Dominated Families of Martingale, Supermartingale and Quasimartingale Laws

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Abstract

Consider a dominated family $\mathcal{Q}$ of probability measures; we investigate the question of whether a single probability $\hat{Q} \in \mathcal{Q}$ equivalent to the whole family $\mathcal{Q}$ exists. We show that for supermartingale, quasimartingale and martingale laws the answer is positive. We then provide a necessary and sufficient condition for the existence of an equivalent (super, quasi) martingale measure and deduce an alternative characterization of semimartingales. We further study this problem in the context of security markets models and generalize the well known fundamental theorem of asset pricing to cover the case of markets with frictions.

KEYWORDS: Semimartingale, Quasimartingale Law, Equivalent Probability Measure, Risk Neutral Measure, Arbitrage, Security Markets.

1 Introduction

Let $\mathcal{L} = \mathcal{L}(\Omega, \mathcal{F})$ be the set of probabilities defined on a measurable space $(\Omega, \mathcal{F})$ and fix a probability measure (pr.m.) $P \in \mathcal{L}$. Let $L^0(\Omega, \mathcal{F}, P)$ be the space of $P - a.s.$ finite random variables on $(\Omega, \mathcal{F})$ and, for any subset $V \subseteq$

$L^0(\Omega, \mathcal{F}, P)$, set $V_+ \triangleq \{f \in V : f \geq 0 \; P - a.s.\}$, $V_{++} \triangleq \{f \in V_+ : P(f > 0) > 0\}$, and analogously for V_-.

Let K be a convex subset of $L^0(\Omega, \mathcal{F}, P)$ with $0 \in K$, and define:

$$\mathcal{Q}_a^K \triangleq \{Q \in \mathcal{L} : Q \ll P, \; K \subseteq L^1(\Omega, \mathcal{F}, Q) \text{ and } \sup_{k \in K} E_Q[k] < +\infty\}, \qquad (\bullet)$$

$$\mathcal{Q}_e^K \triangleq \left\{Q \in \mathcal{Q}_a^K : Q \sim P\right\}.$$

A relevant role for our further analysis is played by the following property:

Definition 1 *We say that a family $\mathcal{Q} \subseteq \mathcal{L}$ is* tamed *if there exists a $\hat{Q} \in \mathcal{Q}$ such that $Q \ll \hat{Q}$ for all $Q \in \mathcal{Q}$.*

A first significant application of this property is provided in theorem 9, where we prove that:

if the family $\mathcal{Q}_a^K$ is tamed, then

$$\mathcal{Q}_e^K \neq \emptyset \qquad (A)$$

if and only if

$$\forall f \in L_{++}^{\infty}(\Omega, \mathcal{F}, P), \; \exists \hat{r} \geq 0 : \forall r > \hat{r} \quad rf \notin \overline{K - L_+^{\infty}(\Omega, \mathcal{F}, P)}^{\tau_K}, \qquad (C)$$

where τ_K is a specific weak topology that we will introduce and explain later (def. 6).

As we now show, the importance of this theorem is revealed from the investigation of almost sure properties of stochastic processes.

Let χ be a finite or countable family of real adapted stochastic processes on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathcal{T}}, P)$, where $\mathcal{T} = [0, +\infty)$.

Definition 2 *We call a probability measure $Q \in \mathcal{L}(\Omega, \mathcal{F})$ a quasimartingale law for χ if each process $X \in \chi$ is a $(\mathcal{F}_t)_{t\in\mathcal{T}}-$quasimartingale under the probability measure Q. Similar definitions apply for semimartingale, submartingale, supermartingale and martingale laws for χ.*

Our first task is to show that the families of quasimartingale, supermartingale, submartingale and martingale laws for χ, that are absolutely continuous with respect to P, may be represented in the form $(\bullet)$ for a suitable choice of the set K (lemma 11).

Then, our second task is to show that the families of quasimartingale, supermartingale, submartingale and martingale laws for χ, absolutely continuous with respect to P, are all *tamed* families (theorem 13 and corollary 14).

So, we can apply theorem 9 above mentioned and conclude that (corollary 15):

condition (C), with the appropriate choice of the set K, is necessary and sufficient for the existence of a quasimartingale, supermartingale or martingale law for χ equivalent to P.

An important consequence of this result is an alternative characterization of semimartingales (theorem 16):

a real adapted stochastic process $X \in \chi$ is a semimartingale if and only if condition (C) holds, where

$$K = \left\{ \int_0^\infty H dX : H \text{ is elementary predictable process, } \|H\|_\infty \leq 1 \right\}.$$

It is possible to interpret these results in the theory of stochastic models of security markets. The so called fundamental theorem of asset pricing (informally) states that in *frictionless markets* there exists an *equivalent martingale measure* (also frequently called risk neutral measure) if and only if the market admits "*no arbitrage*".

Many authors ([1], [2], [3], [5], [6], [9], [12], [13], [14], [15]) have proposed different models of frictionless security markets (essentially assuming different hypotheses on the classes of admissible stochastic processes representing deflated price processes and of admissible trading strategies) in order to give the above theorem a precise and rigorous formulation.

An important step in this direction was formulated by Stricker (1990 [15]) who discovered the pertinence of a theorem of Yan [16] to this subject. Yan's theorem has become an important tool in most subsequent works in the area: and indeed also proposition 8 is clearly inspired by Yan's work.

In a former paper [6] Frittelli and Lakner studied this problem in the context of frictionless market (the martingale case) and provided a new formulation of the fundamental theorem of asset pricing. No one would expect the theorem to be true in *frictional markets* too (see for example Jouini & Kallal [11]), but in this work we analyze to what extent and under what restrictions the theorem may be generalized to cover also the frictional markets cases.

We model a fairly general class of restrictions by requiring the trading strategies to belong to a convex set Θ. The selection of this set determines the class of *admissible* trading strategies. The generality of the specification of Θ provides sufficient flexibility in order to treat different forms of constraints. Moreover, in our model any real adapted stochastic process may be selected for representing the (deflated) price process of an available asset.

In section 4 we interpret condition (C) as a *no extreme arbitrage opportunity* condition and we then deduce that there exists an *equivalent quasi-martingale measure if and only if no extreme arbitrage opportunity exists (*corollary 18*).* This may also be rephrased by saying that a real adapted process *is a semimartingale if and only if no extreme arbitrage opportunity exists.* The special case of *shortsale constraints* is also considered (corollary 18).

2 Tamed families of probabilities

2.1 Properties and examples

We recall that two families $\mathcal{R}$,$\mathcal{Q} \subseteq \mathcal{L}$ of probabilities are *equivalent* ($\mathcal{R} \sim \mathcal{Q}$) iff $\forall A \in \mathcal{F}$ $(R(A) = 0 \; \forall R \in \mathcal{R}) \Leftrightarrow (Q(A) = 0 \; \forall Q \in \mathcal{Q})$ and that a family of probabilities $\mathcal{Q}$ is *dominated* by a probability $\hat{Q}$ if $Q \ll \hat{Q} \; \forall Q \in \mathcal{Q}$. So, a family $\mathcal{Q}$ is *tamed* if and only if it is dominated by a single measure $\hat{Q} \in \mathcal{Q}$. Clearly, a single probability $\hat{Q} \in \mathcal{Q}$ dominates the family $\mathcal{Q}$ if and only if $\hat{Q} \sim \mathcal{Q}$.

Lemma 3 *A dominated family $\mathcal{Q} \subseteq \mathcal{L}$ is tamed if, for any countable family $\{Q_n\}_{n\in\mathbb{N}} \in \mathcal{Q}$, there exists a sequence $\{a_n\}_{n\in\mathbb{N}}$ of positive real numbers such that $\sum\limits_{n=1}^{\infty} a_n Q_n \in \mathcal{Q}$.*

Proof. The Halmos-Savage lemma [8] guarantees that any dominated family of probabilities $\mathcal{Q}$ contains a countable equivalent family. Thus, to show that a dominated family $\mathcal{Q}$ is tamed it is sufficient to prove that any given countable family $\{Q_n\}_{n\in\mathbb{N}} \in \mathcal{Q}$ is dominated by some measure in $\mathcal{Q}$. ■

Remark 1 *A dominated family $\mathcal{Q}$ is tamed if and only if for any given countable family $\{Q_n\}_{n\in\mathbb{N}} \in \mathcal{Q}$, there exists a sequence $\{a_n\}_{n\in\mathbb{N}}$ of set functions $a_n : \mathcal{F} \to (0, +\infty)$ such that $\sum\limits_{n=1}^{\infty} a_n Q_n \in \mathcal{Q}$ (we defer the proof to the Appendix).*

Recall the notation ($\bullet$) given in the Introduction for $\mathcal{Q}_a^K$. Note that if K is positively homogeneous ($k \in K, \lambda \geq 0 \Rightarrow \lambda k \in K$) then $\mathcal{Q}_a^K \equiv \{Q \in \mathcal{L} : Q \ll P,\ K \subseteq L^1(\Omega, \mathcal{F}, Q)$ and $\sup\limits_{k \in K} E_Q[k] = 0\}$ while if K is homogeneous ($k \in K, \lambda \in R \Rightarrow \lambda k \in K$) then $\mathcal{Q}_a^K \equiv \{Q \in \mathcal{L} : Q \ll P,\ K \subseteq L^1(\Omega, \mathcal{F}, Q)$ and $E_Q[k] = 0\ \forall k \in K\}$.

Here are two trivial examples of tamed families (we defer the proofs to the Appendix):

Example 4 *Let $K \subseteq L^\infty(\Omega, \mathcal{F}, P)$ be such that $\mathcal{Q}_a^K$ is not empty, then $\mathcal{Q}_a^K$ is tamed.*

Example 5 *Let $K \subseteq L^0(\Omega, \mathcal{F}, P)$ be countable, then $\mathcal{Q}_a^K$ is tamed.*

Note that from a well known theorem of C. Dellacherie ([4] Ch.VII, Th. 57) one may deduce that, for K countable, there always exists a probability measure Q equivalent to P such that $K \subset L^1(\Omega, \mathcal{F}, Q)$ and $\sup_{k \in K} E_Q[k] < +\infty$. This implies that $\mathcal{Q}_e^K$ is not empty and $\mathcal{Q}_a^K$ is tamed.

2.2 The topology

We denote with $\Sigma(\Omega, \mathcal{F})$ the space of finite signed measures on $(\Omega, \mathcal{F})$. Let $K \subseteq L^0(\Omega, \mathcal{F}, P)$ be an arbitrary set. Define:

$$S_K \triangleq Lin \{K \cup L^\infty(\Omega, \mathcal{F}, P)\},$$

$$
\begin{aligned}
S'_K &\triangleq \left\{ \mu \in \Sigma(\Omega, \mathcal{F}) : \mu \ll P \text{ and } \int_\Omega |s| d|\mu| < +\infty \ \forall s \in S_K \right\} \\
&= \left\{ s' \in L^1(\Omega, \mathcal{F}, P) : E_P[|s's|] < +\infty \ \forall s \in S_K \right\}.
\end{aligned}
$$

Note that S'_K is a subspace of the algebraic dual of S_K and so (S_K, S'_K) defines a dual system.

Definition 6 *Let τ_K be any topology on S_K compatible with the dual system (S_K, S'_K).*

(S_K, τ_K) is a locally convex topological vector space (see Grothendieck [7], Ch. II, Sec. 8 & 13) that, in general, is not metrizable (see Frittelli & Lakner [6], [13], for further details on this topology).

One example of a topology compatible with (S_K, S'_K) is the weakest topology $\hat{\tau}$ on S_K which is stronger than each $L^1(\Omega, \mathcal{F}, Q)-$norm topology, for all $Q \sim P$ such that $K \subseteq L^1(\Omega, \mathcal{F}, Q)$ (again see [6] for more details).

Note also that if one requires $K \subseteq L^\infty(\Omega, \mathcal{F}, P)$ then $S_K = L^\infty(\Omega, \mathcal{F}, P)$, $S'_K = L^1(\Omega, \mathcal{F}, P)$ and one may, for instance, choose as τ_K the more familiar weak* topology $\sigma(L^\infty, L^1)$ on $L^\infty(\Omega, \mathcal{F}, P)$.

Remark 2 *S_K, S'_K and τ_K are invariant under substitution of P with an equivalent probability measure: this implies that also the geometric condition (C) will have this remarkable property.*

2.3 A basic theorem

Let $K \subseteq L^0(\Omega, \mathcal{F}, P)$. For any $V \subseteq S_K$ we denote with $\overline{V}^{\tau_K}$ its closure in the τ_K topology. Consider the following conditions:

(A) $\mathcal{Q}_e^K \neq \emptyset$.

(B) $\forall A \in \mathcal{F} : P(A) > 0\ \exists Q_A \in \mathcal{Q}_a^K : Q_A(A) > 0$.

(C) $\forall f \in L_{++}^\infty(\Omega, \mathcal{F}, P),\ \exists \hat{r} \geq 0 : \forall r > \hat{r} \quad rf \notin \overline{K - L_+^\infty(\Omega, \mathcal{F}, P)}^{\tau_K}$.

(C1) $L_{++}^\infty(\Omega, \mathcal{F}, P) \cap \overline{K - L_+^\infty(\Omega, \mathcal{F}, P)}^{\tau_K} = \emptyset$.

(T) $\mathcal{Q}_a^K$ is tamed (i.e. $\exists \hat{Q} \in \mathcal{Q}_a^K : Q \ll \hat{Q} \quad \forall Q \in \mathcal{Q}_a^K$).

Proposition 7 *If the family $\mathcal{Q}_a^K$ is tamed, then (A) and (B) are equivalent.*

If the set K is positively homogeneous then (C) and $(C1)$ are equivalent.

Proof. It is straightforward to check that: $(A) \Rightarrow (B)$; $(A) \Rightarrow (T)$; (B) and $(T) \Rightarrow (A)$. The second statement is obvious. ■

Proposition 8 *Let $K \subseteq L^0(\Omega, \mathcal{F}, P)$. Then (A) implies (C). If K is convex and $0 \in K$, then (C) implies $\mathcal{Q}_a^K \neq \emptyset$ and (B).*

Proof. $(A) \Rightarrow (C)$: Let $\hat{Q} \in \mathcal{Q}_e^K$ and $\hat{c} \triangleq \sup_K \hat{E}[k] < +\infty$ where $\hat{E}[k] \triangleq E_P[\hat{y}k]$ and $\hat{y} \triangleq \frac{d\hat{Q}}{dP} \in S_K'$. Define $G \triangleq \left\{ s \in S_K : \hat{E}[s] \leq \hat{c} \right\}$. G is closed in (S_K, τ_K) since $\hat{y} \in S_K'$. If $g \in (K - L_+^\infty)$ then $\hat{E}[g] \leq \hat{c}$ and therefore

$\overline{(K - L^\infty_+)}^{\tau_K} \subseteq G$. Let $f \in L^\infty_{++}$; then $\hat{E}[f] > 0$ (since $\hat{Q} \sim P$) and we may select $\hat{r} = \frac{\hat{c}}{\hat{E}[f]}$. For $r > \hat{r}$ we have: $\hat{E}[rf] > \hat{r}\hat{E}[f] = \hat{c}$; hence $rf \notin G$.

$(C) \Rightarrow (B)$: This part of the proof, which is inspired by Yan [16], is a simple application of the separation theorem in locally convex T.V.S. Fix $A \in \mathcal{F}$ such that $P(A) > 0$ so that $1_A \in L^\infty_{++}$. Fix also $r > \hat{r}$. Since the compact set $\{r1_A\}$ is disjoint from the closed convex set $\overline{(K - L^\infty_+)}^{\tau_K}$ the separation theorem (Grothendieck, [7] , Ch. II, Sec. 7, Prop. 10) guarantees the existence of a non trivial functional $s' \in S'_K$ such that

$$\sup_{k \in K, h \in L^\infty_+} E_P[s'(k-h)] < E_P[s' r 1_A] < +\infty.$$

Take $k = 0$; then $\sup\limits_{h \in L^\infty_+} E_P[-s'h] < +\infty$ implies $s' \geq 0$ $P - a.s.$. Define Q_A by $\frac{dQ_A}{dP} = \frac{s'}{\|s'\|_{L^1}}$, so then $Q_A \ll P$.

Take $h = 0$; then $\sup\limits_{k \in K} E_{Q_A}[k] < E_{Q_A}[r1_A] = rQ_A(A)$. Hence $Q_A \in \mathcal{Q}^K_a$ and $\mathcal{Q}^K_a$ is not empty. Now, $0 \in K$ implies $Q_A(A) > 0$. ■

From propositions 7 and 8 we deduce:

Theorem 9 *If $\mathcal{Q}^K_a$ is a tamed family and $K \subseteq L^0(\Omega, \mathcal{F}, P)$ is a convex set with $0 \in K$ then conditions $(A), (B)$ and (C) are equivalent.*

Remark 3 *Note that one may replace L^∞_{++} with $(S_K)_{++}$ and L^∞_+ with $(S_K)_+$ in condition (C).*

Furthermore, let $f \in L^\infty_{++}$ *and let the set* $U(f) \subseteq [0, +\infty]$ *be defined by:*

$$U(f) \triangleq \left\{ \alpha = \left\{ \begin{array}{ll} \frac{c}{E_Q[f]} & if\ E_Q[f] > 0 \\ +\infty & if\ E_Q[f] = 0 \end{array} \right. ,\ for\ Q \in \mathcal{Q}_a^K\ and\ c \triangleq \sup_{k \in K} E_Q[k] < +\infty \right\}.$$

Set: $r(f) \triangleq \inf U(f) \leq +\infty$*. Clearly* (B) *holds if and only if* $r(1_A) < +\infty\ \forall A \in \mathcal{F} : P(A) > 0$*. Moreover one may select* $r(f)$ *to replace* $\hat{r}$ *in condition* (C)*. Indeed:* $(A) \Rightarrow (C)$ *with* $\hat{r} = r(f)$ *(we defer the proof to the Appendix).*

3 Almost sure properties of stochastic processes

3.1 Quasimartingale, supermartingale and martingale laws

Let $\mathcal{T} \triangleq [0, +\infty)$, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathcal{T}}, P)$ be a filtered probability space with $\mathcal{F}_0$ trivial and complete, $\chi \equiv (X^j)_{j\in\mathbb{J}}$ be a family of real adapted stochastic processes $X^j = (X_t^j)_{t\in\mathcal{T}}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathcal{T}}, P)$, where the index set $\mathbb{J} = \{1, 2, ...\} \subseteq \mathbb{N}$ may be finite or countable. Set $J = \sup\{\mathbb{J}\} \leq +\infty$.

Let $\Theta = (\Theta^j)_{j\in\mathbb{J}}$, be a family of subsets $\Theta^j \subseteq L^0(\Omega, \mathcal{F}, P)$ such that $0 \in \Theta^j$. Denote by E the space of bounded elementary predictable processes on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathcal{T}}, P)$ (see Dellacherie & Meyer, [4], Ch. VIII, 2.) and by E^{Θ^j} the class of processes $H \in E$ that have the following representation:

$$H = \sum_{i=0}^{n} H_i 1_{(t_i, t_{i+1}]} \quad \text{where} \quad H_i \in L^\infty(\Omega, \mathcal{F}_{t_i}, P) \cap \Theta^j$$

and $0 \leq t_0 < ... < t_{n+1}$ is a finite sequence of time indices $t_i \in \mathcal{T}$.

For any $j \in \mathbb{J}$ and $X^j \in \chi$ we may define the elementary integral application

$$I_j : E^{\Theta^j} \to L^0(\Omega, \mathcal{F}, P),\ H \to \sum_{i=0}^{n} H_i(X^j_{t_{i+1}} - X^j_{t_i}).$$

Let us denote the image $I_j(E^{\Theta^j})$ of E^{Θ^j} by $I_j[\Theta^j]$.

Define:

$$I[\Theta] \triangleq \sum_{j=1}^{J} I_j[\Theta^j].$$

Note that $f \in I[\Theta]$ if and only if $f = f_{j_1} + ... + f_{j_N}$ is any finite sum of elements f_{j_i} where each f_{j_i} belongs to a different set $I_{j_i}[\Theta^{j_i}]$. Notice also that $0 \in I_j[\Theta^j]$ and that if Θ^j are all convex sets then $I_j[\Theta^j]$ and $I[\Theta]$ are convex as well. We now select four specific sets of special interest; we defer their financial interpretation to section 4.

Definition 10

$M \triangleq I[\Theta]$ *when* $\Theta^j = L^0(\Omega, \mathcal{F}, P)\ \forall j \in \mathbb{J}$

$M_- \triangleq I[\Theta]$ *when* $\Theta^j = L^0_-(\Omega, \mathcal{F}, P)\ \forall j \in \mathbb{J}$

$M_+ \triangleq I[\Theta]$ *when* $\Theta^j = L^0_+(\Omega, \mathcal{F}, P)\ \forall j \in \mathbb{J}$

$M_1 \triangleq I[\Theta]$ *where* $\Theta^j = \Theta_1 \triangleq \{H \in L^\infty(\Omega, \mathcal{F}, P) : \|H\|_\infty \leq 1\}\ \forall j \in \mathbb{J}.$

Similar definitions apply for each component (for example $M^j_+ \triangleq I_j[\Theta^j]$ *when* $\Theta^j = L^0_+(\Omega, \mathcal{F}, P)$.

In the following we will replace the set K appearing in sections 1 and 2 by the convex sets M, M_-, M_+, M_1. Note that 0 belongs to all the above sets, M is a linear space, M_- and M_+ are positively homogeneous.

Lemma 11 *If χ is a finite family, then $\mathcal{Q}_a^{M_1}$ represents the set of quasi-martingale laws for χ that are absolutely continuous w.r.to P. For a countable family χ, $\mathcal{Q}_a^{M_-}$, $\mathcal{Q}_a^{M_+}$, $\mathcal{Q}_a^{M}$ represent respectively the sets of submartingale, supermartingale and martingale laws for χ that are absolutely continuous w.r.to P.*

Proof. First note that the integrability requirements are satisfied for all the pr.m. we are considering (note also that if $\mathcal{F}_0$ was not suppose to be trivial, an integrability condition on X_0 should have been assumed). Let $Q \ll P$. Define:

$$Var_Q[X^j]_t \triangleq \sup_{\pi(t)} \{E_Q[\sum_{i=0}^{n} \left| E_Q[X^j_{t_{i+1}} - X^j_{t_i} | \mathcal{F}_{t_i}] \right|]\},$$

where the sup is taken over all finite partitions $\pi(t)$ of $[0,t] \cap \mathcal{T}$. By definition X^j is a $Q-$quasimartingale if $Var_Q[X^j] \triangleq Var_Q[X^j]_\infty < +\infty$. If $f \in M_1^j$, then:

$$E_Q[f] = E_Q[\sum_{i=0}^{n} H_i^j (X^j_{t_{i+1}} - X^j_{t_i})] = E_Q[\sum_{i=0}^{n} H_i^j E_Q[X^j_{t_{i+1}} - X^j_{t_i} | \mathcal{F}_{t_i}]] \leq Var_Q[X^j]$$

and one gets the equality $\sup_{f \in M_1^j} E_Q[f] = Var_Q[X^j]$ from considering $H_i^j = sgn(E_Q[X^j_{t_{i+1}} - X^j_{t_i} | \mathcal{F}_{t_i}])$. Therefore Q is a quasimartingale law for X^j if and only if $\sup_{f \in M_1^j} E_Q[f] < +\infty$.

For a finite family χ, $\sup\limits_{f\in M_1} E_Q[f] = \sum\limits_{j=1}^{J} Var_Q[X^j]$, and so Q is a quasi-martingale measure for χ if and only if $\sup\limits_{f\in M_1} E_Q[f] < +\infty$.

Q is a submartingale (resp. supermartingale) law for X^j if and only if $E_Q[f] \leq 0$ $\forall f \in M_-^j$ (resp. M_+^j) and hence, if and only if $Q \in \mathcal{Q}_a^{M_-^j}$ (resp. $Q^{M_+^j}$) since M_-^j and M_+^j are positively homogeneous. Therefore Q is a submartingale (resp. supermartingale) law for χ if and only if $Q \in \mathcal{Q}_a^{M_-}$ (resp. Q^{M_+}).

Q is martingale law for X^j if and only if $E_Q[f] = 0$ $\forall f \in M^j$ and hence, if and only if $Q \in \mathcal{Q}_a^{M^j}$, since M^j is homogeneous. Therefore Q is a martingale law for χ if and only if $Q \in \mathcal{Q}_a^{M}$. ∎

3.2 The taming property of (super, quasi) martingale laws

We consider now a well known result (due to J.Jacod, [10], Ch.7, Th.42). The probability measures in theorems 12 and 13 are assumed to be absolutely continuous w.r.to P.

Theorem 12 *(countable convexity of semimartingale laws). Let* $\{Q_n\}_{n\in\mathbb{N}}$ *be semimartingale laws for a process* $X \in \chi$*,* a_n *positive constants such that* $\sum\limits_{n=1}^{\infty} a_n = 1$ *and* $Q \triangleq \sum\limits_{n=1}^{\infty} a_n Q_n$*. Then* Q *is a semimartingale law for* X*.*

From this theorem and lemma 3 we deduce that the family of dominated semimartingale laws is tamed. We now show that also the families of dominated quasimartingale, submartingale, supermartingale and martingale laws, for the *countable* family χ, are tamed:

Theorem 13 *Let $\{Q_n\}_{n\in\mathbb{N}}$ be quasimartingale laws (resp. submartingale, supermartingale, martingale laws) for χ. Then there exist positive constants a_n such that $\sum_{n=1}^{\infty} a_n = 1$ and $Q \triangleq \sum_{n=1}^{\infty} a_n Q_n$ is a quasimartingale law (resp. submartingale, supermartingale, martingale law) for χ.*

Proof. First assume that χ consists only of a single process X^j. $\forall n \in \mathbb{N}$, $\forall 0 \le s \le t$ the inequality $|X^j_s| \le |E_{Q_n}[X^j_t|\mathcal{F}_s]| + |X^j_s - E_{Q_n}[X^j_t|\mathcal{F}_s]| \le E_{Q_n}[|X^j_t||\mathcal{F}_s] + |E_{Q_n}[X^j_t - X^j_s|\mathcal{F}_s]|$ holds and we get $E_{Q_n}[|X^j_s|] \le E_{Q_n}[|X^j_t|] + E_{Q_n}[|E_{Q_n}[X^j_t - X^j_s|\mathcal{F}_s]|] \le E_{Q_n}[|X^j_t|] + Var_{Q_n}[X^j]_t$. Define:

$$b_n \triangleq \frac{2^{-n}}{1 + E_{Q_n}[|X^j_n|] + Var_{Q_n}[X^j]_n}$$

so that for each $s \in \mathcal{T}$, and for n sufficiently large ($n \ge \max(s, j)$) we have:

$$b_n E_{Q_n}[|X^j_s|] \le b_n(E_{Q_n}[|X^j_n|] + Var_{Q_n}[X^j]_n) \le 2^{-n}.$$

Assume that each Q_n is a quasimartingale law for X^j. Set

$$a_n \triangleq \alpha \frac{b_n}{1 + Var_{Q_n}[X^j]}, \quad \text{where } \alpha \in \mathbb{R} \text{ is such that } \sum_{n=1}^{\infty} a_n = 1.$$

Clearly $0 < b_n \le 2^{-n}$ and $0 < a_n \le \alpha b_n$. Since Q is a probability measure, we need only to show that $Var_Q[X^j] < +\infty$. For each $s \in \mathcal{T}$, the series $\sum_{n=1}^{\infty} b_n E_{Q_n}[|X^j_s|]$ converges and so does $\sum_{n=1}^{\infty} a_n E_{Q_n}[|f|]$ for any $f \in$

M_1^j. The monotone convergence theorem and the dominated convergence theorem now imply that $Var_Q[X^j] = \sup_{f \in M_1^j} E_Q[f] = \sup_{f \in M_1^j} \sum_{n=1}^{\infty} a_n E_{Q_n}[f] \leq \sum_{n=1}^{\infty} a_n Var_{Q_n}[X^j] \leq \sum_{n=1}^{\infty} \alpha b_n \leq \alpha$.

Assume now that Q_n are submartingale laws for X^j. Let

$$a_n \triangleq \frac{b_n}{\sum_{m=1}^{\infty} b_m}.$$

Then, by the same argument as above, $\sum_{n=1}^{\infty} a_n E_{Q_n}[|f|]$ converges for each $f \in M_-^j$ and we get $E_Q[f] = \sum_{n=1}^{\infty} a_n E_{Q_n}[f] \leq 0 \; \forall f \in M_-^j$.

If Q_n are supermartingale laws for X^j then Q_n are submartingale laws for $-X^j$. $Q \triangleq \sum_{n=1}^{\infty} a_n Q_n$ is therefore a submartingale law for $-X^j$ and hence a supermartingale law for X^j.

If Q_n are martingale laws for X^j then Q_n are submartingale and supermartingale laws for X^j and therefore so is $Q \triangleq \sum_{n=1}^{\infty} a_n Q_n$. Being both a submartingale and a supermartingale law, Q is a martingale law for X^j.

To prove the theorem when χ is a finite or countable family, just replace b_n with

$$\frac{2^{-n}}{1 + \max_{1 \leq j \leq n \wedge J} \left\{ E_{Q_n}[|X_n^j|] + Var_{Q_n}[X^j]_n \right\}}$$

and, for the quasimartingale case, a_n with $\alpha b_n \left(1 + \max_{1 \leq j \leq n \wedge J} \{ Var_{Q_n}[X^j] \} \right)^{-1}$. ■

From theorem 13, lemma 3 and lemma 11 we immediately get:

Corollary 14 *Each (not empty) family $\mathcal{Q}_a^{M_1}, \mathcal{Q}_a^{M_-}, \mathcal{Q}_a^{M_+}, \mathcal{Q}_a^{M}$ is tamed.*

3.3 A semimartingale characterization

Theorem 9, proposition 7 and corollary 14 now yield:

Corollary 15 *If the family χ is finite, then there exists a quasimartingale law Q for χ equivalent to P if and only if (C) holds with K replaced by M_1. For a countable family χ, there exists a martingale (resp. submartingale, supermartingale) law Q for χ equivalent to P if and only if $(C1)$ holds with K replaced by M (resp. M_-, M_+).*

Remark 4 *Conditions (C) and $(C1)$ are invariant under substitution of the reference measure P with an equivalent one; therefore, the statements in corollary 15 may be interpreted as an almost sure characterization of quasimartingales, supermartingales and martingales. We note that (only) the martingale case was considered in a previous work (Frittelli & Lakner [6]).*

Since a real adapted process $X = (X_t)_{t\in\mathcal{T}}$ is a $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathcal{T}}, P)$ semimartingale up to infinity if and only if there exists a probability measure Q equivalent to P under which X is a quasimartingale (see for example Dellacherie & Meyer [4] Ch. VII, Th. 58 and also Remark 58 for the generalization to vector valued process), from corollary 15 we deduce the following:

Theorem 16 *A vector valued adapted process $\chi = (X_t^1, ..., X_t^J)_{t\in\mathcal{T}}$ is a $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathcal{T}}, P)$-semimartingale up to infinity if and only if condition (C) holds with K replaced by M_1.*

Remark 5 *We have chosen the index set $\mathcal{T} \triangleq [0, +\infty)$ in order to simplify the notation. However, any subset of $[0, +\infty)$ (containing 0) would also be appropriate (we defer the proof to the Appendix).*

4 Security Markets models

We apply the results of previous sections to the study of stochastic security markets models.

Let $\mathcal{T} \triangleq [0, T]$ and let now the *finite* family $\chi = (X_t^1, ..., X_t^J)_{t \in \mathcal{T}}$ of real adapted processes on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathcal{T}}, P)$ represents the (deflated) price processes of J securities available for trading in the market. The set of cumulative gains resulting from trading in the available assets is $I[\Theta]$ where Θ now determines the class of *admissible* strategies:

$$\begin{aligned} I[\Theta] &\triangleq \left\{ \sum_{j=1}^{J} k_j : \ k_j \in I_j[\Theta^j] \ \forall j = 1, ..., N \right\} \\ k_j &= I_{X^j}(H^j) \triangleq \sum_{i=0}^{n} H_i^j (X_{t_{i+1}}^j - X_{t_i}^j) \ for \ H^j \in E^{\Theta^j}. \end{aligned}$$

We see that a trading strategy consists of the selection of a finite number of trading dates $0 \leq t_0 < ... < t_{n+1}$, $t_i \in \mathcal{T}$ and a finite number of a.s. bounded, $F_{t_i}-$measurable r.v. $H_i^j \in \Theta^j$ which determine the amount invested in asset j between date t_i and t_{i+1}.

If Θ^j (for all j) coincides with the whole space $L^0(\Omega, \mathcal{F}, P)$ then no restrictions are imposed on the class of trading strategies and the set M of cumulative gains (or the set of contingent claims replicable with admissible trading strategies with zero initial cost) is a linear space. The limitation of no short selling is modeled by requiring $\Theta^j = L^0_+(\Omega, \mathcal{F}, P)$ and in this case M_+ is a convex cone. Further (convex) constraints on trading strategies may be imposed simply by requiring that Θ^j is a (convex) set, as in the case of Θ_1 and M_1.

It has been already pointed out by many authors (see for example Jouini & Kallal [11]) that, when one allows frictions in security markets models, one may not expect to identify the appropriate pricing functionals with equivalent martingale measures (or risk neutral measures). For example when there are shortsale constraints it is reasonable to look, for the purpose of super-replication and no arbitrage pricing for equivalent supermartingale measures: indeed one would like to bet the supermartingale goes down, but one is unable to do so because of the constraint.

In the sequel we will denote with $K \subseteq L^0(\Omega, \mathcal{F}, P)$ an arbitrary convex set that is to be interpreted as the set $I[\Theta]$ of cumulative gains. We will then replace the generic set K with sets of the form $I[\Theta]$ for specific selections of the convex sets Θ^j.

From the previous sections we know that (A) is the condition of the existence of a quasimartingale (resp. submartingale, supermartingale, martingale) law Q equivalent to P, when K is replaced with M_1 (resp. with

$M_-,\ M_+,\ M)$. From corollary 14 and proposition 7 we know that (A) and (B) are equivalent.

We say that a random variable $k \in K$ is an *arbitrage opportunity* if

$$k \geq 0\ P - a.s. \text{ and } P(k > 0) > 0.$$

Remark 6 *If the set K is positively homogeneous, then, (B) implies that there are no arbitrage opportunities. Indeed, suppose k is an arbitrage opportunity and let $A \triangleq \{k > 0\}$. Then there exists a $Q_A \in \mathcal{Q}_a^K : Q_A(k > 0) > 0$ and $Q_A(k < 0) = 0$ (since $P(k < 0) = 0$). Therefore $E_{Q_A}[k] > 0$ which contradicts the requirement that $\sup_{k \in K} E_{Q_A}[k] = 0$.*

Definition 17 *We say that (K, τ_K) admits* no extreme arbitrage opportunity *if condition (C) holds.*

We say that (K, τ_K) admits no global free lunch *if condition $(C1)$ holds.*

Note that by definition there exists an extreme arbitrage opportunity if and only if

$$\exists f \in L^\infty_{++},\ \exists \{\alpha_n\}_{n \in \mathbb{N}} \in \mathbb{R}_+ \text{ s.t. } \alpha_n \to +\infty \text{ and } (\alpha_n f) \in \overline{K - L^\infty_+}^{\tau_K}\ \forall n \in \mathbb{N}.$$

Here is an (informal) interpretation of these definitions; (we will suppose, in order to simplify the notations, that $\chi \equiv \{X\}$).

A *global free lunch* (for M_+) consists of a $P - a.s.$ non negative and non zero random variable f which may be "approximated" by a random variable

$\hat{f} = g - h \in M_+ - L^\infty_+$ where $g = \sum_{i=0}^{N} H_i(X_{t_{i+1}} - X_{t_i}) \in M_+$ is replicable by a bounded trading strategy $(H_i)_{i=0,\dots,N}$ such that $H_i \in L^\infty(\Omega, \mathcal{F}_{t_i}, P)$ and $H_i \geq 0$ $P-a.s.$ (prohibition of shortselling). Note that, even though each H_i is bounded, the approximation of f may eventually require *arbitrarily large* investments.

An *extreme arbitrage opportunity* (for M_1) consists of a $P-a.s.$ non negative and non zero random variable f and a diverging sequence α_n of positive real numbers such that each (*arbitrarily large*) $(\alpha_n f)$ may be "approximated" by random variables $\hat{f}_n = g_n - h_n \in M_1 - L^\infty_+$ where $g_n = \sum_{i=0}^{N} H_i^n(X_{t_{i+1}} - X_{t_i}) \in M_1$ is replicable by a bounded trading strategy $(H_i^n)_{i=0,\dots,N}$ such that $H_i^n \in L^\infty(\Omega, \mathcal{F}_{t_i}, P)$ and $\|H_i^n\|_\infty \leq 1$ $\forall n \in \mathbb{N}$. This means that in order to approximate each $\alpha_n f$ only investments that do not exceed a *limited amount* are admitted.

Corollary 15 and theorem 16 may now be restated as:

Corollary 18 *There exists a quasimartingale law Q for χ equivalent to P if and only if (M_1, τ_{M_1}) admits no extreme arbitrage opportunity.*

There exists a submartingale (resp. supermartingale, martingale) law Q for χ equivalent to P if and only if (M_-, τ_{M_-}) (resp. (M_+, τ_{M_+}), (M, τ_M)) admits no global free lunch.

A vector valued adapted stochastic process χ is a vector semimartingale if and only if (M_1, τ_{M_1}) admits no extreme arbitrage opportunity.

5 Appendix

Proof. Remark 1. We show that if $\hat{Q} \in \mathcal{Q}$ dominates $\mathcal{Q}$ and hence any countable family $\{Q_n\}_{n\geq 1} \in \mathcal{Q}$ then it is possible to determine set functions a_n with positive values so that $\hat{Q}(A) = \sum_{n=1}^{\infty} a_n(A)Q_n(A),\ A \in \mathcal{F}$. Assume that $\hat{Q} \sim \{Q_n\}_{n\geq 1} \sim \mathcal{Q}$, $\hat{Q}$ and $Q_n \in \mathcal{Q}$ and fix a set $A \in \mathcal{F}$. Given A and $\{Q_n\}_{n\geq 1}$ one uniquely determines the strictly increasing subsequence $\{h_j^A\}_{j=1,\ldots,N} \subseteq \mathbb{N}$, $N \leq +\infty$, such that $Q_{h_j^A}(A) > 0$ (if $Q_n(A) = 0\ \forall n \geq 1$ then $a_n(A) = 1\ \forall n \geq 1$). Define

$$a_n(A) \triangleq \begin{cases} 1 & \text{if } Q_n(A) = 0 \\ x_n \frac{\hat{Q}(A)}{Q_n(A)} & \text{if } Q_n(A) > 0 \end{cases},\ \text{where } x_n \triangleq \begin{cases} \frac{1}{N} & \text{if } N < +\infty \\ 2^{-j} & \text{if } N = +\infty, \quad n = h_j^A. \\ 1 & \text{otherwise} \end{cases}$$

Then: $\sum_{n=1}^{\infty} a_n(A)Q_n(A) = \sum_{j=1}^{N} a_{h_j^A}(A)Q_{h_j^A}(A) = \hat{Q}(A) \sum_{j=1}^{N} x_{h_j^A} = \hat{Q}(A)$. ■

Proof. Example 4. Assume that $\mathcal{Q}_a^K \neq \emptyset$. Let $\{Q_n\}_{n\in\mathbb{N}} \in \mathcal{Q}_a^K$ and $c_n \triangleq \sup_{k\in K} E_{Q_n}[k] \geq 0$ (since $0 \in K$). Define:

$$a_n \triangleq \alpha \frac{2^{-n}}{1+c_n}, \quad \text{where} \quad \alpha \triangleq \left(\sum_{n=1}^{\infty} \frac{2^{-n}}{1+c_n} \right)^{-1}.$$

Clearly $\alpha < +\infty$, $0 < a_n \leq \alpha 2^{-n}$ and $\sum_{n=1}^{\infty} a_n = 1$. Let $Q \triangleq \sum_{n=1}^{\infty} a_n Q_n$. Since $a_n E_{Q_n}[|k|] \leq \alpha 2^{-n} \|k\|_{\infty}$, for each $k \in K$ $\sum_{n=1}^{\infty} a_n E_{Q_n}[|k|]$ is convergent and we get: $E_Q[k] = \sum_{n=1}^{\infty} a_n E_{Q_n}[k] \leq \sum_{n=1}^{\infty} a_n c_n \leq \alpha \sum_{n=1}^{\infty} 2^{-n} = \alpha$, so $Q \in \mathcal{Q}_a^K$. ■

Proof. Example 5. Let $\{Q_n\}_{n\in\mathbb{N}} \in \mathcal{Q}_a^K$, $K \equiv \{k_i\}_{i\in\mathbb{N}}$ and $c_n \triangleq \sup_{i\in\mathbb{N}} E_{Q_n}[k_i] \geq$

0 (since $0 \in K$). Define

$$b_n \triangleq \frac{2^{-n}}{1+\max_{1\le i\le n}(E_{Q_n}[|k_i|])}, \; a_n \triangleq \alpha \frac{b_n}{1+c_n}, \quad \text{where} \quad \alpha \triangleq \left(\sum_{n=1}^{\infty} \frac{b_n}{1+c_n} \right)^{-1}.$$

Clearly $0 < b_n \le 2^{-n}$, $\alpha < +\infty$, $0 < a_n \le \alpha b_n$ and $\sum\limits_{n=1}^{\infty} a_n = 1$. Let $Q \triangleq \sum\limits_{n=1}^{\infty} a_n Q_n$. From $b_n E_{Q_n}[|k_i|] \le 2^{-n}$ for $n \ge i$, we deduce that $\sum\limits_{n=1}^{\infty} a_n E_{Q_n}[|k_i|]$ is convergent for each $i \in \mathbb{N}$ and we get $E_Q[k_i] = \sum\limits_{n=1}^{\infty} a_n E_{Q_n}[k_i] \le \sum\limits_{n=1}^{\infty} a_n c_n \le \alpha \sum\limits_{n=1}^{\infty} 2^{-n} = \alpha$, so $Q \in \mathcal{Q}_a^K$. ■

Proof. Remark 3. Let $f \in L^\infty_{++}$. Then, since $\hat{Q} \in \mathcal{Q}_e^K$, $\hat{E}[f] > 0$ and $r(f) < +\infty$. Take any $r > r(f)$ and let $\varepsilon > 0$ such that $r = r(f)+\varepsilon$. By definition of $r(f)$ there exists $\alpha^\varepsilon \in U(f) : r(f) \le \alpha^\varepsilon < r(f)+\varepsilon$. Let $Q^\varepsilon \in \mathcal{Q}_a^K : \alpha^\varepsilon = \frac{c^\varepsilon}{E^\varepsilon[f]}$ where $c^\varepsilon = \sup_{k\in K} E^\varepsilon[k] < +\infty$, $E^\varepsilon[k] \triangleq E_P[y^\varepsilon k]$ and $y^\varepsilon \triangleq \frac{dQ^\varepsilon}{dP} \in S'_K$ since $Q^\varepsilon \in \mathcal{Q}_a^K$. Define $G^\varepsilon \triangleq \{s \in S_K : E^\varepsilon[s] \le c^\varepsilon\}$. G^ε is closed in (S_K, τ_K) since $y^\varepsilon \in S'_K$. If $g \in (K - L^\infty_+)$ then $E^\varepsilon[g] \le c^\varepsilon$ and therefore $\overline{(K - L^\infty_+)}^{\tau_K} \subseteq G^\varepsilon$. However, $E^\varepsilon[rf] = (r(f)+\varepsilon)E^\varepsilon[f] > \alpha^\varepsilon E^\varepsilon[f] = c^\varepsilon$ so that $(rf) \notin G^\varepsilon$. ■

Proof. Remark 5. Let $\mathcal{T}$ be an arbitrary subset of $[0,+\infty)$ such that $0 \in \mathcal{T}$ and set $T \triangleq \sup\{\mathcal{T}\}$. Replace, in the proof of lemma 11, $Var_Q[X^j]_\infty$ with $Var_Q[X^j]_T$. So only the proof of theorem 13 must be modified. Let $\{\tau_n\}_{n\in\mathbb{N}} \in \mathcal{T}$ such that $\tau_n \uparrow T$. Then $\forall s \in \mathcal{T}\ \exists m = m(s) \in \mathbb{N}$ such that $s \le \tau_m$. The sequence b_n is then given by $b_n \triangleq 2^{-n}\left\{1 + E_{Q_n}[|X^j_{\tau_n}|] + Var_{Q_n}[X^j]_{\tau_n}\right\}^{-1}$. ■

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