

Short Communication: Are Shortfall Systemic Risk Measures One Dimensional?*

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Abstract. Shortfall systemic (multivariate) risk measures ρ defined through an N -dimensional multivariate utility function U and random allocations can be represented as classical (1-dimensional) shortfall risk measures associated to an explicitly determined 1-dimensional function constructed from U . This finding allows for simplifying the study of several properties of ρ , such as dual representations, law invariance, and stability.

Key words. systemic risk measures, shortfall risk measures, sup-convolution

MSC codes. 91G45, 91G70, 91B16, 91G15

DOI. 10.1137/23M1580413

1. Introduction. We consider risky financial positions $(X^1, \dots, X^N) := \mathbf{X}$ and assume, to simplify the exposition in this introduction, that $\mathbf{X} \in (L^\infty(\Omega, \mathcal{F}, P))^N := (L^\infty)^N$. We also let $\pi : (L^\infty)^N \rightarrow L^\infty$ be a pricing functional and $U : \mathbb{R}^N \rightarrow \mathbb{R}$ be a multivariate utility function, and we set $\mathbb{U}(\mathbf{X}) := \mathbb{E}[U(\mathbf{X})]$ and $\mathcal{C} := \{\mathbf{Y} \in (L^\infty)^N \mid \pi(\mathbf{Y}) \in \mathbb{R}\}$. Then the functional $\rho_{\pi, \mathbb{U}}(\mathbf{X}) : (L^\infty)^N \rightarrow [-\infty, +\infty]$ defined by

$$(1) \quad \rho_{\pi, \mathbb{U}}(\mathbf{X}) := \inf \{ \pi(\mathbf{Y}) \mid \mathbf{Y} \in \mathcal{C}, \mathbb{U}(\mathbf{X} + \mathbf{Y}) \geq 0 \}, \quad \mathbf{X} \in (L^\infty)^N,$$

represents a general capital requirement, as introduced in [12], as well as a shortfall systemic risk measure, as extensively analyzed in [1], [2], and [3]. A related but alternative approach, based on set-valued maps, is considered in [10]. Observe that the amount $\pi(\mathbf{Y})$ is enforced to be deterministic, even though the terminal-time allocations $\mathbf{Y} \in (L^\infty)^N$ are allowed to be scenario-dependent. The map $\pi(\mathbf{Y}) = \sum_{i=1}^N Y^i$ is a classical example for a pricing functional. Frequently used multivariate utility functions have the form $U(\mathbf{x}) = \sum_{i=1}^N U^i(x^i)$ for univariate utility functions $U^i : \mathbb{R} \rightarrow \mathbb{R}$, as in, e.g., [3], where also a detailed discussion on scenario-dependent allocation can be found. A conditional version of (1) was treated in [7].

In this paper, we aim at establishing whether the functional $\rho_{\pi, \mathbb{U}}$ can be reduced to a classical *univariate* shortfall risk measure

$$(2) \quad \rho_{\mathbb{E}[g]}(X) := \inf \{ \alpha \in \mathbb{R} \mid \mathbb{E}[g(X + \alpha)] \geq 0 \}, \quad X \in L^\infty,$$

*Received by the editors June 19, 2023; accepted for publication (in revised form) October 13, 2023; published electronically January 4, 2024. All authors are members of GNAMPA-INDAM.

<https://doi.org/10.1137/23M1580413>

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based on some function $g : \mathbb{R} \rightarrow \mathbb{R}$ that can be *explicitly recovered* from U and π . As a corollary of one of our main results, Theorem 2.9, we show that under suitable conditions on π and U

$$(3) \quad \rho_{\pi, \mathbb{U}}(\mathbf{X}) = \rho_{\mathbb{E}[g]}(\pi(\mathbf{X})) = \inf \{ \alpha \in \mathbb{R} \mid \mathbb{E}[g(\pi(\mathbf{X}) + \alpha)] \geq 0 \}, \quad \mathbf{X} \in (L^\infty)^N,$$

for $g(x) = \square_\pi U(x) := \sup \{ U(\mathbf{w}) \mid \mathbf{w} \in \mathbb{R}^N, \pi(\mathbf{w}) = x \}$, $x \in \mathbb{R}$. Moreover, we prove that $\rho_{\pi, \mathbb{U}}(\mathbf{X})$ admits a (unique) optimum $\mathbf{Y}_{\mathbf{X}}$, given by $\mathbf{Y}_{\mathbf{X}} = -\mathbf{X} + \Theta(\pi(\mathbf{X}) + \rho_{\pi, \mathbb{U}}(\mathbf{X}))$ for a continuous explicit function $\Theta : \mathbb{R} \rightarrow \mathbb{R}^N$ depending on U and π .

The fact that a particular class of shortfall systemic risk measures (based on the particular choices $U(\mathbf{x}) = \sum_{i=1}^N U^i(x^i)$ and $\pi(\mathbf{Y}) = \sum_{i=1}^N Y^i$) could be reduced to univariate ones was already observed in [2, Proposition 5.3], [3, Proposition 3.1(ii)], and [13, Theorem 3.16]. One main difference from our work is that we provide very explicit representations (i.e., (3) and the more general case in Theorem 2.9 below) and produce explicit formulas for the function g and for the optimum.

The representation (3) is obtained as a particular instance of a more general result. Indeed, in (1) $\mathbb{U}$ need not be an expected utility, but can rather be taken to be a general (multivariate) utility functional $\mathbb{U} : (L^\infty)^N \rightarrow \mathbb{R}$. In such a case, we prove that (3) takes the form

$$(4) \quad \rho_{\pi, \mathbb{U}}(\mathbf{X}) = \inf \{ \alpha \in \mathbb{R} \mid \square_\pi \mathbb{U}(\pi(\mathbf{X}) + \alpha) \geq 0 \}, \quad \mathbf{X} \in (L^\infty)^N,$$

where $\square_\pi \mathbb{U}(Z) := \sup \{ \mathbb{U}(\mathbf{W}) : \mathbf{W} \in (L^\infty)^N, \pi(\mathbf{W}) = Z \}$, $Z \in L^\infty$, is the functional counterpart of $\square_\pi U$. Our results cover also cases when (i) we allow for unbounded positions $\mathbf{X}$ and (ii) π is $(L^\infty)^M$ -valued for some $M \geq 1$. When $M \geq 2$, the analogue of (3), namely (13) or its functional counterpart (8) in the more general case, can be seen as a dimensionality reduction property induced by random allocations. In this case, the right-hand side (RHS) of (3) takes the form of a shortfall-type systemic risk measure with deterministic allocations, i.e., in the form (5) below, such as those treated, e.g., in [1]. The case $M \geq 2$ also covers grouping examples, in which terminal-time exchanges are allowed only within certain subgroups of the whole system (see section 2.1).

The dual representation of systemic risk measures is an immediate byproduct of (4). A second consequence is the law invariance of multivariate shortfall risk measures in the form (1) (see section 3.1). section 3.2 is then devoted to establishing a Law of Large Numbers-type result in the style of [15] for systemic shortfall risk measures in the form (1). Our approach here is inspired by the one of [4].

2. Systemic risk measures can be reduced to univariate risk measures. We consider a vector subspace L of $L^0(\Omega, \mathcal{F}, P)$, with $\mathbb{R} \subseteq L$, and we induce on the Cartesian product L^N , $N \geq 1$, the order from the standard componentwise P -a.s. ordering from $(L^0(\Omega, \mathcal{F}, P))^N$. For $N \geq M \geq 1$, let $\pi = (\pi^1, \dots, \pi^M)^T : L^N \rightarrow L^M$ and set $\mathcal{C} := \{ \mathbf{Y} \in L^N \mid \pi(\mathbf{Y}) \in \mathbb{R}^M \}$.

Given the functions $\mathbb{U} : L^N \rightarrow [-\infty, +\infty)$ and $G : L^M \rightarrow [-\infty, +\infty)$, we define $\rho_{\pi, \mathbb{U}} : L^N \rightarrow [-\infty, +\infty]$ and $\rho_G : L^M \rightarrow [-\infty, +\infty]$ by

$$(5) \quad \begin{aligned} \rho_{\pi, \mathbb{U}}(\mathbf{X}) &:= \inf \left\{ \sum_{m=1}^M \pi^m(\mathbf{Y}) \mid \mathbf{Y} \in \mathcal{C}, \mathbb{U}(\mathbf{X} + \mathbf{Y}) \geq 0 \right\}, \quad \mathbf{X} \in L^N, \\ \rho_G(\mathbf{X}) &:= \inf \left\{ \sum_{m=1}^M \alpha^m \mid \alpha \in \mathbb{R}^M, G(\mathbf{X} + \alpha) \geq 0 \right\}, \quad \mathbf{X} \in L^M. \end{aligned}$$

Let $\mathbf{e}^i$ be the i th element of the canonical basis of $\mathbb{R}^N$. We say that $\mathbb{U} : L^N \rightarrow [-\infty, +\infty)$ is strictly increasing in some component if for any $\mathbf{X} \in L^N$ there exists some $j \in \{1, \dots, N\}$ such that the function $x \rightarrow \mathbb{U}(\mathbf{X} + \mathbf{e}^j x)$, $x \in \mathbb{R}$, is strictly increasing.

Assumption 2.1. (a) $\pi : L^N \rightarrow L^M$ is linear and satisfies, for $L_+ = L \cap L_+^0(\Omega, \mathcal{F}, P)$,

$$(6) \quad \pi(L_+^N) = L_+^M \quad \text{and} \quad \pi(\mathbb{R}^N) = \mathbb{R}_+^M.$$

(b) $\mathbb{U} : L^N \rightarrow \mathbb{R}$ is concave, increasing, and strictly increasing in some component.

Remark 2.2. One may easily check that (6) implies that π is monotone and that

$$(7) \quad \pi(L^N) = L^M \quad \text{and} \quad \pi(\mathbb{R}^N) = \mathbb{R}^M.$$

In the following we adopt the conventions $\inf \emptyset = +\infty$, $\sup \emptyset = -\infty$.

Definition 2.3. We define the functional $\square_\pi \mathbb{U} : L^M \rightarrow [-\infty, +\infty]$ by

$$\square_\pi \mathbb{U}(\mathbf{Z}) := \sup \{ \mathbb{U}(\mathbf{W}) \mid \mathbf{W} \in L^N, \pi(\mathbf{W}) = \mathbf{Z} \}, \quad \mathbf{Z} \in L^M.$$

In section 5 the analogous concept, defined for *convex* functions f , is there named *image function of f under π* , a terminology adopted from [14]. The functional $\square_\pi \mathbb{U}$ plays the role of a *generalized sup-convolution*. Take indeed $M = 1$, suppose that $\mathbb{U}^i : L \rightarrow [-\infty, \infty)$, $i = 1, \dots, N$, are N given univariate utility functions, and consider the multivariate utility $\mathbb{U} : L^N \rightarrow [-\infty, \infty)$ defined by $\mathbb{U}(\mathbf{W}) = \sum_{i=1}^N \mathbb{U}^i(W^i)$ and the functional $\pi : L^N \rightarrow L$ given by the sum of the components, namely $\pi(\mathbf{W}) = \sum_{i=1}^N W^i$. Then by computing the classical sup-convolution $\mathbb{U}^1 \square \dots \square \mathbb{U}^N$ of the functions $\mathbb{U}^i$ we get

$$(\mathbb{U}^1 \square \dots \square \mathbb{U}^N)(Z) = \square_\pi \mathbb{U}(Z), \quad Z \in L.$$

The following proposition, whose proof can be found in section 4, is our first main finding.

Proposition 2.4. Suppose that Assumption 2.1 holds true and that $\square_\pi \mathbb{U}(\mathbf{Z}) < +\infty$ for every $\mathbf{Z} \in L^M$. Then

1. the functional $\square_\pi \mathbb{U}$ is finite valued, concave and increasing on L^M ;
2. if $\mathbf{X} \in L^N$ satisfies $\sup \{ \mathbb{U}(\mathbf{X} + \mathbf{y}) \mid \mathbf{y} \in \mathbb{R}^N \} > 0$, then

$$(8) \quad \rho_{\pi, \mathbb{U}}(\mathbf{X}) = \rho_{(\square_\pi \mathbb{U})}(\pi(\mathbf{X})).$$

Observe that in case $M = 1$,

$$\rho_{(\square_\pi \mathbb{U})}(\pi(\mathbf{X})) = \inf \{ \alpha \in \mathbb{R} \mid \square_\pi \mathbb{U}(\pi(\mathbf{X}) + \alpha) \geq 0 \}$$

is a *classical (univariate) risk measure*. Proposition 2.4 shows that *any* systemic risk measure in the form $\rho_{\pi, \mathbb{U}}(\mathbf{X})$ can be written as a univariate risk measure $\rho_{(\square_{\pi} \mathbb{U})}(\pi(\mathbf{X}))$ associated to $\square_{\pi} \mathbb{U}$, namely to an explicitly determined univariate function. For $1 \leq M < N$ we analogously obtain a reduction in dimensionality.

Remark 2.5. The assumption in item 2 of Proposition 2.4 is automatic if $L = L^{\infty}$ and $\sup\{\mathbb{U}(\mathbf{y}) \mid \mathbf{y} \in \mathbb{R}^N\} > 0$, by monotonicity of $\mathbb{U}$.

Remark 2.6. Recall the general notion of a capital requirement $\rho_{\pi, \mathcal{A}} : L^N \rightarrow [-\infty, +\infty]$ (see [12]) and of a monetary risk measure $\rho_{\mathcal{B}} : L \rightarrow [-\infty, +\infty]$ (see [11]):

$$(9) \quad \rho_{\pi, \mathcal{A}}(\mathbf{X}) := \inf \{ \pi(\mathbf{Y}) \mid \mathbf{Y} \in \mathcal{C}, \mathbf{X} + \mathbf{Y} \in \mathcal{A} \}, \quad \mathbf{X} \in L^N,$$

$$(10) \quad \rho_{\mathcal{B}}(Z) := \inf \{ \alpha \in \mathbb{R} \mid Z + \alpha \in \mathcal{B} \}, \quad Z \in L,$$

for some acceptance sets $\mathcal{A} \subseteq L^N$ and $\mathcal{B} \subseteq L$. For a linear $\pi : L^N \rightarrow L$ (so that $M = 1$) and for $\mathcal{C} := \{ \mathbf{Y} \in L^N \mid \pi(\mathbf{Y}) \in \mathbb{R} \}$ we have for all $\mathbf{X} \in L^N$

$$\begin{aligned} \rho_{\pi, \mathcal{A}}(\mathbf{X}) &= \inf \{ \alpha \in \mathbb{R} \mid \pi(\mathbf{Y}) = \alpha, \mathbf{X} + \mathbf{Y} \in \mathcal{A} \} \\ &= \inf \{ \alpha \in \mathbb{R} \mid \exists \mathbf{W} \in \mathcal{A} \text{ s.t. } \pi(\mathbf{W}) = \pi(\mathbf{X}) + \alpha \} \\ &= \inf \{ \alpha \in \mathbb{R} \mid \pi(\mathbf{X}) + \alpha \in \pi(\mathcal{A}) \} = \rho_{\pi(\mathcal{A})}(\pi(\mathbf{X})). \end{aligned}$$

Hence, any capital requirement (or systemic multivariate risk measure) of dimension N in the form (9) with π linear can be reduced to a classical univariate risk measure in the form (10). We notice that a different reduction formula was also considered in [9].

In the remainder of this section we work in the following setting.

Setting 2.7.

1. We select $L = L^{\infty}$.
2. The linear functional $\pi : (L^{\infty})^N \rightarrow (L^{\infty})^M$ is assigned by

$$\pi(\mathbf{X}) = A\mathbf{X},$$

where the (deterministic) matrix A in $\mathbb{R}^{M \times N}$ satisfies $A(\mathbb{R}_+^N) = \mathbb{R}_+^M$.

3. The multivariate utility function $U : \mathbb{R}^N \rightarrow \mathbb{R}$ is nondecreasing (w.r.t. the component-wise order), differentiable, strictly concave throughout all $\mathbb{R}^N$ with $\sup\{U(\mathbf{y}) \mid \mathbf{y} \in \mathbb{R}^N\} > 0$.
4. The functional $\mathbb{U} : (L^{\infty})^N \rightarrow \mathbb{R}$ has the form

$$\mathbb{U}(\mathbf{X}) := \mathbb{E}[U(\mathbf{X})], \quad \mathbf{X} \in (L^{\infty})^N.$$

We point out that $A(\mathbb{R}_+^N) = \mathbb{R}_+^M$ implies that A has full rank, $\text{rank}(A) = M$, and that π in item 2 satisfies $\pi((L^{\infty})_+^N) = (L^{\infty})_+^M$. Moreover, the function $\mathbb{U}$ in item 4 is also strictly increasing (in any component). Thus, in Setting 2.7, Assumption 2.1 holds true. The choices made in Setting 2.7 lead to the classical shortfall systemic risk measure:

$$(11) \quad \rho_{\pi, \mathbb{U}}(\mathbf{X}) := \inf \left\{ \sum_{m=1}^M \pi^m(\mathbf{Y}) \mid \mathbf{Y} \in \mathcal{C}, \mathbb{E}[U(\mathbf{X} + \mathbf{Y})] \geq 0 \right\}, \quad \mathbf{X} \in (L^{\infty})^N,$$

which is a monotone increasing, convex, cash additive map. By definition, $\square_\pi U : \mathbb{R}^M \rightarrow \mathbb{R}$ is assigned by

$$(12) \quad \square_\pi U(\mathbf{y}) := \sup\{U(\mathbf{x}) \mid \mathbf{x} \in \mathbb{R}^N, \pi(\mathbf{x}) = \mathbf{y}\}, \quad \mathbf{y} \in \mathbb{R}^M.$$

Assumption 2.8. For some $\mathbf{y} \in \mathbb{R}^M$, the problem in (12) admits an optimum; namely there exists $\mathbf{x} = \mathbf{x}(\mathbf{y}) \in \mathbb{R}^N$ such that $\pi(\mathbf{x}) = \mathbf{y}$ and $\square_\pi U(\mathbf{y}) = U(\mathbf{x})$.

We provide in Lemma A.1 in the appendix mild conditions which guarantee the validity of Assumption 2.8.

The main result of this note is described in the following theorem. Shortfall systemic risk measures $\rho_{\pi, \mathbb{U}}(\mathbf{X})$ defined through an N -dimensional multivariate utility function U can be represented as a shortfall risk measure $\rho_{(\mathbb{E}[\square_\pi U])}(\pi(\mathbf{X}))$ associated to the M -dimensional function $\square_\pi U$. Additionally, we provide the explicit formula for the optimum. The proof is deferred to section 6.

Theorem 2.9. Suppose that Assumption 2.8 is satisfied. Then the following hold:

1. $\square_\pi \mathbb{U}(\mathbf{Z}) = \mathbb{E}[(\square_\pi U)(\mathbf{Z})]$ for every $\mathbf{Z} \in (L^\infty)^M$.
2. For every $\mathbf{X} \in (L^\infty)^N$ we have

$$(13) \quad \rho_{\pi, \mathbb{U}}(\mathbf{X}) = \rho_{(\mathbb{E}[\square_\pi U])}(\pi(\mathbf{X})) := \inf \left\{ \sum_{m=1}^M \alpha^m \mid \alpha \in \mathbb{R}^M, \mathbb{E}[\square_\pi U(\pi(\mathbf{X}) + \alpha)] \geq 0 \right\}.$$

If additionally there exists an optimum $\hat{\alpha} \in \mathbb{R}^M$ attaining the infimum in the RHS of (13), then also $\rho_{\pi, \mathbb{U}}(\mathbf{X})$ admits a (unique) optimum $\mathbf{Y}_\mathbf{X}$, given by

$$(14) \quad \mathbf{Y}_\mathbf{X} = -\mathbf{X} - \nabla U^*(\pi^T \cdot \nabla(\square_\pi U)(\pi(\mathbf{X}) + \hat{\alpha})) = -\mathbf{X} + \Theta(\pi(\mathbf{X}) + \hat{\alpha}),$$

where U^* is the concave conjugate of U , π^T is the transposed map of π , and $\Theta : \mathbb{R}^M \rightarrow \mathbb{R}^N$, $\Theta(\mathbf{y}) := -\nabla U^*(\pi^T \cdot \nabla(\square_\pi U)(\mathbf{y}))$, is continuous.

In case $M = 1$, $\rho_{\pi, \mathbb{U}}$ is finite valued, the optimum in the RHS of (13) always exists, and it is given by $\hat{\alpha} = \rho_{(\mathbb{E}[\square_\pi U])}(\pi(\mathbf{X}))$.

In the case $M = 1$, as an immediate byproduct of Theorem 2.9, the dual representation for the systemic risk measure $\rho_{\pi, \mathbb{U}} (= \rho_{(\mathbb{E}[\square_\pi U])})$ can be directly obtained from the well-known classical dual representation of the univariate convex risk measure $\rho_{(\mathbb{E}[\square_\pi U])}$. Indeed, letting $\ell(x) = -\square_\pi U(-x)$, $x \in \mathbb{R}$, the dual representation of $\rho_{(\mathbb{E}[\square_\pi U])}$ follows from [11, Theorem 4.115] with minimal penalty function $\alpha^{\min}$ in [11, Theorem 4.115] explicitly given, since $\ell^*(z) = -U^*(A^T z)$, $z \in \mathbb{R}$.

2.1. Grouping case. In Example 5.2 of [3] and Definition 5.1 of [7] the systemic risk measure is the sum of M minimal cash amount that can be allocated within M subgroups in order to make the system safe. Such a risk measure is in the form $\rho_{\pi, \mathbb{U}}(\mathbf{X})$ with the appropriate choice of π and with

$$(15) \quad \mathcal{C} := \{\mathbf{Y} \in L^N \mid \pi(\mathbf{Y}) \in \mathbb{R}^M\} = \left\{ \mathbf{Y} \in (L^\infty)^N \mid \sum_{n \in I_m} Y^n \in \mathbb{R} \quad \forall m = 1, \dots, M \right\}.$$

In Lemma 2.10 below, whose simple proof is omitted, we show how the dimensionality reduction put in evidence in (13) also covers such a grouping case.

Lemma 2.10. Let $I_1, \dots, I_M \subseteq \{1, \dots, N\}$ be a partition of $\{1, \dots, N\}$, clearly with $M \leq N$. Define the matrix $A = (a_{mn})_{mn} \in \mathbb{R}^{M \times N}$ via

$$a_{mn} = \begin{cases} 1 & \text{if } n \in I_m, \\ 0 & \text{otherwise,} \end{cases} \quad n = 1, \dots, N; m = 1, \dots, M.$$

Set $L = L^\infty$ and $\pi(\mathbf{X}) = A\mathbf{X}$ (as a matrix-vector product). Then A is full rank, the second item in Setting 2.7 is satisfied, and (15) holds.

3. Applications. Proposition 2.4 and Theorem 2.9 allow us to apply the results established in the literature for univariate risk measures to systemic risk measures. From this perspective, we illustrate the cases of law invariance and of stability of risk measures.

3.1. Law invariance. Using the same notation of section 2, we write P_X (resp., $P_{\mathbf{X}}$) for the law of a random variable X (resp., vector $\mathbf{X}$) on $\mathbb{R}$ (resp., $\mathbb{R}^N$) and $X \stackrel{P}{\sim} Y$ if the random variables (or vectors) X, Y have the same law under P . We recall that a set $\mathcal{B}$ is then law invariant if $X \stackrel{P}{\sim} Y$ with $X \in \mathcal{B}$ implies that also $Y \in \mathcal{B}$, while law invariance of a risk measure is fulfilled when it assigns the same riskiness to random variables (or vectors) with the same law. Law invariance is a desirable property not only from an axiomatic point of view but also for statistical/empirical purposes, justifying indeed the use of numerical simulations. In this respect, we refer the reader to the next section, where stability will be investigated.

Proposition 3.1. Assume that $\pi(\mathbf{X}) \stackrel{P}{\sim} \pi(\mathbf{Y})$ with $\mathbf{X}, \mathbf{Y} \in L^N$. Then the following hold:

1. For $M = 1$ and if $\pi(\mathcal{A})$ is law invariant, then $\rho_{\pi, \mathcal{A}}(\mathbf{X}) = \rho_{\pi, \mathcal{A}}(\mathbf{Y})$.
2. For $M \geq 1$, if (8) holds and if $\mathcal{B} = \{\mathbf{Z} \in L^M \mid \square_\pi \mathbb{U}(\mathbf{Z}) \geq 0\}$ is law invariant, then $\rho_{\pi, \mathbb{U}}(\mathbf{X}) = \rho_{\pi, \mathbb{U}}(\mathbf{Y})$.

Proof. Item 1 is an immediate consequence of $\rho_{\pi, \mathcal{A}}(\mathbf{X}) = \rho_{\pi(\mathcal{A})}(\pi(\mathbf{X}))$ proven in Remark 2.6. As for item 2, from $\pi(\mathbf{X}) \stackrel{P}{\sim} \pi(\mathbf{Y})$ and the law invariance of $\mathcal{B}$ we get $\square_\pi \mathbb{U}(\pi(\mathbf{X}) + \alpha) \geq 0$ iff $\square_\pi \mathbb{U}(\pi(\mathbf{Y}) + \alpha) \geq 0$. From $\rho_{\pi, \mathbb{U}}(\mathbf{X}) = \rho_{(\square_\pi \mathbb{U})}(\pi(\mathbf{X}))$, we deduce that $\rho_{\pi, \mathbb{U}}(\mathbf{X}) = \rho_{\pi, \mathbb{U}}(\mathbf{Y})$. ■

Remark 3.2. Obviously, if π is law invariant and $\mathbf{X} \stackrel{P}{\sim} \mathbf{Y}$, then the assumption in the previous proposition holds, so that Proposition 3.1 gives in particular sufficient conditions for the law invariance of the systemic risk measures $\rho_{\pi, \mathcal{A}}$ and $\rho_{\pi, \mathbb{U}}$.

Corollary 3.3. Let $U: \mathbb{R}^N \rightarrow \mathbb{R}$ be nondecreasing, differentiable, strictly concave throughout all $\mathbb{R}^N$, and satisfying $\sup\{U(\mathbf{y}) \mid \mathbf{y} \in \mathbb{R}^N\} > 0$ and Assumption 2.8. Then ρ defined by

$$(16) \quad \rho(\mathbf{X}) := \inf \left\{ \sum_{i=1}^N Y^i \mid \mathbf{Y} \in (L^\infty)^N, \sum_{i=1}^N Y^i \in \mathbb{R}, \mathbb{E}[U(\mathbf{X} + \mathbf{Y})] \geq 0 \right\}, \quad \mathbf{X} \in (L^\infty)^N,$$

is finite valued and law invariant.

Proof. Observe that $\rho = \rho_{\pi, \mathbb{U}}(\mathbf{X})$ for $\mathbb{U}(\cdot) := \mathbb{E}[U(\cdot)]$ and $\pi(\mathbf{Y}) := Y_1 + \dots + Y_N$, the latter being law invariant. Thus the assumptions in Setting 2.7 hold. By Proposition 5.7, $\square_\pi \mathbb{U}$ is continuous on $\mathbb{R}$ and, by Theorem 2.9, item 1, $\square_\pi \mathbb{U}(Z) = \mathbb{E}[\square_\pi \mathbb{U}(Z)]$. Thus, $\mathcal{B} := \{Z \in L^\infty \mid \square_\pi \mathbb{U}(Z) \geq 0\} = \{Z \in L^\infty \mid \mathbb{E}[\square_\pi \mathbb{U}(Z)] \geq 0\}$ is law invariant. The conclusion follows from Theorem 2.9 and Proposition 3.1, item 2. ■

3.2. Stability. Let $(\Omega, \mathcal{F}, P)$ be an atomless standard probability space. If $\rho : (L^\infty)^N \rightarrow \mathbb{R}$ is a law invariant functional, then it is possible to think of ρ as defined on the class of probability measures on $\mathbb{R}^N$. Indeed, whenever $P_{\mathbf{X}} = P_{\mathbf{Y}}$ on $\mathbb{R}^N$ we have $\mathbf{X} \stackrel{P}{\sim} \mathbf{Y}$ and $\rho(\mathbf{X}) = \rho(\mathbf{Y})$, and since the underlying space is nonatomic, every probability measure on $\mathbb{R}^N$ is realized as the law under P of some random vector $\mathbf{Z}$ defined on Ω by the Skorokhod theorem. By a slight abuse of notation we write $\rho(Q)$ meaning $\rho(\mathbf{Z})$ for every $\mathbf{Z}$ having law Q on $\mathbb{R}^N$. In this subsection, we exploit the law invariance established in Corollary 3.3 to ensure stability/consistency (see, e.g., [5]) properties of ρ , defined in (16), considered as a function of laws rather than random variables. More precisely, we demonstrate how a Law of Large Numbers-type result can be obtained for ρ . Such a result provides a rather sound theoretical basis for a Monte Carlo or historical estimation of both ρ and the associated optima. For a neural network approach to the approximation of primal (and dual) optima of ρ , we refer the reader to [6], which is indeed based on Monte Carlo estimations.

Corollary 3.4. *In Setting 2.7 with $M = 1$ and $A\mathbf{x} = \sum_{j=1}^N x^j$ for every $\mathbf{x} \in \mathbb{R}^N$, suppose that the assumptions of Corollary 3.3 are satisfied. Take probability measures $\{Q_n\}_n$ on $\mathbb{R}^N$ for $n = 1, \dots, +\infty$, such that for some positive radius $r > 0$ we have $Q_n(B_r) = 1$ for every $1 \leq n \leq +\infty$, B_r being the ball of radius r centered in $\mathbf{0}$ in $\mathbb{R}^N$. Suppose additionally that Q_n converges to Q_∞ in the weak sense for probability measures. Then ρ defined in (16) satisfies*

$$\lim_{n \rightarrow +\infty} \rho(Q_n) = \rho(Q_\infty).$$

Proof. By the Skorokhod representation theorem there exist N -dimensional random vectors $\mathbf{Z}_n, 1 \leq n \leq +\infty$, on $(\Omega, \mathcal{F}, P)$ such that Q_n is the law of $\mathbf{Z}_n$ under P and $\mathbf{Z}_n \rightarrow \mathbf{Z}_\infty$ P -a.s. In particular then $\mathbf{Z}_n \in (L^\infty)^N$, $\pi(\mathbf{Z}_n) \rightarrow \pi(\mathbf{Z}_\infty)$, and $\|\pi(\mathbf{Z}_n)\|_\infty \leq Nr$ P -a.s. for every n . Then for $1 \leq n \leq +\infty$ we have $\rho(Q_n) = \rho(\mathbf{Z}_n) = \rho_{(\mathbb{E}[\square_\pi U])}(\pi(\mathbf{Z}_n))$ by Theorem 2.9, where $\square_\pi U : \mathbb{R} \rightarrow \mathbb{R}$ is increasing and nonconstant by Proposition 5.7. We now show that $\rho_{(\mathbb{E}[\square_\pi U])}(\pi(\mathbf{Z}_n)) \rightarrow_n \rho_{(\mathbb{E}[\square_\pi U])}(\pi(\mathbf{Z}_\infty))$. By [11, Proposition 4.113] $\rho_{(\mathbb{E}[\square_\pi U])}$ is continuous from below. Then it has the Lebesgue property [11, Corollary 4.35], and the desired convergence follows. ■

Take now $\mathbf{X} \in (L^\infty)^N$. Replacing the law $P_{\mathbf{X}}$ in $\rho(P_{\mathbf{X}})$ with the empirical measure $\hat{P}_n$ based on an i.i.d. sample $(\mathbf{X}_1, \dots, \mathbf{X}_n)$, we obtain the empirical estimate/historical estimate $\rho(\hat{P}_n)$ of $\rho(P_{\mathbf{X}}) = \rho(\mathbf{X})$. Under the assumptions of Corollary 3.4, we have in particular $\lim_n \rho(\hat{P}_n) = \rho(P_{\mathbf{X}})$ P -a.s. by weak convergence (a.s.) of $\hat{P}_n$ to $P_{\mathbf{X}}$. This can be exploited in conjunction with the explicit formula for the optima (14), since in the case $M = 1$ we know $\hat{\alpha} = \rho(\mathbf{X})$, to guarantee a.s. convergence of the approximated optimal allocation functions $\mathbf{x} \mapsto -\mathbf{x} + \Theta(\pi(\mathbf{x}) + \rho(\hat{P}_n))$.

4. Proof of Proposition 2.4. The functional $\square_\pi U$ is finite valued since, by (7), $\square_\pi U(\mathbf{Z}) > -\infty$ for any $\mathbf{Z} \in L^M$. Moreover, monotonicity and concavity of $\square_\pi U$ can be easily checked. We now prove item 2. Observe first that under the additional assumption in item 2, we have $\rho_{\pi, U}(\mathbf{X}) < +\infty$. From the linearity of π and the definition of $\mathcal{C}$, we have for any $\mathbf{X} \in L^N$

$$\begin{aligned}
\rho_{\pi, \mathbb{U}}(\mathbf{X}) &= \inf \left\{ \sum_{m=1}^M \pi^m(\mathbf{Y}) \mid \mathbf{Y} \in \mathcal{C}, \mathbb{U}(\mathbf{X} + \mathbf{Y}) \geq 0 \right\} \\
(17) \quad &= \inf \left\{ \sum_{m=1}^M \pi^m(\mathbf{Y}) \mid \mathbf{Y} \in \mathcal{C}, \mathbb{U}(\mathbf{X} + \mathbf{Y}) > 0 \right\} \\
&= \inf \left\{ \sum_{m=1}^M \alpha^m \mid \alpha \in \mathbb{R}^M \text{ satisfies } \pi(\mathbf{Y}) = \alpha \text{ for some } \mathbf{Y} \in L^N, \mathbb{U}(\mathbf{X} + \mathbf{Y}) > 0 \right\} \\
&= \inf \left\{ \sum_{m=1}^M \alpha^m \mid \alpha \in \mathbb{R}^M, \exists \mathbf{W} \in L^N, \mathbb{U}(\mathbf{W}) > 0 \text{ s.t. } \pi(\mathbf{W}) = \pi(\mathbf{X}) + \alpha \right\} \\
&= \inf \left\{ \sum_{m=1}^M \alpha^m \mid \alpha \in \mathbb{R}^M, \sup \{ \mathbb{U}(\mathbf{W}) \mid \mathbf{W} \in L^N, \pi(\mathbf{W}) = \pi(\mathbf{X}) + \alpha \} > 0 \right\} \\
&= \inf \left\{ \sum_{m=1}^M \alpha^m \mid \alpha \in \mathbb{R}^M, \square_{\pi} \mathbb{U}(\pi(\mathbf{X}) + \alpha) > 0 \right\} \\
(18) \quad &= \inf \left\{ \sum_{m=1}^M \alpha^m \mid \alpha \in \mathbb{R}^M, \square_{\pi} \mathbb{U}(\pi(\mathbf{X}) + \alpha) \geq 0 \right\} \\
&= \rho_{(\square_{\pi} \mathbb{U})}(\pi(\mathbf{X})),
\end{aligned}$$

where only the equalities (17) and (18) are not evident. To prove the equality in (17), observe first that, obviously,

$$\rho_{\pi, \mathbb{U}}(\mathbf{X}) \leq \inf \left\{ \sum_{m=1}^M \pi^m(\mathbf{Y}) \mid \mathbf{Y} \in \mathcal{C}, \mathbb{U}(\mathbf{X} + \mathbf{Y}) > 0 \right\} := a$$

with $a \in \mathbb{R}$. Suppose by contradiction that $\rho_{\pi, \mathbb{U}}(\mathbf{X}) < a$ and let $0 < \varepsilon < \frac{a - \rho_{\pi, \mathbb{U}}(\mathbf{X})}{2}$. Then there exists $\mathbf{Y} \in \mathcal{C}$ such that $\mathbb{U}(\mathbf{X} + \mathbf{Y}) \geq 0$ and

$$(19) \quad \sum_{m=1}^M \pi^m(\mathbf{Y}) < \rho_{\pi, \mathbb{U}}(\mathbf{X}) + \varepsilon.$$

Since $\mathbb{U}$ is strictly increasing on one component, say component i , take $\hat{\mathbf{Y}} := \mathbf{Y} + \varepsilon \frac{\mathbf{e}^i}{\sum_{m=1}^M \pi^m(\mathbf{e}^i) + 1}$, noticing that this is well defined as $\pi^m(\mathbf{e}^i) \geq 0$, for all m , by (6). Since π is linear and $\pi(\mathbb{R}^N) \subseteq \mathbb{R}^M$ (Remark 2.2), $\sum_{m=1}^M \pi^m(\hat{\mathbf{Y}}) = (\sum_{m=1}^M \pi^m(\mathbf{Y}) + \varepsilon \frac{\sum_{m=1}^M \pi^m(\mathbf{e}^i)}{\sum_{m=1}^M \pi^m(\mathbf{e}^i) + 1}) \in \mathbb{R}$ and $\hat{\mathbf{Y}} \in \mathcal{C}$. Moreover, $\mathbb{U}(\mathbf{X} + \hat{\mathbf{Y}}) = \mathbb{U}(\mathbf{X} + \mathbf{Y} + \varepsilon \frac{\mathbf{e}^i}{\sum_{m=1}^M \pi^m(\mathbf{e}^i) + 1}) > \mathbb{U}(\mathbf{X} + \mathbf{Y}) \geq 0$ so that $a \leq \sum_{m=1}^M \pi^m(\hat{\mathbf{Y}})$. But this is a contradiction using (19): $a \leq \sum_{m=1}^M \pi^m(\hat{\mathbf{Y}}) = \sum_{m=1}^M \pi^m(\mathbf{Y}) + \varepsilon \frac{\sum_{m=1}^M \pi^m(\mathbf{e}^i)}{\sum_{m=1}^M \pi^m(\mathbf{e}^i) + 1} < \rho_{\pi, \mathbb{U}}(\mathbf{X}) + 2\varepsilon < a$. To prove the equality in (18), we set

$$g_{\mathbf{X}}(\alpha) := \square_{\pi} \mathbb{U}(\pi(\mathbf{X}) + \alpha), \quad \alpha \in \mathbb{R}^M,$$

and show that

$$\inf \left\{ \sum_{m=1}^M \alpha^m \mid \alpha \in \mathbb{R}^M, g_{\mathbf{X}}(\alpha) > 0 \right\} = \inf \left\{ \sum_{m=1}^M \alpha^m \mid \alpha \in \mathbb{R}^M, g_{\mathbf{X}}(\alpha) \geq 0 \right\}.$$

The fact that $\text{LHS} \geq \text{RHS}$ is clear, and the equality is trivial if the set in RHS is empty. Then we assume this is not the case and prove $\text{LHS} \leq \text{RHS}$. Take a minimizing sequence $(\alpha_n)_n$ such that $g_{\mathbf{X}}(\alpha_n) \geq 0$ for each n and $\sum_{m=1}^M \alpha_n^m \downarrow_n \inf \left\{ \sum_{m=1}^M \alpha^m \mid \alpha \in \mathbb{R}^M, g_{\mathbf{X}}(\alpha) \geq 0 \right\}$.

Case 1. $(\alpha_n)_n$ admits a subsequence $(\alpha_{n_k})_k$ with $g_{\mathbf{X}}(\alpha_{n_k}) > 0$ for each k . Clearly we get $\text{LHS} \leq \sum_{m=1}^M \alpha_{n_k}^m \downarrow_k \text{RHS}$, which is the desired remaining inequality.

Case 2. $g_{\mathbf{X}}(\alpha_n) = 0$ definitely in n . We assume the equality holds for each n w.l.o.g. Define now the functions $h_n(\beta) := g_{\mathbf{X}}(\alpha_n + \beta \mathbf{1})$, $\beta \in \mathbb{R}$, and observe that Proposition 2.4, item 1 ensures that, for each n , $h_n : \mathbb{R} \rightarrow \mathbb{R}$ is increasing and concave on $\mathbb{R}$ and thus continuous. Moreover,

$$g_{\mathbf{X}}(\alpha) \leq h_n \left(\max_m |\alpha_n^m| + \max_m |\alpha^m| \right) \quad \forall \alpha \in \mathbb{R}^M,$$

guaranteeing that $\sup_{\beta \in \mathbb{R}} h_n(\beta) = \sup_{\alpha \in \mathbb{R}^M} g_{\mathbf{X}}(\alpha) := \widehat{g}$. By the linearity of π we then get

$$\begin{aligned} \sup_{\beta \in \mathbb{R}} h_n(\beta) &= \sup_{\alpha \in \mathbb{R}^M} g_{\mathbf{X}}(\alpha) = \sup_{\alpha \in \mathbb{R}^M} \left\{ \sup \{ \mathbb{U}(\mathbf{W}) \mid \mathbf{W} \in L^N, \pi(\mathbf{W}) = \pi(\mathbf{X}) + \alpha \} \right\} \\ &= \sup_{\alpha \in \mathbb{R}^M} \left\{ \sup \{ \mathbb{U}(\mathbf{X} + \mathbf{Y}) \mid \mathbf{Y} \in \mathcal{C}, \pi(\mathbf{Y}) = \alpha \} \right\} \\ (20) \quad &= \sup \{ \mathbb{U}(\mathbf{X} + \mathbf{Y}) \mid \mathbf{Y} \in \mathcal{C} \} \end{aligned}$$

$$(21) \quad \geq \sup \{ \mathbb{U}(\mathbf{X} + \mathbf{y}) \mid \mathbf{y} \in \mathbb{R}^N \} > 0,$$

where we used the following: in (20) the equality $\pi(\mathcal{C}) = \mathbb{R}^M$ (a consequence of (7)); in the first inequality in (21) the fact that $\pi(\mathbb{R}^N) \subseteq \mathbb{R}^M$, and the last strict inequality is guaranteed by hypothesis. Observe that since $g_{\mathbf{X}}(\alpha_n) = 0$, we also have $h_n(0) = 0 < \sup_{\beta \in \mathbb{R}} h_n(\beta) = \widehat{g}$. Let $\widehat{\beta}_n = \inf \{ \beta \in \mathbb{R} \mid h_n(\beta) = \widehat{g} \} \leq +\infty$. From $h_n(0) < \widehat{g}$, the continuity and monotonicity of h_n , we have $\widehat{\beta}_n > 0$ for every n . Additionally, h_n as a univariate concave and increasing function is strictly increasing on $(-\infty, \widehat{\beta}_n)$. Thus, for some $0 < \varepsilon_n < \min(\frac{1}{n}, \widehat{\beta}_n)$ we have $0 = h_n(0) < h_n(\varepsilon_n) = g_{\mathbf{X}}(\alpha_n + \varepsilon_n \mathbf{1})$. Thus $\beta_n := \alpha_n + \varepsilon_n \mathbf{1}$ defines a minimizing sequence, with $g_{\mathbf{X}}(\beta_n) > 0$:

$$\sum_{m=1}^M \beta_n^m = \sum_{m=1}^M \alpha_n^m + M\varepsilon_n \downarrow_n \inf \left\{ \sum_{m=1}^M \alpha^m \mid \alpha \in \mathbb{R}^M, g_{\mathbf{X}}(\alpha) \geq 0 \right\},$$

and one can argue as in Case 1.

5. Image functions on $\mathbb{R}^N$. We now present some key properties of image functions. All definitions, as well as the notation, are adopted from [14]. For convenience of the reader and to simplify the comparison with this reference, we thus opted to present the concepts and results for *convex* functions f and linear maps A , which replace the function $(-U)$ and the linear map π in the previous sections.

In section 5 we work again in Setting 2.7 without further mention. We set $f = -U : \mathbb{R}^N \rightarrow \mathbb{R}$ and denote by f^* the usual convex conjugate function of f ,

$$f^*(\mathbf{z}) := \sup_{\mathbf{x} \in \mathbb{R}^N} \left(\sum_{j=1}^N x^j z^j - f(\mathbf{x}) \right) \in (-\infty, +\infty], \quad \mathbf{z} \in \mathbb{R}^N.$$

Definition 5.1. The image function of f under A is the function $\square^A f : \mathbb{R}^M \rightarrow [-\infty, +\infty]$,

$$(22) \quad \square^A f(\mathbf{y}) := \inf \{ f(\mathbf{x}) \mid \mathbf{x} \in \mathbb{R}^N, A\mathbf{x} = \mathbf{y} \}, \quad \mathbf{y} \in \mathbb{R}^M,$$

with the usual convention $\inf \emptyset = +\infty$. We say that, for a given $\mathbf{y} \in \mathbb{R}^M$, the problem in (22) admits an optimum if there exists $\mathbf{x} = \mathbf{x}(\mathbf{y}) \in \mathbb{R}^N$ such that $A\mathbf{x} = \mathbf{y}$ and $\square^A f(\mathbf{y}) = f(\mathbf{x})$.

In [14] the image function $\square^A f$ is denoted by Af . Observe that $\square^A f = -\square_\pi U$, as in (12), for $f = -U$ and $A = \pi$ and that $\square^A f$ is a convex function (see Theorem 5.7 in [14]), and, since f is real valued on the whole $\mathbb{R}^N$ and A has full range, $\square^A f(\mathbf{y}) < +\infty$ for every $\mathbf{y} \in \mathbb{R}^M$. We stress that by strict convexity of f , the problem (22) admits at most one optimum.

Remark 5.2. Under Assumption 2.8, we have that for some $\mathbf{y} \in \mathbb{R}^M$, the problem in (22) admits an optimum.

We refer the reader to [14, Chapter 26] for definitions of essentially smooth functions and Legendre-type pairs. In our setting, f is essentially smooth on the whole $\mathbb{R}^N$ and thus $(f, \mathbb{R}^N)$ is of Legendre type. We also briefly recall the following key results.

Theorem 5.3 (see [14, Theorem 26.5]). Let $h : \mathbb{R}^D \rightarrow (-\infty, +\infty]$ be a closed (i.e., proper lower semicontinuous w.r.t. the usual Euclidean topology) convex function. Set $C := \text{int dom}(h)$, $C^* := \text{int dom}(h^*)$. Then (h, C) is a convex function of Legendre type iff (h^*, C^*) is. When these conditions hold, ∇h is one-to-one from the open convex set C onto C^* and continuous in both directions, and $(\nabla h)^{-1} = \nabla h^*$.

We now reformulate [14, Theorem 16.3] adopting our notation.

Theorem 5.4 (see [14, Theorem 16.3]). Suppose that there exists $\lambda \in \mathbb{R}^M$ such that $A^T \lambda \in \text{ridom}(f^*)$. Then for every $\mathbf{y} \in \mathbb{R}^M$ there exists an optimum for (22).

Proposition 5.5. Take $\mathbf{y} \in \mathbb{R}^M$ such that $\square^A f(\mathbf{y}) > -\infty$. Then $\square^A f(\mathbf{y}) \in \mathbb{R}$ and the following are equivalent:

1. there exists the optimum $\mathbf{x} = \mathbf{x}(\mathbf{y}) \in \mathbb{R}^N$ for (22);
2. there exist $(\mathbf{x}, \lambda) = (\mathbf{x}(\mathbf{y}), \lambda(\mathbf{y})) \in \mathbb{R}^N \times \mathbb{R}^M$ solving

$$(23) \quad \begin{cases} \nabla f(\mathbf{x}) = A^T \lambda, \\ A\mathbf{x} = \mathbf{y}. \end{cases}$$

Proof. We already know that $\square^A f(\mathbf{y}) < +\infty$ for every $\mathbf{y} \in \mathbb{R}^M$; thus $\square^A f(\mathbf{y}) \in \mathbb{R}$. Observe that, once $\mathbf{y} \in \mathbb{R}^M$ is fixed, (22) is what is called in [14, Chapter 28] an ordinary convex program admitting a feasible solution (since f is real valued). Its set of constraints is given by $\mathbf{y} - A\mathbf{x} = 0$, and $C = \text{ri}(C) = \mathbb{R}^N$ in the notation of [14]. By the Kuhn–Tucker theorem (see [14, Corollary 28.3.1], whose hypotheses are met since we are assuming $\square^A f(\mathbf{y}) > -\infty$),

item 1 in the statement is equivalent to the following: there exists a pair $(\mathbf{x}, \lambda) \in \mathbb{R}^N \times \mathbb{R}^N$ satisfying conditions (a),(b),(c) in [14, Theorem 28.3] (with $\mathbf{x}$ in place of $\bar{\mathbf{x}}$ and λ in place of $\mathbf{u}^*$). By the discussion following the proof of [14, Theorem 28.3], since f and the functions enforcing the constraints are differentiable, condition (c) can be rewritten as $\nabla f(\mathbf{x}) - A^T \lambda = \mathbf{0}$. Condition (b) is $A\mathbf{x} = \mathbf{y}$, and conditions (a) can actually be ignored since we have no inequality constraints. This proves that items 1 and 2 are equivalent. ■

Proposition 5.6. *The following are equivalent:*

1. *There exists the optimum for (22) for **all** $\mathbf{y} \in \mathbb{R}^M$.*
2. *There exists the optimum for (22) for **some** $\hat{\mathbf{y}} \in \mathbb{R}^M$.*

Proof. Clearly item 1 implies item 2. As for the converse, observe that since f is essentially smooth, by Theorem 5.3 $\nabla f(\mathbb{R}^N) = \text{int dom}(f^*) \subseteq \text{ri dom}(f^*)$. Then, by the equivalence established in Proposition 5.5, $A^T \lambda(\hat{\mathbf{y}}) = \nabla f(\mathbf{x}(\hat{\mathbf{y}})) \in \text{int dom}(f^*) \subseteq \text{ri dom}(f^*)$, and Theorem 5.4 yields the existence of the optimum for every $\mathbf{y} \in \mathbb{R}^M$. ■

Proposition 5.7. *Under Assumption 2.8, the map $\square^A f : \mathbb{R}^M \rightarrow \mathbb{R}$ is continuously differentiable and strictly convex on $\mathbb{R}^M$. Its conjugate is given by $(\square^A f)^*(\mathbf{z}) = f^*(A^T \mathbf{z})$, $\mathbf{z} \in \mathbb{R}^M$, which is continuously differentiable on the interior of its domain. The gradient $\nabla \square^A f(\cdot)$ is a homeomorphism between $\mathbb{R}^M$ and $\text{int dom}((\square^A f)^*) = \text{int}\{\mathbf{z} \in \mathbb{R}^M \mid A^T \mathbf{z} \in \text{dom}(f^*)\} =: \mathcal{O} \subseteq \mathbb{R}^M$, and its (continuous) inverse is given by $\nabla(\square^A f)^*(\mathbf{z})$, $\mathbf{z} \in \mathcal{O}$. Finally, the unique optimum $\mathbf{x} = \mathbf{x}(\mathbf{y})$ of problem (22) is given by*

$$(24) \quad \mathbf{x} = \Theta(\mathbf{y}) = \nabla f^*(A^T \cdot \nabla(\square^A f)(\mathbf{y})),$$

where $\Theta : \mathbb{R}^M \rightarrow \mathbb{R}^N$ is continuous on $\mathbb{R}^M$.

Proof. By Remark 5.2 and Proposition 5.6 there exists an optimum of (22) for all $\mathbf{y} \in \mathbb{R}^M$. In particular $\square^A f(\mathbf{y}) \in \mathbb{R} \forall \mathbf{y} \in \mathbb{R}^M$. As argued in the proof of Proposition 5.6 there exists $\lambda \in \mathbb{R}^M$ s.t. $A^T \lambda \in \text{int dom}(f^*) \subseteq \text{ri dom}(f^*)$. Since f is essentially smooth on the whole $\mathbb{R}^N$, by [14, Corollary 26.3.3] $\square^A f$ is itself essentially smooth throughout the whole $\mathbb{R}^M$. Existence of optima yields by standard arguments the strict convexity of $\square^A f$, which is induced by the one of f , and $(\square^A f, \mathbb{R}^M)$ is then of Legendre type. By Theorem 5.3 applied to $h = \square^A f$, $\square^A f : \mathbb{R}^M \rightarrow \mathbb{R}$ is continuously differentiable on $\mathbb{R}^M$. Its conjugate $(\square^A f)^*$ is continuously differentiable on $\mathcal{O}$, the gradient $\nabla(\square^A f)(\cdot)$ is a homeomorphism between $\mathbb{R}^M$ and $\mathcal{O}$, and its (continuous) inverse is given by $\nabla(\square^A f)^*(\cdot)$. Now, fix $\mathbf{y} \in \mathbb{R}^M$ and take $(\mathbf{x}, \lambda) = (\mathbf{x}(\mathbf{y}), \lambda(\mathbf{y})) \in \mathbb{R}^N \times \mathbb{R}^M$ solving (23). In particular $A^T \lambda \in \text{int dom}(f^*)$, and $\mathbf{x} = (\nabla f)^{-1}(A^T \lambda) = \nabla f^*(A^T \lambda)$, by Theorem 5.3. Since $A\mathbf{x} = \mathbf{y}$, we get $\mathbf{y} = A \nabla f^*(A^T \lambda)$. The last step is to prove that $\lambda \in \text{int dom}((\square^A f)^*) = \mathcal{O}$ and $A \nabla f^*(A^T \lambda) = \nabla(\square^A f)^*(\lambda)$, as this would give $\lambda = (\nabla(\square^A f)^*)^{-1}(\mathbf{y}) = \nabla(\square^A f)(\mathbf{y})$ by Theorem 5.3 so that $\mathbf{x} = \nabla f^*(A^T \lambda) = \nabla f^*(A^T \cdot \nabla(\square^A f)(\mathbf{y}))$.

We come to these verifications. First, observe that as argued before $A^T \lambda \in \text{int dom}(f^*)$, which is open. Then, λ belongs to the pre-image of $\text{int dom}(f^*)$ under A^T , which is open by continuity of A^T . Set now $\mathcal{O}' := (A^T)^{-1}(\text{int dom}(f^*))$. Since $A^T(\mathcal{O}') = \text{int dom}(f^*) \subseteq \text{dom}(f^*)$, then $\lambda \in \mathcal{O}' \subseteq \mathcal{O}$. To conclude we prove that

$$\nabla(\square^A f)^*(\mathbf{z}) = A \nabla f^*(A^T \mathbf{z}), \quad \mathbf{z} \in \mathcal{O}' \subseteq \mathcal{O}.$$

First, by [14, Theorem 16.3] $(\square^A f)^*(\mathbf{z}) = f^*(A^T \mathbf{z})$ for all $\mathbf{z} \in \mathbb{R}^M$. The map $(\square^A f)^*$ on $\mathcal{O}'$ is then the composition of the map A^T , differentiable on $\mathcal{O}'$ and taking values in $\text{int dom}(f^*)$, and f^* , differentiable on the latter set. The formula is then the usual chain rule. Continuity of Θ follows by observing that for every $\mathbf{y} \in \mathbb{R}^M$, $A^T \lambda(\mathbf{y}) \in \text{int dom}(f^*)$, where ∇f^* is continuous, and that $\lambda(\mathbf{y}) = \nabla(\square^A f)(\mathbf{y})$, the latter being continuous on $\mathbb{R}^M$. ■

6. Proof of Theorem 2.9. We work again in Setting 2.7 under Assumption 2.8. By Remark 5.2 we may apply the results in Proposition 5.7. The proof is indeed based on the following two facts that are proven in Proposition 5.7, using there the notation $f := -U$, $A\mathbf{X} := \pi(\mathbf{X})$, and $\square^A f = -\square_\pi U$.

- (a) The function $\square_\pi U : \mathbb{R}^M \rightarrow \mathbb{R}$ is continuous on $\mathbb{R}^M$;
- (b) Fix any $\mathbf{z} \in \mathbb{R}^M$. There exists a unique optimum $\mathbf{x} = \Theta(\mathbf{z}) \in \mathbb{R}^N$ for the problem $\square_\pi U(\mathbf{z})$ with $\Theta : \mathbb{R}^M \rightarrow \mathbb{R}^N$ being the continuous function on $\mathbb{R}^M$ defined in (24). In particular we have (b1) $\square_\pi U(\mathbf{z}) = U(\Theta(\mathbf{z}))$ and (b2) $\pi(\Theta(\mathbf{z})) = \mathbf{z}$.

Proof of item 1. Fix $\mathbf{Z} \in (L^\infty)^M$. We now prove

$$(25) \quad \sup \{ \mathbb{E}[U(\mathbf{Y})] \mid \mathbf{Y} \in (L^\infty)^N, \pi(\mathbf{Y}) = \mathbf{Z} \} =: \square_\pi \mathbb{U}(\mathbf{Z}) = \mathbb{E}[\square_\pi U(\mathbf{Z})].$$

Fix $\mathbf{z} \in \mathbb{R}^M$ and observe that if $\mathbf{y} \in \mathbb{R}^N$ satisfies $\pi(\mathbf{y}) = \mathbf{z}$, then

$$U(\mathbf{y}) \leq \sup \{ U(\mathbf{x}) \mid \mathbf{x} \in \mathbb{R}^N, \pi(\mathbf{x}) = \mathbf{z} \} = \square_\pi U(\mathbf{z}).$$

Now, we can plug in $\mathbf{Y} \in (L^\infty)^N$ s.t. $\pi(\mathbf{Y}) = \mathbf{Z}$ to get $U(\mathbf{Y}) \leq \square_\pi U(\mathbf{Z})$. From (a) we know that $\square_\pi U$, as well as U , is a continuous function and so no measurability issues arise and both $U(\mathbf{Y})$ and $\square_\pi U(\mathbf{Z})$ are bounded random variables. We can then take expectations on both sides of the latter inequality and deduce $\square_\pi \mathbb{U}(\mathbf{Z}) \leq \mathbb{E}[(\square_\pi U)(\mathbf{Z})]$. We prove the converse inequality. Consider the continuous function Θ in (b) and set $\hat{\mathbf{Y}} := \Theta(\mathbf{Z})$. Then $\hat{\mathbf{Y}} \in (L^\infty)^N$ and by (b2) above, $\pi(\hat{\mathbf{Y}}) = \pi(\Theta(\mathbf{Z})) = \mathbf{Z}$. Thus, $\hat{\mathbf{Y}}$ satisfies the constraints in the LHS of (25). Consequently,

$$\sup \{ \mathbb{E}[U(\mathbf{Y})] \mid \mathbf{Y} \in (L^\infty)^N, \pi(\mathbf{Y}) = \mathbf{Z} \} \geq \mathbb{E}[U(\hat{\mathbf{Y}})] = \mathbb{E}[U(\Theta(\mathbf{Z}))] = \mathbb{E}[\square_\pi U(\mathbf{Z})],$$

by (b1), which concludes the proof of (25).

Proof of item 2. Recall that in Setting 2.7, Assumption 2.1 holds true. Fix $\mathbf{Z} \in (L^\infty)^M$ and $\mathbf{X} \in (L^\infty)^N$. From (a) we deduce that $\mathbb{E}[\square_\pi U(\mathbf{Z})] < \infty$ and by (25), $\square_\pi \mathbb{U}(\mathbf{Z}) = \mathbb{E}[\square_\pi U(\mathbf{Z})] < \infty$. By the assumption $\sup \{ U(\mathbf{y}) \mid \mathbf{y} \in \mathbb{R}^N \} > 0$ we obtain $\sup \{ \mathbb{U}(\mathbf{X} + \mathbf{y}) \mid \mathbf{y} \in \mathbb{R}^N \} > 0$. Thus all the assumptions in Proposition 2.4 are satisfied and hence $\rho_{\pi, \mathbb{U}}(\mathbf{X}) = \rho_{(\square_\pi \mathbb{U})}(\pi(\mathbf{X})) = \rho_{(\mathbb{E}[\square_\pi U])}(\pi(\mathbf{X}))$, by (25). Recalling the definition in (5), we thus proved (13). Regarding the optimality of $\mathbf{Y}_\mathbf{X} := -\mathbf{X} + \Theta(\pi(\mathbf{X}) + \hat{\alpha})$, observe that $\mathbb{U}(\mathbf{X} + \mathbf{Y}_\mathbf{X}) = \mathbb{U}(\Theta(\pi(\mathbf{X}) + \hat{\alpha})) = \mathbb{E}[U(\Theta(\pi(\mathbf{X}) + \hat{\alpha}))] = \mathbb{E}[\square_\pi U(\pi(\mathbf{X}) + \hat{\alpha})] \geq 0$, where in the last equality we used (b1), and the inequality follows from the optimality of $\hat{\alpha}$ in (13). Thus $\mathbf{Y}_\mathbf{X}$ satisfies the inequality constraint in (11). Moreover, using the linearity of π and (b2), we get

$$\pi(\mathbf{Y}_\mathbf{X}) = \pi(-\mathbf{X} + \Theta(\pi(\mathbf{X}) + \hat{\alpha})) = -\pi(\mathbf{X}) + \pi(\Theta(\pi(\mathbf{X}) + \hat{\alpha})) = -\pi(\mathbf{X}) + \pi(\mathbf{X}) + \hat{\alpha} = \hat{\alpha}$$

so that $\mathbf{Y}_\mathbf{X} \in \mathcal{C}$. Finally, $\sum_{m=1}^M \pi^m(\mathbf{Y}_\mathbf{X}) = \sum_{m=1}^M \hat{\alpha}^m = \rho_{\pi, \mathbb{U}}(\mathbf{X})$, by optimality of $\hat{\alpha}$. Thus $\mathbf{Y}_\mathbf{X}$ is the desired optimum, which is unique by the strict concavity of $\mathbb{U}$.

Conclusion for the case $M = 1$. If $\rho_{\pi, U}(\mathbf{X})$ is finite, then the optimality of $\hat{\alpha} = \rho_{\pi, U}(\mathbf{X})$ is directly checked by the monotone convergence theorem, considering that $\square_{\pi} U$ is continuous on $\mathbb{R}$ and nondecreasing by Proposition 5.7. Thus, we only need to show that $\rho_{\pi, U}(\mathbf{X}) \in \mathbb{R}$ for every $\mathbf{X} \in (L^{\infty})^N$. Since we are in Setting 2.7, by Remark 2.5 we have $\sup\{U(\mathbf{X} + \mathbf{y}) \mid \mathbf{y} \in \mathbb{R}^N\} > 0$, which yields $\rho_{\pi, U}(\mathbf{X}) < +\infty$. Suppose now by contradiction that $\rho_{\pi, U}(\mathbf{X}) = -\infty$, and take a minimizing sequence $\mathbf{Y}_n \in \mathcal{C}$ with $\pi(\mathbf{Y}_n) \downarrow_n -\infty$ and $\mathbb{E}[U(\mathbf{X} + \mathbf{Y}_n)] \geq 0$ for every n . By Proposition 5.5, since we are under Assumption 2.8 and Remark 5.2 applies, there exists a $\lambda \in \mathbb{R}$ such that $A^T \lambda \in \nabla f(\mathbb{R}^N)$. By Theorem 5.3 we have $A^T \lambda \in \text{intdom}(f^*) \subseteq (-\infty, 0)^N$, the latter following from monotonicity of $f = -U$, which implies $\lambda < 0$: indeed all the components of A are nonnegative, as $A(\mathbb{R}_+^N) = \mathbb{R}_+$. Now by the Fenchel inequality we have $-f^*(A^T \lambda) - f(\mathbf{x}) \leq (-\lambda)A\mathbf{x}$. Substituting $\mathbf{x}$ with $\mathbf{X} + \mathbf{Y}_n$ and taking expectations yields a contradiction, as $f^*(A^T \lambda) \in \mathbb{R}$ and $\mathbb{E}[-f(\mathbf{X} + \mathbf{Y}_n)] \geq 0$ for each n , while the RHS tends to $-\infty$, as $\pi(\mathbf{Y}_n) \downarrow_n -\infty$.

Appendix A. A function $\Phi : [0, +\infty)^N \rightarrow \mathbb{R}$ is called a multivariate Orlicz function if it is null in 0, convex, continuous, and increasing in the usual componentwise order and satisfies the following: there exist $A > 0$, B constants such that $\Phi(\mathbf{x}) \geq A \sum_{j=1}^N x^j - B$ for every $\mathbf{x} \in [0, +\infty)^N$. We refer the reader to [1] and [8] for further details. Inspired by [8, Definition 3.4], we say that a function $U : \mathbb{R}^N \rightarrow \mathbb{R}$ is well controlled if there exist a multivariate Orlicz function $\hat{\Phi} : \mathbb{R}^N \rightarrow \mathbb{R}$ and a function $h : [0, +\infty) \rightarrow \mathbb{R}$ such that $U(\mathbf{x}) \leq -\hat{\Phi}((x)^-) + \varepsilon \sum_{j=1}^N |x^j| + h(\varepsilon)$ for every $\varepsilon > 0$. Using a simplified version of the techniques applied in [8, Theorem 6.1, step 2a], one can obtain the following result.

Lemma A.1. *Suppose $U : \mathbb{R}^N \rightarrow \mathbb{R}$ is strictly concave, strictly increasing in the componentwise order, and also well controlled. Suppose also that π satisfies Assumption 2.1(a) and that $\sum_{m=1}^M \pi^m(\mathbf{x}) = \sum_{j=1}^N x^j$ for every $\mathbf{x} \in \mathbb{R}^N$. Then Assumption 2.8 is satisfied.*

REFERENCES

- [1] Y. ARMENTI, S. CRÉPEY, S. DRAPEAU, AND A. PAPAPANTOLEON, *Multivariate shortfall risk allocation and systemic risk*, SIAM J. Financial Math., 9 (2018), pp. 90–126, <https://doi.org/10.1137/16M1087357>.
- [2] F. BIAGINI, J.-P. FOUQUE, M. FRITTELLI, AND T. MEYER-BRANDIS, *A unified approach to systemic risk measures via acceptance sets*, Math. Finance, 29 (2019), pp. 329–367, <https://doi.org/10.1111/mafi.12170>.
- [3] F. BIAGINI, J.-P. FOUQUE, M. FRITTELLI, AND T. MEYER-BRANDIS, *On fairness of systemic risk measures*, Finance Stoch., 24 (2020), pp. 513–564, <https://doi.org/10.1007/s00780-020-00417-4>.
- [4] M. BURZONI, A. DOLDI, AND E. MONZIO COMPAGNONI, *Risk Sharing with Deep Neural Networks*, preprint, <https://arxiv.org/abs/2212.11752v2>, 2023.
- [5] R. CONT, R. DEGUET, AND G. SCANDOLO, *Robustness and sensitivity analysis of risk measurement procedures*, Quant. Finance, 10 (2010), pp. 593–606, <https://doi.org/10.1080/14697681003685597>.
- [6] A. DOLDI, Y. FENG, J.-P. FOUQUE, AND M. FRITTELLI, *Multivariate systemic risk measures and computation by deep learning algorithms*, Quant. Finance, 23 (2023), pp. 1431–1444, <https://doi.org/10.1080/14697688.2023.2231505>.
- [7] A. DOLDI AND M. FRITTELLI, *Conditional systemic risk measures*, SIAM J. Financial Math., 12 (2021), pp. 1459–1507, <https://doi.org/10.1137/20M1370616>.
- [8] A. DOLDI AND M. FRITTELLI, *Multivariate systemic optimal risk transfer equilibrium*, Ann. Oper. Res., (2022), <https://doi.org/10.1007/s10479-022-04652-0>.

- [9] W. FARKAS, P. KOCH-MEDINA, AND C. MUNARI, *Measuring risk with multiple eligible assets*, Math. Financ. Econ., 9 (2015), pp. 3–27, <https://doi.org/10.1007/s11579-014-0118-0>.
- [10] Z. FEINSTEIN, B. RUDLOFF, AND S. WEBER, *Measures of systemic risk*, SIAM J. Financial Math., 8 (2017), pp. 672–708, <https://doi.org/10.1137/16M1066087>.
- [11] H. FÖLLMER AND A. SCHIED, *Stochastic Finance: An Introduction in Discrete Time*, 4th revised and extended ed., De Gruyter Textbook, De Gruyter, Berlin, 2016, <https://doi.org/10.1515/9783110463453>.
- [12] M. FRITTELLI AND G. SCANDOLO, *Risk measures and capital requirements for processes*, Math. Finance, 16 (2006), pp. 589–612, <https://doi.org/10.1111/j.1467-9965.2006.00285.x>.
- [13] L. OVERBECK AND F. SCHINDLER, *Scalar Systemic Risk Measures and Aumann-Shapley Allocations*, preprint, <https://arxiv.org/abs/2112.06534v2>, 2022.
- [14] R. T. ROCKAFELLAR, *Convex Analysis*, Princeton Math. Ser. 28, Princeton University Press, Princeton, NJ, 1970.
- [15] A. SHAPIRO, *Consistency of sample estimates of risk averse stochastic programs*, J. Appl. Probab., 50 (2013), pp. 533–541.