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A unified approach to systemic risk measures via acceptance sets

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Abstract

We specify a general methodological framework for systemic risk measures via multidimensional acceptance sets and aggregation functions. Existing systemic risk measures can usually be interpreted as the minimal amount of cash needed to secure the system *after aggregating* individual risks. In contrast, our approach also includes systemic risk measures that can be interpreted as the minimal amount of cash that secures the aggregated system by allocating capital to the single institutions *before aggregating* the individual risks. An important feature of our approach is the possibility of allocating cash according to the future state of the system (scenario-dependent allocation). We also provide conditions that ensure monotonicity, convexity, or quasi-convexity of our systemic risk measures.

KEYWORDS

acceptance set, aggregation, systemic risk, risk measures

1 | INTRODUCTION

The financial crisis has dramatically demonstrated that traditional risk management strategies of financial systems, which predominantly focus on the solvency of individual institutions as if they were in isolation, insufficiently capture the perilous systemic risk that is generated by the interconnectedness of the system entities and the corresponding contagion effects. This has brought awareness of the urgent need for novel approaches that capture systemic riskiness. A large part of the current literature on systemic financial risk is concerned with the modeling structure of financial networks, the analysis of the contagion, and the spread of a potential exogenous (or even endogenous) shock into the system. For a given financial (possibly random) network and a given random shock, one then

determines the “cascade” mechanism that generates possibly many defaults. This mechanism often requires a detailed description of the balance sheet of each institution; assumptions on the interbank network and exposures, on the recovery rate at default, on the liquidation policy; the analysis of direct liabilities, bankruptcy costs, cross-holdings, leverage structures, fire sales, and liquidity freezes.

These approaches may also be relevant from the viewpoint of a policy maker that has to intervene and regulate the banking system to reduce the risk that, in case of an adverse (local) shock, a substantial part or even the complete system breaks down. However, once such a model for the financial network has been identified and the mechanism for the spread of the contagion determined, one still has to understand how to compare the possible final outcomes in a reasonable way or, in other words, how to measure the risk carried by the global financial system. This is the focus of our approach, as we measure the risk embedded in a financial system taking as primitive the vector $\mathbf{X} = (X^1, \dots, X^N)$ of changes in the value to the horizon T , where X^i is interpreted as the future gain if X^i is positive or as the future loss if X^i is negative of institution i . Such profit and loss¹ is typically uncertain and therefore it will be modeled by a random variable $X^i(\omega)$ on some space of possible scenarios $\omega \in \Omega$.

Our approach is close in spirit to the “classical” conceptual framework of univariate monetary risk measures initiated by the seminal paper by Artzner et al. (1999) and the aim of this paper is to extend and generalize such univariate framework to a multivariate setting that takes into account not only one single institution but a complete system. More precisely, we specify a general methodological framework to design *systemic risk measures via multivariate acceptance sets*, which is very flexible and comprises most of the existing systemic risk measures proposed in the literature. In particular, it contains the popular class of systemic risk measures that can be interpreted as the minimal amount of cash needed to secure the system *after aggregating* the N individual risks $\mathbf{X}$ into some univariate system-wide risk by some aggregation rule; see Chen, Iyengar, and Moallemi (2013), Kromer, Overbeck, and Zilch (2013), and Hoffmann, Meyer-Brandis, and Svindland (2015, 2016) for a structural analysis of this class. However, beyond this class, our approach offers various other interesting ways and extensions to design systemic risk measures. In particular, one of its most interesting features is that it also includes systemic risk measures that can be interpreted as the minimal amount of cash that secures the aggregated system by allocating capital to the single institutions *before aggregating* the individual risks. Further, we include the possibility of generalized (random) admissible assets to secure the system that allows for the allocation of cash according to the future state of the system (scenario-dependent allocation). This is particularly interesting from the viewpoint of a lender of last resort who might be interested in determining today the total capital needed to secure the system, and then, in the future, can inject the capital where it serves the most in response to which scenario has been realized. Throughout the paper, we will pay special attention to the analysis and examples of this family of systemic risk measures. The examples will also show that scenario-dependent allocations permit to take into account in a natural way the dependence structure of the risk vector $\mathbf{X}$.

There is a huge literature on systemic risk, which takes into account different points of view on the subject. For empirical studies on banking networks, we refer, for example, to Craig and von Peter (2014), Boss, Elsinger, Summer, and Thurner (2004), and Cont, Moussa, and Santos (2013). Interbank lending has been studied via interacting diffusions and mean field approach in several papers such as Fouque and Sun (2013), Fouque and Ichiba (2013), Carmona, Fouque, and Sun (2015), Kley, Klüppelberg, and Reichel (2015), and Battiston, Delli Gatti, Gallegati, Greenwald, and Stiglitz (2012). Among the many contributions on systemic risk modeling, we mention here the classical contagion model proposed by Eisenberg and Noe (2001); the default model of Gai and Kapadia (2010b); the illiquidity cascade models of Gai and Kapadia (2010a), Hurd, Cellai, Melnik, and Shao (2014), and Lee (2013); the asset fire-sale cascade model by Cifuentes, Ferrucci, and Shin (2005) and Caccioli, Shrestha, Moore, and Farmer (2012); as well as the model in Awiszus and Weber (2015) that

additionally includes cross-holdings. Further works on network modeling are Amini, Cont, and Minca (2016), Rogers and Veraart (2013), Amini, Filipovic, and Minca (2014), Gleeson, Hurd, Melnik, and Hackett (2013), Battiston and Caldarelli (2013), Detering, Meyer-Brandis, and Panagiotou (2016), and Detering, Meyer-Brandis, Panagiotou, and Ritter (2016). See also the references therein. For an exhaustive overview on the literature on systemic risk, we refer the reader to the recent volumes of Hurd (2016) and of Fouque and Langsam (2013).

The structure of the paper is the following. In Section 2, starting from the well-known formulation of univariate risk measures, we gradually motivate and develop our framework for systemic risk measures defined via multivariate acceptance sets. In Section 3, we lay the theoretical foundations of our approach. In particular, we study the properties of monotonicity and quasi-convexity (or convexity) of systemic risk measures. In Section 4, we analyze various families of systemic risk measures where the risk measurement is defined by aggregating the vector of risk factors into some system-wide univariate risk and then testing acceptability with respect to some one-dimensional acceptance set as in (2.16). Section 5 investigates the interesting class of systemic risk measures that are defined in terms of a set $\mathcal{C}$ of scenario-dependent allocations as in (2.10). Then, we present two concrete examples within this class of systemic risk measures in Sections 6 and 7. In Section 6, we look at Gaussian systems and consider both deterministic cash allocations as well as a certain class of random cash allocations. In Section 7, we introduce an example on a finite probability space, where we are able to compute explicitly systemic risk measures for very general scenario-dependent cash allocations.

2 | SYSTEMIC RISK MEASURES

2.1 | From one-dimensional to N -dimensional risk profiles

In this subsection, we review the literature on risk measurement based on acceptance sets, both in the traditional one-dimensional setting and in the case of N interacting financial institutions. Here, we denote with $\mathcal{L}^0(\mathbb{R}^N) := \mathcal{L}^0(\Omega, \mathcal{F}; \mathbb{R}^N)$, $N \in \mathbb{N}$, the space of $\mathbb{R}^N$ -valued random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Traditional risk management strategies evaluate the risk $\eta(X^i)$ of each institution $i \in \{1, \dots, N\}$ by applying a *univariate monetary risk measure* η to the single financial position X^i . A monetary risk measure (see Föllmer & Schied, 2004) is a map $\eta : \mathcal{L}^0(\mathbb{R}) \rightarrow \mathbb{R}$ that can be interpreted as the minimal capital needed to secure a financial position with payoff $X \in \mathcal{L}^0(\mathbb{R})$, i.e., the minimal amount $m \in \mathbb{R}$ that must be added to X in order to make the resulting (discounted) payoff at time T acceptable

$$\eta(X) := \inf \{m \in \mathbb{R} \mid X + m \in \mathbb{A}\}, \quad (2.1)$$

where the acceptance set $\mathbb{A} \subseteq \mathcal{L}^0(\mathbb{R})$ is assumed to be monotone; i.e., $X \geq Y \in \mathbb{A}$ implies $X \in \mathbb{A}$. In addition to decreasing monotonicity, the characterizing feature of these maps is the cash additivity property

$$\eta(X + m) = \eta(X) - m, \quad \text{for all } m \in \mathbb{R}. \quad (2.2)$$

Under the assumption that the set $\mathbb{A}$ is convex (respectively, is a convex cone) the maps in (2.1) are convex (respectively, convex and positively homogeneous) and are called *convex* (respectively, coherent) *risk measures*; see Artzner et al. (1999), Föllmer and Schied (2002), and Frittelli and Rosazza Gianin

(2002). The principle that diversification should not increase the risk is mathematically translated not necessarily with the convexity property but with the weaker condition of quasi-convexity

$$\eta(\lambda X + (1 - \lambda)Y) \leq \eta(X) \vee \eta(Y).$$

As a result, in Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2010) and Frittelli and Maggis (2014), the only properties assumed in the definition of a *quasi-convex risk measure* are monotonicity and quasi-convexity. Such risk measures can always be written as

$$\eta(X) := \inf \{m \in \mathbb{R} \mid X \in \mathbb{A}^m\}, \quad (2.3)$$

where each set $\mathbb{A}^m \subseteq \mathcal{L}^0(\mathbb{R})$ is monotone and convex, for each m . Here, $\mathbb{A}^m$ is interpreted as the class of payoffs carrying the same risk level m . Contrary to the convex, cash additive case where each random variable is binary cataloged as acceptable or as not acceptable, in the quasi-convex case one admits various degrees of acceptability, described by the risk level m ; see Cherny and Madan (2009). Furthermore, in the quasi-convex case, the cash additivity property will not hold in general and one loses a direct interpretation of m as the minimal capital required to secure the payoff X , but preserves the interpretation of $\mathbb{A}^m$ as the set of positions acceptable for the given risk level m . By selecting $\mathbb{A}^m := \mathbb{A} - m$, the risk measure in (2.1) is clearly a particular case of the one in (2.3).

However, as mentioned in the introduction, traditional risk management by univariate risk measures insufficiently captures systemic risk, and a rapidly growing literature is concerned with designing more appropriate risk measures for financial systems. A *systemic risk measure* is then a map $\rho : \mathcal{L}^0(\mathbb{R}^N) \rightarrow \mathbb{R}$ that evaluates the risk $\rho(\mathbf{X})$ of the complete system $\mathbf{X} = (X^1, \dots, X^N)$. Most of the systemic risk measures in the existing literature are of the form

$$\rho(\mathbf{X}) = \eta(\Lambda(\mathbf{X})), \quad (2.4)$$

where $\eta : \mathcal{L}^0(\mathbb{R}) \rightarrow \mathbb{R}$ is a univariate risk measure and

$$\Lambda : \mathbb{R}^N \rightarrow \mathbb{R}$$

is an aggregation rule that aggregates the N -dimensional risk factor $\mathbf{X}$ into a univariate risk factor $\Lambda(\mathbf{X})$. Some examples of aggregation rules found in the literature are the following:

- (i) One of the most common ways to aggregate multivariate risk is to simply sum the single risk factors: $\Lambda(\mathbf{x}) = \sum_{i=1}^N x_i$, $\mathbf{x} = (x_1, \dots, x_N) \in \mathbb{R}^N$. Also, in the literature on systemic risk measures, there are examples using this aggregation rule, like, for example, the *systemic expected shortfall* introduced in Acharya, Pedersen, Philippon, and Richardson (2010), or the *contagion value at risk* (CoVaR) introduced in Adrian and Brunnermeier (2011). However, while summing up profit and loss positions might be reasonable from the viewpoint of a portfolio manager where the portfolio components compensate each other, this aggregation rule seems inappropriate for a financial system where cross-subsidization between institutions is rather unrealistic. Further, if the sum was a suitable aggregation of risk in financial systems, then the traditional approach of applying a univariate coherent risk measure η to the single risk factors would be sufficiently prudent in the sense that by sublinearity it holds that $\eta(\sum_{i=1}^N X_i) \leq \sum_{i=1}^N \eta(X_i)$.
- (ii) One possible aggregation that takes the lack of cross-subsidization between financial institutions into account is to sum up losses over a threshold only: $\Lambda(\mathbf{x}) = \sum_{i=1}^N -(x_i - d_i)^-$, where $x^- := -\min(x, 0)$. This kind of aggregation is for example used in Huang, Zhou, and Zhu (2009) and

Lehar (2005). See also Brunnermeier and Cheridito (2013) for an extension of this type of aggregation rule that also considers a certain effect of gains, as $\Lambda(\mathbf{x}) = \sum_{i=1}^N -\alpha_i x_i^- + \sum_{i=1}^N \beta_i (x_i - v_i)^+$ for some $\alpha_i, \beta_i, v_i \in \mathbb{R}^+, i = 1, \dots, N$.

- (iii) Beside the lack of cross-subsidization in a financial system, the aggregation rule may also account for contagion effects that can considerably accelerate system-wide losses resulting from an initial shock. Motivated by the structural contagion model of Eisenberg and Noe (2001), Chen et al. (2013) introduce an aggregation function that explicitly models the net systemic cost of the contagion in a financial system by defining the aggregation rule

$$\Lambda_{CM}(x) = \min_{y_i \geq x_i + \sum_{j=1}^N \Pi_{ij} y_j, \forall i=1, \dots, N, y \in \mathbb{R}_+^N} \left\{ \sum_{i=1}^N y_i \right\}.$$

Here, $\Pi = (\Pi_{ij})_{i,j=1, \dots, N}$ represents the relative liability matrix; i.e., firm i has to pay the proportion Π_{ij} of its total liabilities to firm j .

An axiomatic characterization of systemic risk measures of the form (2.4) on a finite state space is provided in Chen et al. (2013); see also Kromer et al. (2013) for the extension to a general probability space and Hoffmann et al. (2016) for a further extension to a conditional setting. Also, in these references, further examples of possible aggregation functions can be found.

Remark 2.1. The above examples of aggregation functions clarify that, depending on the model and aggregation approach, the primitive $\mathbf{X}$ in our systemic risk measure might be interpreted as future profit and loss positions. In the context of the first two aggregation rules, $\mathbf{X}$ denotes the vector that already includes the potential risk of a contagion spread into the system. Thus, in this situation, $\mathbf{X}$ already includes a reduced-form modeling of the potential contagion channels. On the other hand, in the context of the third aggregation rule, $\mathbf{X}$ is interpreted as the future profit and loss positions before the contagion takes place and the contagion mechanism will explicitly be embedded in the risk measure via the aggregation function. Either way, our scope is to provide a consistent criterion to assess whether one possible vector $\mathbf{X}$ is riskier than another.

If η in (2.4) is a monetary risk measure, it follows from (2.1) that we can rewrite the systemic risk measure ρ in (2.4) as

$$\rho(\mathbf{X}) := \inf \{ m \in \mathbb{R} \mid \Lambda(\mathbf{X}) + m \in \mathbb{A} \}. \quad (2.5)$$

Thus, presuming $\Lambda(\mathbf{X})$ represents some loss, systemic risk can again be interpreted as the minimal cash amount that secures the system when it is added to the total aggregated system loss $\Lambda(\mathbf{X})$. If $\Lambda(\mathbf{X})$ does not allow for an interpretation as cash, the risk measure in (2.5) has to be understood as some general risk level of the system rather than some capital requirement. Similarly, from (2.3), if η is a quasi-convex risk measure, the systemic risk measure ρ in (2.4) can be rewritten as

$$\rho(\mathbf{X}) := \inf \{ m \in \mathbb{R} \mid \Lambda(\mathbf{X}) \in \mathbb{A}^m \}. \quad (2.6)$$

Again, one first aggregates the risk factors via the function Λ and in a second step one computes the minimal risk level associated to $\Lambda(\mathbf{X})$.

Example 2.2. Most of the existing systemic risk measures in the literature can be embedded in our framework by the formulation (2.5) (first aggregation, then applying a univariate risk measure). For example, the distress insurance premium (DIP; Black, Correa, Huang, & Zhou, 2016;

see also Huang et al., 2009) given by $E^Q[\sum_i X_i \mathbb{1}_{\{\sum_i X_i \leq L_{min}\}}]$ is obtained from (2.5) by putting $\Lambda(\mathbf{x}) = \sum_i x_i \mathbb{1}_{\{\sum_i x_i \leq L_{min}\}}$ and taking $\mathbb{A}$ as the acceptance set associated to the univariate risk measure η given by the expected shortfall with respect to the risk-neutral measure Q . So, DIP measures the risk-neutral expected systemic losses that exceed a certain threshold L_{min} .

The CoVar (Adrian & Brunnermeier, 2011) with respect to institution j is given by $Var(\sum_i X_i | X_j = Var(X_j))$. This can be computed for $(X_1, \dots, X_n)$ from (2.5) by putting $\Lambda(\mathbf{x}) = \sum_i x_i$ and taking $\mathbb{A}$ as the acceptance set associated to the univariate risk measure η given by the VaR with respect to the conditional probability distribution given the event $\{X_j = Var(X_j)\}$.

The marginal expected shortfall (MES; Acharya et al., 2010) with respect to institution j is defined as $E[X_j | \sum_i X_i \leq Var(\sum_i X_i)]$. Here, $\Lambda(\mathbf{x}) = x_j$ and $\mathbb{A}$ is the acceptance set associated with the univariate risk measure η given by the expectation with respect to the condition probability distribution given the event $\{\sum_i X_i \leq Var(\sum_i X_i)\}$. Note that DIP measures systemic risk in an unconditional setting; i.e., it takes the probability of a systemic loss into account, while CoVar and MES measure systemic risk conditional on the event that a loss has occurred. More precisely, CoVar measures the impact of realized distress of institution j on the system, while MES measures the impact of realized distress of the system on institution j . For other examples of systemic risk measures, we also refer to Benoit, Colliard, Hurlin, and Perignon (2017).

While the approach prescribed in (2.5) and (2.6) defines an interesting class of systemic risk measures, one could think about meaningful alternative or extended procedures of measuring systemic risk not captured by (2.5) or (2.6). In the following subsections, we extend the conceptual framework for systemic risk measures via acceptance sets step by step in order to gradually include certain novel key features before we reach our general formulation of systemic risk measures via multivariate acceptance sets.

2.2 | First add capital, then aggregate

The interpretation of (2.5) to measure systemic risk as minimal capital needed to secure the system *after aggregating individual risks* is, for example, meaningful in the situation where some kind of rescue fund will be installed to repair damage from systemic loss. However, for instance, from the viewpoint of a regulator who has the possibility to intervene at the level of the single institutions before contagion effects generate further losses, it might be more relevant to measure systemic risk as the minimal capital that secures the aggregated system by injecting the capital into the single institutions *before aggregating the individual risks*. This way of measuring systemic risk can be expressed by

$$\rho(\mathbf{X}) := \inf \left\{ \sum_{i=1}^N m_i \mid \mathbf{m} = (m_1, \dots, m_N) \in \mathbb{R}^N, \Lambda(\mathbf{X} + \mathbf{m}) \in \mathbb{A} \right\}. \quad (2.7)$$

Here, the amount m_i is added to the financial position X^i of institution $i \in \{1, \dots, N\}$ before the corresponding total loss $\Lambda(\mathbf{X} + \mathbf{m})$ is computed. For example, considering the aggregation function Λ_{CM} from above it becomes clear that injecting cash first might prevent further losses that would be generated by contagion effects. The systemic risk is then measured as the minimal total amount $\sum_{i=1}^N m_i$ injected into the institutions to secure the system.

Independently, a related concept in the context of set-valued systemic risk measures has been developed in Feinstein, Rudloff, and Weber (2015). They also use acceptance sets in building the notion of systemic risk measures and admit the possibility of injecting capital before aggregating individual risks (aggregation mechanism sensitive to capital levels, in the language of Feinstein et al., 2015). However,

the two approaches are difficult to compare essentially because they rely on different risk measures (set-valued risk measures in Feinstein et al., 2015, vs. real-valued risk measures in our paper). Furthermore, only deterministic allocations are allowed in Feinstein et al. (2015), while we also admit scenario-dependent allocations. Notice also that Feinstein et al. (2015) recognize the need for extracting a real-valued risk measure with minimal properties (called “orthant risk measure”) from a set-valued risk measure. Even in this form, such orthant risk measure cannot include our formulation of risk measures, for the reason explained above. As illustrated in the examples of Sections 6 and 7, one of the consequences of our approach is that the dependence structure of the risk vector $\mathbf{X}$ can be taken into account due to the flexibility of selecting scenario-dependent allocations, a feature illustrated in the next section and absent in Feinstein et al. (2015).

Remark 2.3. Mainly, the literature on systemic risk can be divided into contributions to two different challenges: The first one concerns the definition and identification of an appropriate measure of the overall systemic risk in a financial system. Having identified the systemic risk, a second question is then how to allocate fair shares of the total systemic risk to the individual institutions and to establish a ranking of the institutions in terms of systemic relevance. The main purpose of this paper is to contribute to the former question and to introduce a general methodology to design systemic risk measures. However, an interesting question that arises for a systemic risk measure defined as in (2.7) is whether it delivers, in addition to a measure of total systemic risk, also a potential ranking of the institutions. Indeed, if $\mathbf{m}^* = (m_1^*, \dots, m_N^*)$ is such that $\rho(\mathbf{X}) = \sum_{i=1}^N m_i^*$, one could be tempted to interpret the ordered cash allocations $m_{i_1}^* \geq \dots \geq m_{i_N}^*$ as a ranking of individual risk shares, and the essential question is whether this ranking can be justified as being *fair*. We will touch on this aspect in the examples in Sections 6 and 7, but a comprehensive analysis of this interesting and challenging question is beyond the scope of this paper and is left for future research.

2.3 | First add scenario-dependent allocation, then aggregate

One main novelty of this paper is that we want to allow for the possibility of adding to $\mathbf{X}$ not merely a vector $\mathbf{m} = (m_1, \dots, m_N) \in \mathbb{R}^N$ of cash but a random vector

$$\mathbf{Y} \in C \subseteq \mathcal{L}^0(\mathbb{R}^N),$$

which represents admissible assets with possibly random payoffs at time T , in the spirit of Frittelli and Scandolo (2006). To each $\mathbf{Y} \in C$, we assign a measure $\pi(\mathbf{Y})$ of the risk (or cost) associated with $\mathbf{Y}$ determined by a monotone increasing map

$$\pi : C \rightarrow \mathbb{R}. \quad (2.8)$$

This leads to the following extension of (2.7):

$$\rho(\mathbf{X}) := \inf \{ \pi(\mathbf{Y}) \in \mathbb{R} \mid \mathbf{Y} \in C, \Lambda(\mathbf{X} + \mathbf{Y}) \in \mathbb{A} \}. \quad (2.9)$$

Considering a general set C in (2.9) allows for more general measurement of systemic risk than the cash needed today for each institution to secure the system. For example, C could be a set of (vectors of) general admissible financial assets that can be used to secure a system by adding $\mathbf{Y}$ to $\mathbf{X}$ component-wise, and $\pi(\mathbf{Y})$ is a valuation of $\mathbf{Y}$. Another example that we analyze in more detail in this paper and

that is particularly interesting from the viewpoint of a lender of last resort is the following class of sets C :

$$C \subseteq \left\{ \mathbf{Y} \in \mathcal{L}^0(\mathbb{R}^N) \mid \sum_{n=1}^N Y^n \in \mathbb{R} \right\} =: C_{\mathbb{R}}, \quad (2.10)$$

and $\pi(\mathbf{Y}) = \sum_{n=1}^N Y^n$. Here, the notation $\sum_{n=1}^N Y^n \in \mathbb{R}$ means that $\sum_{n=1}^N Y^n$ is equal to some deterministic constant in $\mathbb{R}$, even though each single Y^n , $n = 1, \dots, N$, is a random variable. Then, as in (2.7), the systemic risk measure

$$\rho(\mathbf{X}) := \inf \left\{ \sum_{n=1}^N Y^n \mid \mathbf{Y} \in C, \Lambda(\mathbf{X} + \mathbf{Y}) \in \mathbb{A} \right\} \quad (2.11)$$

can still be interpreted as the minimal total cash amount $\sum_{n=1}^N Y^n \in \mathbb{R}$ needed today to secure the system by distributing the cash at the future time T among the components of the risk vector $\mathbf{X}$. However, contrary to (2.7), in general, the allocation $Y^i(\omega)$ to institution i does not need to be decided today, but depends on the scenario ω that has been realized at time T . This corresponds to the situation of a lender of last resort who is equipped with a certain amount of cash today and who will allocate it according to where it serves the most depending on the scenario that has been realized at T . Restrictions on the possible distributions of cash are given by the set C . For example, for $C = \mathbb{R}^N$, the situation corresponds to (2.7) where the distribution is already determined today, while for $C = C_{\mathbb{R}}$, the distribution can be chosen completely freely depending on the scenario ω that has been realized (including negative amounts, i.e., withdrawals of cash from certain components).

Sections 5–7 will be devoted to the analysis and concrete examples of the class of systemic risk measures using a set C as in (2.10). We will see that in the case $C = C_{\mathbb{R}}$ where unrestricted cross-subsidization is possible, the canonical way of measuring systemic risk measure is of the form (2.4) with aggregation rule $\Lambda(\mathbf{x}) = \sum_{i=1}^N x_i$, $\mathbf{x} \in \mathbb{R}^N$, i.e., to apply a univariate risk measure to the sum of the risk factors. Another interesting feature of allowing scenarios depending allocations of cash $\mathbf{Y} \in C \subseteq C_{\mathbb{R}}$ is that, in general, the systemic risk measure will take the dependence structure of the components of $\mathbf{X}$ into account even though acceptable positions might be defined in terms of the marginal distributions of X^i , $i = 1, \dots, N$ only. For instance, the example in Section 6 employs the aggregation rule $\Lambda(\mathbf{x}) = \sum_{i=1}^N -(x_i - d_i)^-$, $\mathbf{x} \in \mathbb{R}^N$, $d_i \in \mathbb{R}$, for $i = 1, \dots, N$, and the acceptance set $\mathbb{A}_{\gamma} := \{Z \in \mathcal{L}^0(\mathbb{R}) \mid E[Z] \geq \gamma\}$, $\gamma \in \mathbb{R}$. Then, a risk vector $\mathbf{Z} = (Z_1, \dots, Z_N) \in \mathcal{L}^0(\mathbb{R}^N)$ is acceptable if and only if $\Lambda(\mathbf{Z}) \in \mathbb{A}$, i.e.,

$$\sum_{i=1}^N -E[(Z_i - d_i)^-] \geq \gamma,$$

which only depends on the marginal distributions of $\mathbf{Z}$. Thus, if we choose $C = \mathbb{R}^N$, then it is obvious that in this case also the systemic risk measure $\rho(\mathbf{X})$ in (2.11) depends on the marginal distributions of $\mathbf{X}$ only. If, however, one allows for more general allocations $\mathbf{Y} \in C \subseteq C_{\mathbb{R}}$ that might differ from scenario to scenario, the systemic risk measure will, in general, depend on the multivariate distribution of $\mathbf{X}$, as it can play on the dependence of the components of $\mathbf{X}$ to minimize the costs. In the examples of Sections 6 and 7, this feature is explicitly shown, as we provide a comparison between the risk measure computed with deterministic and with scenario-dependent allocation.

2.4 | Multidimensional acceptance sets

Until now, we have always defined systemic risk measures in terms of acceptability of an aggregated, one-dimensional system-wide risk. However, not necessarily every relevant systemic risk measure is of this aggregated type. Consider, for instance, the popular approach (though possibly problematic for financial systems as explained above) to add single univariate monetary risk measures η_i , $i = 1, \dots, N$, i.e.,

$$\rho(\mathbf{X}) := \sum_{i=1}^N \eta_i(X^i). \quad (2.12)$$

In general, the systemic risk measure in (2.12) cannot be expressed in the form (2.9). Denoting by $\mathbb{A}_i \subseteq \mathcal{L}^0(\mathbb{R})$ the acceptance set of η_i , $i = 1, \dots, N$, one easily sees from (2.1), however, that ρ in (2.12) can be written in terms of the multivariate acceptance set $\mathbb{A}_1 \times \dots \times \mathbb{A}_N$

$$\rho(\mathbf{X}) := \inf \left\{ \sum_{i=1}^N m_i \mid \mathbf{m} = (m_1, \dots, m_N) \in \mathbb{R}^N, \mathbf{X} + \mathbf{m} \in \mathbb{A}_1 \times \dots \times \mathbb{A}_N \right\}.$$

Motivated by this example, we extend (2.9) to the formulation of systemic risk measures as the minimal cost of admissible asset vectors $\mathbf{Y} \in C$ that, when added to the vector of financial positions $\mathbf{X}$, makes the augmented financial positions $\mathbf{X} + \mathbf{Y}$ acceptable in terms of a general multidimensional acceptance set $\mathcal{A} \subseteq \mathcal{L}^0(\mathbb{R}^N)$

$$\rho(\mathbf{X}) := \inf \{ \pi(\mathbf{Y}) \in \mathbb{R} \mid \mathbf{Y} \in C, \mathbf{X} + \mathbf{Y} \in \mathcal{A} \}. \quad (2.13)$$

Note that by putting $\mathcal{A} := \{ \mathbf{Z} \in \mathcal{L}^0(\mathbb{R}^N) \mid \Lambda(\mathbf{Z}) \in \mathbb{A} \}$, Definition (2.9) is a special case of (2.13). Also, in analogy to (2.2), we remark that for linear valuation rules π , the systemic risk measure given in (2.13) exhibits an extended type of cash invariance in the sense that

$$\rho(\mathbf{X} + \mathbf{Y}) = \rho(\mathbf{X}) + \pi(\mathbf{Y}) \quad (2.14)$$

for $\mathbf{Y} \in C$ such that $\mathbf{Y}' \pm \mathbf{Y} \in C$ for all $\mathbf{Y}' \in C$; see Frittelli and Scandolo (2006).

2.5 | Degree of acceptability

In order to reach the final, most general formulation of systemic risk measures, we assign, in analogy to (2.3), to each $\mathbf{Y} \in C$ a set $\mathcal{A}^{\mathbf{Y}} \subseteq \mathcal{L}^0(\mathbb{R}^N)$ of risk vectors that are acceptable for the given (random) vector $\mathbf{Y}$, and define the systemic risk measure by

$$\rho(\mathbf{X}) := \inf \{ \pi(\mathbf{Y}) \in \mathbb{R} \mid \mathbf{Y} \in C, \mathbf{X} \in \mathcal{A}^{\mathbf{Y}} \}. \quad (2.15)$$

Note that analogously to the one-dimensional quasi-convex case (2.3), the systemic risk measures (2.15) cannot necessarily be interpreted as cash added to the system but, in general, represent some minimal aggregated risk level $\pi(\mathbf{Y})$ at which the system $\mathbf{X}$ is acceptable. The approach in (2.15) is very flexible and unifies a variety of different features in the design of systemic risk measures. In particular, it includes all previous cases if we set

$$\mathcal{A}^{\mathbf{Y}} := \mathcal{A} - \mathbf{Y},$$

where the set $\mathcal{A} \subseteq \mathcal{L}^0(\mathbb{R}^N)$ represents acceptable risk vectors. Then, obviously, (2.13) is obtained from (2.15).

Another advantage of the formulation in terms of general acceptance sets is the possibility to design systemic risk measures via general aggregation rules. Indeed, formulation (2.15) includes the case

$$\rho(\mathbf{X}) := \inf \{ \pi(\mathbf{Y}) \in \mathbb{R} \mid \mathbf{Y} \in \mathcal{C}, \Theta(\mathbf{X}, \mathbf{Y}) \in \mathbb{A} \}, \quad (2.16)$$

where $\Theta : \mathcal{L}^0(\mathbb{R}^N) \times \mathcal{C} \rightarrow \mathcal{L}^0(\mathbb{R})$ denotes some aggregation function jointly in $\mathbf{X}$ and $\mathbf{Y}$. Just select $\mathcal{A}^{\mathbf{Y}} := \{ \mathbf{Z} \in \mathcal{L}^0(\mathbb{R}^N) \mid \Theta(\mathbf{Z}, \mathbf{Y}) \in \mathbb{A} \}$. In particular, (2.16) includes both the case “injecting capital before aggregation” as in (2.7) and (2.9) by putting $\Theta(\mathbf{X}, \mathbf{Y}) = \Lambda(\mathbf{X} + \mathbf{Y})$, and the case “aggregation before injecting capital” as in (2.5) by putting $\Theta(\mathbf{X}, \mathbf{Y}) := \Lambda_1(\mathbf{X}) + \Lambda_2(\mathbf{Y})$, where $\Lambda_1 : \mathcal{L}^0(\mathbb{R}^N) \rightarrow \mathcal{L}^0(\mathbb{R})$ is an aggregation function and $\Lambda_2 : \mathcal{C} \rightarrow \mathcal{L}^0(\mathbb{R})$ could be, for example, the discounted cost of $\mathbf{Y}$.

Also, again, in analogy to the one-dimensional case (2.3), the more general dependence of the acceptance set on $\mathbf{Y}$ in (2.15) allows for multidimensional quasi-convex risk measures. Note that the cash additivity property (2.14) is then lost in general.

3 | DEFINITION OF SYSTEMIC RISK MEASURES AND PROPERTIES

In this section, we provide the definitions and properties of the systemic risk measures in our setting. As in Section 2, we consider the set of random vectors

$$\mathcal{L}^0(\mathbb{R}^N) := \{ \mathbf{X} = (X^1, \dots, X^N) \mid X^n \in \mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P}), \quad n = 1, \dots, N \},$$

on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that $\mathcal{L}^0(\mathbb{R}^N)$ is equipped with an order relation $\geq$ such that it is a vector lattice. One such example is provided by the order relation: $\mathbf{X}_1 \geq \mathbf{X}_2$ if $X_1^i \geq X_2^i$ for all components $i = 1, \dots, N$, where for random variables in $\mathcal{L}^0(\mathbb{R})$, the order relation is determined by $\mathbb{P}$ -a.s inequality.

Definition 3.1. Let $\mathbf{X}_1, \mathbf{X}_2 \in \mathcal{L}^0(\mathbb{R}^N)$.

1. A set $\mathcal{A} \subset \mathcal{L}^0(\mathbb{R}^N)$ is $\geq$ -monotone if $\mathbf{X}_1 \in \mathcal{A}$ and $\mathbf{X}_2 \geq \mathbf{X}_1$ implies $\mathbf{X}_2 \in \mathcal{A}$.
2. A map $f : \mathcal{L}^0(\mathbb{R}^N) \rightarrow \mathcal{L}^0(\mathbb{R})$ is $\geq$ -monotone decreasing if $\mathbf{X}_2 \geq \mathbf{X}_1$ implies $f(\mathbf{X}_1) \geq f(\mathbf{X}_2)$. Analogously for functions $f : \mathcal{L}^0(\mathbb{R}^N) \rightarrow \overline{\mathbb{R}}$.
3. A map $f : \mathcal{L}^0(\mathbb{R}^N) \rightarrow \overline{\mathbb{R}}$ is quasi-convex if

$$f(\lambda \mathbf{X}_1 + (1 - \lambda) \mathbf{X}_2) \leq f(\mathbf{X}_1) \vee f(\mathbf{X}_2).$$

A vector $\mathbf{X} = (X^1, \dots, X^N) \in \mathcal{L}^0(\mathbb{R}^N)$ denotes a configuration of risky factors at a future time T associated with a system of N entities. Let

$$\mathcal{C} \subseteq \mathcal{L}^0(\mathbb{R}^N).$$

To each $\mathbf{Y} \in \mathcal{C}$, we assign a set $\mathcal{A}^{\mathbf{Y}} \subseteq \mathcal{L}^0(\mathbb{R}^N)$. The set $\mathcal{A}^{\mathbf{Y}}$ represents the risk vectors $\mathbf{X}$ that are acceptable for the given random vector $\mathbf{Y}$. Let also consider a map

$$\pi : \mathcal{C} \rightarrow \mathbb{R},$$

so that $\pi(\mathbf{Y})$ represents the risk (or cost) associated with $\mathbf{Y}$.

We now introduce the concept of monotone and (quasi-) convex systemic risk measure.

Definition 3.2. The *systemic risk measure* associated with $C, \mathcal{A}^Y$ and π is a map $\rho : \mathcal{L}^0(\mathbb{R}^N) \rightarrow \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$, defined by

$$\rho(\mathbf{X}) := \inf \{ \pi(\mathbf{Y}) \in \mathbb{R} \mid \mathbf{Y} \in C, \mathbf{X} \in \mathcal{A}^Y \}. \quad (3.1)$$

Moreover ρ is called a quasi-convex (respectively, convex) systemic risk measure if it is $\geq$ -monotone decreasing and quasi-convex (respectively, convex on $\{\rho(\mathbf{X}) < +\infty\}$).

In other words, the systemic risk of a random vector $\mathbf{X}$ is measured by the minimal risk (cost) of those random vectors $\mathbf{Y}$ that make $\mathbf{X}$ acceptable.

We now focus on several examples of systemic risk measures of the type (3.1). To guarantee that such maps are finite-valued, one could consider their restriction to some vector subspaces of $\mathcal{L}^0(\mathbb{R}^N)$ (for examples $\mathcal{L}^p(\mathbb{R}^N)$, $p \in [1, \infty]$) and impose further conditions on the defining ingredients $(\pi, C, \mathcal{A}^Y)$ of ρ . For example, suppose that C and $\mathcal{A}^Y$ satisfy the two conditions

1. $\{m\mathbf{1} \in \mathbb{R}^N \mid m \in \mathbb{R}_+, \mathbf{1} := (1, \dots, 1)\} \subseteq C$,
2. $-m\mathbf{1} \in \mathcal{A}^{m\mathbf{1}}$ and $\mathcal{A}^{m\mathbf{1}}$ is a monotone set for each $m \in \mathbb{R}_+$,

then $\rho : \mathcal{L}^\infty(\mathbb{R}^N) \rightarrow \overline{\mathbb{R}}$ defined by (3.1) satisfies $\rho(\mathbf{X}) < +\infty$ for all $\mathbf{X} \in \mathcal{L}^\infty(\mathbb{R}^N)$. Indeed, for $m := \max_i \|X^i\|_\infty$, $\mathbf{X} \geq -m\mathbf{1} \in \mathcal{A}^{m\mathbf{1}}$ implies that $\mathbf{X} \in \mathcal{A}^{m\mathbf{1}}$ and $\pi(m\mathbf{1}) < +\infty$.

Clearly, other sufficient conditions may be obtained in each specific example of systemic risk measures considered in the subsequent sections.

We opt to accept the possibility that such maps ρ may assume values $\pm\infty$. However, it is not difficult to find simple sufficient conditions assuring that the systemic risk measure in (3.1) is proper (not identically equal to $+\infty$). One such example is the condition

$$\text{if } \mathbf{0} \in C \text{ and } \mathbf{0} \in \mathcal{A}^{\mathbf{0}} \text{ then } \rho(\mathbf{0}) \leq \pi(\mathbf{0}) < +\infty,$$

and in this case, we can always obtain $\rho(\mathbf{0}) = 0$ by replacing $\rho(\cdot)$ with $\rho(\cdot) - \rho(\mathbf{0})$. We now consider the “structural properties” (i.e., monotonicity, quasi-convexity, convexity) of our systemic risk measures and introduce two sets of conditions (properties (P1), (P2), and (P3) below and the alternative properties (P2a) and (P3a)) that guarantee that the map in (3.1) is a quasi-convex (or convex) risk measure. In Section 4, we show that these sets of conditions can be easily checked in some relevant examples of maps in the form (3.1), where the set $\mathcal{A}^Y$ is determined from aggregation and one-dimensional acceptance sets.

We introduce the following properties:

- (P1) For all $\mathbf{Y} \in C$, the set $\mathcal{A}^Y \subset \mathcal{L}^0(\mathbb{R}^N)$ is $\geq$ -monotone.
- (P2) For all $m \in \mathbb{R}$, for all $\mathbf{Y}_1, \mathbf{Y}_2 \in C$ such that $\pi(\mathbf{Y}_1) \leq m$ and $\pi(\mathbf{Y}_2) \leq m$ and for all $\mathbf{X}_1 \in \mathcal{A}^{\mathbf{Y}_1}$, $\mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}_2}$ and all $\lambda \in [0, 1]$, there exists $\mathbf{Y} \in C$ such that $\pi(\mathbf{Y}) \leq m$ and $\lambda\mathbf{X}_1 + (1 - \lambda)\mathbf{X}_2 \in \mathcal{A}^Y$.
- (P3) For all $\mathbf{Y}_1, \mathbf{Y}_2 \in C$ and all $\mathbf{X}_1 \in \mathcal{A}^{\mathbf{Y}_1}$, $\mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}_2}$ and all $\lambda \in [0, 1]$, there exists $\mathbf{Y} \in C$ such that $\pi(\mathbf{Y}) \leq \lambda\pi(\mathbf{Y}_1) + (1 - \lambda)\pi(\mathbf{Y}_2)$ and $\lambda\mathbf{X}_1 + (1 - \lambda)\mathbf{X}_2 \in \mathcal{A}^Y$.

It is clear that property (P3) implies property (P2). Moreover, we have

Lemma 3.3.

- (i) If the systemic risk measure ρ defined in (3.1) satisfies the properties (P1) and (P2), then ρ is $\geq$ -monotone decreasing and quasi-convex.

- (ii) If the systemic risk measure ρ defined in (3.1) satisfies the properties (P1) and (P3), then ρ is $\succeq$ -monotone decreasing and convex on $\{\rho(\mathbf{X}) < +\infty\}$.

Proof. Set

$$B(\mathbf{X}) := \{\mathbf{Y} \in C \mid \mathbf{X} \in \mathcal{A}^{\mathbf{Y}}\}.$$

First assume that property (P1) holds and w.l.o.g. suppose $\mathbf{X}_2 \succeq \mathbf{X}_1$ and $B(\mathbf{X}_1) \neq \emptyset$. Then, property (P1) implies that if $\mathbf{X}_1 \in \mathcal{A}^{\mathbf{Y}}$ and $\mathbf{X}_2 \succeq \mathbf{X}_1$, then: $B(\mathbf{X}_1) \subseteq B(\mathbf{X}_2)$. Hence,

$$\rho(\mathbf{X}_1) = \inf\{\pi(\mathbf{Y}) \mid \mathbf{Y} \in B(\mathbf{X}_1)\} \geq \inf\{\pi(\mathbf{Y}) \mid \mathbf{Y} \in B(\mathbf{X}_2)\} = \rho(\mathbf{X}_2),$$

so that ρ is $\succeq$ -monotone decreasing.

- (i) Now assume that property (P2) holds and let $\mathbf{X}_1, \mathbf{X}_2 \in \mathcal{L}^0(\mathbb{R}^N)$ be arbitrarily chosen. For the quasi-convexity, we need to prove, for any $m \in \mathbb{R}$, that

$$\rho(\mathbf{X}_1) \leq m \quad \text{and} \quad \rho(\mathbf{X}_2) \leq m \Rightarrow \rho(\lambda \mathbf{X}_1 + (1 - \lambda) \mathbf{X}_2) \leq m.$$

By definition of the infimum in the definition of $\rho(\mathbf{X}_i)$, $\forall \varepsilon > 0$, there exist $\mathbf{Y}_i \in C$ such that $\mathbf{X}_i \in \mathcal{A}^{\mathbf{Y}_i}$ and

$$\pi(\mathbf{Y}_i) \leq \rho(\mathbf{X}_i) + \varepsilon \leq m + \varepsilon, \quad i = 1, 2.$$

Take any $\lambda \in [0, 1]$. Property (P2) guarantees the existence of $\mathbf{Z} \in C$ such that $\pi(\mathbf{Z}) \leq m + \varepsilon$ and $\lambda \mathbf{X}_1 + (1 - \lambda) \mathbf{X}_2 \in \mathcal{A}^{\mathbf{Z}}$. Hence,

$$\begin{aligned} \rho(\lambda \mathbf{X}_1 + (1 - \lambda) \mathbf{X}_2) &= \inf\{\pi(\mathbf{Y}) \mid \mathbf{Y} \in C, \lambda \mathbf{X}_1 + (1 - \lambda) \mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}}\} \\ &\leq \pi(\mathbf{Z}) \leq m + \varepsilon. \end{aligned}$$

As this holds for any $\varepsilon > 0$, we obtain the quasi-convexity.

- (ii) Assume that property (P3) holds and that $\mathbf{X}_1, \mathbf{X}_2 \in \mathcal{L}^0(\mathbb{R}^N)$ satisfy $\rho(\mathbf{X}_i) < +\infty$. Then, $B(\mathbf{X}_i) \neq \emptyset$ and, as before, $\forall \varepsilon > 0$, there exists $\mathbf{Y}_i \in \mathcal{L}^0(\mathbb{R}^N)$ such that $\mathbf{Y}_i \in C$, $\mathbf{X}_i \in \mathcal{A}^{\mathbf{Y}_i}$ and

$$\pi(\mathbf{Y}_i) \leq \rho(\mathbf{X}_i) + \varepsilon, \quad i = 1, 2. \quad (3.2)$$

By property (P3), there exists $\mathbf{Z} \in C$ such that $\pi(\mathbf{Z}) \leq \lambda \pi(\mathbf{Y}_1) + (1 - \lambda) \pi(\mathbf{Y}_2)$ and $\lambda \mathbf{X}_1 + (1 - \lambda) \mathbf{X}_2 \in \mathcal{A}^{\mathbf{Z}}$. Hence,

$$\begin{aligned} \rho(\lambda \mathbf{X}_1 + (1 - \lambda) \mathbf{X}_2) &= \inf\{\pi(\mathbf{Y}) \mid \mathbf{Y} \in C, \lambda \mathbf{X}_1 + (1 - \lambda) \mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}}\} \\ &\leq \pi(\mathbf{Z}) \leq \lambda \pi(\mathbf{Y}_1) + (1 - \lambda) \pi(\mathbf{Y}_2) \\ &\leq \lambda \rho(\mathbf{X}_1) + (1 - \lambda) \rho(\mathbf{X}_2) + \varepsilon, \end{aligned}$$

from (3.2). As this holds for any $\varepsilon > 0$, the map ρ is convex on $\{\rho(\mathbf{X}) < +\infty\}$. $\square$

We now consider the following alternative properties:

- (P2a) For all $\mathbf{Y}_1, \mathbf{Y}_2 \in C$, $\mathbf{X}_1 \in \mathcal{A}^{\mathbf{Y}_1}$, $\mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}_2}$ and $\lambda \in [0, 1]$, there exists $\alpha \in [0, 1]$ such that $\lambda \mathbf{X}_1 + (1 - \lambda) \mathbf{X}_2 \in \mathcal{A}^{\alpha \mathbf{Y}_1 + (1 - \alpha) \mathbf{Y}_2}$.

(P3a) For all $\mathbf{Y}_1, \mathbf{Y}_2 \in C$, $\mathbf{X}_1 \in \mathcal{A}^{\mathbf{Y}_1}$, $\mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}_2}$ and $\lambda \in [0, 1]$, it holds that $\lambda \mathbf{X}_1 + (1 - \lambda) \mathbf{X}_2 \in \mathcal{A}^{\lambda \mathbf{Y}_1 + (1 - \lambda) \mathbf{Y}_2}$.

It is clear that property (P3a) implies property (P2a). Furthermore, we introduce the following properties for C and π :

(P4) C is convex,

(P5) π is quasi-convex, and

(P6) π is convex.

We have the following:

Lemma 3.4.

- (i) Under the conditions (P1), (P2a), (P4), and (P5), the map ρ defined in (3.1) is a quasi-convex systemic risk measure.
- (ii) Under the conditions (P1), (P3a), (P4), and (P6), the map ρ defined in (3.1) is a convex systemic risk measure.

Proof.

- (i) It follows from Lemma 3.3 and the fact that the properties (P2a), (P4), and (P5) imply (P2). Indeed, let $\mathbf{Y}_1, \mathbf{Y}_2 \in C$ such that $\pi(\mathbf{Y}_1) \leq m$, $\pi(\mathbf{Y}_2) \leq m$ and let $\mathbf{X}_1 \in \mathcal{A}^{\mathbf{Y}_1}$, $\mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}_2}$ and $\lambda \in [0, 1]$. Then there exists $\alpha \in [0, 1]$ such that $\lambda \mathbf{X}_1 + (1 - \lambda) \mathbf{X}_2 \in \mathcal{A}^{\alpha \mathbf{Y}_1 + (1 - \alpha) \mathbf{Y}_2}$. If we set: $\mathbf{Y} := \alpha \mathbf{Y}_1 + (1 - \alpha) \mathbf{Y}_2 \in C$, then $\lambda \mathbf{X}_1 + (1 - \lambda) \mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}}$ and $\pi(\alpha \mathbf{Y}_1 + (1 - \alpha) \mathbf{Y}_2) \leq \max(\pi(\mathbf{Y}_1), \pi(\mathbf{Y}_2)) \leq m$.
- (ii) It follows from Lemma 3.3 and the fact that the properties (P3a), (P4), and (P6) imply (P3). Indeed, let $\mathbf{Y}_1, \mathbf{Y}_2 \in C$ and let $\mathbf{X}_1 \in \mathcal{A}^{\mathbf{Y}_1}$, $\mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}_2}$ and $\lambda \in [0, 1]$. If we set: $\mathbf{Y} := \lambda \mathbf{Y}_1 + (1 - \lambda) \mathbf{Y}_2 \in C$, then $\lambda \mathbf{X}_1 + (1 - \lambda) \mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}}$ and $\pi(\mathbf{Y}) \leq \lambda \pi(\mathbf{Y}_1) + (1 - \lambda) \pi(\mathbf{Y}_2)$. $\square$

4 | SYSTEMIC RISK MEASURES VIA AGGREGATION AND ONE-DIMENSIONAL ACCEPTANCE SETS

In this section, we study four classes of systemic risk measures in the form (3.1), which differ from each other by the definition of their aggregation functions and their acceptance sets. However, these four classes, defined in equations (4.2), (4.3), (4.6), and (4.7), all satisfy the structural properties of monotonicity and quasi-convexity (or convexity). We consider the following definitions and assumptions, which will hold true throughout this section:

1. the aggregation functions are

$$\Lambda : \mathcal{L}^0(\mathbb{R}^N) \times C \rightarrow \mathcal{L}^0(\mathbb{R}),$$

$$\Lambda_1 : \mathcal{L}^0(\mathbb{R}^N) \rightarrow \mathcal{L}^0(\mathbb{R}),$$

and we assume that Λ_1 is $\geq$ -increasing and concave;

2. the acceptance family

$$(\mathcal{B}^x)_{x \in \mathbb{R}}$$

is an increasing family with respect to x and each set $\mathcal{B}^x \subseteq \mathcal{L}^0(\mathbb{R})$ is assumed monotone and convex;

3. the acceptance subset

$$\mathbb{A} \subseteq \mathcal{L}^0(\mathbb{R})$$

is assumed monotone and convex.

The convexity of the acceptance set $\mathbb{A} \subseteq \mathcal{L}^0(\mathbb{R})$ (or of the acceptance family $(B^x)_{x \in \mathbb{R}}$) are the standard conditions that have been assumed as the origin of the theory of risk measures. The concavity of the aggregation functions is justified, not only from the many relevant examples in the literature, but also by the preservation of the convexity from one-dimensional acceptance sets to multidimensional ones. Indeed, let $\Theta : \mathcal{L}^0(\mathbb{R}^N) \rightarrow \mathcal{L}^0(\mathbb{R})$ be an aggregation function, $\mathbb{A} \subseteq \mathcal{L}^0(\mathbb{R})$ a one-dimensional acceptance set, and define $\mathcal{A} \subseteq \mathcal{L}^0(\mathbb{R}^N)$ as the inverse image $\mathcal{A} := \Theta^{-1}(\mathbb{A})$. Suppose that Θ is increasing and concave. It can be easily checked that if $\mathbb{A}$ is monotone and convex, then $\mathcal{A}$ is monotone and convex.

We note that in all results of this section, the selection of the set $C \subseteq \mathcal{L}^0(\mathbb{R}^N)$ of permitted vectors is left as general as possible (in some cases, we require the convexity of C and only in Proposition 4.6, we further ask that $C + \mathbb{R}_+^N \in C$). Therefore, we are very flexible in the choice of C and we may interpret its elements as vectors of admissible or safe financial assets, or merely as cash vectors. Only in the next section, we attribute a particular structure to C .

In the conclusive statements of the following propositions in this section, we apply Lemmas 3.3 and 3.4 without explicit mention.

Proposition 4.1. *Let*

$$\mathcal{A}^Y := \{Z \in \mathcal{L}^0(\mathbb{R}^N) \mid \Lambda(Z, Y) \in \mathbb{A}\}, \quad Y \in C, \quad (4.1)$$

where Λ be concave and $\Lambda(\cdot, Y)$ be $\geq$ -increasing for all $Y \in C$. Then, $\mathcal{A}^Y$ satisfies properties (P1) and (P3a) (and (P2a)). The map ρ defined in (3.1) is given by

$$\rho(X) := \inf \{\pi(Y) \in \mathbb{R} \mid Y \in C, \Lambda(X, Y) \in \mathbb{A}\}, \quad (4.2)$$

and is a quasi-convex systemic risk measure, under the assumptions (P4) and (P5); it is a convex systemic risk measure under the assumptions (P4) and (P6).

Proof. Property (P1): Let $X_1 \in \mathcal{A}^Y$ and $X_2 \geq X_1$. Note that $X_1 \in \mathcal{A}^Y$ implies $\Lambda(X_1, Y) \in \mathbb{A}$ and $X_2 \geq X_1$ implies $\Lambda(X_2, Y) \geq \Lambda(X_1, Y)$. Because $\mathbb{A}$ is monotone, we have $\Lambda(X_2, Y) \in \mathbb{A}$ and $X_2 \in \mathcal{A}^Y$.

Property (P3a): Let $Y_1, Y_2 \in C$, $X_1 \in \mathcal{A}^{Y_1}$, $X_2 \in \mathcal{A}^{Y_2}$, and $\lambda \in [0, 1]$. Then, $\Lambda(X_1, Y_1) \in \mathbb{A}$ and $\Lambda(X_2, Y_2) \in \mathbb{A}$ and the convexity of $\mathbb{A}$ guarantees

$$\lambda \Lambda(X_1, Y_1) + (1 - \lambda) \Lambda(X_2, Y_2) \in \mathbb{A}.$$

From the concavity of $\Lambda(\cdot, \cdot)$ we obtain

$$\Lambda(\lambda X_1 + (1 - \lambda) X_2, \lambda Y_1 + (1 - \lambda) Y_2) \geq \lambda \Lambda(X_1, Y_1) + (1 - \lambda) \Lambda(X_2, Y_2) \in \mathbb{A}.$$

The monotonicity of $\mathbb{A}$ implies

$$\Lambda(\lambda X_1 + (1 - \lambda) X_2, \lambda Y_1 + (1 - \lambda) Y_2) = \Lambda(\lambda X_1, Y_1) + (1 - \lambda) \Lambda(X_2, Y_2) \in \mathbb{A},$$

and therefore, $\lambda X_1 + (1 - \lambda) X_2 \in \mathcal{A}^{\lambda Y_1 + (1 - \lambda) Y_2}$. □

The class of systemic risk measures defined in (4.2) is a fairly general representation as the aggregation function Λ only needs to be concave and increasing in one of its arguments and the acceptance set $\mathbb{A}$ is only required to be monotone and convex. As shown in the following Corollary 4.2, such a risk measure may describe either the possibility of “first aggregate and second add the capital” (e.g., if $\Lambda(\mathbf{X}, \mathbf{Y}) := \Lambda_1(\mathbf{X}) + \Lambda_2(\mathbf{Y})$, where $\Lambda_2(\mathbf{Y})$ could be interpreted as the discounted cost of $\mathbf{Y}$) or the case of “first add and second aggregate” (e.g., if $\Lambda(\mathbf{X}, \mathbf{Y}) := \Lambda_1(\mathbf{X} + \mathbf{Y})$).

Corollary 4.2. *Let $\Lambda_2 : C \rightarrow \mathcal{L}^0(\mathbb{R})$ be concave, let $\mathcal{A}^{\mathbf{Y}}$ be defined in (4.1), where the function Λ has one of the following forms:*

$$\Lambda(\mathbf{Z}, \mathbf{Y}) = \Lambda_1(\mathbf{Z}) + \Lambda_2(\mathbf{Y}),$$

$$\Lambda(\mathbf{Z}, \mathbf{Y}) = \Lambda_1(\mathbf{Z} + \mathbf{Y}).$$

Then, $\mathcal{A}^{\mathbf{Y}}$ fulfills properties (P1) and (P3a). Therefore, the map ρ defined in (4.2) is a quasi-convex systemic risk measure under the assumptions (P4) and (P5); it is a convex systemic risk measure under the assumptions (P4) and (P6).

We now turn to the class of truly quasi-convex systemic risk measures defined by (4.3), which represents the generalization of the quasi-convex risk measure in (2.6) in the one-dimensional case.

Proposition 4.3. *Let $\theta : C \rightarrow \mathbb{R}$. Then, the set*

$$\mathcal{A}^{\mathbf{Y}} := \{\mathbf{Z} \in \mathcal{L}^0(\mathbb{R}^N) \mid \Lambda_1(\mathbf{Z}) \in \mathcal{B}^{\theta(\mathbf{Y})}\}, \quad \mathbf{Y} \in C,$$

satisfies properties (P1) and (P2a). The map ρ defined in (3.1) is given by

$$\rho(\mathbf{X}) := \inf \{\pi(\mathbf{Y}) \in \mathbb{R} \mid \mathbf{Y} \in C, \Lambda_1(\mathbf{X}) \in \mathcal{B}^{\theta(\mathbf{Y})}\}, \quad (4.3)$$

and under the assumptions (P4) and (P5), it is a quasi-convex systemic risk measure.

Proof. Property (P1): Let $\mathbf{X}_1 \in \mathcal{A}^{\mathbf{Y}}$ and $\mathbf{X}_2 \geq \mathbf{X}_1$. Note that $\mathbf{X}_1 \in \mathcal{A}^{\mathbf{Y}}$ implies $\Lambda_1(\mathbf{X}_1) \in \mathcal{B}^{\theta(\mathbf{Y})}$ and $\mathbf{X}_2 \geq \mathbf{X}_1$ implies $\Lambda_1(\mathbf{X}_2) \geq \Lambda_1(\mathbf{X}_1)$. As $\mathcal{B}^x$ is a monotone set for all x , we have $\Lambda_1(\mathbf{X}_2) \in \mathcal{B}^{\theta(\mathbf{Y})}$ and $\mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}}$.

Property (P2a): Fix $\mathbf{Y}_1, \mathbf{Y}_2 \in C$, $\mathbf{X}_1 \in \mathcal{A}^{\mathbf{Y}_1}$, $\mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}_2}$ and $\lambda \in [0, 1]$. Then, $\Lambda_1(\mathbf{X}_1) \in \mathcal{B}^{\theta(\mathbf{Y}_1)}$ and $\Lambda_1(\mathbf{X}_2) \in \mathcal{B}^{\theta(\mathbf{Y}_2)}$. From the concavity of Λ_1 , we obtain

$$\Lambda_1(\lambda \mathbf{X}_1 + (1 - \lambda) \mathbf{X}_2) \geq \lambda \Lambda_1(\mathbf{X}_1) + (1 - \lambda) \Lambda_1(\mathbf{X}_2) \in \lambda \mathcal{B}^{\theta(\mathbf{Y}_1)} + (1 - \lambda) \mathcal{B}^{\theta(\mathbf{Y}_2)}.$$

As $(\mathcal{B}^x)_{x \in \mathbb{R}}$ is an increasing family and each $\mathcal{B}^x$ is convex, we deduce

$$\lambda \mathcal{B}^{\theta(\mathbf{Y}_1)} + (1 - \lambda) \mathcal{B}^{\theta(\mathbf{Y}_2)} \subseteq \mathcal{B}^{\max\{\theta(\mathbf{Y}_1), \theta(\mathbf{Y}_2)\}}. \quad (4.4)$$

Suppose that $\max(\theta(\mathbf{Y}_1), \theta(\mathbf{Y}_2)) = \theta(\mathbf{Y}_1)$, using the monotonicity of the set $\mathcal{B}^{\theta(\mathbf{Y}_1)}$, we deduce

$$\Lambda_1(\lambda \mathbf{X}_1 + (1 - \lambda) \mathbf{X}_2) \in \mathcal{B}^{\theta(\mathbf{Y}_1)},$$

and $\lambda \mathbf{X}_1 + (1 - \lambda) \mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}_1}$, so that property (P2a) is satisfied with $\alpha = 1$. □

In the risk measure (4.3), we are not allowing to add capital to $\mathbf{X}$ before the aggregation takes place, as the quasi-convexity property of ρ would be lost in general. Next, we contemplate this possibility (i.e., we consider conditions of the type $\Lambda(\mathbf{X}, \mathbf{Y}) \in \mathcal{B}^{\theta(\mathbf{Y})}$) in the systemic risk measures (4.6) and (4.7) but only under some nontrivial restrictions: For the case (4.6), we impose conditions on the aggregation

function Λ that are made explicit in equation (4.5) and in Example 4.5. In contrast, for the case (4.7), we consider a general aggregation function Λ , but we restrict the family of acceptance sets to $\mathcal{B}^{\pi(\mathbf{Y})}$, where π is positively linear and represents the risk level of the acceptance family.

Proposition 4.4. *Let $\theta : C \rightarrow \mathbb{R}$ and*

$$\mathcal{A}^{\mathbf{Y}} := \{\mathbf{Z} \in \mathcal{L}^0(\mathbb{R}^N) \mid \Lambda(\mathbf{Z}, \mathbf{Y}) \in \mathcal{B}^{\theta(\mathbf{Y})}\}, \quad \mathbf{Y} \in C,$$

where $\Lambda(\cdot, \mathbf{Y}) : \mathcal{L}^0(\mathbb{R}^N) \rightarrow \mathcal{L}^0(\mathbb{R})$ is $\geq$ -increasing and concave for all $\mathbf{Y} \in C$. Assume in addition that

$$\theta(\mathbf{Y}_2) \geq \theta(\mathbf{Y}_1) \Rightarrow \Lambda(\mathbf{X}, \mathbf{Y}_2) \geq \Lambda(\mathbf{X}, \mathbf{Y}_1) \quad \text{for all } \mathbf{X} \in \mathcal{L}^0(\mathbb{R}^N). \quad (4.5)$$

Then properties (P1) and (P2) hold. The map ρ defined in (3.1) is given by

$$\rho(\mathbf{X}) := \inf \{ \pi(\mathbf{Y}) \in \mathbb{R} \mid \mathbf{Y} \in C, \Lambda(\mathbf{X}, \mathbf{Y}) \in \mathcal{B}^{\theta(\mathbf{Y})} \} \quad (4.6)$$

and is a quasi-convex systemic risk measure.

Proof. Property (P1): Let $\mathbf{X}_1 \in \mathcal{A}^{\mathbf{Y}}$ and $\mathbf{X}_2 \geq \mathbf{X}_1$. Note that $\mathbf{X}_1 \in \mathcal{A}^{\mathbf{Y}}$ implies $\Lambda(\mathbf{X}_1, \mathbf{Y}) \in \mathcal{B}^{\theta(\mathbf{Y})}$ and $\mathbf{X}_2 \geq \mathbf{X}_1$ implies $\Lambda(\mathbf{X}_2, \mathbf{Y}) \geq \Lambda(\mathbf{X}_1, \mathbf{Y})$. As $\mathcal{B}^{\theta(\mathbf{Y})}$ is a monotone set, we have $\Lambda(\mathbf{X}_2, \mathbf{Y}) \in \mathcal{B}^{\theta(\mathbf{Y})}$ and $\mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}}$.

Property (P2): Fix $m \in \mathbb{R}$, $\mathbf{Y}_1, \mathbf{Y}_2 \in C$ such that $\pi(\mathbf{Y}_1) \leq m$ and $\pi(\mathbf{Y}_2) \leq m$, $\lambda \in [0, 1]$ and take $\mathbf{X}_1 \in \mathcal{A}^{\mathbf{Y}_1}$ and $\mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}_2}$. Then, $\Lambda(\mathbf{X}_1, \mathbf{Y}_1) \in \mathcal{B}^{\theta(\mathbf{Y}_1)}$ and $\Lambda(\mathbf{X}_2, \mathbf{Y}_2) \in \mathcal{B}^{\theta(\mathbf{Y}_2)}$. Then, w.l.o.g. we may assume that $\theta(\mathbf{Y}_2) \geq \theta(\mathbf{Y}_1)$. As $(\mathcal{B}^x)_{x \in \mathbb{R}}$ is an increasing family, we have $\mathcal{B}^{\theta(\mathbf{Y}_1)} \subseteq \mathcal{B}^{\theta(\mathbf{Y}_2)}$. Condition (4.5) implies $\Lambda(\mathbf{X}_1, \mathbf{Y}_2) \geq \Lambda(\mathbf{X}_1, \mathbf{Y}_1) \in \mathcal{B}^{\theta(\mathbf{Y}_1)} \subseteq \mathcal{B}^{\theta(\mathbf{Y}_2)}$, so that $\Lambda(\mathbf{X}_1, \mathbf{Y}_2) \in \mathcal{B}^{\theta(\mathbf{Y}_2)}$. From the concavity of $\Lambda(\cdot, \mathbf{Y}_2)$ and the convexity of $\mathcal{B}^{\theta(\mathbf{Y}_2)}$, we obtain

$$\Lambda(\lambda \mathbf{X}_1 + (1 - \lambda) \mathbf{X}_2, \mathbf{Y}_2) \geq \lambda \Lambda(\mathbf{X}_1, \mathbf{Y}_2) + (1 - \lambda) \Lambda(\mathbf{X}_2, \mathbf{Y}_2) \in \mathcal{B}^{\theta(\mathbf{Y}_2)}.$$

Hence, $\Lambda(\lambda \mathbf{X}_1 + (1 - \lambda) \mathbf{X}_2, \mathbf{Y}_2) \in \mathcal{B}^{\theta(\mathbf{Y}_2)}$, which means that $\lambda \mathbf{X}_1 + (1 - \lambda) \mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}_2}$. As $\pi(\mathbf{Y}_2) \leq m$, property (P2) holds with $\mathbf{Y} = \mathbf{Y}_2$. $\square$

Example 4.5. Let $\theta : C \rightarrow \mathbb{R}$ and let Λ be defined by

$$\Lambda(\mathbf{X}, \mathbf{Y}) = g(\mathbf{X}, \theta(\mathbf{Y})),$$

where $g(\cdot, z) : \mathbb{R}^N \rightarrow \mathbb{R}$ is increasing and concave for all $z \in \mathbb{R}$ and $g(x, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ is increasing for all $x \in \mathbb{R}^N$. Then, Λ satisfies all the assumptions in Proposition 4.4. Examples of functions g satisfying these conditions are

$$g(x, z) = f(x) + h(z),$$

with f increasing and concave and h increasing, or

$$g(x, z) = f(x)h(z)$$

with f increasing, concave, and positive and h increasing and positive.

Proposition 4.6. *Suppose that $C \subseteq \mathcal{L}^0(\mathbb{R}^N)$ is a convex set such $\mathbf{0} \in C$ and $C + \mathbb{R}_+^N \in C$. Assume in addition that $\pi : C \rightarrow \mathbb{R}$ satisfies $\pi(u) = 1$ for a given $u \in \mathbb{R}_+^N$, $u \neq \mathbf{0}$, and*

$$\pi(\alpha_1 Y_1 + \alpha_2 Y_2) = \alpha_1 \pi(Y_1) + \alpha_2 \pi(Y_2)$$

for all $\alpha_i \in \mathbb{R}_+$ and $Y_i \in C$. Let

$$\mathcal{A}^Y := \{\mathbf{Z} \in \mathcal{L}^0(\mathbb{R}^N) \mid \Lambda(\mathbf{Z}, \mathbf{Y}) \in \mathcal{B}^{\pi(\mathbf{Y})}\},$$

where Λ is concave and $\Lambda(\mathbf{X}, \cdot) : C \rightarrow \mathcal{L}^0(\mathbb{R})$ is increasing (with respect to the componentwise ordering) for all $\mathbf{X} \in \mathcal{L}^0(\mathbb{R}^N)$. Then, the family of sets $\mathcal{A}^Y$ fulfill properties (P1) and (P2). The map ρ defined in (3.1) is given by

$$\rho(\mathbf{X}) = \inf \{\pi(\mathbf{Y}) \in \mathbb{R} \mid \mathbf{Y} \in C, \Lambda(\mathbf{X}, \mathbf{Y}) \in \mathcal{B}^{\pi(\mathbf{Y})}\} \quad (4.7)$$

and is a quasi-convex systemic risk measure.

Proof. Property (P1): it follows immediately from the monotonicity of $\mathcal{B}^x$, $x \in \mathbb{R}$.

Property (P2): Let $\mathbf{Y}_1, \mathbf{Y}_2 \in C$, $m \in \mathbb{R}$ and assume w.l.o.g. that $\pi(\mathbf{Y}_1) \leq \pi(\mathbf{Y}_2) \leq m$. Let $\mathbf{X}_1 \in \mathcal{A}^{\mathbf{Y}_1}$, $\mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}_2}$ and $\lambda \in [0, 1]$. Then, $\Lambda(\mathbf{X}_1, \mathbf{Y}_1) \in \mathcal{B}^{\pi(\mathbf{Y}_1)}$ and $\Lambda(\mathbf{X}_2, \mathbf{Y}_2) \in \mathcal{B}^{\pi(\mathbf{Y}_2)}$. Because $(\mathcal{B}^x)_{x \in \mathbb{R}}$ is increasing, we get $\Lambda(\mathbf{X}_1, \mathbf{Y}_1) \in \mathcal{B}^{\pi(\mathbf{Y}_2)}$. Set

$$\hat{\mathbf{Y}}_1 := \mathbf{Y}_1 + (\pi(\mathbf{Y}_2) - \pi(\mathbf{Y}_1))\mathbf{u} \in C.$$

Then, $\hat{\mathbf{Y}}_1 \geq \mathbf{Y}_1$ and, as $\Lambda(\mathbf{X}, \cdot)$ is increasing, $\Lambda(\mathbf{X}_1, \hat{\mathbf{Y}}_1) \geq \Lambda(\mathbf{X}_1, \mathbf{Y}_1) \in \mathcal{B}^{\pi(\mathbf{Y}_2)}$ and

$$\Lambda(\mathbf{X}_1, \hat{\mathbf{Y}}_1) \in \mathcal{B}^{\pi(\mathbf{Y}_2)}$$

because of the monotonicity of $\mathcal{B}^{\pi(\mathbf{Y}_2)}$. Letting

$$\mathbf{Y} := \lambda \hat{\mathbf{Y}}_1 + (1 - \lambda)\mathbf{Y}_2 \in C,$$

and using the properties of π , we obtain

$$\begin{aligned} \pi(\mathbf{Y}) &= \pi(\lambda[\mathbf{Y}_1 + (\pi(\mathbf{Y}_2) - \pi(\mathbf{Y}_1))\mathbf{u}] + (1 - \lambda)\mathbf{Y}_2) \\ &= \lambda\pi(\mathbf{Y}_1 + (\pi(\mathbf{Y}_2) - \pi(\mathbf{Y}_1))\mathbf{u}) + (1 - \lambda)\pi(\mathbf{Y}_2) = \pi(\mathbf{Y}_2) \leq m. \end{aligned}$$

From the concavity of $\Lambda(\cdot, \cdot)$ and the convexity of $\mathcal{B}^{\pi(\mathbf{Y}_2)}$, we obtain

$$\begin{aligned} \Lambda(\lambda\mathbf{X}_1 + (1 - \lambda)\mathbf{X}_2, \mathbf{Y}) &= \Lambda(\lambda\mathbf{X}_1 + (1 - \lambda)\mathbf{X}_2, \lambda\hat{\mathbf{Y}}_1 + (1 - \lambda)\mathbf{Y}_2) \\ &= \Lambda(\lambda(\mathbf{X}_1, \hat{\mathbf{Y}}_1) + (1 - \lambda)(\mathbf{X}_2, \mathbf{Y}_2)) \\ &\geq \lambda\Lambda(\mathbf{X}_1, \hat{\mathbf{Y}}_1) + (1 - \lambda)\Lambda(\mathbf{X}_2, \mathbf{Y}_2) \\ &\in \mathcal{B}^{\pi(\mathbf{Y}_2)} = \mathcal{B}^{\pi(\mathbf{Y})}, \end{aligned}$$

and the monotonicity of $\mathcal{B}^{\pi(\mathbf{Y})}$ implies

$$\Lambda(\lambda\mathbf{X}_1 + (1 - \lambda)\mathbf{X}_2, \mathbf{Y}) \in \mathcal{B}^{\pi(\mathbf{Y})},$$

which means: $\lambda\mathbf{X}_1 + (1 - \lambda)\mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}}$. Hence, property (P2) is satisfied. $\square$

Remark 4.7. Suppose that an aggregation function $\Lambda_0 : \mathcal{L}^0(\mathbb{R}^N) \rightarrow \mathcal{L}^0(\mathbb{R})$ is assigned and set $\Lambda_d(\mathbf{X}) := \Lambda_0(\mathbf{X} - \mathbf{d})$, for a vector $\mathbf{d} \in \mathbb{R}^N$. Consider the associated risk measures, as in (2.9)

$$\rho_d(\mathbf{X}) := \inf \{\pi(\mathbf{Y}) \mid \mathbf{Y} \in C, \Lambda_d(\mathbf{X} + \mathbf{Y}) \in \mathbb{A}\}, \quad \mathbf{d} \in \mathbb{R}^N,$$

$$\rho_0(\mathbf{X}) := \inf \{ \pi(\mathbf{Y}) \mid \mathbf{Y} \in C, \Lambda_0(\mathbf{X} + \mathbf{Y}) \in \mathbb{A} \}, \quad \mathbf{d} = (0, \dots, 0).$$

Then,

$$\rho_d(\mathbf{X}) = \rho_0(\mathbf{X} - \mathbf{d}).$$

Indeed, setting $\mathbf{W} := \mathbf{X} - \mathbf{d}$, one obtains

$$\begin{aligned} \rho_d(\mathbf{X}) &= \inf \{ \pi(\mathbf{Y}) \mid \mathbf{Y} \in C, \Lambda_d(\mathbf{X} + \mathbf{Y}) \in \mathbb{A} \} \\ &= \inf \{ \pi(\mathbf{Y}) \mid \mathbf{Y} \in C, \Lambda_0(\mathbf{W} + \mathbf{Y}) \in \mathbb{A} \} = \rho_0(\mathbf{W}) = \rho_0(\mathbf{X} - \mathbf{d}). \end{aligned}$$

5 | SCENARIO-DEPENDENT ALLOCATIONS

We will now focus on the particularly interesting family of sets C of risk-level vectors $\mathbf{Y}$ defined by

$$C \subseteq \left\{ \mathbf{Y} \in \mathcal{L}^0(\mathbb{R}^N) \mid \sum_{n=1}^N Y^n \in \mathbb{R} \right\} =: C_{\mathbb{R}}. \quad (5.1)$$

A vector $\mathbf{Y} \in C$ as in (5.1) can be interpreted as the cash amount $\sum_{n=1}^N Y^n \in \mathbb{R}$ (which is known today because it is deterministic) that at the future time horizon T is allocated to the financial institutions according to the realized scenario. That is, for $i = 1, \dots, N$, $Y^i(\omega)$ is allocated to institution i in case scenario ω has been realized at T , but the total allocated cash amount $\sum_{n=1}^N Y^n$ stays constant over the different scenarios. One could think about a lender of last resort or a regulator who at time T has a certain amount of cash at disposal to distribute among financial institutions in the most efficient way (with respect to systemic risk) according to the scenario that has been realized. Restrictions on the admissible distributions of cash are implied by the choice of set C . For example, choosing $C = \mathbb{R}^N$ corresponds to the fact that the distribution is deterministic, i.e., the allocation to each institution is already determined today, whereas for $C = C_{\mathbb{R}}$, the distribution can be chosen completely freely depending on the scenario ω that has been realized. Note that the latter case includes potential negative cash allocations, i.e., withdrawals of cash from certain components, which allows for cross-subsidization between financial institutions. The (more realistic) situation of scenario-dependent cash distribution without cross-subsidization is represented by the set

$$C := \{ \mathbf{Y} \in C_{\mathbb{R}} \mid Y^i \geq 0, \quad i = 1, \dots, N \}.$$

In this section, we give some structural results and examples concerning systemic risk measures defined in terms of sets C as in (5.1). In Sections 6 and 7, we then present two more extensive examples of systemic risk measures that employ specific sets C of type (5.1).

In the following, we always assume the componentwise order relation on $\mathcal{L}^0(\mathbb{R}^N)$, i.e., $\mathbf{X}_1 \geq \mathbf{X}_2$ if $\mathbf{X}_1^i \geq \mathbf{X}_2^i$ for all components $i = 1, \dots, N$, and we start by specifying a general class of quasi-convex systemic risk measures that allow the interpretation of the minimal total amount needed to secure the system by scenario-dependent cash allocations as described above. To this end, let $C \subseteq C_{\mathbb{R}}$ be such that

$$C + \mathbb{R}_+^N \in C. \quad (5.2)$$

Let the valuation $\pi(\mathbf{Y})$ of a $\mathbf{Y} \in C$ be given by $\tilde{\pi}(\sum_{n=1}^N Y^n)$ for $\tilde{\pi} : \mathbb{R} \rightarrow \mathbb{R}$ increasing (e.g., the present value of the total cash amount $\sum_{n=1}^N Y^n$ at time T). Further, let $(\mathcal{A}^x)_{x \in \mathbb{R}}$ be an increasing family (w.r.t. x) of monotone, convex subsets $\mathcal{A}^x \subseteq \mathcal{L}^0(\mathbb{R}^N)$, and let $\theta : \mathbb{R} \rightarrow \mathbb{R}$ be an increasing function. We can then define the following family of systemic risk measures:

$$\rho(\mathbf{X}) := \inf \left\{ \pi(\mathbf{Y}) \in \mathbb{R} \mid \mathbf{Y} \in C, \mathbf{X} + \mathbf{Y} \in \mathcal{A}^{\theta(\sum Y^n)} \right\}; \quad (5.3)$$

i.e., the risk measure can be interpreted as the valuation of the minimal total amount needed at time T to secure the system by distributing the cash in the most effective way among institutions. Note that here, the criteria whether a system is safe or not after injecting a vector $\mathbf{Y}$ is given by the acceptance set $\mathcal{A}^{\theta(\sum Y^n)}$ that itself depends on the total amount $\sum_{n=1}^N Y^n$. This gives, for example, the possibility of modeling an increasing level of prudence when defining safe systems for higher amounts of the required total cash. This effect will lead to truly quasi-convex systemic risk measures as the next proposition shows.

Proposition 5.1. *The family of sets*

$$\mathcal{A}^{\mathbf{Y}} := \mathcal{A}^{\theta(\sum Y^n)} - \mathbf{Y}, \quad \mathbf{Y} \in C,$$

fulfills properties (P1) and (P2) with respect to the componentwise order relation on $\mathcal{L}^0(\mathbb{R}^N)$. Hence, the map (5.3) is a quasi-convex risk measure. If, furthermore, $\tilde{\pi}$ is convex and θ is constant, then the map (5.3) is even a convex risk measure.

Proof. Property (P1) follows immediately from the monotonicity of $\mathcal{A}^x$, $x \in \mathbb{R}$. To show property (P2), let $\mathbf{Y}_1, \mathbf{Y}_2 \in C$, $m \in \mathbb{R}$, and $\tilde{\pi}(\sum_{n=1}^N Y_1^n) \leq \tilde{\pi}(\sum_{n=1}^N Y_2^n) \leq m$, where w.l.o.g. $\sum Y_1^n \leq \sum Y_2^n$. Further, let $\mathbf{X}_1 \in \mathcal{A}^{\mathbf{Y}_1}$, $\mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}_2}$, and $\lambda \in [0, 1]$. Because $(\mathcal{A}^x)_{x \in \mathbb{R}}$ and θ are increasing, we get $\mathbf{X}_1 + \mathbf{Y}_1 \in \mathcal{A}^{\theta(\sum Y_2^n)}$. Set

$$\hat{\mathbf{Y}}_1 := \mathbf{Y}_1 + \left(\sum Y_2^n - \sum Y_1^n, 0, \dots, 0 \right) \in C.$$

Then,

$$\mathbf{X}_1 + \hat{\mathbf{Y}}_1 \in \mathcal{A}^{\theta(\sum Y_2^n)}$$

because of the monotonicity of $\mathcal{A}^{\theta(\sum Y_2^n)}$, and

$$\lambda(\mathbf{X}_1 + \hat{\mathbf{Y}}_1) + (1 - \lambda)(\mathbf{X}_2 + \mathbf{Y}_2) \in \mathcal{A}^{\theta(\sum Y_2^n)}$$

because of the convexity of $\mathcal{A}^{\theta(\sum Y_2^n)}$. Furthermore, with

$$\mathbf{Y} := \lambda \hat{\mathbf{Y}}_1 + (1 - \lambda) \mathbf{Y}_2,$$

we get $\lambda \mathbf{X}_1 + (1 - \lambda) \mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}}$ and $\pi(\mathbf{Y}) = \pi(\mathbf{Y}_2) \leq m$ as $\sum_{n=1}^N Y^n = \sum_{n=1}^N Y_2^n$. Hence, property (P2) is satisfied. The final statement follows from Frittelli and Scandolo (2006). $\square$

Note that the quasi-convex risk measures in (5.3) are obtained in a similar way as the ones in (4.6), the main difference being that the risk measures in (4.6) are defined on an aggregated level in terms of one-dimensional acceptance sets, while the ones in (5.3) are defined in terms of general multidimensional acceptance sets. However, in the case $C = C_{\mathbb{R}}$, the next proposition shows that every systemic risk measure of type (5.3) can be written as a univariate quasi-convex risk measure applied to the sum of

the risk factors. That is, when free scenario-dependent allocations with unlimited cross-subsidization between the financial institutions are possible, the sum as aggregation rule not only is acceptable as mentioned in Section 1 but also is the canonical way to aggregate and the canonical way to measure systemic risk of type (2.4). However, while this situation and in sight is relevant for a portfolio manager, the typical financial system does not allow for unlimited cross-subsidization and more restricted sets C together with more appropriate aggregation rules have to be considered.

Proposition 5.2. *Let $C = C_{\mathbb{R}}$. Then ρ in (5.3) is of the form*

$$\rho(\mathbf{X}) = \tilde{\rho}\left(\sum_{n=1}^N X^n\right) \quad (5.4)$$

for some quasi-convex risk measure

$$\tilde{\rho} : \mathcal{L}^0(\mathbb{R}) \rightarrow \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{\infty\}.$$

Proof. Let $\mathbf{X}_1, \mathbf{X}_2 \in \mathcal{L}^0(\mathbb{R}^N)$ be such that $\sum_{n=1}^N X_1^n = \sum_{n=1}^N X_2^n$. In the notation of the proof of Lemma 3.3, let $\mathbf{Y}^1 \in B(\mathbf{X}_1)$ and set

$$\mathbf{Y}_2 := \mathbf{Y}_1 + (\mathbf{X}_1 - \mathbf{X}_2) \in C.$$

Then, $\mathbf{X}_1 + \mathbf{Y}_1 = \mathbf{X}_2 + \mathbf{Y}_2$, and thus $\mathbf{Y}_2 \in B(\mathbf{X}_2)$ because $\sum Y_1^n = \sum Y_2^n$, which implies $\mathcal{A}^{\theta(\sum Y_1^n)} = \mathcal{A}^{\theta(\sum Y_2^n)}$. As $\pi(\mathbf{Y}_1) = \pi(\mathbf{Y}_2)$, this implies $\rho(\mathbf{X}_1) \geq \rho(\mathbf{X}_2)$. Interchanging the roles of $\mathbf{X}_1$ and $\mathbf{X}_2$ yields $\rho(\mathbf{X}_1) = \rho(\mathbf{X}_2)$, and the map $\tilde{\rho} : \mathcal{L}^0(\mathbb{R}) \rightarrow \overline{\mathbb{R}}$ given by

$$\tilde{\rho}(X) := \rho(\mathbf{X}),$$

where $\mathbf{X} \in \mathcal{L}^0(\mathbb{R}^N)$ is such that $X = \sum_{n=1}^N X^n$ is well defined. For $X_1, X_2 \in \mathcal{L}^0(\mathbb{R})$, define

$$\mathbf{X}_i := (X_i, 0, \dots, 0) \in \mathcal{L}^0(\mathbb{R}^N), \quad i = 1, 2.$$

Then,

$$\begin{aligned} \tilde{\rho}(\lambda X_1 + (1 - \lambda)X_2) &= \rho(\lambda \mathbf{X}_1 + (1 - \lambda)\mathbf{X}_2) \leq \max\{\rho(\mathbf{X}_1), \rho(\mathbf{X}_2)\} \\ &= \max\{\tilde{\rho}(X_1), \tilde{\rho}(X_2)\}. \end{aligned}$$

Further, if $X_1 \leq X_2$, then $\mathbf{X}_1 \leq \mathbf{X}_2$ and

$$\tilde{\rho}(X_1) = \rho(\mathbf{X}_1) \geq \rho(\mathbf{X}_2) = \tilde{\rho}(X_2).$$

So, $\tilde{\rho} : \mathcal{L}^0(\mathbb{R}) \rightarrow \overline{\mathbb{R}}$ is a quasi-convex risk measure and $\rho(\mathbf{X}) = \tilde{\rho}(\sum_{n=1}^N X^n)$. $\square$

We conclude this section by two examples that compare the risk measurement by “injecting after aggregation” as in (2.5) versus the risk measurement by “injecting before aggregation” as in (2.9) for different sets $C \subset C_{\mathbb{R}}$ in the situation of the worst case and the expected shortfall acceptance sets, respectively.

5.1 | Example: Worst-case acceptance set

In this example, we measure systemic risk by considering aggregated risk factors defined in terms of the aggregation rule

$$\Lambda_d(\mathbf{X}) := \sum_{i=1}^N -(X_i - d_i)^-. \quad (5.5)$$

When we derive explicit formulas for $\rho_0(X)$, for example, for $\rho^{ag}(\mathbf{X})$, $\rho^{\mathbb{R}^N}(\mathbf{X})$, $\rho^\gamma(\mathbf{X})$ in this and the next section, we can immediately compute $\rho_d(X) = \rho_0(X - d)$, as noted in Remark 4.7, and obtain explicitly the dependence of the risk measure on the capital buffer d . Therefore, in the sequel, we will only consider $\mathbf{d} = (0, \dots, 0)$ and use the aggregate function Λ_0 . Further, we consider the acceptance set $\mathbb{A}^W$ associated with the worst case risk measure, that is, a system $\mathbf{X}$ is acceptable (or safe) if $\sum_{i=1}^N -(X_i)^- \in \mathbb{A}^W$ where $\mathbb{A}^W := \mathcal{L}_+^0(\mathbb{R})$, and we denote by $\rho_W : \mathcal{L}^0(\mathbb{R}) \rightarrow \overline{\mathbb{R}}$ the univariate worst case risk measure defined by

$$\rho_W(X) := \inf \{m \in \mathbb{R} \mid X + m \in \mathbb{A}^W\}.$$

The possible sets $\mathcal{C}$ are, on the one hand, the deterministic allocations $\mathcal{C} = \mathbb{R}^N$ and, on the other hand, the family of constrained scenario-dependent cash allocations of the form

$$\mathcal{C}_\gamma := \{\mathbf{Y} \in \mathcal{C}_{\mathbb{R}} \mid Y_i \geq \gamma_i, \quad i = 1, \dots, N\},$$

where $\gamma := (\gamma_1, \dots, \gamma_N)$, $\gamma_i \in [-\infty, 0]$. Note that for $\gamma := (-\infty, \dots, -\infty)$, this family of subsets includes $\mathcal{C}_\infty = \mathcal{C}_{\mathbb{R}}$. Finally, we let the valuation be

$$\pi(\mathbf{Y}) := \sum_{i=1}^n Y_i.$$

The objective of the following proposition is to analyze and relate the systemic risk measurement by “injecting cash after aggregation”

$$\rho^{ag}(\mathbf{X}) := \inf \{y \in \mathbb{R} \mid \Lambda_0(\mathbf{X}) + y \in \mathbb{A}^W\} = \rho_W \left(\sum_{i=1}^N -(X^i)^- \right),$$

to the systemic risk measurement by “injecting cash before aggregation,” both in the case of deterministic cash allocations

$$\rho^{\mathbb{R}^N}(\mathbf{X}) := \inf \{\pi(\mathbf{Y}) \mid \mathbf{Y} \in \mathbb{R}^N, \Lambda_0(\mathbf{X} + \mathbf{Y}) \in \mathbb{A}^W\},$$

as well as in the case of scenario-dependent cash allocations

$$\rho^\gamma(\mathbf{X}) := \inf \{\pi(\mathbf{Y}) \mid \mathbf{Y} \in \mathcal{C}_\gamma, \Lambda_0(\mathbf{X} + \mathbf{Y}) \in \mathbb{A}^W\}.$$

Proposition 5.3. *It holds that*

$$\begin{aligned} \rho^{\mathbb{R}^N}(\mathbf{X}) &= \sum_{i=1}^N \rho_W(X^i) \geq \rho^{ag}(\mathbf{X}) \\ \rho^\gamma(\mathbf{X}) &= \rho_W \left(\sum_{i=1}^N \left(X^i \mathbb{I}_{\{X^i \leq -\gamma_i\}} - \gamma_i \mathbb{I}_{\{X^i \geq -\gamma_i\}} \right) \right) \leq \rho^{ag}(\mathbf{X}). \end{aligned}$$

In particular, for $\gamma = 0 := (0, \dots, 0)$, we get $\rho^0(\mathbf{X}) = \rho^{ag}(\mathbf{X})$, and for $\gamma = -\infty := (-\infty, \dots, -\infty)$, we get $\rho^{-\infty}(\mathbf{X}) = \rho_W(\sum_{i=1}^N X^i)$.

Before we prove the proposition, we make some comments on the results. We see that if we interpret the risk measure as capital requirement (which in this situation also is possible for ρ^{ag} as the aggregation $\Lambda_0(\mathbf{X})$ can be interpreted as a monetary amount), the capital requirement when “injecting before aggregation” with deterministic allocations is higher than the one when “injecting after aggregation.” When allowing for “injecting before aggregation” with scenario-dependent cash allocations, the flexibility gained in allocating the cash leads to decreasing capital requirements. For fully flexible allocations, the minimum amount $\rho^{-\infty}(\mathbf{X}) = \rho_W(\sum_{i=1}^N X^i)$ is obtained, which corresponds to the representation given in Proposition 5.2 in terms of the sum as aggregation rule. Obviously, here, the relations between ρ^{ag} , $\rho^{\mathbb{R}^N}$, and ρ^γ depend on the choice of the acceptance set in conjunction with the aggregation function as it is illustrated in the next example.

Further, from the proof below, it follows that in the case $C = \mathbb{R}^N$, there exists a unique allocation $\mathbf{Y}^* \in \mathbb{R}^N$ for a given $\mathbf{X} \in \mathcal{L}^0(\mathbb{R}^N)$ such that $\rho^{\mathbb{R}^N}(\mathbf{X}) = \pi(\mathbf{Y}^*)$. On the other hand, in the case $C = C_\gamma$, there generically exist infinitely many scenario-dependent allocations $\mathbf{Y}^* \in C_\gamma$ for a given $\mathbf{X} \in \mathcal{L}^0(\mathbb{R}^N)$ for which the infimum of the risk measure $\rho^\gamma(\mathbf{X}) = \pi(\mathbf{Y}^*)$ is obtained.

Proof. Note that for $\mathbf{X} \in \mathcal{L}^0(\mathbb{R}^N)$, it holds that $\Lambda_0(\mathbf{X}) \in \mathbb{A}^W$ iff $X^i \in \mathbb{A}^W$, $i = 1, \dots, N$. Thus, we can rewrite

$$\rho^{\mathbb{R}^N}(\mathbf{X}) := \inf \left\{ \sum_{i=1}^N Y^i \mid \mathbf{Y} \in \mathbb{R}^N, \mathbf{X} + \mathbf{Y} \in (\mathbb{A}^W)^N \right\},$$

and obviously get

$$\rho^{\mathbb{R}^N}(\mathbf{X}) = \sum_{i=1}^N -\text{ess.inf}(X^i) = \sum_{i=1}^N \rho_W(X^i),$$

and for $\mathbf{X} \in \mathcal{L}^0(\mathbb{R}^N)$ the allocation $\hat{\mathbf{Y}} := (\text{ess.inf}(X^1), \dots, \text{ess.inf}(X^N))$ is the unique $\hat{\mathbf{Y}} \in \mathbb{R}^N$ such that $\rho^{\mathbb{R}^N}(\mathbf{X}) = \pi(\hat{\mathbf{Y}})$.

For ρ^γ , we analogously rewrite

$$\rho^\gamma(\mathbf{X}) := \inf \left\{ \sum_{i=1}^N Y^i \mid \mathbf{Y} \in C_\gamma, \mathbf{X} + \mathbf{Y} \in (\mathbb{A}^W)^N \right\}.$$

Now consider first the optimization problem

$$\tilde{\rho}(\mathbf{X}) := \inf \left\{ \text{ess.sup} \left(\sum_{i=1}^N Y^i \right) \mid \mathbf{Y} \in \mathcal{L}^0(\mathbb{R}^N), Y^i \geq \gamma_i, \mathbf{X} + \mathbf{Y} \in (\mathbb{A}^W)^N \right\}. \quad (5.6)$$

Then, clearly, $\tilde{\rho} \leq \rho^\gamma$ and $\mathbf{Y}^* := -(X^i \mathbb{I}_{\{X^i \leq -\gamma_i\}} - \gamma_i \mathbb{I}_{\{X^i \geq -\gamma_i\}})_{i=1, \dots, N}$ is an optimal solution of (5.6). Now define

$$\tilde{\mathbf{Y}} := \mathbf{Y}^* + \left(\text{ess.sup} \left(\sum_{i=1}^N Y_i^* \right) - \sum_{i=1}^N Y_i^*, 0, \dots, 0 \right).$$

Then, $\tilde{\mathbf{Y}} \in C_\gamma$ and $\rho^\gamma(\mathbf{X}) \leq \pi(\tilde{\mathbf{Y}}) = \text{ess.sup}(\sum_{i=1}^N Y_i^*) = \tilde{\rho}(\mathbf{X}) \leq \rho^\gamma(\mathbf{X})$, and thus

$$\begin{aligned} \rho^\gamma(\mathbf{X}) &= \sum_{i=1}^N \tilde{Y}_i = \text{ess.sup} \left(\sum_{i=1}^N Y_i^* \right) \\ &= \text{ess.sup} \left(\sum_{i=1}^N \left(X^i \mathbb{I}_{\{X^i \leq -\gamma_i\}} - \gamma_i \mathbb{I}_{\{X^i \geq -\gamma_i\}} \right) \right). \end{aligned}$$

Finally, we remark that generically for a given $\mathbf{X} \in \mathcal{L}^0(\mathbb{R}^N)$, the above allocation $\tilde{\mathbf{Y}} \in C_\gamma$ is not unique such that $\rho^\gamma(\mathbf{X}) = \pi(\tilde{\mathbf{Y}})$. In fact, any allocation of the form

$$\mathbf{Y}^* + (Z_1, \dots, Z_N)$$

with $(Z_1, \dots, Z_N) \in \mathcal{L}^0(\mathbb{R}^N)$ such that $\sum_{i=1}^N Z_i = \text{ess.sup}(\sum_{i=1}^N Y_i^*) - \sum_{i=1}^N Y_i^*$ will satisfy the desired property. $\square$

5.2 | Example: Expected shortfall acceptance set

We now consider the “expected shortfall” risk measure ρ_{ES} (at some given quantile level $\alpha \in (0, 1]$) given by $\rho_{ES}(X) := \frac{1}{\alpha} \int_0^\alpha \text{VaR}_\gamma(X) d\gamma$, where VaR_γ denotes the value at risk of order $\gamma \in [0, 1]$; see, e.g., Föllmer and Schied (2004) for further details. The acceptance set associated with ρ_{ES} is

$$\mathbb{A}^{ES} := \{X \in \mathcal{L}^0(\mathbb{R}) \mid \rho_{ES}(X) \leq 0\}.$$

Everything else is assumed to be as in Example 5.1. Then,

$$\rho^{ag}(\mathbf{X}) = \rho_{ES} \left(\sum_{i=1}^N -(X^i)^- \right).$$

For $\rho^{\mathbb{R}^N}$ and ρ^γ , however, $\mathbb{A}^{ES}$ gives the same result as $\mathbb{A}^W$, i.e.,

$$\rho^{\mathbb{R}^N}(\mathbf{X}) = \sum_{i=1}^N \rho_W(X^i) \geq \rho^{ag}(\mathbf{X}), \quad (5.7)$$

$$\rho^\gamma(\mathbf{X}) = \rho_W \left(\sum_{i=1}^N \left(X^i \mathbb{I}_{\{X^i \leq -\gamma_i\}} - \gamma_i \mathbb{I}_{\{X^i \geq -\gamma_i\}} \right) \right). \quad (5.8)$$

Indeed, by the definition of ρ_{ES} , it immediately follows that $\sum_{i=1}^N -(X^i)^- \in \mathbb{A}^{ES}$ if and only if $X^i \in \mathbb{A}^W$, $i = 1, \dots, N$, and (5.7) and (5.8) are then obtained from Proposition 5.3. So, opposite to the situation in Example 5.1, here the risk measure when “injecting before aggregation” even with scenario-dependent allocations might be higher than the one when “injecting after aggregation.” Indeed, we easily see that we always have $\rho^0 \geq \rho^{ag}$, and generically even $\rho^{-\infty} \geq \rho^{ag}$ holds. This illustrates that these kinds of relations highly depend on the interplay between aggregation and acceptance set.

6 | GAUSSIAN SYSTEMS

In this section, we assume a Gaussian financial system; i.e., we let $\mathbf{X} = (X^1, \dots, X^N)$ be an N -dimensional Gaussian random vector with covariance matrix $\mathbf{Q}$, where $[\mathbf{Q}]_{ii} := \sigma_i^2$, $i = 1, \dots, N$, and $[\mathbf{Q}]_{ij} := \rho_{i,j}$ for $i \neq j$, $i, j = 1, \dots, N$, and mean vector $\mu := (\mu_1, \dots, \mu_N)$, i.e., $\mathbf{X} \sim N(\mu, \mathbf{Q})$. The systemic risk measure we consider is given by

$$\rho(\mathbf{X}) := \inf \left\{ \sum_{i=1}^N Y^i \mid \mathbf{Y} \in \mathcal{C} \subseteq \mathcal{C}_{\mathbb{R}}, \Lambda(\mathbf{X} + \mathbf{Y}) \in \mathbb{A}_{\gamma} \right\}, \quad (6.1)$$

where the set $\mathcal{C}_{\mathbb{R}}$ of scenario-dependent cash allocations is defined in (5.1), the aggregation rule is given by $\Lambda(\mathbf{X}) := \sum_{i=1}^N -(X^i - d_i)^-$ for $d_i \in \mathbb{R}$, and the acceptance set is

$$\mathbb{A}_{\gamma} := \{Z \in \mathcal{L}^0(\mathbb{R}) \mid \mathbb{E}[Z] \geq -\gamma\} \quad (6.2)$$

for some $\gamma \in \mathbb{R}^+$. Here, d_i in the aggregation rule denotes the capital buffer of institution i , $i = 1, \dots, N$, and the risk measure is concerned with the expected total shortfall below these levels in the system.

In Subsection 6.1, we compute the allocations and the systemic risk measure in case of deterministic cash allocations $\mathcal{C} := \mathbb{R}^N$. This computation serves several purposes:

- (i) it illustrates the allocations m_i s in a case where an explicit computation is possible,
- (ii) it allows us to show that these allocations are fair in the sense that an increase in the mean μ_i (respectively, in the volatility σ_i) would decrease (respectively, increase) the corresponding m_i , and
- (iii) it gives a benchmark for comparison with the case with scenario-dependent allocations as introduced in Subsection 6.2.

In Subsection 6.2, we allow for more flexible scenario-dependent allocations of the form

$$\mathcal{C} := \left\{ \mathbf{Y} \in \mathcal{L}^0(\mathbb{R}^n) \mid \mathbf{Y} = \mathbf{m} + \alpha I_D, \mathbf{m}, \alpha \in \mathbb{R}^N, \sum_{i=1}^N \alpha_i = 0 \right\} \subseteq \mathcal{C}_{\mathbb{R}}, \quad (6.3)$$

where I_D is the indicator function of the event $D := \{\sum_{i=1}^N X^i \leq d\}$ for some $d \in \mathbb{R}$. Note that the condition $\sum_{i=1}^N \alpha_i = 0$ implies that $\sum_{i=1}^N Y^i$ is constant almost surely. Cash allocations in (6.3) can be interpreted as the flexibility to let the allocation depend on whether the system at time T is in trouble or not, represented by the events that $\sum_{i=1}^N X^i$ is less or greater than some critical level d , respectively. The correlation structure $\mathbf{Q}$ enters in the computation of the allocations and in Table 6.1, we illustrate this dependence with a numerical example. As expected, the systemic risk measure with scenario-dependence is lower than with deterministic allocations, and that is more pronounced in the case of small or negative correlation between banks.

Note that such “naive” Gaussian systems arise as Nash equilibria of specific linear-quadratic stochastic differential games in the context of interbank borrowing and lending models studied in Carmona et al. (2015).

TABLE 6.1 Systemic risk measures for Gaussian systems (dependence on the correlation)

$\rho_{1,2} \downarrow$		Deterministic	Random
-0.8	m_1	0.5772	0.1597
	m_2	1.7316	1.7230
	α	0	2.8704
	$\rho = m_1 + m_2$	2.3088	1.8827
-0.5	m_1	0.5772	0.2908
	m_2	1.7316	1.7776
	α	0	2.3161
	$\rho = m_1 + m_2$	2.3088	2.0683
0	m_1	0.5772	0.4490
	m_2	1.7316	1.7796
	α	0	1.7208
	$\rho = m_1 + m_2$	2.3088	2.2286
0.5	m_1	0.5772	0.5463
	m_2	1.7316	1.7461
	α	0	1.3389
	$\rho = m_1 + m_2$	2.3088	2.2924
0.8	m_1	0.5772	0.5737
	m_2	1.7316	1.7314
	α	0	0.7905
	$\rho = m_1 + m_2$	2.3088	2.3053

Notes: In this table, we compare the risk measures with deterministic and scenario-dependent (or random) allocations, discussed, respectively, in Subsections 6.1 and 6.2, for different values of the correlation coefficient $\rho_{1,2}$. We report the optimal values m_1 and m_2 of the allocations as well as their sum ρ . The parameters are: means $\mu_i = 0$ for $i = 1, 2$, standard deviations $\sigma_1 = 1$, $\sigma_2 = 3$, acceptance level $\gamma = 0.7$, and critical level $d = 2$.

6.1 | Deterministic cash allocations

We now consider the case $C = \mathbb{R}^N$ and compute the systemic risk measure

$$\rho(\mathbf{X}) := \inf \left\{ \sum_{i=1}^N m_i \mid \mathbf{m} = (m_1, \dots, m_N) \in \mathbb{R}^N, \Lambda(\mathbf{X} + \mathbf{m}) \in \mathbb{A}_\gamma \right\}, \quad (6.4)$$

where for notational clarity, we write $\mathbf{m}$ instead of $\mathbf{Y}$ for deterministic cash allocations. We thus need to minimize the objective function $\sum_{i=1}^N m_i$ over $\mathbb{R}^N$ under the constrained $\Lambda(\mathbf{X} + \mathbf{m}) \in \mathbb{A}_\gamma$, which clearly is equivalent to the constraint

$$\sum_{i=1}^N \mathbb{E}[(X^i + m_i - d_i)^-] = \gamma. \quad (6.5)$$

This constrained optimization problem can be solved with the associated Lagrangian

$$L(m_1, \dots, m_N, \lambda) := \sum_{i=1}^N m_i + \lambda \left(\sum_{i=1}^N \psi_i(m_i) - \gamma \right), \quad (6.6)$$

where $\psi_i(m_i) := \mathbb{E}[(X^i + m_i - d_i)^-]$. As $X^i \sim N(\mu_i, \sigma_i^2)$, one obtains for $i = 1, \dots, N$ that

$$\psi_i(m_i) = \frac{\sigma_i}{\sqrt{2\pi}} \exp \left[-\frac{(d_i - \mu_i - m_i)^2}{2\sigma_i^2} \right] - (m_i + \mu_i - d_i) \Phi \left(\frac{d_i - \mu_i - m_i}{\sigma_i} \right), \quad (6.7)$$

where $\Phi(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$. By direct computation, this leads to

$$\frac{\partial L(m_1, \dots, m_N, \lambda)}{\partial m_i} = 1 - \lambda \Phi \left(\frac{d_i - \mu_i - m_i}{\sigma_i} \right). \quad (6.8)$$

By solving the Lagrangian system, we then obtain the critical point $\mathbf{m}^* = (m_1^*, \dots, m_N^*)$ given by

$$m_i^* = d_i - \mu_i - \sigma_i R,$$

where R solves the equation

$$P(R) := R\Phi(R) + \frac{1}{\sqrt{2\pi}} \exp \left[-\frac{R^2}{2} \right] = \frac{\gamma}{\sum_{i=1}^N \sigma_i}. \quad (6.9)$$

It is easily verified that $\mathbf{m}^*$ is indeed a global minimum and thus the optimal cash allocation associated with the risk measure (6.4).

We now investigate the sensitivity of the optimal solution m_i^* to changes in the underlying drift and volatility. From now on, we assume that $\frac{\gamma}{\sum_{i=1}^N \sigma_i} < P(0) = \frac{1}{\sqrt{2\pi}}$, which ensures that $R < 0$. This condition is satisfied as soon as γ is small enough or N is large enough if volatilities are uniformly bounded away from zero, for instance. In particular, we obtain the following:

1. $\frac{\partial m_i^*}{\partial \mu_i} = -1$: the systemic riskiness decreases with increasing mean.
2. $\frac{\partial m_i^*}{\partial \sigma_i} > 0$: the systemic riskiness increases with increasing volatility. This is obtained as follows. We have

$$\frac{\partial m_i^*}{\partial \sigma_i} = -R - \sigma_i \frac{\partial R}{\partial \sigma_i}. \quad (6.10)$$

By differentiating (6.9), we obtain

$$\frac{\partial P}{\partial \sigma_i} = \frac{\partial P}{\partial R} \frac{\partial R}{\partial \sigma_i} = -\frac{\gamma}{\left(\sum_{k=1}^N \sigma_k \right)^2}. \quad (6.11)$$

As $\frac{\partial P}{\partial R} = \Phi(R)$, we can compute $\frac{\partial R}{\partial \sigma_i}$ and substitute it in (6.10)

$$\begin{aligned} \frac{\partial m_i^*}{\partial \sigma_i} &= -R + \frac{\sigma_i \gamma}{\left(\sum_{k=1}^N \sigma_k \right)^2} \frac{1}{\Phi(R)} \\ &= -R + \frac{\sigma_i \left(\sum_{k=1}^N \sigma_k \right) P(R)}{\left(\sum_{k=1}^N \sigma_k \right)^2 \Phi(R)} \end{aligned}$$

$$\begin{aligned}
&= -R + \frac{\sigma_i \left(R\Phi(R) + \frac{1}{\sqrt{2\pi}} \exp \left[-\frac{R^2}{2} \right] \right)}{\left(\sum_{k=1}^N \sigma_k \right) \Phi(R)} \\
&= \left(\frac{\sigma_i}{\sum_{k=1}^N \sigma_k} - 1 \right) R + \frac{\sigma_i}{\sum_{k=1}^N \sigma_k} \frac{1}{\sqrt{2\pi} \Phi(R)} \exp \left[-\frac{R^2}{2} \right],
\end{aligned}$$

which is positive as $R < 0$.

6.2 | A class of scenario-dependent allocations

We now allow for different allocations of the total capital at disposal depending on which state the system is in. More precisely, we differentiate between the two states that $D := \{S \leq d\}$ and $D^c = \{S > d\}$ for some level $d \in \mathbb{R}$ and $S := \sum_{i=1}^N X^i$, and consider allocations C given in (6.3). The systemic risk measure now becomes

$$\rho(\mathbf{X}) := \inf \left\{ \sum_{i=1}^N m_i \mid \mathbf{m} + \alpha I_D \in C, \Lambda(\mathbf{X} + \mathbf{m} + \alpha I_D) \in \mathbb{A}_\gamma \right\}.$$

To compute the risk measure in this case, we now need to minimize the objective function $\sum_{i=1}^N m_i$ over $(\mathbf{m}, \alpha) \in \mathbb{R}^{2N}$ under the constraints

$$\sum_{i=1}^N \alpha_i = 0 \quad \text{and} \quad \sum_{i=1}^N \mathbb{E} \left[(X^i + m_i + \alpha_i I_D - d_i)^- \right] = \gamma.$$

In analogy to the Subsection 6.1, we apply the method of Lagrange multipliers to minimize the function

$$\begin{aligned}
L(m_1, \dots, m_N, \alpha_1, \dots, \alpha_{N-1}, \lambda) = \\
\sum_{i=1}^N m_i + \lambda \left(\Psi(m_1, \dots, m_N, \alpha_1, \dots, \alpha_{N-1}) - \gamma \right),
\end{aligned} \tag{6.12}$$

where

$$\begin{aligned}
\Psi(m_1, \dots, m_N, \alpha_1, \dots, \alpha_{N-1}) := \\
\sum_{i=1}^{N-1} \mathbb{E} \left[(X^i + m_i + \alpha_i I_D - d_i)^- \right] + \mathbb{E} \left[\left(X^N + m_N - \sum_{j=1}^{N-1} \alpha_j I_D - d_N \right)^- \right],
\end{aligned}$$

as follows.

1. By computing the derivatives with respect to α_i , $i = 1, \dots, N-1$: $\frac{\partial L}{\partial \alpha_i} = 0$ if and only if

$$F_{i,S}(d_i - m_i - \alpha_i, d) = F_{N,S} \left(d_N - m_N + \sum_{j=1}^{N-1} \alpha_j, d \right) \tag{6.13}$$

for $i = 1, \dots, N-1$, where $F_{i,S}$ and $F_{N,S}$ are the joint distribution functions of (X^i, S) and (X^N, S) , respectively.

2. By computing the derivatives with respect to m_i , for $i = 1, \dots, N-1$: $\frac{\partial L}{\partial m_i} = 0$ if and only if

$$\Phi\left(\frac{d_i - \mu_i - m_i}{\sigma_i}\right) + F_{i,S}(d_i - m_i, d) = \Phi\left(\frac{d_N - \mu_N - m_N}{\sigma_N}\right) + F_{N,S}(d_N - m_N, d), \quad (6.14)$$

for $i = 1, \dots, N-1$.

3. By computing the derivatives with respect to λ : we have $\frac{\partial L}{\partial \lambda} = 0$ if and only if $\Psi(m_1, \dots, m_N, \alpha_1, \dots, \alpha_{N-1}) = \gamma$, where

$$\begin{aligned} \Psi(m_1, \dots, m_N, \alpha_1, \dots, \alpha_{N-1}) &= \sum_{i=1}^N \psi_i(m_i) \\ &+ \sum_{i=1}^{N-1} \left[(m_i - d_i)F_{N,S}(d_i - m_i, d) - (m_i + \alpha_i - d_i)F_{i,S}(d_i - m_i - \alpha_i, d) \right. \\ &\quad \left. + \int_{d_i - m_i - \alpha_i}^{d_i - m_i} \int_{-\infty}^d x F_{i,S}(x, y) dy dx \right] + (m_N - d_N)F_{N,S}(d_N - m_N, d) \\ &- \left(m_N - \sum_{j=1}^{N-1} \alpha_j - d_N \right) F_{N,S} \left(d_N - m_N + \sum_{j=1}^{N-1} \alpha_j, d \right) \\ &+ \int_{d_N - m_N + \sum_{j=1}^{N-1} \alpha_j}^{d_N - m_N} \int_{-\infty}^d x F_{N,S}(x, y) dy dx, \end{aligned}$$

and $\psi_i, i = 1, \dots, N$, are defined in (6.7).

From (6.12) and (6.13), we immediately obtain that if the $X^i, i = 1, \dots, N$, are identically distributed, then the optimal solution is obtained for $\alpha_i = 0, i = 1, \dots, N$, and corresponds to the one obtained explicitly in Subsection 6.1 for deterministic injections.

We now present numerical illustrations of our results in the simple case with two banks. In Table 6.1, we set the means $\mu_i = 0$ for $i = 1, 2$, the standard deviations $\sigma_1 = 1, \sigma_2 = 3$, the acceptance level $\gamma = 0.7$, and the critical level $d = 2$. The last two columns show the sensitivities with respect to the correlation for deterministic allocation (case $\alpha = 0$, computed in Subsection 6.1) and for scenario-dependent allocation, respectively. We observe that for highly positively correlated banks, the scenario-dependent allocation does not change the total capital requirement $m_1 + m_2$. Indeed, as expected, if the banks are moving together, one may have to subsidize both of them. However, when they are negatively correlated, one benefits from scenario-dependent allocation as the total allocation $m_1 + m_2$ is lower in that case.

In Table 6.2 with means $\mu_i = 0$ for $i = 1, 2$, correlation $\rho = -0.5$, standard deviation $\sigma_1 = 1$, acceptance level $\gamma = 0.7$, and critical level $d = 2$, we investigate the sensitivity with respect to the standard deviation σ_2 of the second bank. We note here that the assumption of negative correlation is of theoretical interest, but difficult to be justified empirically,² as in this case, systemic risk would have a much smaller impact on the system. We observe that for equal marginals ($\sigma_1 = \sigma_2 = 1$), random allocation does not change the total capital requirement, as already stated in Subsection 6.2. As σ_2 increases, the systemic risk measure as well as the allocation increase with increasing standard deviation in agreement

TABLE 6.2 Systemic risk measures for Gaussian systems (dependence on the standard deviation)

$\sigma_2 \downarrow$		Deterministic	Random
1	m_1	0.1008	0.1008
	m_2	0.1031	0.1031
	α	0	0.0002
	$\rho = m_1 + m_2$	0.2039	0.2039
	m_1	0.8168	0.3167
5	m_2	4.0816	4.1295
	α	0	3.5987
	$\rho = m_1 + m_2$	4.8984	4.4462
	m_1	1.1417	0.4631
	m_2	11.3964	11.4333
10	α	0	6.9909
	$\rho = m_1 + m_2$	12.5381	11.8963

Notes: In this table, we report the optimal values of the risk measures with deterministic and scenario-dependent (or random) allocations for different values of the standard deviation. The parameters are: means $\mu_i = 0$ for $i = 1, 2$, correlation $\rho = -0.5$, standard deviation $\sigma_1 = 1$, acceptance level $\gamma = 0.7$, and critical level $d = 2$.

with the sensitivity analysis presented in Subsection 6.1 for the deterministic case. Also, we observe that scenario-dependent allocation allows for smaller total capital requirement $m_1 + m_2$.

7 | EXAMPLE ON A FINITE PROBABILITY SPACE

We now consider a financial system $\mathbf{X} = (X^1, \dots, X^N)$ that is defined on a finite probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\Omega = \{\omega_1, \dots, \omega_M\}$, $\mathcal{F} = 2^\Omega$, $\mathbb{P}(\omega_j) = p_j \in (0, 1)$, $j = 1, \dots, M$. The systemic risk measure we are interested in here is given by

$$\rho(\mathbf{X}) := \inf \left\{ \sum_{i=1}^N Y^i \mid \mathbf{Y} = (Y^1, \dots, Y^N) \in \mathcal{C}^h, \Lambda(\mathbf{X} + \mathbf{Y}) \in \mathbb{A}_\gamma \right\}, \quad (7.1)$$

where as in Section 6, the acceptance set is $\mathbb{A}_\gamma = \{Z \in \mathcal{L}^0(\mathbb{R}) \mid \mathbb{E}[Z] \geq -\gamma\}$ for $\gamma > 0$ and the admissible allocations $\mathcal{C}^h$ are introduced below. The aggregation is defined by

$$\Lambda(x_1, \dots, x_N) := \sum_{i=1}^N -\exp(-\alpha_i x_i) \quad (7.2)$$

for $\alpha_i > 0$, $i = 1, \dots, N$. Compared to the aggregation in Section 6, the aggregation in (7.2) is more risk averse with respect to bigger losses but also takes benefits of gains into account.

Due to the finite probability space, the computation of the optimal allocation associated with the risk measure (7.1) reduces to solving a finite-dimensional system of equations even for most general scenario-dependent allocation. More precisely, let $n_0 = 0$ and, for a given $h \in \{1, \dots, N\}$, let $\mathbf{n} := (n_1, \dots, n_h)$, with $n_{m-1} < n_m$ for all $m = 1, \dots, h$ and $n_h := N$, represent some partition of $\{1, \dots, N\}$. We then introduce the following family of allocations:

$$\mathcal{C}^h = \left\{ \mathbf{Y} \in \mathcal{L}^0(\mathbb{R}^N) \mid \exists d = (d_1, \dots, d_h) \in \mathbb{R}^h \text{ such that } \sum_{i=1}^{n_1} Y^i(w_j) = d_1, \right.$$

$$\left\{ \sum_{i=n_1+1}^{n_2} Y^i(w_j) = d_2, \dots, \sum_{i=n_{h-1}+1}^{n_h} Y^i(w_j) = d_h, \quad \text{for } j = 1, \dots, M \right\} \subseteq C_{\mathbb{R}}. \quad (7.3)$$

This corresponds to the situation when the regulator is constrained in the way that she cannot distribute cash freely among all financial institutions but only within h subgroups that are induced by the partition $\mathbf{n}$. In other words, the risk measure is the sum of h minimal cash amounts $d_1, \dots, d_h$ determined today, which at time T can be allocated within the h subgroups in order to make the system safe. Note that this family spans from deterministic allocations $C = \mathbb{R}^N$ for $h = N$ to $C_{\mathbb{R}}$ for $h = 1$.

For a given partition of h subgroups, one can now explicitly compute a unique optimal allocation $\mathbf{Y}$ and the corresponding systemic risk $\rho(\mathbf{X}) = \sum_{i=1}^N Y^i = \sum_{m=1}^h d_m$ in (7.1) by solving the corresponding Lagrangian system. Let $j \in \{1, \dots, M\}$, $i \in \{1, \dots, N\}$ and set $y_j^i := Y^i(w_j)$, $x_j^i := X^i(w_j)$. The optimal values are given by

$$d_m := -\beta_m \log \left(\frac{\gamma}{\beta \zeta_m \alpha_{n_{m-1}+1}} \right), \quad \text{for } m = 1, \dots, h, \quad (7.4)$$

$$y_j^k := -x_j^k + \frac{1}{\beta_m \alpha_k} [\bar{x}_{j,m} + A_m + d_m] - A_m^k, \quad \text{for } k = n_{m-1} + 1, \dots, n_m, \quad (7.5)$$

where, for $m = 1, \dots, h$,

$$\begin{aligned} \beta_m &:= \sum_{i=n_{m-1}+1}^{n_m} \frac{1}{\alpha_i}, \quad \beta := \sum_{i=1}^N \frac{1}{\alpha_i}, \\ \bar{x}_{j,m} &:= \sum_{k=n_{m-1}+1}^{n_m} x_j^k, \quad \zeta_m := \sum_{j=1}^M p_j \exp \left\{ -\frac{\bar{x}_{j,m} + A_m}{\beta_m} \right\}, \\ A_m^k &:= \frac{1}{\alpha_k} \log \frac{\alpha_{n_{m-1}+1}}{\alpha_k}, \quad A_m := \sum_{k=n_{m-1}+1}^{n_m} A_m^k. \end{aligned}$$

The proof relies on a straightforward but tedious verification that the formulas (7.4) and (7.5) are the solutions of the system $\nabla L = 0$, where the Lagrangian is given by

$$L \left(d_1, \dots, d_h, \left(y_j^k \right)_{j,k}, \lambda, \delta \right) = d + \lambda \left[\sum_{j=1}^M p_j \sum_{m=1}^h \left(\sum_{k=n_{m-1}+1}^{n_m} e^{-\alpha_k (x_j^k + y_j^k)} - \gamma \right) \right] + \delta \left(\sum_{i=1}^N Y^i - d \right),$$

with $\lambda = \frac{\beta}{\gamma}$ and $d = \sum_{m=1}^h d_m$.

The value of the risk measure $\rho(\mathbf{X})$ explicitly depends on the values $\alpha_i > 0$, $i = 1, \dots, N$, on the parameter γ , on the probability $\mathbb{P}$, and on x_j^i , $i = 1, \dots, N$, $j = 1, \dots, M$. These computations are useful to compare the risk measure $\rho^N(\mathbf{X})$ with deterministic allocations ($h = N$) with the risk measure $\rho^1(\mathbf{X})$ with fully unconstrained scenario-dependent allocations ($h = 1$). The difference $\rho^N(\mathbf{X}) - \rho^1(\mathbf{X})$ represents the potential amount that can be saved by adopting the scenario-dependent allocations that we propose in this paper, in contrast to the standard practice of deterministic allotments. In addition, with the above formulas, we can show that the risk measure with scenario-dependent allocations is very sensible to the dependence structure of the components of the vector $\mathbf{X} = (X^1, \dots, X^N)$, even though the set $\mathbb{A}_\gamma$ of acceptable positions is defined via the marginal distributions of X^i , $i = 1, \dots, N$, only. This fact is pointed out in the next simple example, where we compare the values of the risk

TABLE 7.1 Systemic risk measures for Example 7.1

Systemic Risk Measure	Case	Systemic Risk Measure	Case
-26.36	$h = 1$		
-0.56	$h = 3, \{X_1, X_3\}$	-15.86	$h = 2, \{X_1, X_3\}, \{X_2, X_4\}$
4.44	$h = 3, \{X_2, X_3\}$	-5.13	$h = 2, \{X_2, X_3\}, \{X_1, X_4\}$
63.71	$h = 3, \{X_2, X_4\}$		
68.36	$h = 3, \{X_1, X_4\}$		
72.96	$h = 3, \{X_3, X_4\}$	68.37	$h = 2, \{X_3, X_4\}, \{X_1, X_2\}$
74.48	$h = 3, \{X_1, X_2\}$		
79.02	$h = 4$		

Notes In this table, we report the values of the systemic risk measure (first and third columns) for different grouping of the four banks of the Example 7.1. The cases $h = 1$ and $h = 4$ correspond, respectively, to one large group and to the four individual banks. In the case $h = 3$, there are three groups, two of which consist of a single bank and the third one is composed of the two banks shown in the second column. For $h = 2$, the two groups are shown in the fourth column and the corresponding risk measure in the third one.

measures $\rho(\mathbf{X})$ for the six possible different choices of grouping four given banks $X^1, \dots, X^4$ into three subgroups ($h = 3$), as, for example, $\{X^1\}, \{X^2, X^3\}, \{X^4\}$. The outcome strongly depends on whether the two banks that we group together are independent, comonotone, or countermonotone.

Example 7.1. We consider a system of four banks represented by the random variables X^1, X^2, X^3 , and X^4 on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = (\omega_1, \omega_2, \omega_3, \omega_4)$, $\mathcal{F} = 2^\Omega$, and $\mathbb{P}(\omega_1) = 0.64$, $\mathbb{P}(\omega_2) = \mathbb{P}(\omega_3) = 0.16$, and $\mathbb{P}(\omega_4) = 0.04$. We assume that X^4 is independent of X^1, X^2, X^3 , that X^2 is comonotone with X^1 and that X^3 is countermonotone with X^1 . Furthermore, $X^1(w_1) = X^1(w_3) = 100$, $X^1(w_2) = X^1(w_4) = -50$, $X^2(w_1) = X^2(w_3) = 50$, $X^2(w_2) = X^2(w_4) = -25$, $X^3(w_1) = X^3(w_3) = -25$, $X^3(w_2) = X^3(w_4) = 50$ and $X^4(w_1) = X^4(w_2) = 50$, $X^4(w_3) = X^4(w_4) = -25$. We set $\alpha_i = 0.3$ for $i = 1, \dots, 4$ and $\gamma = 50$ and consider the set C^h defined above. Note that here $\rho(\mathbf{0})$ is not zero, but it could be normalized by replacing $\rho(\cdot)$ with $\rho(\cdot) - \rho(\mathbf{0})$.

In Table 7.1, we provide the systemic risk measures when the number of subgroups is $h = 1$ or $h = 2$ or $h = 3$ or $h = 4$. From Table 7.1, we note that the maximum and minimum values of ρ are obtained, respectively, in the deterministic ($h = 4$) and the fully unconstrained scenario-dependent ($h = 1$) cases.

For $h = 3$, we always have three groups, where two are singleton and one is composed of two banks (which are shown in the table). In this case, whenever one groups X^1 and X^3 , or X^2 and X^3 , ρ is substantially reduced (-0.56 or 4.44), compared to the deterministic case (79.02), as these couples of vectors are countermonotone; whenever one groups X^4 with any of the X^1, X^2 , and X^3 , there is little difference (68.36, 63.71, 72.96) with respect to the deterministic case (79.02), as X^4 is independent from the others; grouping X^1 and X^2 has very little effect (74.48) compared to the deterministic case (79.02), as X^1 and X^2 are comonotone.

For $h = 2$, we have two groups each one with two banks. We always obtain a reduction of ρ when compared to grouping the same two banks but leaving the other two as singleton (compare the cases $h = 3$ and $h = 2$ on the same line). As expected, the size of the reduction is associated with the dependence structure of the component of X .

Remark 7.2. One could extend the above setting further by considering the possibility to limit cross-subsidization in the allocations. This can be done by adding the constraint $Y^i \geq b_i$, $i = 1, \dots, N$, into the family (7.3) of cash allocations, where $(b_1, \dots, b_N) \in \mathbb{R}^N$. For example, putting $b := (0, \dots, 0)$ excludes cash withdrawals from institutions and in this sense, does not allow for any cross-subsidization. The systemic risk measure and corresponding optimal allocations solution can now be

computed by resorting to the Karush Kuhn Tucker conditions (Boyd & Vandenberghe, 2009); see the computations in Pastore (2014).

7.1 | Allocating systemic risk shares

In this subsection, we propose to determine an allocation of systemic risk by computing the expectations $E_{\mathbb{Q}}[Y^i]$, $i = 1, \dots, N$, of the scenario-dependent cash allocations in (7.5). When the probability measure $\mathbb{Q}$ is appropriately selected, such allocation has some nice fairness properties that are illustrated below in (7.10), (7.11), and (7.12).

To ease notation, we consider the case of one single group ($h = 1$) and we assume that $\alpha_i = \alpha$ for some $\alpha > 0$, $i = 1, \dots, N$, but the analysis can be generalized to any other grouping and heterogeneous α_i . Then, (7.5) reduces to

$$y_j^i := \left[\frac{\bar{x}_j}{N} - x_j^i \right] + \frac{d}{N}, \quad i = 1, \dots, N, \quad (7.6)$$

where $y_j^i := Y^i(\omega_j)$, $x_j^i := X^i(\omega_j)$, $\bar{x}_j := \sum_{i=1}^N x_j^i$ and

$$d := \rho(\mathbf{X}) = \frac{N}{\alpha} \log \left(\frac{N}{\gamma} E \left[\exp \left(-\alpha \frac{\bar{X}}{N} \right) \right] \right) \quad (7.7)$$

with $\bar{X} := \sum_{i=1}^N X^i$. The term d/N in (7.6) represents the scenario-independent part of allocation y_j^i , while the first parenthesis can be interpreted as some scenario-dependent mean-field term that increases or decreases allocation y_j^i according to how much the individual risk x_j^i is below or above the group average $\bar{x}_j/N$.

Let $\mathbb{Q}$ be a probability measure and let $Y^i \in \mathcal{L}^0(\mathbb{R})$ be the scenario-dependent cash allocations given in (7.6). We may define the systemic risk share of bank i by the real number $E_{\mathbb{Q}}[Y^i]$. Then, for each $\mathbb{Q}$, $(E_{\mathbb{Q}}[Y^i])_{i=1, \dots, N}$, constitutes an allocation of the total systemic risk

$$\sum_{i=1}^N E_{\mathbb{Q}}[Y^i] = E_{\mathbb{Q}} \left[\sum_{i=1}^N Y^i \right] = E_{\mathbb{Q}}[\rho(\mathbf{X})] = \rho(\mathbf{X}).$$

Now let us consider a change in the vector $\mathbf{X}$ by adding another vector $\epsilon \mathbf{Z}$, for some $\epsilon \in \mathbb{R}$ and $\mathbf{Z} = (Z^1, \dots, Z^N) \in \mathcal{L}^0(\mathbb{R}^N)$. We put $X^{i,\epsilon} := X^i + \epsilon Z^i$, $i = 1, \dots, N$, and $\mathbf{X}^\epsilon := (X^{1,\epsilon}, \dots, X^{N,\epsilon})$. We then compare the optimal solutions $y_j^i(\mathbf{X} + \epsilon \mathbf{Z})$, given in equation (7.6), with $y_j^i(\mathbf{X})$ and the risk measure $\rho(\mathbf{X} + \epsilon \mathbf{Z})$ with $\rho(\mathbf{X})$, respectively.

- (i) We first consider the situation in which the institutions add some cash amounts to their positions, i.e., we set $\mathbf{Z} = \mathbf{c} = (c^1, \dots, c^N) \in \mathbb{R}^N$ and $\epsilon = 1$. A straightforward computation then gives the following cash invariance of both the risk measure ρ (see also (2.14)) and the components y_j^i for all $i = 1, \dots, N$ and $j = 1, \dots, M$

$$\rho(\mathbf{X} + \mathbf{c}) = \rho(\mathbf{X}) - \sum_{i=1}^N c^i, \quad y_j^i(\mathbf{X} + \mathbf{c}) = y_j^i(\mathbf{X}) - c^i, \quad \mathbf{c} \in \mathbb{R}^N. \quad (7.8)$$

In particular, if only institution $l \in \{1, \dots, N\}$ changes its position by a cash amount, i.e., $c^l = 0$ for $i \neq l$, the change $\rho(\mathbf{X} + \mathbf{c}) - \rho(\mathbf{X}) = -c^l$ in total systemic risk is exactly covered by the change in the risk allocation $E_{\mathbb{Q}}[Y^l(\mathbf{X} + \mathbf{c})] - E_{\mathbb{Q}}[Y^l(\mathbf{X})] = -c^l$ of institution l , whereas the risk allocations

of the other institutions are unchanged: $E_{\mathbb{Q}}[Y^i(\mathbf{X} + \mathbf{c})] - E_{\mathbb{Q}}[Y^i(\mathbf{X})] = 0$ for $i \neq l$ (whatever $\mathbb{Q}$ might be). This full responsibility for one's own changes in the financial position is an allocation property referred to as causal responsibility (CR) in the literature; see Brunnermeier and Cheridito (2013). In particular, for an institution the incentive to change its financial position (by cash) is with respect to its own risk allocation only and not with respect to possible manipulation of the other institutions' risk allocations.

- (ii) Motivated by the these considerations, the objective now is to identify a probability measure $\mathbb{Q}$ such that the corresponding risk allocations $(E_{\mathbb{Q}}[Y^i])_{i=1,\dots,N}$ fulfill the (CR) property not only for changes in the underlying financial positions in the direction of cash $\mathbf{Z} = \mathbf{c}$ but in the direction of general changes $\mathbf{Z} \in \mathcal{L}^0(\mathbb{R}^N)$, at least in some local sense. To this end, we consider the sensitivity of the total systemic risk with respect to the change of the system in the direction of $\mathbf{Z}$. From (7.7), we obtain

$$\frac{d}{d\epsilon} \rho(\mathbf{X}^\epsilon)|_{\epsilon=0} = - \frac{\sum_{i=1}^N E \left[\exp \left(-\alpha \frac{\bar{X}}{N} \right) Z^i \right]}{E \left[\exp \left(-\alpha \frac{\bar{X}}{N} \right) \right]}.$$

We then define the probability measure $\mathbb{Q} \sim \mathbb{P}$ by

$$\mathbb{Q}[\omega_j] := p_j \frac{\exp \left(-\alpha \frac{\bar{X}_j}{N} \right)}{E \left[\exp \left(-\alpha \frac{\bar{X}}{N} \right) \right]}, \quad (7.9)$$

and note that we can rewrite

$$\frac{d}{d\epsilon} \rho(\mathbf{X}^\epsilon)|_{\epsilon=0} = - \sum_{i=1}^N E_{\mathbb{Q}}[Z^i]. \quad (7.10)$$

Now we define the systemic risk allocation by assigning the systemic risk share $E_{\mathbb{Q}}[Y^i(\mathbf{X})]$ to institution i , for each $i = 1, \dots, N$. From (7.6), we then compute the sensitivities of the risk allocations with respect to the change of the system in the direction of $\mathbf{Z}$

$$\frac{d}{d\epsilon} E_{\mathbb{Q}}[Y^i(\mathbf{X}^\epsilon)]|_{\epsilon=0} = -E_{\mathbb{Q}}[Z^i] \quad \text{for } i = 1, \dots, N. \quad (7.11)$$

Thus, employing the measure $\mathbb{Q}$, defined in (7.9), to compute the risk allocation, we have derived in (7.10) and (7.11) for general changes $\mathbf{Z}$ a marginal version of what we have derived for cash changes in (7.8). In particular, if only institution $l \in \{1, \dots, N\}$ changes its position in the direction $\mathbf{Z}^l$, i.e., $Z^i = 0$ for $i \neq l$, we (marginally) get CR of institution l for this change

$$\begin{aligned} \frac{d}{d\epsilon} E_{\mathbb{Q}}[Y^l(\mathbf{X}^\epsilon)]|_{\epsilon=0} &= -E_{\mathbb{Q}}[Z^l] = \frac{d}{d\epsilon} \rho(\mathbf{X}^\epsilon)|_{\epsilon=0}, \\ \frac{d}{d\epsilon} E_{\mathbb{Q}}[Y^i(\mathbf{X}^\epsilon)]|_{\epsilon=0} &= 0, \quad \text{for } i \neq l. \end{aligned}$$

We conclude this section by comparing the risk allocations $(E_{\mathbb{Q}}[Y^i(\mathbf{X})])_{i=1,\dots,N}$, obtained in the scenario-dependent setting ($Y^i(\mathbf{X}) \in \mathcal{L}^0(\mathbb{R})$) and using the probability $\mathbb{Q}$ defined in (7.9), with the cash allocation $y^i(\mathbf{X}) \in \mathbb{R}$ in the deterministic setting ($h = N$), which is now interpreted as risk allocation to institution i . It is easily verified from (7.5), taking $h = N$, that the deterministic risk

allocation of institution i is given by $y^i(\mathbf{X}) = \frac{1}{\alpha} \log\left(\frac{N}{\gamma} E_{\mathbb{P}}[e^{-\alpha X^i}]\right)$. We also know that when going from deterministic to scenario-dependent allocations, the total systemic risk decreases. It is then desirable that each institution profits from this decrease in total systemic risk in the sense that its individual risk share also decreases

$$E_{\mathbb{Q}}[Y^i(\mathbf{X})] \leq y^i(\mathbf{X}), \quad \text{for each } i = 1, \dots, N. \quad (7.12)$$

The opposite would clearly be perceived as unfair. The following computation shows that our allocation fulfills this property. Indeed, from (7.6) and (7.7), Jensen's inequality, and (7.9), we obtain

$$\begin{aligned} E_{\mathbb{Q}}[Y^i(\mathbf{X})] &= E_{\mathbb{Q}}\left[\frac{\bar{X}}{N} - X^i\right] + \frac{1}{\alpha} \log\left(\frac{N}{\gamma} E_{\mathbb{P}}\left[e^{-\alpha \frac{\bar{X}}{N}}\right]\right) \\ &= \frac{1}{\alpha} \log \circ \exp\left\{\alpha E_{\mathbb{Q}}\left[\frac{\bar{X}}{N} - X^i\right]\right\} + \frac{1}{\alpha} \log\left(\frac{N}{\gamma} E_{\mathbb{P}}\left[e^{-\alpha \frac{\bar{X}}{N}}\right]\right) \\ &\leq \frac{1}{\alpha} \log\left\{E_{\mathbb{Q}}\left[e^{\alpha\left(\frac{\bar{X}}{N} - X^i\right)}\right]\right\} + \frac{1}{\alpha} \log\left(\frac{N}{\gamma} E_{\mathbb{P}}\left[e^{-\alpha \frac{\bar{X}}{N}}\right]\right) \\ &= \frac{1}{\alpha} \log\left(\frac{N}{\gamma} E_{\mathbb{P}}\left[e^{-\alpha X^i}\right]\right) = y^i(\mathbf{X}). \end{aligned}$$

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ENDNOTES

¹ Even though we defined X^i as the change in the portfolio's i th value, one may also interpret X^i as the future value of firm i , as done, for example, in the seminal paper Artzner, Delbaen, Eber, and Heath (1999).

² We note that the Basel II/III internal-ratings-based approach to bank capital assumes that X^1 and X^2 are comonotonic; see, for example, Gordy (2003).

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APPENDIX

A.1 Gaussian case with random injections

We provide here the computations necessary to minimize the function (6.12). We first consider

$$\mathbb{E}[(X^i + Y^i - d_i)^-] = \mathbb{E}[(X^i + m_i + \alpha_i I_D - d_i)^-]$$

$$\begin{aligned}
&= \mathbb{E}[(X^i + m_i + \alpha_i - d_i)^- I_D] + \mathbb{E}[(X^i + m_i - d_i)^- I_{A^c}] \\
&= \mathbb{E}[\{(X^i + m_i + \alpha_i - d_i)^- - (X^i + m_i - d_i)^-\} I_D] + \mathbb{E}[(X^i + m_i - d_i)^-] \quad (A.1)
\end{aligned}$$

for $i = 1, \dots, N$. To compute (A.1), we distinguish between the cases $\alpha_i > 0$ and $\alpha_i < 0$. Note that by the definition of $\mathcal{C}$, we cannot a priori argue on the sign of α . For $\alpha_i > 0$, we have that $\{X^i \leq d_i - m_i\} = \{X^i \leq d_i - m_i - \alpha_i\} \cup \{d_i - m_i - \alpha_i < X^i \leq d_i - m_i\}$. Here, we set $A_1 := \{X^i \leq d_i - m_i - \alpha_i\}$ and $A_2 := \{d_i - m_i - \alpha_i < X^i \leq d_i - m_i\}$. Then,

$$(X^i + m_i + \alpha_i - d_i)^- - (X^i + m_i - d_i)^- = -\alpha_i I_{A_1} + (X^i + m_i - d_i) I_{A_2},$$

and

$$\begin{aligned}
&\mathbb{E}[(X^i + Y^i - d_i)^-] \\
&= -\alpha_i \mathbb{E}[I_{A_1} I_D] + \mathbb{E}[(X^i + m_i - d_i) I_{A_2} I_D] + \mathbb{E}[(X^i + m_i - d_i)^-] \\
&= (m_i - d_i) F_{i,S}(d_i - m_i, d) - (m_i + \alpha_i - d_i) F_{i,S}(d_i - m_i - \alpha_i, d) \\
&\quad + \int_{d_i - m_i - \alpha_i}^{d_i - m_i} \int_{-\infty}^d x f_{i,S}(x, y) dy dx + \mathbb{E}[(X^i + m_i - d_i)^-],
\end{aligned}$$

where $F_{i,S}$ and $f_{i,S}$ are the joint distribution function and the density of (X^i, S) , respectively. Recall that in our setting $(X^i, S) \sim N_2(\bar{\mu}^i, \bar{Q}^i)$ with mean vector $\bar{\mu}^i = (\mu^i, \sum_{j=1}^n \mu_j)$ and covariance matrix

$$\bar{Q}^i = \begin{pmatrix} \sigma_i^2 & \sigma_i^2 + \sum_{j \neq i} \rho_{i,j} \\ \sigma_i^2 + \sum_{j \neq i} \rho_{i,j} & \sum_{j=1}^n \sigma_j^2 + \sum_{j,k=1}^n \rho_{j,k} \end{pmatrix}.$$

Analogous computations hold in the case $\alpha_i < 0$. Summing up, we obtain that

$$\begin{aligned}
&\mathbb{E} \left[\sum_{i=1}^N (X^i + Y^i - d_i)^- \right] = \sum_{i=1}^N \mathbb{E} [(X^i + Y^i - d_i)^-] \\
&= \sum_{i=1}^N \mathbb{E} [(X^i + m_i - d_i)^-] \\
&\quad + \sum_{i=1}^N I_{\alpha_i \geq 0} \left[(m_i - d_i) F_{i,S}(d_i - m_i, d) - (m_i + \alpha_i - d_i) F_{i,S}(d_i - m_i - \alpha_i, d) \right. \\
&\quad \left. + \int_{d_i - m_i - \alpha_i}^{d_i - m_i} \int_{-\infty}^d x f_{i,S}(x, y) dy dx \right] \\
&\quad + \sum_{i=1}^N I_{\alpha_i < 0} \left[(m_i - d_i) F_{i,S}(d_i - m_i, d) - (m_i + \alpha_i - d_i) F_{i,S}(d_i - m_i - \alpha_i, d) \right. \\
&\quad \left. + \int_{d_i - m_i - \alpha_i}^{d_i - m_i} \int_{-\infty}^d x f_{i,S}(x, y) dy dx \right]
\end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^N \mathbb{E} [(X^i + m_i - d_i)^-] \\
&\quad + \sum_{i=1}^N \left[(m_i - d_i) F_{i,S}(d_i - m_i, d) - (m_i + \alpha_i - d_i) F_{i,S}(d_i - m_i - \alpha_i, d) \right. \\
&\quad \left. + \int_{d_i - m_i - \alpha_i}^{d_i - m_i} \int_{-\infty}^d x f_{i,S}(x, y) dy dx \right] \\
&= \sum_{i=1}^N \mathbb{E} [(X^i + m_i - d_i)^-] \\
&\quad + \sum_{i=1}^{N-1} \left[(m_i - d_i) F_{i,S}(d_i - m_i, d) - (m_i + \alpha_i - d_i) F_{i,S}(d_i - m_i - \alpha_i, d) \right. \\
&\quad \left. + \int_{d_i - m_i - \alpha_i}^{d_i - m_i} \int_{-\infty}^d x f_{i,S}(x, y) dy dx \right] \\
&\quad + (m_N - d_N) F_{N,S}(d_N - m_N, d) - \left(m_N - \sum_{j=1}^{N-1} \alpha_j - d_N \right) F_{N,S} \left(d_N - m_N \right. \\
&\quad \left. + \sum_{j=1}^{N-1} \alpha_j, d \right) + \int_{d_N - m_N + \sum_{j=1}^{N-1} \alpha_j}^{d_N - m_N} \int_{-\infty}^d x f_{N,S}(x, y) dy dx,
\end{aligned}$$

where in the last equality, we have used the constraint $\sum_{j=1}^N \alpha_j = 0$. We now denote by μ_i , σ_i the mean and the quadratic variation of X^i , $i = 1, \dots, N$, and $\Phi(x) = \int_{+\infty}^x \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$. Set $\bar{f}_{i,S}(x, y) = \int_{-\infty}^y f_{i,S}(x, s) ds$.

1. By computing the derivatives with respect to α_i , $i = 1, \dots, N - 1$, we obtain $\frac{\partial L}{\partial \alpha_i} = 0$ if and only if

$$\begin{aligned}
0 &= \lambda \left[(m_i + \alpha_i - d_i) \bar{f}_{i,S}(d_i - m_i - \alpha_i, d) - F_{i,S}(d_i - m_i - \alpha_i, d) \right. \\
&\quad + (d_i - m_i - \alpha_i) \int_{-\infty}^d f_{i,S}(d_i - m_i - \alpha_i, y) dy \\
&\quad + F_{N,S} \left(d_N - m_N + \sum_{j=1}^{N-1} \alpha_j, d \right) \\
&\quad - \left(m_N - \sum_{j=1}^{N-1} \alpha_j - d_N \right) \bar{f}_{N,S} \left(d_N - m_N + \sum_{j=1}^{N-1} \alpha_j, d \right) \\
&\quad \left. + \left(m_N - \sum_{j=1}^{N-1} \alpha_j - d_N \right) \int_{-\infty}^d f_{N,S} \left(d_N - m_N + \sum_{j=1}^{N-1} \alpha_j, y \right) dy \right]
\end{aligned}$$

$$= \lambda \left(F_{N,S}(d_N - m_N + \sum_{j=1}^{N-1} \alpha_j, d) - F_{i,S}(d_i - m_i - \alpha_i, d) \right).$$

We then obtain that the equation above has a solution if $\lambda = 0$ or when

$$F_{i,S}(d_i - m_i - \alpha_i, d) = F_{N,S} \left(d_N - m_N + \sum_{j=1}^{N-1} \alpha_j, d \right) \quad (\text{A.2})$$

for $i = 1, \dots, N - 1$.

2. By computing the derivatives with respect to m_i , for $i = 1, \dots, N$, we obtain $\frac{\partial L}{\partial m_i} = 0$ if and only if

$$\begin{aligned} 0 &= 1 + \lambda \left(\Phi \left(\frac{d_i - \mu_i - m_i}{\sigma_i} \right) - (m_i - d_i) \bar{f}_{i,S}(d_i - m_i, d) + F_{i,S}(d_i - m_i, d) \right. \\ &\quad \left. - F_{i,S}(d_i - m_i - \alpha_i, d) + (m_i + \alpha_i - d_i) \bar{f}_{i,S}(d_i - m_i - \alpha_i, d) \right. \\ &\quad \left. + (d_i - m_i - \alpha_i) \int_{-\infty}^d f_{i,S}(d_i - m_i - \alpha_i, y) - (d_i - m_i) \int_{-\infty}^d f_{i,S}(d_i - m_i, y) dy \right) \\ &= 1 + \lambda \left(\Phi \left(\frac{d_i - \mu_i - m_i}{\sigma_i} \right) + F_{i,S}(d_i - m_i, d) - F_{N,S}(d_N - m_N + \sum_{j=1}^{N-1} \alpha_j, d) \right), \end{aligned}$$

where we have used (6.8), (A.2), and the notation above. In particular,

$$\lambda = - \left(\Phi \left(\frac{d_N - \mu_N - m_N}{\sigma_N} \right) + F_{N,S}(d_N - m_N, d) - F_{N,S}(d_N - m_N + \sum_{j=1}^{N-1} \alpha_j, d) \right)^{-1}, \quad (\text{A.2})$$

if the denominator is different from zero. By (A.2), we then obtain

$$\begin{aligned} &\Phi \left(\frac{d_i - \mu_i - m_i}{\sigma_i} \right) + F_{i,S}(d_i - m_i, d) \\ &= \Phi \left(\frac{d_N - \mu_N - m_N}{\sigma_N} \right) + F_{N,S}(d_N - m_N, d), \end{aligned} \quad (\text{A.4})$$

for $i = 1, \dots, N - 1$.