}_N(\tau_n) \cdot X)_T) \right) \\ &= \lim_{n \rightarrow \infty} (\bar{Y}_{\tau_n} - \ell_{\tau_n}). \end{aligned}$$

Then $(\bar{Y}_{\tau_n} - \ell_{\tau_n})_{n \in \mathbb{N}} \subseteq (\mathcal{Y} - (L_+^0(\Omega, \mathcal{F}, P))^N)$ converges almost surely and by Assumption 1) in Theorem 6.9, its limit also belongs to $\mathcal{Y} - (L_+^0(\Omega, \mathcal{F}, P))^N$. This implies that $W_\infty - ((\eta_1 \cdot X)_T, \dots, (\eta_N \cdot X)_T) = Y_\infty - \ell_\infty$ for some $Y_\infty \in \mathcal{Y}$ and some $\ell_\infty \in (L_+^0(\Omega, \mathcal{F}, P))^N$, which concludes the proof.

We now prove that $P[A] > 0$ is in contradiction with $\text{NCA}(\mathcal{Y})$. Set

$$A_n := A \cap \left\{ \sum_{i=1}^N |\hat{H}_i(n)| > 0 \right\} \in \mathcal{F}_0,$$

$$Z_n := \frac{1}{\sum_{i=1}^N |\hat{H}_i(n)|} 1_{A_n} \in L^0(\Omega, \mathcal{F}_0, P).$$

Observe that for every n , we have $Z_n(\hat{H}_1(n), \dots, \hat{H}_N(n)) \in \mathcal{H}_{\mathcal{Y}}^\perp$ by (6.4) and $Z_n \bar{Y}_n \in \mathcal{Y}$ by Assumption 2) in Theorem 6.9. Additionally,

$$|Z_n(\hat{H}_1(n), \dots, \hat{H}_N(n))| = \frac{1}{\sum_{i=1}^N |\hat{H}_i(n)|} \sum_{i=1}^N |\hat{H}_i(n)| 1_{A_n} = 1_{A_n}$$

which in particular implies that $Z_n(\hat{H}_1(n), \dots, \hat{H}_N(n))$ is P -a.s. bounded and that

$$\lim_{n \rightarrow \infty} |Z_n(\hat{H}_1(n), \dots, \hat{H}_N(n))| = \lim_{n \rightarrow \infty} 1_{A_n} = 1_A \quad P\text{-a.s.} \quad (6.7)$$

By applying again [14, Lemma 1.64] to the sequence

$$(\xi_n)_{n \in \mathbb{N}} := \left(Z_n(\hat{H}_1(n), \dots, \hat{H}_N(n)) \right)_{n \in \mathbb{N}},$$

we deduce the existence of a strictly increasing sequence $(\sigma_n)_{n \in \mathbb{N}}$ of $\mathcal{F}_0$ -measurable integer-valued random variables and an element $\xi \in \times_{i=1}^N (L^0(\Omega, \mathcal{F}_0, P))^{d_i}$ with

$$\lim_{n \rightarrow \infty} Z_{\sigma_n}(\hat{H}_1(\sigma_n), \dots, \hat{H}_N(\sigma_n)) = \xi \quad P\text{-a.s.} \quad (6.8)$$

In particular, by (6.7) and since $\sigma_n \uparrow \infty$ as $n \rightarrow \infty$, we get

$$\begin{aligned} |\xi| &= \lim_{n \rightarrow \infty} |Z_{\sigma_n}(\hat{H}_1(\sigma_n), \dots, \hat{H}_N(\sigma_n))| \\ &= \lim_{n \rightarrow \infty} |Z_n(\hat{H}_1(n), \dots, \hat{H}_N(n))| = 1_A. \end{aligned} \quad (6.9)$$

As in (6.6), we can use the decomposition

$$Z_{\sigma_n}(\hat{H}_1(\sigma_n), \dots, \hat{H}_N(\sigma_n)) = \lim_{K \rightarrow \infty} \sum_{k=0}^K 1_{\{\sigma_n=k\}} Z_k(\hat{H}_1(k), \dots, \hat{H}_N(k)) \quad P\text{-a.s.},$$

the facts that $1_{\{\sigma_n=k\}} Z_k \in L^0(\Omega, \mathcal{F}_0, P)$ and $(\hat{H}_1(k), \dots, \hat{H}_N(k)) \in \mathcal{H}_{\mathcal{Y}}^\perp$ and (6.4) to deduce $1_{\{\sigma_n=k\}} Z_k(\hat{H}_1(k), \dots, \hat{H}_N(k)) \in \mathcal{H}_{\mathcal{Y}}^\perp$. Arguing as after (6.6), we then deduce that $\xi \in \mathcal{H}_{\mathcal{Y}}^\perp$ and similarly also $\ell_{\sigma_n} \in L_+^0(\Omega, \mathcal{F}, P)$ and $Z_{\sigma_n} \bar{Y}_{\sigma_n} \in \mathcal{Y}$. Now (6.8) yields that P -a.s., as $n \rightarrow \infty$,

$$Z_{\sigma_n} \hat{k}_{\sigma_n} = (Z_{\sigma_n} \hat{H}_1(\sigma_n) \cdot X)_T, \dots, (Z_{\sigma_n} \hat{H}_N(\sigma_n) \cdot X)_T \longrightarrow ((\xi_1 \cdot X)_T, \dots, (\xi_N \cdot X)_T)$$

with $\xi \in \mathcal{H}_{\mathcal{Y}}^{\perp}$. Moreover, observe that $Z_{\sigma_n} W_{\sigma_n} \rightarrow 0$ P -a.s. as $n \rightarrow \infty$ and then, using (6.5),

$$-((\xi_1 \cdot X)_T, \dots, (\xi_N \cdot X)_T) = \lim_{n \rightarrow \infty} (Z_{\sigma_n} W_{\sigma_n} - Z_{\sigma_n} \hat{K}_{\sigma_n}) = \lim_{n \rightarrow \infty} (Z_{\sigma_n} \bar{Y}_{\sigma_n} - Z_{\sigma_n} \ell_{\sigma_n})$$

Because $Z_{\sigma_n} \bar{Y}_{\sigma_n} - Z_{\sigma_n} \ell_{\sigma_n}$ belongs to $(\mathcal{Y} - (L_+^0(\Omega, \mathcal{F}, P))^N)^N$ for each n and $\mathcal{Y} - (L_+^0(\Omega, \mathcal{F}, P))^N$ is closed in probability by Assumption 1) in Theorem 6.9, we conclude that $-((\xi_1 \cdot X)_T, \dots, (\xi_N \cdot X)_T) \in \mathcal{Y} - (L_+^0(\Omega, \mathcal{F}, P))^N$. In particular, there exist $Y_{\infty} \in \mathcal{Y}$, $\ell_{\infty} \in (L_+^0(\Omega, \mathcal{F}, P))^N$ with

$$((\xi_1 \cdot X)_T, \dots, (\xi_N \cdot X)_T) + Y_{\infty} = \ell_{\infty} \geq 0. \quad (6.10)$$

Now $\text{NCA}(\mathcal{Y})$ yields $((\xi_1 \cdot X)_T, \dots, (\xi_N \cdot X)_T) + Y_{\infty} = 0$. Since $\xi \in \mathcal{H}_{\mathcal{Y}}^{\perp}$, the condition in (6.3) then implies $\xi = 0$, which yields a contradiction with (6.9) if $P[A] > 0$.

The above reasoning shows that $K^{\mathcal{Y}}$ is closed in probability. To show that also $\times_{i=1}^N K_i + \mathcal{Y}$ is closed in probability, we apply the same argument as in the second step of the proof of Theorem 6.3 and in particular the observation that $\times_{i=1}^N K_i + \mathcal{Y}$ can be obtained from $K^{\mathcal{Y}}$ by replacing in $K^{\mathcal{Y}}$ the cone $L_+^0(\Omega, \mathbf{F}_0, P)$ with $\{0\}$. Thus going through the above proof and replacing the elements of $L_+^0(\Omega, \mathbf{F}_0, P)$ with 0, one can conclude similarly to (6.10) that

$$((\xi_1 \cdot X)_T, \dots, (\xi_N \cdot X)_T) + Y_{\infty} = \ell_{\infty} = 0,$$

which yields the desired contradiction without even using $\text{NCA}(\mathcal{Y})$. $\square$

6.3.2 The multiperiod market with general $\mathcal{Y} \subseteq (L^0(\Omega, \mathcal{F}, P))^N$

In this section, we suppose that $\mathcal{T} = \{0, \dots, T\}$ and $\mathbb{F}^i = \mathbb{F}^j = \mathbb{F}$ for all i, j . We now extend the notation introduced in Sect. 2: for $0 \leq s < S \leq T$, we set

$$(H \cdot X)_s^S := \sum_{h \in (i)} (H^h \cdot X^h)_s^S \quad \text{for } H \in \mathcal{H}_i, \quad (6.11)$$

$$K_{i,s:S} := \{(H \cdot X)_s^S : H \in \mathcal{H}_i\} \subseteq K_i,$$

$$K_{s:S} := \{(H \cdot X)_s^S : H \in \mathcal{H}\} \subseteq K, \quad (6.12)$$

where $(H^h \cdot X^h)_s^S := \sum_{t=s+1}^S H_t^h (X_t^h - X_{t-1}^h)$. Observe in particular that $K_i = K_{i,0:T}$ and that $(H^h \cdot X^h)_s^S$ is the stochastic integral of H^h with respect to X^h on the time set $\{s, s+1, \dots, S\}$. We consider for some fixed $\hat{t} \in \{1, \dots, T\}$ a vector subspace $\mathcal{Y} \subseteq L^0(\Omega, \mathcal{F}_{\hat{t}}, P)^N$ and set

$$K_{s:S}^{\mathcal{Y}} := \times_{i=1}^N K_{i,s:S} - (L_+^0(\Omega, \mathcal{F}, P))^N + \mathcal{Y}, \quad 0 \leq s < S \leq T.$$

We consider the following conditions:

- Classical no-arbitrage condition for agent i in time period $\{s, \dots, S\}$:

$$\text{NA}_{i,s:S} : K_{i,s:S} \cap L_+^0(\Omega, \mathcal{F}, P) = \{0\}.$$

- Classical no-global-arbitrage condition in time period $\{s, \dots, S\}$:

$$\text{NA}_{s:S} : K_{s:S} \cap L_+^0(\Omega, \mathcal{F}, P) = \{0\}.$$

- No collective arbitrage in time period $\{s, \dots, S\}$:

$$\text{NCA}_{s:S}(\mathcal{Y}) : \left(\bigtimes_{i=1}^N K_{i,s:S} + \mathcal{Y} \right) \cap (L_+^0(\Omega, \mathcal{F}, P))^N = \{0\},$$

which is equivalent to $K_{s:S}^{\mathcal{Y}} \cap (L_+^0(\Omega, \mathcal{F}, P))^N = \{0\}$.

In this setting, we have $K_{0:T}^{\mathcal{Y}} = K^{\mathcal{Y}}$ and $\text{NCA}(\mathcal{Y})$ from (3.3) can be rewritten as

$$K_{0:T}^{\mathcal{Y}} \cap (L_+^0(\Omega, \mathcal{F}, P))^N = \{0\}.$$

Theorem 6.11 Suppose that $\mathcal{T} = \{0, \dots, T\}$ and $\mathbb{F}^i = \mathbb{F}^j = \mathbb{F}$ for all i, j . Fix $\hat{t} \in \{1, \dots, T\}$, assume that $\mathcal{Y} \subseteq (L^0(\Omega, \mathcal{F}_{\hat{t}}, P))^N$ is a vector space and that

1) $\mathcal{Y}$ is closed in probability and $\mathcal{Y} \subseteq \mathcal{Y}_0$.

2) for any $Z \in L^0(\Omega, \mathcal{F}_{\hat{t}-1}, P)$ and $Y \in \mathcal{Y}$, we have $ZY \in \mathcal{Y}$.

If $\text{NCA}(\mathcal{Y})$ holds, then $K_{0:T}^{\mathcal{Y}}$ and $\bigtimes_{i=1}^N K_{i,0:T} + \mathcal{Y}$ are closed in probability. If $\mathcal{Y}$ additionally satisfies (3.11), then

$$\begin{aligned} \text{NCA}(\mathcal{Y}) &\iff \text{NCA}_{t-1:t}(\mathcal{Y}), \forall t = 1, \dots, T \\ &\iff \begin{cases} \text{NA}_{i,t:t+1}, \forall i = 1, \dots, N, t = \hat{t}, \dots, T-1, \\ \text{NCA}_{\hat{t}-1:\hat{t}}(\mathcal{Y}), \\ \text{NA}_{t,t+1}, \forall t = 0, \dots, \hat{t}-2. \end{cases} \end{aligned} \quad (6.13)$$

Proof By Theorem 6.3 and Remark 6.2,

$$K_{\hat{t}:T}^{\mathcal{Y}} \text{ and } \bigtimes_{i=1}^N K_{i,\hat{t}:T} + \mathcal{Y} \text{ are closed in probability.} \quad (6.14)$$

Note that $Zk \in \bigtimes_{i=1}^N K_{i,\hat{t}:T}$ for all $Z \in L^0(\Omega, \mathcal{F}_{\hat{t}-1}, P)$ and $k \in \bigtimes_{i=1}^N K_{i,\hat{t}:T}$. Consider the vector space $\overline{\mathcal{Y}} := \bigtimes_{i=1}^N K_{i,\hat{t}:T} + \mathcal{Y}$ and set $\overline{\mathcal{Y}} - (L_+^0(\Omega, \mathcal{F}, P))^N = K_{\hat{t}:T}^{\mathcal{Y}}$. Then by (6.14), $\overline{\mathcal{Y}}$ and $\overline{\mathcal{Y}} - (L_+^0(\Omega, \mathcal{F}, P))^N$ are closed in probability and so $\overline{\mathcal{Y}}$ satisfies 1) and 2) in Theorem 6.9 for the time period $\{\hat{t}-1, \hat{t}\}$. Thus by Theorem 6.9, $\bigtimes_{i=1}^N K_{i,\hat{t}-1:\hat{t}} + \overline{\mathcal{Y}}$ is closed in probability. One can check that $\text{NCA}(\mathcal{Y})$ implies

$\text{NCA}_{\hat{t}-1, \hat{t}}(\bar{\mathcal{Y}})$. Thus again by Theorem 6.9, also $K_{\hat{t}-1, \hat{t}}^{\bar{\mathcal{Y}}}$ is closed in probability. But

$$\begin{aligned} K_{\hat{t}-1, \hat{t}}^{\bar{\mathcal{Y}}} &= \bigotimes_{i=1}^N K_{i, \hat{t}-1, \hat{t}} + \bar{\mathcal{Y}} - (L_+^0(\Omega, \mathcal{F}, P))^N \\ &= \bigotimes_{i=1}^N K_{i, \hat{t}-1, \hat{t}} + \bigotimes_{i=1}^N K_{i, \hat{t}, T} + \mathcal{Y} - (L_+^0(\Omega, \mathcal{F}, P))^N \\ &= \bigotimes_{i=1}^N K_{i, \hat{t}-1, T} + \mathcal{Y} - (L_+^0(\Omega, \mathcal{F}, P))^N = K_{\hat{t}-1, T}^{\mathcal{Y}}, \end{aligned}$$

which shows that $K_{\hat{t}-1, T}^{\mathcal{Y}}$ is closed in probability.

Similarly, as $\bigotimes_{i=1}^N K_{i, \hat{t}-1, \hat{t}} + \bar{\mathcal{Y}} = \bigotimes_{i=1}^N K_{i, \hat{t}-1, T} + \mathcal{Y}$, also $\bigotimes_{i=1}^N K_{i, \hat{t}-1, T} + \mathcal{Y}$ is closed in probability. Thus by backward induction, $K_{0, T}^{\mathcal{Y}}$ and $\bigotimes_{i=1}^N K_{i, 0, T} + \mathcal{Y}$ are both closed in probability. As to the equivalence (6.13), we claim that

$$\text{NCA}(\mathcal{Y}) \iff \begin{cases} \text{NA}_{i, \hat{t}, T}, \forall i = 1, \dots, N, \\ \text{NCA}_{\hat{t}-1, \hat{t}}(\mathcal{Y}), \\ \text{NA}_{0, \hat{t}-1}. \end{cases} \quad (6.15)$$

Furthermore, by standard no-arbitrage arguments (see [14, Proposition 5.11]), it can be shown that

$$\begin{cases} \text{NA}_{i, \hat{t}, T}, \forall i = 1, \dots, N, \\ \text{NCA}_{\hat{t}-1, \hat{t}}(\mathcal{Y}), \\ \text{NA}_{0, \hat{t}-1}. \end{cases} \iff \begin{cases} \text{NA}_{i, t, t+1}, \forall i = 1, \dots, N, t = \hat{t}, \dots, T-1, \\ \text{NCA}_{\hat{t}-1, \hat{t}}(\mathcal{Y}), \\ \text{NA}_{t, t+1}, \forall t = 0, \dots, \hat{t}-2. \end{cases} \quad (6.16)$$

From (6.15) and (6.16), the equivalence “ $\text{NCA}(\mathcal{Y}) \Leftrightarrow \text{NCA}_{t-1, t}(\mathcal{Y}), \forall t = 1, \dots, T$ ” follows by applying Theorem 6.5 in one period for the case $t = \hat{t}, \dots, T-1$ and Proposition 6.6 for $t = 0, \dots, \hat{t}-2$.

We come to the proof of (6.15). To begin with, observe that since $\mathbb{R}_0^N \subseteq \mathcal{Y}$, we have using (3.10) and 2) that

$$(\mathcal{Y}_0 \cap (L^0(\Omega, \mathcal{F}_{\hat{t}-1}, P))^N) \subseteq \mathcal{Y}. \quad (6.17)$$

Let us now show “ $\Rightarrow$ ” in (6.15). Assuming $\text{NCA}(\mathcal{Y})$, we have $\text{NCA}_{\hat{t}-1, \hat{t}}(\mathcal{Y})$ directly by inclusion arguments, and $\text{NA}_{i, \hat{t}, T}, \forall i = 1, \dots, N$, by using (3.7). To show $\text{NA}_{0, \hat{t}-1}$, observe that by (6.17), we have again by inclusion arguments that

$$\left(\bigotimes_{i=1}^N K_{i, 0, \hat{t}-1} + (\mathcal{Y}_0 \cap (L^0(\Omega, \mathcal{F}_{\hat{t}-1}, P))^N) \right) \cap (L_+^0(\Omega, \mathcal{F}, P))^N = \{0\}.$$

This means that $\text{NCA}_{0, \hat{t}-1}(\mathcal{Y}_0 \cap (L^0(\Omega, \mathcal{F}_{\hat{t}-1}, P))^N)$ holds. Invoking Proposition 6.6, we get that also $\text{NA}_{0, \hat{t}-1}$ holds.

We now prove “ $\Leftarrow$ ” in (6.15) by showing that the contraposition holds. Let us say that for an N -dimensional random vector Z , property CA holds if $Z^i \geq 0$ a.s. for every $i = 1, \dots, N$ and $P[Z^j > 0] > 0$ for some j . For later use, we stress that CA does *not* hold for a given random vector Z if either $P[Z^j < 0] > 0$ for some j , or $Z^i \leq 0$ P -a.s. for every i . Take a collective arbitrage, i.e., $k \in \times_{i=1}^N K_i$, $Y \in \mathcal{Y}$ such that property CA holds for $k + Y$. Based on (6.11) and (6.12), we write $k = \sum_{t=0}^{T-1} k_{t,t+1}$ for $k_{t,t+1} \in \times_{i=1}^N K_{i,t:t+1}$, $t = 0, \dots, T-1$. Observe that by setting $\hat{k} = \frac{1}{N} \sum_{i=1}^N \sum_{t=0}^{\hat{T}-2} k_{t,t+1}^i \in K_{0:\hat{T}-1}$ and taking the vector $\mathbf{1} \in \mathbb{R}^N$ with all components equal to 1, we have $\sum_{t=0}^{\hat{T}-2} k_{t,t+1} = \hat{k}\mathbf{1} + \sum_{t=0}^{\hat{T}-2} k_{t,t+1} - \hat{k}\mathbf{1}$ and

$$\hat{Y} = \sum_{t=0}^{\hat{T}-2} k_{t,t+1} - \hat{k}\mathbf{1} \in \mathcal{Y}_0 \cap (L^0(\Omega, \mathcal{F}_{\hat{T}-1}, P))^N \subseteq \mathcal{Y},$$

where the latter inclusion is by (6.17). Then

$$\hat{k}\mathbf{1} + k_{\hat{T}-1,\hat{T}} + \sum_{t=\hat{T}}^{T-1} k_{t,t+1} + (\hat{Y} + Y) = k + Y$$

satisfies property CA, and $\hat{Y} + Y \in \mathcal{Y}$ since the latter is a convex cone. We now apply an iterative procedure backwards.

Case 1: $Z := \hat{k}\mathbf{1} + k_{\hat{T}-1,\hat{T}} + (\hat{Y} + Y)$ does not satisfy property CA. Then by the definition of the property CA, there are two alternatives:

- $P[Z^j < 0] > 0$ for some j : In this case, since $\{Z^j < 0\} \in \mathcal{F}_{\hat{T}}$, we have

$$h^j := 1_{\{Z^j < 0\}} \sum_{t=\hat{T}}^{T-1} k_{t,t+1}^j \in K_{j,\hat{T}:T}$$

and $Z^j + \sum_{t=\hat{T}}^{T-1} k_{t,t+1}^j = k^j + Y^j$. Thus

$$\begin{aligned} h^j &= 1_{\{Z^j < 0\}} \left(Z^j + \sum_{t=\hat{T}}^{T-1} k_{t,t+1}^j \right) - 1_{\{Z^j < 0\}} Z^j \\ &= 1_{\{Z^j < 0\}} (k^j + Y^j) - 1_{\{Z^j < 0\}} Z^j \geq 0 \end{aligned}$$

since $k + Y$ satisfies property CA, and so $k^j + Y^j \geq 0$. From $P[Z^j < 0] > 0$, we now obtain $P[h^j > 0] > 0$, violating $\text{NA}_{j,\hat{T}:T}$.

- $Z^j \leq 0$ P -a.s. for every j : In this case, $h^j := \sum_{t=\hat{T}}^{T-1} k_{t,t+1}^j \in K_{j,\hat{T}:T}$ and additionally $h^j = k^j + Y^j - Z^j \geq k^j + Y^j$ so that h^j violates $\text{NA}_{j,\hat{T}:T}$ for every j .

Case 2: $\hat{k}\mathbf{1} + k_{\hat{T}-1,\hat{T}} + (\hat{Y} + Y)$ satisfies property CA.

Case 2.1: $\hat{k}\mathbf{1}$ does not satisfy property CA. Then there are two alternatives:

• $P[\widehat{k} < 0] > 0$: Then we get $1_{\{\widehat{k} < 0\}} k_{\widehat{t}-1, \widehat{t}} \in \times_{i=1}^N K_{i, \widehat{t}-1, \widehat{t}}$ from $\{\widehat{k} < 0\} \in \mathcal{F}_{\widehat{t}-1}$, and $1_{\{\widehat{k} < 0\}} (\widehat{Y} + Y) \in \mathcal{Y}$ by 2). So $h := 1_{\{\widehat{k} < 0\}} k_{\widehat{t}-1, \widehat{t}} + 1_{\{\widehat{k} < 0\}} (\widehat{Y} + Y) \in K_{\widehat{t}-1, \widehat{t}}^{\mathcal{Y}}$ and

$$\begin{aligned} h &= 1_{\{\widehat{k} < 0\}} k_{\widehat{t}-1, \widehat{t}} + 1_{\{\widehat{k} < 0\}} (\widehat{Y} + Y) \\ &= 1_{\{\widehat{k} < 0\}} (\widehat{k}\mathbf{1} + k_{\widehat{t}-1, \widehat{t}} + (\widehat{Y}^j + Y^j)) - 1_{\{\widehat{k} < 0\}} \widehat{k}\mathbf{1} \geq 0 \end{aligned}$$

componentwise by the condition in Case 2. Moreover, as $P[\widehat{k} < 0] > 0$, we also have $P[h^j > 0] > 0$ for each j , violating $\text{NCA}_{\widehat{t}-1, \widehat{t}}(\mathcal{Y})$.

• $\widehat{k} \leq 0$ P -a.s.: Note that $h = k_{\widehat{t}-1, \widehat{t}} + (\widehat{Y} + Y) \in K_{\widehat{t}-1, \widehat{t}}^{\mathcal{Y}}$ and

$$h = (\widehat{k}\mathbf{1} + k_{\widehat{t}-1, \widehat{t}} + (\widehat{Y} + Y)) - \widehat{k}\mathbf{1} \geq 0$$

componentwise by the condition in Case 2 which also implies that some component of h is positive with positive probability. Thus h violates $\text{NCA}_{\widehat{t}-1, \widehat{t}}(\mathcal{Y})$.

Case 2.2: $\widehat{k}\mathbf{1}$ satisfies property CA. Then it is immediate to see that $\widehat{k}$ violates $\text{NA}_{0, \widehat{t}-1}$. $\square$

As a corollary of Theorems 3.12 and 6.11, we deduce the following collective FTAP.

Theorem 6.12 Suppose that $\mathcal{T} = \{0, \dots, T\}$ and $\mathbb{F}^i = \mathbb{F}^j = \mathbb{F}$ for all i, j . Fix $\widehat{t} \in \{1, \dots, T\}$ and let $\mathcal{Y} \subseteq (L^0(\Omega, \mathcal{F}_{\widehat{t}}, P))^N$ be a vector space containing $\mathbb{R}_0^N$. If 1) and 2) in Theorem 6.11 are satisfied, then

$$\text{NCA}(\mathcal{Y}) \quad \text{if and only if} \quad (C^{\mathcal{Y}})_1^0 \cap \mathcal{P}_e^N \neq \emptyset$$

and

$$(C^{\mathcal{Y}})_1^0 = \{(Q^1, \dots, Q^N) \in (C^{\mathcal{Y}})_1^0 : Q^i = Q^j \text{ on } \mathcal{F}_{\widehat{t}-1}, \forall i, j = 1, \dots, N\}.$$

If additionally $\mathcal{Y} \subseteq L^1(\Omega, \mathbf{F}_T, P)$, then

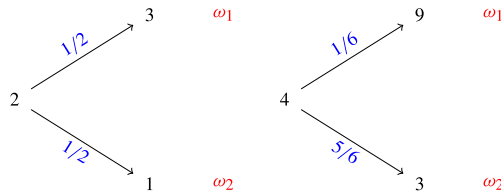
$$\text{NCA}(\mathcal{Y}) \quad \text{if and only if} \quad \mathcal{M}^{\mathcal{Y}} \cap \mathcal{P}_e^N \neq \emptyset.$$

Proof We only need to prove (6.12). Recall that e^i, e^j are elements in the canonical basis of $\mathbb{R}^N$ and so $e^i - e^j \in \mathbb{R}_0^N$ for all i, j . Observe that (6.17) holds under the assumptions of Theorem 6.11. From (6.17), we have $1_A(e^i - e^j) \in \mathcal{Y}$ for any $A \in \mathcal{F}_{\widehat{t}-1}$. This in turns implies, since $\mathcal{Y}$ is a vector space, that $Q^i[A] = Q^j[A]$ for every i, j . $\square$

Example 6.13 In Example 7.2.3, we provide an example of a two-period market with two assets and two agents where there exists a (global) arbitrage, $\text{NCA}(\mathcal{Y})$ holds true and it is possible to profit from cooperation thanks to (6.12) when superreplicating claims.

Table 1 Overview of some examples. We set $\widehat{\mathcal{Y}} = \mathcal{Y} + \mathbb{R}_0^N$

NA	NCA($\mathcal{Y}$)	NCA($\widehat{\mathcal{Y}}$)	NA $_i \forall i$	$M_{\mathcal{Y}} \neq \emptyset$	$\pi_+^{\mathcal{Y}} < \pi_+^N$	$\rho_+^{\mathcal{Y}} < \rho_+^N$	Example
✗	✓	✓	✓	✓	✓	✗	7.1.1
✗	✓	✓	✓	✓	✓	✓	7.2.3
✗	✓	✗	✓	✗	✓	✓	7.1.3–7.1.5

**Fig. 1** Tree for the stocks (X^1, X^2) at times $t = 0, 1$

7 Examples

In Sects. 7.1 and 7.2, the set Ω is finite and therefore the convex cone $K^{\mathcal{Y}}$ is closed in probability by the results of Sect. 5. We present in Table 1 an overview of some key features of the subsequent examples.

7.1 A one-period toy example

Example 7.1.1 below is possibly the simplest example of a market where there exists a (global) arbitrage, NCA($\mathcal{Y}$) holds true and it is possible to profit from cooperation in computing the superreplication $\pi_+^{\mathcal{Y}}(g)$ of any claim.

Let $\mathcal{T} = \{0, 1\}$, $|\Omega| = 2$ with $P[\{\omega_1\}] > 0$ and $P[\{\omega_2\}] > 0$, $\mathcal{F}_0$ the trivial σ -algebra and $\mathcal{F}_1^1 = \mathcal{F}_1^2 := \mathcal{F}_1$ the σ -algebra of all subsets of Ω . As before, assume zero interest rate. In addition to the riskless asset, consider two assets with (discounted) price processes $X^1 = (X_0^1, X_1^1)$, $X^2 = (X_0^2, X_1^2)$ adapted to $(\mathcal{F}_0, \mathcal{F}_1)$ and with $X_t^i > 0$ for all i and t . Recall also the notation for the spaces $K_1 := \{H^1(X_1^1 - X_0^1) : H^1 \in \mathbb{R}\}$, $K_2 := \{H^2(X_1^2 - X_0^2) : H^2 \in \mathbb{R}\}$ and finally $K := \{H^1(X_1^1 - X_0^1) + H^2(X_1^2 - X_0^2) : (H^1, H^2) \in \mathbb{R}^2\} = K_1 + K_2$. Suppose that $M_1 \cap \mathcal{P}_e = \{Q^1\}$ and $M_2 \cap \mathcal{P}_e = \{Q^2\}$, so that NA $_1$ and NA $_2$ hold true and each single market is complete. Suppose also that $Q^1 \neq Q^2$; so $M \cap \mathcal{P}_e = M_1 \cap M_2 \cap \mathcal{P}_e = \emptyset$, which implies the existence of a (global) arbitrage in the market K . This is the case for example when

$$X_0^1 = 2, X_1^1 = (3, 1), \quad X_0^2 = 4, X_1^2 = (9, 3)$$

as in Fig. 1, for which $Q^1 = (\frac{1}{2}, \frac{1}{2})$ and $Q^2 = (\frac{1}{6}, \frac{5}{6})$.

We now consider two agents, where agent i is allowed to invest in the single market K_i , $i = 1, 2$. In this setup, we make several choices for the set $\mathcal{Y}$ describing different ways the agents may cooperate.

Example 7.1.1 This is the setting of Sects. 5 and 6.1. Take

$$\mathcal{Y} = \{Y \in (L^0(\Omega, \mathcal{F}_0, P))^2 = \mathbb{R}^2 : Y^1 + Y^2 = 0\}.$$

By Theorem 6.1, we have “NCA($\mathcal{Y}$) $\Leftrightarrow$ (NA₁ and NA₂)”. From the above setup, we thus conclude that NCA($\mathcal{Y}$) holds true. Observe that $\mathcal{M}^{\mathcal{Y}} \cap \mathcal{P}_e^2 := \{(Q^1, Q^2)\}$ is not empty, and thus we may also deduce that NCA($\mathcal{Y}$) holds true from the collective FTAP in Theorem 6.5.

So now we can apply the pricing–hedging duality from Sect. 4. Each single agent may superreplicate, and actually replicate since each single market is complete, any single contingent claim $f \in L^1(\Omega, \mathcal{F}_1, P)$ and we have

$$\rho_{i,+}(f) = \sup_{Q \in M_i} E_Q[f] = E_{Q^i}[f], \quad \forall f \in L^1(\Omega, \mathcal{F}_1, P), i = 1, 2.$$

From the pricing–hedging duality in Theorem 4.12 and from (4.3)–(4.5), we deduce for any $g = (g^1, g^2) \in (L^1(\Omega, \mathcal{F}_1, P))^2$ that

$$\begin{aligned} \rho_+^{\mathcal{Y}}(g) &= \sup_{Q \in \mathcal{M}^{\mathcal{Y}}} \sum_{i=1}^2 E_{Q^i}[g^i] = E_{Q^1}[g^1] + E_{Q^2}[g^2] = \rho_+^N(g), \\ \pi_+^N(g) &= \max_i \rho_{i,+}(g^i) = \max\{E_{Q^1}[g^1], E_{Q^2}[g^2]\}, \\ \pi_+^{\mathcal{Y}}(g) &= \frac{1}{2} \rho_+^{\mathcal{Y}}(g) = \frac{1}{2} E_{Q^1}[g^1] + \frac{1}{2} E_{Q^2}[g^2]. \end{aligned}$$

The fact that $\rho_+^N = \rho_+^{\mathcal{Y}}$ is obvious, as now the elements in $\mathcal{Y}$ are vectors in $\mathbb{R}^2$ and so the two definitions coincide. The value of ρ_+^N is in agreement with intuition, as each agent i is allowed to (super)replicate the single claim g^i investing only in the corresponding market, and then the total cost for superreplicating both claims must be $E_{Q^1}[g^1] + E_{Q^2}[g^2]$.

We also see that it is possible to profit from cooperation since $\pi_+^{\mathcal{Y}}(g)$ is always strictly smaller than $\pi_+^N(g)$ except when $E_{Q^1}[g^1] = E_{Q^2}[g^2]$, as can be seen from the dual formulations above.

Example 7.1.2 This is the setting of Sects. 5 and 6.2. Take

$$\mathcal{Y} = \mathcal{Y}_0 = \{Y \in (L^1(\Omega, \mathcal{F}_1, P))^2 : Y^1 + Y^2 = 0\}.$$

By Proposition 6.6, we have “NCA($\mathcal{Y}$) $\Leftrightarrow$ NA”. From the above setup, we thus conclude that there exists a collective arbitrage. Observe that for $(Q^1, Q^2) \in \mathcal{M}^{\mathcal{Y}}$, it is necessary that $E_{Q^1}[Y^1] + E_{Q^2}[Y^2] \leq 0$ for all $(Y^1, Y^2) \in \mathcal{Y}$, which implies $Q^1 = Q^2$. Thus $\mathcal{M}^{\mathcal{Y}} \cap \mathcal{P}_e^2 = \emptyset$ and we can also deduce the existence of a collective arbitrage from the collective FTAP in Theorem 5.1.

Example 7.1.3 This is a case where there exists a global arbitrage, $\text{NCA}(\mathcal{Y})$ holds true, but at the same time $(C^{\mathcal{Y}})_1^0 = \mathcal{M}^{\mathcal{Y}} = \emptyset$. Take

$$\mathcal{Y} := \{(\lambda X_1^1 + \mu X_1^2, -\lambda X_1^1 - \mu X_1^2), \lambda, \mu \in \mathbb{R}\} \subseteq \mathcal{Y}_0,$$

which is a closed vector space not fulfilling (3.11). Recall that we assume $Q^1 \neq Q^2$. Then we have $E_{Q^2}[\frac{X_1^1}{X_0^1}] \neq 1$ and $E_{Q^1}[\frac{X_2^2}{X_0^2}] \neq 1$. We show that in this simple setting, the condition

$$E_{Q^2}\left[\frac{X_1^1}{X_0^1}\right]E_{Q^1}\left[\frac{X_2^2}{X_0^2}\right] = 1 \quad (7.1)$$

discriminates between $\text{NCA}(\mathcal{Y})$ and $\text{CA}(\mathcal{Y})$. One may check that if (7.1) is not satisfied, then $(C^{\mathcal{Y}})^0 = \{(0, 0)\}$. In this case, there exists a collective arbitrage by Proposition 3.9. Otherwise, if (7.1) is satisfied, then

$$\begin{aligned} (C^{\mathcal{Y}})^0 &= \left\{ \alpha \left(z^1 E_{Q^2}\left[\frac{X_1^1}{X_0^1}\right], z^2 \right) : \alpha \in \mathbb{R}, \alpha \geq 0 \right\}, \\ (C^{\mathcal{Y}})_1^0 &= \mathcal{M}^{\mathcal{Y}} = \emptyset \end{aligned} \quad (7.2)$$

for $z^i := \frac{dQ^i}{dP}$. Observe that the explicit formulation of $(C^{\mathcal{Y}})^0$ immediately yields $(C^{\mathcal{Y}})_1^0 = \emptyset$ as the expectations under P of two components $(z^1 E_{Q^2}[\frac{X_1^1}{X_0^1}], z^2)$ cannot be equal due to $E_{Q^2}[\frac{X_1^1}{X_0^1}] \neq 1$. Since there exists a positive element in $(C^{\mathcal{Y}})^0$, we know by Proposition 3.11 that $\text{NCA}(\mathcal{Y})$ holds true. To summarise, this is an example where $\mathcal{Y}$ does not fulfil (3.11) and where, if $Q^1 \neq Q^2$ and (7.1) is satisfied, there exists a global arbitrage, $\text{NCA}(\mathcal{Y})$, NA_1 and NA_2 hold true, and $(C^{\mathcal{Y}})_1^0 = \mathcal{M}^{\mathcal{Y}} = \emptyset$.

Example 7.1.4 This is a case where $\mathcal{Y}$ does not fulfil (3.11), $\rho_+^{\mathcal{Y}} = -\infty$ and $\rho_+^{\mathcal{Y}} < \rho_+^N$ and $\pi_+^{\mathcal{Y}} < \pi_+^N$. We use the same setting as in Example 7.1.3 and suppose that (7.1) is satisfied so that $\text{NCA}(\mathcal{Y})$ holds true and $(C^{\mathcal{Y}})_1^0 = \mathcal{M}^{\mathcal{Y}} = \emptyset$. Regarding the pricing-hedging duality, we cannot use (4.9), but we apply (4.7) to deduce by using (7.2) that

$$\begin{aligned} \rho_+^{\mathcal{Y}}(g) &= \inf \left\{ \sum_{i=1}^2 m^i : m \in \mathbb{R}^2, \sum_{i=1}^2 m^i E[z^i] \geq \sum_{i=1}^2 E[z^i g^i], \forall z \in (C^{\mathcal{Y}})^0 \right\} \\ &= \inf \left\{ m^1 + m^2 : E_{Q^2}\left[\frac{X_1^1}{X_0^1}\right]m^1 + m^2 \geq E_{Q^2}\left[\frac{X_1^1}{X_0^1}\right]E_{Q^1}[g^1] + E_{Q^2}[g^2] \right\} \\ &= \inf\{m^1 + m^2 : \beta m^1 + m^2 \geq \beta a + c\} = -\infty \end{aligned} \quad (7.3)$$

for $\beta := E_{Q^2}[\frac{X_1^1}{X_0^1}] \neq 1$, $a := E_{Q^1}[g^1]$, $c := E_{Q^2}[g^2]$. To show (7.3), we may assume without loss of generality that $\beta < 1$. For $\delta \in \mathbb{R}$, define $m^1 := a - \delta$, $m^2 := \beta\delta + c$. We compute $m^1 + m^2 = a + c + \delta(\beta - 1) \rightarrow -\infty$ if $\delta \rightarrow \infty$. The fact that $\rho_+^{\mathcal{Y}} = -\infty$

shows that if $\mathcal{Y}$ does not fulfil (3.11), then Lemma 4.4, 1), namely $\rho_+^{\mathcal{Y}}(0) = 0$, cannot hold true. As $\rho_+^{\mathcal{Y}}(g) = -\infty$ and at the same time $\rho_+^N(g) = E_{Q^1}[g^1] + E_{Q^2}[g^2] \in \mathbb{R}$ for all $g \in L^0 \times L^0$, this is an (extreme) example where $\rho_+^{\mathcal{Y}}(g) < \rho_+^N(g)$ for all $g \in L^0 \times L^0$ (recall that here Ω is finite). Computing $\pi_+^{\mathcal{Y}}(g)$ by using (4.6) and (7.2), one obtains

$$\pi_+^{\mathcal{Y}}(g) = \gamma E_{Q^1}[g^1] + (1 - \gamma) E_{Q^2}[g^2] \quad \text{for } \gamma = \frac{\beta}{1 + \beta} \in (0, 1) \setminus \left\{ \frac{1}{2} \right\},$$

so that $\pi_+^{\mathcal{Y}}(g) < \pi_+^N(g) = \max\{E_{Q^1}[g^1], E_{Q^2}[g^2]\}$ whenever $E_{Q^1}[g^1] \neq E_{Q^2}[g^2]$.

Example 7.1.5 To illustrate the relation between $\text{NCA}(\mathcal{Y})$ and $\text{NCA}(\mathcal{Y} + \mathbb{R}_0^N)$, we use the same setting as Example 7.1.3 and consider the convex cone $\widehat{\mathcal{Y}} := \mathcal{Y} + \mathbb{R}_0^N$ which, differently from $\mathcal{Y}$, fulfils (3.11). It can be checked that $(C^{\widehat{\mathcal{Y}}})_1^0 = (C^{\mathcal{Y}})_1^0$. Thus under (7.1), we have $(C^{\widehat{\mathcal{Y}}})_1^0 = \emptyset$, which then implies by the last statement in Proposition 3.9 the existence of a collective arbitrage for $\widehat{\mathcal{Y}}$. This proves that $\text{NCA}(\mathcal{Y})$ and $\text{NCA}(\mathcal{Y} + \mathbb{R}_0^N)$ are in general not equivalent. Observe that when there exists a collective arbitrage for $\widehat{\mathcal{Y}}$, we cannot apply Lemma 4.4, 1) and conclude that $\rho_+^{\widehat{\mathcal{Y}}}(0) = 0$. Indeed, by Lemma 4.4, 6) and 4), we know in this example that $\rho_+^{\widehat{\mathcal{Y}}} = \rho_+^{\mathcal{Y}} = -\infty$.

7.2 A two-period example with exchanges $\mathcal{Y}$ that are $\mathcal{F}_1$ -measurable

We provide in Example 7.2.3 below an example of a two-period market with two assets and two agents where there exists a (global) arbitrage, $\text{NCA}(\mathcal{Y})$ holds true and it is possible to profit from cooperation when superreplicating two claims.

Let $\mathcal{T} = \{0, 1, 2\}$ and assume zero interest rate. In addition to the riskless asset, consider two assets with (discounted) price processes $X^1 = (X_0^1, X_1^1, X_2^1)$ and $X^2 = (X_0^2, X_1^2, X_2^2)$, and let $\mathbb{F}^i = (\mathcal{F}_t^i)_{t \in \mathcal{T}}$ be the filtration generated by X^i . We assume that $\mathbb{F}^1 = \mathbb{F}^2$, so set $\mathcal{F}_t^1 = \mathcal{F}_t^2 =: \mathcal{F}_t$ for all t , and that $\mathcal{F}_0$ is the trivial σ -algebra. We let $\mathcal{H}$ be the class of one-dimensional predictable processes. We then consider the three spaces of terminal gains from trading strategies given by

$$\begin{aligned} K_1 &:= \{H_1^1(X_1^1 - X_0^1) + H_2^1(X_2^1 - X_1^1) : H^1 \in \mathcal{H}\}, \\ K_2 &:= \{H_1^2(X_1^2 - X_0^2) + H_2^2(X_2^2 - X_1^2) : H^2 \in \mathcal{H}\}, \\ K &:= \{(H^1 \cdot X^1)_2 + (H^2 \cdot X^2)_2 : (H^1, H^2) \in \mathcal{H} \times \mathcal{H}\} = K_1 + K_2. \end{aligned}$$

We consider two agents, with agent i allowed to invest in the single market K_i . Suppose that $|M_1 \cap \mathcal{P}_e| = |M_2 \cap \mathcal{P}_e| = \infty$; so NA_1 and NA_2 hold true and each market is incomplete. Each agent i can superreplicate any contingent claim $f \in L^1(\Omega, \mathcal{F}_2, P)$, and $\rho_{i,+}(f) = \sup_{Q \in M_i} E_Q[f]$ for all $f \in L^1(\Omega, \mathcal{F}_2, P)$ holds for $i = 1, 2$. Suppose additionally that $M \cap \mathcal{P}_e = M_1 \cap M_2 \cap \mathcal{P}_e = \emptyset$, so that there exists a (global) arbitrage in the market K .

To be concrete, we provide an example of a market satisfying these conditions. Take $|\Omega| = 6$ with $P[\{\omega_j\}] > 0$ for all $j = 1, \dots, 6$; let $\mathcal{F}_0$ be the trivial σ -algebra,

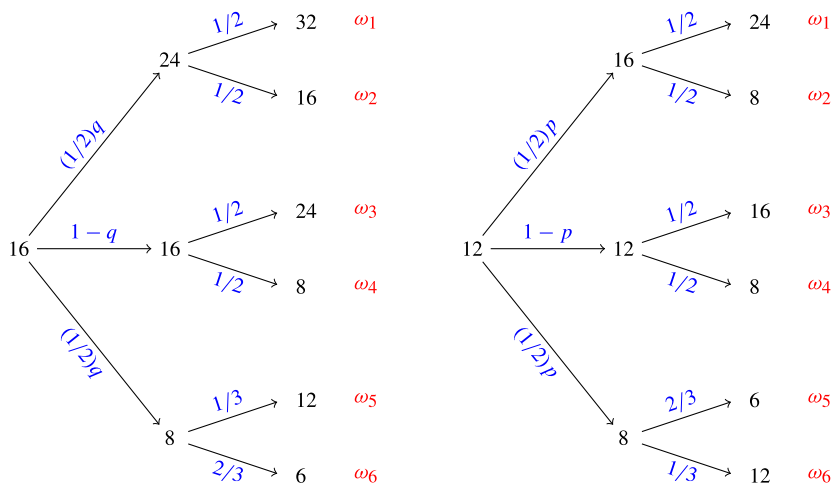


Fig. 2 Tree for the stocks (X^1, X^2) at times $t = 0, 1, 2$

$\mathcal{F}_1^1 = \mathcal{F}_1^2 := \mathcal{F}_1$ the σ -algebra generated by the subsets $\{\omega_1, \omega_2\}$, $\{\omega_3, \omega_4\}$, $\{\omega_5, \omega_6\}$, and $\mathcal{F}_2^1 = \mathcal{F}_2^2 := \mathcal{F}_2$ the σ -algebra of all subsets of Ω . Let

$$\begin{aligned} X_0^1 &= 16, & X_1^1 &= (24, 16, 8), & X_2^1 &= (32, 16, 24, 8, 12, 6), \\ X_0^2 &= 12, & X_1^2 &= (16, 12, 8), & X_2^2 &= (24, 8, 16, 8, 6, 12), \end{aligned} \quad (7.4)$$

with the obvious notations, i.e., $X_1^1(\{\omega_1, \omega_2\}) = 24$, $X_1^1(\{\omega_6\}) = 6$ and so on. In other words, the stocks follow the tree in Fig. 2.

In this case, we have

$$\begin{aligned} M_1 \cap \mathcal{P}_e &= \left\{ \left(\frac{1}{4}q, \frac{1}{4}q, \frac{1}{2}(1-q), \frac{1}{2}(1-q), \frac{1}{6}q, \frac{2}{6}q \right) : 0 < q < 1 \right\}, \\ M_2 \cap \mathcal{P}_e &= \left\{ \left(\frac{1}{4}p, \frac{1}{4}p, \frac{1}{2}(1-p), \frac{1}{2}(1-p), \frac{2}{6}p, \frac{1}{6}p \right) : 0 < p < 1 \right\}. \end{aligned}$$

Then NA_1 and NA_2 hold true and each single market K_i is incomplete. Moreover, $M \cap \mathcal{P}_e = M_1 \cap M_2 \cap \mathcal{P}_e = \emptyset$, so that there exists a (global) arbitrage in the market K .

Now we consider two agents, where agent i is allowed to invest in the single market K_i . In this setup, we make several choices for the set $\mathcal{Y}$ describing different ways the agents may cooperate.

Example 7.2.1 This is the setting of Sects. 5 and 6.1. Take

$$\mathcal{Y} = \{Y \in (L^0(\Omega, \mathcal{F}_0, P))^2 = \mathbb{R}^2 : Y^1 + Y^2 = 0\}.$$

By Theorem 6.1, we have “NCA($\mathcal{Y}$) $\Leftrightarrow$ (NA₁ and NA₂)”. From the above setup, we thus conclude that NCA($\mathcal{Y}$) holds true. Observe that

$$\mathcal{M}^{\mathcal{Y}} \cap \mathcal{P}_e^2 = \{(Q^1, Q^2) : Q^i \in M_i \cap \mathcal{P}_e, i = 1, 2\}$$

is not empty and thus we may also deduce that NCA($\mathcal{Y}$) holds true from the collective FTAP in Theorem 6.5.

Example 7.2.2 This is the setting of Sects. 5 and 6.2. Take now

$$\mathcal{Y} = \mathcal{Y}_0 = \{Y \in (L^1(\Omega, \mathcal{F}_2, P))^2 : Y^1 + Y^2 = 0\}.$$

By Proposition 6.6, we have “NCA($\mathcal{Y}$) $\Leftrightarrow$ NA”. From the above setup, we thus conclude that there exists a collective arbitrage. Observe that for $(Q^1, Q^2) \in \mathcal{M}^{\mathcal{Y}}$, it is necessary that $E_{Q^1}[Y^1] + E_{Q^2}[Y^2] \leq 0$ for all $(Y^1, Y^2) \in \mathcal{Y}$, which implies $Q^1 = Q^2$. From the assumption $M_1 \cap M_2 \cap \mathcal{P}_e = \emptyset$, we deduce $\mathcal{M}^{\mathcal{Y}} \cap \mathcal{P}_e^2 = \emptyset$ and thus the existence of a collective arbitrage also follows from the collective FTAP in Theorem 5.1.

Example 7.2.3 This is the setting of Sects. 5 and 6.3. Take now

$$\mathcal{Y} = \{Y \in (L^1(\Omega, \mathcal{F}_1, P))^2 : Y^1 + Y^2 = 0\}.$$

Observe that for $(Q^1, Q^2) \in \mathcal{M}^{\mathcal{Y}}$, it is necessary that $E_{Q^1}[Y^1] + E_{Q^2}[Y^2] \leq 0$ for all $(Y^1, Y^2) \in \mathcal{Y}$, which implies $Q^1 = Q^2$ on $(\Omega, \mathcal{F}_1)$. One may thus verify that

$$\mathcal{M}^{\mathcal{Y}} \cap \mathcal{P}_e^2 = \{(Q^1, Q^2) : Q^i \in M_i \cap \mathcal{P}_e, i = 1, 2, \text{ and } Q^1 = Q^2 \text{ on } (\Omega, \mathcal{F}_1)\},$$

to be compared with (6.12). Depending on the choice of the price processes X^1, X^2 , one obtains either $\mathcal{M}^{\mathcal{Y}} \cap \mathcal{P}_e^2 \neq \emptyset$ or $\mathcal{M}^{\mathcal{Y}} \cap \mathcal{P}_e^2 = \emptyset$. Assume $\mathcal{M}^{\mathcal{Y}} \cap \mathcal{P}_e^2 \neq \emptyset$, which is equivalent to NCA($\mathcal{Y}$) by Theorem 6.12 or Theorem 5.1. We prove below that this assumption holds in the concrete example (7.4). Thus from the pricing–hedging duality in Theorem 4.12 and (4.3), we deduce for any $g = (g^1, g^2) \in (L^1(\Omega, \mathcal{F}_2, P))^2$ that

$$\begin{aligned} \rho_+^{\mathcal{Y}}(g) &= \sup_{(Q^1, Q^2) \in \mathcal{M}^{\mathcal{Y}}} \sum_{i=1}^2 E_{Q^i}[g^i] \\ &\leq \sup_{(Q^1, Q^2) \in M_1 \times M_2} \sum_{i=1}^2 E_{Q^i}[g^i] \\ &= \sup_{Q^1 \in M_1} E_{Q^1}[g^1] + \sup_{Q^2 \in M_2} E_{Q^2}[g^2] = \rho_+^N(g), \end{aligned}$$

and the inequality is in general strict whenever $\mathcal{M}^{\mathcal{Y}}$ is strictly contained in the set $M_1 \times M_2$. When $\rho_+^{\mathcal{Y}}(g) < \rho_+^N(g)$, cooperation thus allows the agents to save money

when superreplicating the claims (g^1, g^2) . In this case, as $\mathcal{Y}$ fulfils (3.11), we also deduce from (4.5) and (4.4) that $\pi_+^{\mathcal{Y}}(g) < \pi_+^N(g)$.

Coming back to the concrete example (7.4), one can verify that

$$\mathcal{M}^{\mathcal{Y}} \cap \mathcal{P}_e^2 = \left\{ \left(\left(\frac{1}{4}q, \frac{1}{4}q, \frac{1}{2}(1-q), \frac{1}{2}(1-q), \frac{1}{6}q, \frac{2}{6}q \right), \left(\frac{1}{4}q, \frac{1}{4}q, \frac{1}{2}(1-q), \frac{1}{2}(1-q), \frac{2}{6}q, \frac{1}{6}q \right) \right) : 0 < q < 1 \right\}.$$

Thus $\mathcal{M}^{\mathcal{Y}} \cap \mathcal{P}_e^2$ is not empty and strictly contained in $(M_1 \times M_2) \cap \mathcal{P}_e^2$. If we select the two contingent claims

$$g^1 = (26, 18, 24, 20, 12, 9) \quad \text{and} \quad g^2 = (12, 8, 6, 6, 24, 18),$$

it is easy to verify that

$$\begin{aligned} \rho_{1,+}(g^1) &= \sup_{Q \in M_1} E_Q[g^1] = 22, \\ \rho_{2,+}(g^2) &= \sup_{Q \in M_2} E_Q[g^2] = 16, \\ \rho_+^N(g) &= \rho_{1,+}(g^1) + \rho_{2,+}(g^2) = 38, \end{aligned}$$

where the supremum in computing $\rho_{1,+}(g^1)$ (respectively $\rho_{2,+}(g^2)$) is obtained at $q = 0$ (respectively $p = 1$). Thus $\rho_+^N(g) = 38$ is the cost to superreplicate both claims (g^1, g^2) without cooperation. On the other hand, the cost to superreplicate both claims (g^1, g^2) with cooperation is given by

$$\rho_+^{\mathcal{Y}}(g) = \sup_{(Q^1, Q^2) \in \mathcal{M}^{\mathcal{Y}}} \sum_{i=1}^2 E_{Q^i}[g^i] = 32,$$

with the supremum over $\mathcal{M}^{\mathcal{Y}}$ attained at $q = 1$. For $Y^1 = -Y^2$, one verifies that one exchange $(Y^1, Y^2) \in (L^1(\Omega, \mathcal{F}_1, P))^2$ that allows to obtain $\rho_+^{\mathcal{Y}}(g) = 32$ is given by

$$Y^1 = \begin{cases} 0 & \text{on } \{\omega_1, \omega_2\}, \\ 6 & \text{on } \{\omega_3, \omega_4\}, \\ 0 & \text{on } \{\omega_5, \omega_6\}. \end{cases}$$

Thus no exchanges are required at the nodes $\{\omega_1, \omega_2\}$ and $\{\omega_5, \omega_6\}$, while at the node $\{\omega_3, \omega_4\}$, agent 1 receives 6 units of money from agent 2. In order to superreplicate both claims (g^1, g^2) , the cooperation between the two agents allows them to save at the initial time 6 units of money and requires them to exchange 6 units of money at a later time only at one node. Also observe that $E_{\hat{Q}^i}[Y^i] = 0$ with this choice of (Y^1, Y^2) .

Appendix

A.1 Finitely generated convex cones

Let V be a topological vector space. By definition, a convex cone $S \subseteq V$ is finitely generated if

$$S = \left\{ \sum_{i=1}^M \lambda_i v_i : \lambda_i \geq 0 \text{ for each } i \right\}$$

for some finite number of vectors $v_1, \dots, v_M$ in V . Any finite-dimensional vector space $\mathbb{K} \subseteq V$ is given by

$$\mathbb{K} = \left\{ \sum_{i=1}^N \alpha_i k_i : \alpha_i \in \mathbb{R} \right\}$$

for some finite number of vectors $k_1, \dots, k_N \in V$. Then the vector space $\mathbb{K}$ is the finitely generated convex cone generated by the vectors $\pm k_1, \dots, \pm k_N$, as

$$\mathbb{K} = \left\{ \sum_{i=1}^N \alpha_i k_i + \sum_{i=1}^N \beta_i (-k_i) : \alpha_i \geq 0, \beta_i \geq 0 \text{ for each } i \right\}.$$

Clearly, the sum of two (or a finite number of) finitely generated convex cones is a finitely generated convex cone. Indeed, if $S^1 = \{\sum_{i=1}^{M_1} \lambda_i^1 v_i^1 : \lambda_i^1 \geq 0 \text{ for each } i\}$ and $S^2 = \{\sum_{j=1}^{M_2} \lambda_j^2 v_j^2 : \lambda_j^2 \geq 0 \text{ for each } j\}$, then

$$S^1 + S^2 = \left\{ \sum_{i=1}^{M_1} \lambda_i^1 v_i^1 + \sum_{j=1}^{M_2} \lambda_j^2 v_j^2 : \lambda_i^1 \geq 0, \lambda_j^2 \geq 0 \text{ for each } i \text{ and } j \right\}.$$

A.2 Properties of full exchanges

Lemma A.1 Suppose that $\mathbb{F}^i = \mathbb{F}^j = \mathbb{F}$ for all i, j . Let $k \in K$ (see Definition 2.5). Then there exist $k^i \in K_i, i = 1, \dots, N$, such that $k = \sum_{i=1}^N k^i$.

Proof Let $k = (H \cdot X)_T$, where $H = (H^1, \dots, H^J)$. Then

$$k = \sum_{a \in (1)} (H^a \cdot X^a)_T + \sum_{a \in (2) \setminus (1)} (H^a \cdot X^a)_T + \sum_{a \in (3) \setminus ((1) \cup (2))} (H^a \cdot X^a)_T + \dots,$$

where summations over empty sets are set to zero. Since $\mathbb{F}^i = \mathbb{F}^j = \mathbb{F}$ for all i, j , we get $k^1 := \sum_{a \in (1)} (H^a \cdot X^a)_T \in K_1, k^2 := \sum_{a \in (2) \setminus (1)} (H^a \cdot X^a)_T \in K_2$ and so on, and so we have the desired decomposition. $\square$

Lemma A.2 Suppose that $\mathbb{F}^i = \mathbb{F}^j = \mathbb{F}$ for all $i, j = 1, \dots, N$, and that $\mathcal{F}_T = \mathcal{F}$. Then for K introduced in Definition 2.5 and $\mathcal{Y}_0$ as in (3.6), we have

$$\bigtimes_{i=1}^N K_i + \mathcal{Y}_0 = \bigtimes_{i=1}^N K + \mathcal{Y}_0.$$

Proof The inclusion “ $\subseteq$ ” is clear as $K_i \subseteq K$ for every i by the assumption on the filtrations. For “ $\supseteq$ ”, take any $k^1, \dots, k^N \in K$ with $(k^1, \dots, k^N) \in \bigtimes_{i=1}^N K$ and any $Y \in \mathcal{Y}_0$. By applying Lemma A.1 to every $k^j, j = 1, \dots, N$, we ensure that there exist $k_i^j \in K_i, i = 1, \dots, N$, such that $k^j = \sum_{i=1}^N k_i^j$. Defining $\widehat{k}^i := \sum_{j=1}^N k_i^j$, we have $\widehat{k}^i \in K_i$ by the assumption on the filtrations. Setting $\widehat{Y}^i := k^i - \widehat{k}^i$, one can verify that $\sum_{j=1}^N \widehat{Y}^j = \sum_{j=1}^N k^j - \sum_{j=1}^N \widehat{k}^j = 0$, i.e., $\widehat{Y} \in \mathcal{Y}_0$. This implies that $k = \widehat{k} + \widehat{Y}$, and $k + Y = \widehat{k} + (\widehat{Y} + Y) \in \bigtimes_{i=1}^N K_i + \mathcal{Y}_0$. $\square$

A.3 The multidimensional Kreps–Yan theorem

We assume that $\mathbf{P} := (P^1, \dots, P^N)$ with $P^i \in \mathcal{P}_e$ for all i , and so all inequalities among random variables are meant to hold P -a.s.. Recall also Notation 2.4 for $L^1(\Omega, \mathbf{F}_T, \mathbf{P})$. We state without proof the following multidimensional version of the classical one-dimensional Kreps–Yan theorem.

Theorem A.3 Let $G \subseteq L^1(\Omega, \mathbf{F}_T, \mathbf{P})$ be a closed convex cone. Suppose that G satisfies $-L_+^1(\Omega, \mathbf{F}_T, \mathbf{P}) \subseteq G$ and $G \cap L_+^1(\Omega, \mathbf{F}_T, \mathbf{P}) = \{0\}$. Then there exists $z \in L^\infty(\Omega, \mathbf{F}_T, \mathbf{P})$ satisfying $z^i > 0$ for all i and $\sum_{i=1}^N E_{P^i}[z^i g^i] \leq 0$ for all $g \in G$.

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Declarations

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