Fluctuations of Photon Beams and their Correlations

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MS. received 3rd March 1958, *and in revised form 28th April* 1958

Abstract. The distribution of counts from a photoelectric detector illuminated by light of bandwidth $\Delta\nu_0$ is analysed by associating the photons with Gaussian random waves. This is shown to lead to a full statistical description of the counts. It is shown that the number n_T in a time interval $T \ll 1/\Delta\nu_0$ obeys pure Bose–Einstein statistics, and that the fluctuations in longer intervals $T \gg 1/\Delta\nu_0$ are simply the density fluctuations of a boson assembly in a phase space of $\sim \Delta\nu_0 T$ cells. The correlation coefficient ρ of the fluctuations of counts from two detectors illuminated by partially coherent beams is found to be proportional to the local time average of the square of the coherence function $\langle\gamma_{12}^2\rangle$. The correlation is shown to depend on the degeneracy of the beams in such a way that $\rho \to 2\langle\gamma_{12}^2\rangle$ for highly degenerate beams. The results are all consistent with those obtained by Hanbury Brown and Twiss in 1957.

§ 1. Introduction

Hanbury Brown and Twiss (1956, 1958) and Twiss, Little and Hanbury Brown (1957) have described experiments in which they detected correlation between the arrival times of photons in two coherent light beams, but not in incoherent beams. It was first shown by Purcell (1956) that such correlation could be understood in terms of the non-classical fluctuations of photons.

The fluctuations in light beams and the resulting correlations have since been discussed by several authors. Jánossy (1957) and Wolf (1957) have examined the problem classically in terms of waves, while Hanbury Brown and Twiss (1957) have shown that identical results follow both from a classical analysis and a quantum analysis in terms of photons.

Here it is proposed to discuss the problem from the point of view of Purcell, in terms of the number of photons arriving in a certain time interval and to show that expressions for the variances and correlations are simply derivable by considering a photon beam as statistically associated with Gaussian random waves. This assumption, which is also implicit in Purcell's discussion and supported by the results of Hanbury Brown and Twiss (1957), is sufficient for a full statistical description of the fluctuations. It will be shown that it is consistent with the uncertainty principle, that it leads to the Bose–Einstein statistics in the appropriate case and that the fluctuations in short and long time intervals are the density fluctuations of an assembly of bosons in one and several cells of phase space respectively (Fürth 1928 a, b). The correlation between fluctuations in partially coherent beams will be seen to depend on the degeneracy of the beams, so that the effect should basically be regarded as a 'wave effect', as has already been suggested by Hanbury Brown and Twiss (1957).

The results obtained are consistent with those of Purcell (1956) and Hanbury Brown and Twiss (1957) but not with those of Fellgett (1949) and Clark Jones (1953).

§ 2. Fluctuations in an Homogeneous Beam

Consider a beam of light falling on some photoelectric detector, where n_T photoelectrons are ejected in a certain time interval T. Only the photoelectrons and not the photons are, of course, observable and our discussion must therefore be confined to the statistical behaviour of the photoelectrons. Although it is tempting to associate the ejection of a photoelectron with the arrival of a photon, this picture becomes inadmissible by the uncertainty principle for time intervals shorter than the reciprocal frequency spread of the light. We shall suppose that the light comes from a Gaussian random source (Jánossy 1957) emitting a narrow spectral line centred on the frequency ν_0, and that the line shape is describable by the normalized spectral density $\phi(\nu)$ with

$$\int_{-\infty}^{\infty} \phi(\nu)\, d\nu = 1 \quad \text{and} \quad \phi(\nu) = \phi(-\nu).$$

While the shape of $\phi(\nu)$ is arbitrary we shall assume that its effective width $\Delta\nu_0$ defined by

$$\Delta{\nu_0}^2 = 2\int_0^{\infty} (\nu - \nu_0)^2 \phi(\nu)\, d\nu$$

is small compared with ν_0. We shall suppose for the moment that the light is plane polarized, and denote the instantaneous amplitude by $y(t)$ and the corresponding intensity, i.e. the square of $y(t)$ averaged over a few cycles, by $P(t)$. Thus

$$P(t) = \frac{\nu_0}{r}\int_{t-r/2\nu_0}^{t+r/2\nu_0} Q(t')\, dt', \qquad \ldots\ldots(1)$$

where $Q(t) = y^2(t)$ and where r is a small integer. We shall denote this form of average as a 'local' average and write

$$\frac{\nu_0}{r}\int_{t-r/2\nu_0}^{t+r/2\nu_0} Q(t')\, dt' = \langle Q(t) \rangle.$$

Thus, from (1), $P(t) = \langle Q(t) \rangle$.

The restriction to a 'homogeneous' beam ensures that there are no large phase differences between different elements of the beam. More specifically, it is assumed that the phase difference between any two elements is much less than $2\pi\nu_0/\Delta\nu_0$, so that their intensity cross-correlation function decreases with increasing delay.

We shall now associate photons with the Gaussian random wave $y(t)$, by defining a probability that a photoelectron is ejected in a short time interval between t and $t + dt$. If we consider first order transitions only significant, in which one photon gives rise to one photoelectron, then this probability will be given by $\alpha P(t)\, dt$, where α is the quantum sensitivity of the photoelectric detector, assumed constant over the narrow frequency range $\Delta\nu_0$. The observable $P(t)$ provides the only link between the wave and the particle descriptions of the beam.

The fluctuations of the number of particles n_T therefore have two causes. There are first of all the fluctuations of the wave intensity $P(t)$, determined by the spectral line shape and there is the stochastic association of particles with the wave intensity. This two-fold source of the fluctuations results in the departure from classical statistics, as we shall show.

Let $p_n(t, T)$ denote the probability that n photoelectrons are ejected in the interval between t and $t+T$. This probability is therefore itself a stochastic function of time. In particular, from the definition,

$$p_1(t, dt) = \alpha P(t)\, dt \qquad \ldots\ldots (2)$$

and the expectation value of n in the interval t to $t+T$ will be

$$\alpha \int_t^{t+T} P(t')\, dt'.$$

The condition (2) leads from first principles in the usual way to the Poisson distribution in n:

$$p_n(t, T) = \frac{1}{n!}\left[\alpha \int_t^{t+T} P(t')\, dt'\right]^n \exp\left[-\alpha \int_t^{t+T} P(t')\, dt'\right]. \qquad \ldots\ldots (3)$$

$p_n(t, T)$ is not, however, a distribution that can be found experimentally. For, when $P(t)$ fluctuates at random, the given interval t to $t+T$ is unique and only the ensemble averages, which are equal to the time averages for a stationary process, are observable. But the operation of averaging $p_n(t, T)$ over time when $P(t)$ is fluctuating will not, in general, result in another Poisson distribution. This departure of the observed distribution from the classical form can therefore be seen to be a general consequence of the association of particles with fluctuating waves. We may say that the fluctuations of the waves lead to the non-classical fluctuations of the quanta.

The limiting case $T \to \infty$ is exceptional, for, in that case,

$$\int_t^{t+T} P(t')\, dt'$$

is practically independent of t. The operation of averaging over time does not therefore substantially alter $p_n(t, T)$, which remains Poissonian.

In order to determine the mean number of counts in a fixed interval T we have to use equation (3) and average over n and t. If the time average is denoted by a bar we find:

$$\overline{n_T} = \sum_{n=0}^{\infty} \overline{n p_n(t, T)}$$

and, from the properties of the Poisson distribution,

$$= \overline{\alpha \int_t^{t+T} P(t')\, dt'}$$

$$= \alpha \bar{P} T. \qquad \ldots\ldots (4)$$

Also

$$\overline{n_T^2} = \sum_{n=0}^{\infty} \overline{n^2 p_n(t, T)}$$

and, again using the well-known properties of the Poisson distribution,

$$= \overline{\alpha \int_t^{t+T} P(t')\, dt'} + \overline{\left[\alpha \int_t^{t+T} P(t')\, dt'\right]^2}$$

$$= \overline{n_T} + \alpha^2 \int_0^T \int_0^T R_p(y-x)\, dy\, dx,$$

where $R_P(\tau)$ is the autocorrelation function of $P(t)$.

Instead of integrating over the xy plane we can convert the double integral into a single integral by putting $\tau=y-x$ (e.g. see Rice 1945). Thus:

$$\overline{n_T^2}=\overline{n_T}+2\alpha^2\int_0^T (T-\tau)\,R_P(\tau)\,d\tau, \qquad \ldots\ldots(5)$$

where we have made use of the symmetry of $R_P(\tau)$. Before proceeding further we shall examine the relation between $R_P(\tau)$ and the normalized spectral density $\phi(\nu)$ of the light.

If $R_y(\tau)$ is the autocorrelation function of $y(t)$ and $R_y(\tau)/\overline{y^2}=\gamma(\tau)$, then it is well known (e.g. Rice 1944) that $\gamma(\tau)$, the normalized autocorrelation function, and $\phi(\nu)$, the normalized spectral density, are related by a Fourier transformation. Thus

$$\gamma(\tau)=\int_{-\infty}^{\infty}\phi(\nu)\exp(2\pi i\nu\tau)\,d\nu \qquad \ldots\ldots(6)$$

in which $\phi(\nu)=\phi(-\nu)$.

We also have the result (Lawson and Uhlenbeck 1950) that, for a Gaussian random process,

$$R_Q(\tau)=\bar{Q}^2[1+2\gamma^2(\tau)], \qquad \ldots\ldots(7)$$

where $R_Q(\tau)$ is the autocorrelation function of $Q(\tau)$. Now, from the definition of $P(t)$,

$$\begin{aligned} R_P(\tau) &= \overline{\langle Q(t+\tau)\rangle\langle Q(t)\rangle} \\ &= \left(\frac{\nu_0}{r}\right)^2\int_{-r/2\nu_0}^{r/2\nu_0}\int_{-r/2\nu_0}^{r/2\nu_0}\overline{Q(t+\tau+x)\,Q(t+y)}\,dx\,dy \\ &= \langle R_Q(\tau)\rangle, \qquad \ldots\ldots(8) \end{aligned}$$

whence $$R_P(\tau)=\bar{P}^2[1+2\langle\gamma^2(\tau)\rangle], \quad \text{since} \quad \bar{Q}=\bar{P}. \qquad \ldots\ldots(9)$$

It is shown in the Appendix that $\langle\gamma^2(\tau)\rangle$ is a slowly varying function of τ, which does not change much in an interval short compared with $1/\Delta\nu_0$, i.e. short compared with the characteristic coherence time of the light (e.g. Forrester 1956). In particular, $\langle\gamma^2(0)\rangle=\frac{1}{2}$. $R_P(\tau)$ is therefore also a slowly varying function and, like $\langle\gamma^2(\tau)\rangle$, it is appreciably different from zero only in a range of a few times $1/\Delta\nu_0$. We shall make use of these properties of $\langle\gamma^2(\tau)\rangle$ in evaluating the mean square fluctuations of n_T.

§ 3. The Mean Squared Variation of n_T

On introducing (9) into (5) we obtain

$$\begin{aligned} \overline{\Delta n_T^2} &= \overline{n_T^2}-\overline{n_T}^2 \\ &= \overline{n_T}+4\alpha^2\bar{P}^2\int_0^T (T-\tau)\langle\gamma^2(\tau)\rangle\,d\tau. \qquad \ldots\ldots(10) \end{aligned}$$

Since the integrand $(T-\tau)\langle\gamma^2(\tau)\rangle$ is never negative, it follows at once that the fluctuations of n_T are greater than predicted by the classical particle statistics. This result, which is of course characteristic of the so-called bunching of bosons, is here seen to follow directly from associating photons with Gaussian random waves.

If we denote the integral in (10) by $\frac{1}{4}T\xi$, where ξ has the dimension of time, we can write

$$\overline{\Delta n_T^2}=\overline{n_T}\{1+\overline{n_T}(\xi/T)\}. \qquad \ldots\ldots(11)$$

Since $\langle\gamma^2(\tau)\rangle\leqslant\frac{1}{2}$ as shown in the Appendix, it follows from the definition of ξ that $\xi\leqslant T$ and

$$\overline{\Delta n_T^2}\leqslant\overline{n_T}\{1+\overline{n_T}\}.$$

The relation (11) holds generally but, by making use of the properties of $\langle\gamma^2(\tau)\rangle$ derived in the Appendix, we can immediately evaluate ξ in two limiting cases.

Case (a): $\Delta\nu_0 T \ll 1$.

This condition would be extremely difficult to satisfy experimentally. Thus, the narrowest line width obtainable from a ^{198}Hg discharge corresponds to a $\Delta\nu_0$ of the order of 6×10^8 c/s (von Klüber 1958), so that, even with this source, we should be dealing with time intervals T less than about 10^{-9} sec. Nevertheless, the result for this case is interesting.

Since $\langle\gamma^2(\tau)\rangle$ does not depart much from $\frac{1}{2}$ in an interval $T \ll 1/\Delta\nu_0$,

$$\tfrac{1}{4}T\xi \simeq \int_0^T \tfrac{1}{2}(T-\tau)\,d\tau = \tfrac{1}{4}T^2,$$

so that $\xi = T$ and

$$\overline{\Delta n_T{}^2} = \overline{n_T}\{1+\overline{n_T}\}. \qquad \ldots\ldots(12)$$

This is the well-known formula for the fluctuations of the occupation numbers of a single cell in phase space for an assembly of bosons. The photoelectrons in the interval T therefore obey ' pure ' Bose–Einstein statistics. The reason for this can be seen at once if we examine the size of the elementary cell in phase space. In the direction of the beam this extends over a distance $c\Delta t$, where $\Delta t \sim 1/\Delta\nu_0$. Thus, the photons in an interval $T \ll 1/\Delta\nu_0$ as above, i.e. much shorter than the so-called coherence time of the light (e.g. Forrester 1956), occupy the same cell in phase space. By the uncertainty principle they are therefore intrinsically indistinguishable and n_T obeys pure Bose–Einstein statistics. We shall see in the next section that, when $\Delta\nu_0 T \ll 1$, the complete probability distribution follows very simply from equation (3).

The departure from the classical particle statistics is expressible by the 'degeneracy' factor $1+\overline{n_T}$ which is a measure of the extent to which photons share the same cell in phase space. The excess fluctuations correspond to what Hanbury Brown and Twiss (1957) called the wave interaction noise, although these authors did not consider the case $\Delta\nu_0 T \ll 1$. The degeneracy is also indicative of whether the wave or the particle properties of the beam predominate. In the visible region of the spectrum $\overline{n_T}$, and therefore the degeneracy, are normally small and the non-classical behaviour of a single photon beam is difficult to observe. Finally we note from (12) that the percentage fluctuation defined by

$$(\overline{\Delta n_T{}^2})^{1/2}/\overline{n_T} = (1+1/\overline{n_T})^{1/2}$$

is always greater than one, even at high intensities. This feature is again characteristic of a boson assembly and quite different from the behaviour of classical particles, for which the percentage fluctuation tends to zero at high intensities.

Case (b): $\Delta\nu_0 T \gg 1$.

This is the condition assumed to hold in the analyses of both Purcell (1956) and Hanbury Brown and Twiss (1957). It is of course almost invariably satisfied in experiments as shown above, but corresponds to a slightly more complicated situation.

Since $\langle\gamma^2(\tau)\rangle$ extends appreciably only over an interval of the order a few times $1/\Delta\nu_0$, the integral in (10) can be written

$$\tfrac{1}{4}T\xi \simeq \int_0^\infty T\langle\gamma^2(\tau)\rangle\,d\tau = \tfrac{1}{4}T\int_{-\infty}^{\infty} 2\langle\gamma^2(\tau)\rangle\,d\tau.$$

Although the exact value of ξ therefore depends on the shape of the spectral line, we can see that it will be of the order $1/\Delta\nu_0$, since $\langle\gamma^2(\tau)\rangle$ changes only slowly and $\langle\gamma^2(0)\rangle = \frac{1}{2}$. We can therefore write

$$\xi = \kappa/\Delta\nu_0$$

where κ is a number depending on the spectral density $\phi(\nu)$ but of the order 1. Equation (11) therefore becomes

$$\overline{\Delta n_T{}^2} = \overline{n_T}\{1 + \kappa\,\overline{n_T}/\Delta\nu_0 T\}. \qquad \ldots\ldots(13)$$

This is the relation first obtained by Purcell (1956). It can also be seen to be equivalent to that of Hanbury Brown and Twiss (1957), when the integral over time defining ξ is converted to one over frequency.

Since $\gamma(\tau)$ and $\phi(\nu)$ are a Fourier transform pair, we have from Parseval's theorem

$$\int_{-\infty}^{\infty} \gamma^2(\tau)\,d\tau = \int_{-\infty}^{\infty} \phi^2(\nu)\,d\nu,$$

so that

$$\xi = 4\int_0^{\infty} \phi^2(\nu)\,d\nu = \kappa/\Delta\nu_0.$$

From (13) we note that the degeneracy factor is smaller than before for the same $\overline{n_T}$. The reason for this can again be seen if we remember that we are now dealing with a volume of phase space containing roughly $\Delta\nu_0 T = s$ cells.† The mean number of photons per cell is therefore $\overline{n_T}/s = \overline{m}$ and (13) can be written

$$\overline{\Delta n_T{}^2} = \overline{n_T}(1 + \kappa\,\overline{m}) = \overline{n_T}(1 + \kappa s\,\overline{m}^2/\overline{n_T}).$$

This is the expression in the conventional form for the density fluctuations in a larger volume of phase space (Fürth, 1928 a, b). The degeneracy is less because we are dealing with a time interval in which not all the photons are intrinsically indistinguishable. The departure from the classical statistics is again due to those photons which share one cell and therefore would become important only at very great intensities.

The excess fluctuations given by (13) correspond to the wave interaction noise of Hanbury Brown and Twiss (1957). The results do not agree with those of Fellgett (1949, 1957) and Clark Jones (1953) obtained from thermodynamic arguments, who quote larger values. A likely reason for the discrepancy has already been given by Hanbury Brown and Twiss, namely that these authors equate the fluctuations of the detector output to the energy fluctuations of the light, whereas the two are stochastically connected. In any case, the formula of Fellgett is not consistent with that of a boson assembly in a phase space of $\Delta\nu_0 T$ cells.

It is interesting to note that the percentage fluctuation

$$(\overline{\Delta n_T{}^2})^{1/2}/\overline{n_T} = (1/\overline{n_T} + \kappa/\Delta\nu_0 T)^{1/2}$$

could be either greater or less than one, depending on $\overline{n_T}$, since n_T no longer obeys the pure Bose–Einstein statistics.

† By restricting the discussion to what we have called homogeneous beams, we ensure that the spacial extent of the beam over the photo-detector does not include more than one cell.

§ 4. The Distribution Function of n_T when $\Delta\nu_0 T \ll 1$

So far we have been concerned only with the variance of n_T, but, when $\Delta\nu_0 T \ll 1$, it is not difficult to derive the full distribution. This will be given by the time or ensemble average of $p_n(t, T)$ in equation (3). Now, since $P(t)$ does not vary much in an interval $T \ll 1/\Delta\nu_0$, it follows that

$$\alpha \int_t^{t+T} P(t')\,dt' \simeq \alpha P(t) T.$$

We therefore find from (3),

$$\overline{p_n(t, T)} = \frac{1}{n\,!}\,\overline{(\alpha PT)^n \exp(-\alpha PT)}. \qquad \ldots\ldots (14)$$

If the probability distribution of P is known, we can evaluate $\overline{p_n(t, T)}$ by averaging over the ensemble. Since $y(t)$ is a narrow band Gaussian random variable, the local average

$$\langle y^2(t) \rangle = P(t) = \tfrac{1}{2} W^2(t),$$

where $W(t)$ is the envelope of $y(t)$. Now the probability density of $W(t)$ has been shown by Rice (1944) to be of the form:

$$(W/\bar{P}) \exp(-W^2/2\bar{P}).$$

Hence, by transforming from W to P, we arrive at the probability density $p'(P)$ of P. Thus:

$$p'(P)\,dP = (1/\bar{P}) \exp(-P/\bar{P})\,dP.$$

We can now evaluate (14) by averaging over the ensemble and we obtain

$$\overline{p_n(t, T)} = \frac{1}{\bar{P} n\,!} \int_0^\infty (\alpha PT) \exp(-\alpha PT - P/\bar{P})\,dP.$$

The integral is the well-known factorial function integral and leads to

$$\overline{p_n(t, T)} = \{(1 + \alpha\bar{P}T)(1 + 1/\alpha\bar{P}T)^n\}^{-1}.$$

Since $\alpha\bar{P}T = \overline{n_T}$ from (4), we can write this as

$$\overline{p_n(t, T)} = (1 - w) w^n, \qquad \ldots\ldots (15)$$

where $w = \{1 + 1/\overline{n_T}\}^{-1}$. This is the Bose–Einstein distribution function in standard form for the numbers in a single cell of phase space. The distribution can be seen to arise naturally from the association of photons with Gaussian random waves, when $\Delta\nu_0 T \ll 1$.

§ 5. Correlation between Fluctuations in Two Beams

Following the experimental work of Hanbury Brown and Twiss (1956, 1958) and Twiss, Little and Hanbury Brown (1957), the correlation between the fluctuations of two at least partially coherent beams has been studied theoretically by Purcell (1956), Wolf (1957), Jánossy (1957) and Hanbury Brown and Twiss (1957, 1958). The latter authors, in particular, have examined the problem in some detail and shown that the correlation between two beams and the 'excess' photon fluctuations of a single beam are very closely related.

Here it is proposed to show that similar relations are derivable very simply from equation (3) and that the correlation coefficient is very small unless the beams are substantially degenerate. We shall also find that, in sufficiently short time intervals, the correlation is independent of the spectral distribution.

We shall suppose that the two beams have the same spectral density, but not necessarily equal intensities $\overline{P_1}$ and $\overline{P_2}$ and that the degree of coherence is describable by the cross-correlation function $J_{12}(\tau)=\overline{y_1(t+\tau)y_2(t)}$ used by Wolf (1955). The normalized function $J_{12}(\tau)/(\overline{P_1}\,\overline{P_2})^{1/2}$ will be denoted by $\gamma_{12}(\tau)$. Wolf (1955) has shown that $\gamma_{12}(\tau)$ is an observable which is related to the visibility of the fringe system obtained from the superposition of the two beams. In the limit as the beams tend to complete coherence, $\gamma_{12}(\tau)\rightarrow\gamma(\tau)$.

From equation (3) we find

$$\overline{n_1 n_2}=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\overline{n_1 n_2 p_{n_1}(t,T)p_{n_2}(t,T)} \qquad \ldots\ldots(16)$$

which reduces to

$$\alpha^2\int_0^T\int_0^T R_{P_1P_2}(y-x)\,dy\,dx, \qquad \ldots\ldots(17)$$

where $R_{P_1P_2}(\tau)$ is the cross-correlation function of the two intensities.

By using the result of Wolf (1957) that

$$R_{Q_1Q_2}=\overline{Q_1}\,\overline{Q_2}\,[1+2\gamma_{12}^2(\tau)] \qquad \ldots\ldots(18)$$

and taking local averages as before we arrive at

$$R_{P_1P_2}=\overline{P_1}\,\overline{P_2}\,[1+2\langle\gamma_{12}^2(\tau)\rangle], \qquad \ldots\ldots(19)$$

where $\langle\gamma_{12}^2(\tau)\rangle$ is a slowly varying function of τ, as shown in the Appendix.

By substituting in (17) and transforming to a single integral over $\tau=y-x$ as before, we obtain for the cross-correlation function of the fluctuations

$$\overline{\Delta n_1\,\Delta n_2}=4\alpha^2\,\overline{P_1}\,\overline{P_2}\int_0^T (T-\tau)\,\langle\gamma_{12}^2(\tau)\rangle\,d\tau. \qquad \ldots\ldots(20)$$

The value of the integral appears to depend on the detailed form of $\langle\gamma_{12}^2(\tau)\rangle$. We can, however, simplify it appreciably if any path difference between the two partially coherent beams at the two photoelectric detectors is rather less than the coherence length $\mathbf{c}\Delta\nu_0$; in other words, if $\langle\gamma_{12}^2(\tau)\rangle$ is a decreasing function of τ.

Under these conditions P_1 and P_2 will be related by an expression of the form:

$$P_2(t)=aP_2'(t)+bP_1(t),$$

where a and b are positive numbers and $P_2'(t)$ is a function with the same spectral density as $P_1(t)$ but uncorrelated with it. This leads to

$$R_{P_1P_2}(\tau)=\overline{P_1}\,\overline{P_2}+b[R_{P_1}(\tau)-\overline{P_1}^2],$$

so that

$$\langle\gamma_{12}^2(\tau)\rangle=\tfrac{1}{2}\,\frac{R_{P_1P_2}(\tau)-\overline{P_1}\,\overline{P_2}}{\overline{P_1P_2}-\overline{P_1}\,\overline{P_2}}\,\frac{\overline{P_1P_2}-\overline{P_1}\,\overline{P_2}}{\overline{P_1}\,\overline{P_2}}$$
$$=2\langle\gamma^2(\tau)\rangle\langle\gamma_{12}^2(0)\rangle. \qquad \ldots\ldots(21)$$

It follows that, for beams with small path difference, only the zero-time cross-correlation enters into the equations, as has already been pointed out by Wolf (1955). An expression similar to (21) has also been derived by Hanbury Brown and Twiss (1957) from more detailed considerations.

Under these conditions also the integral in equation (20) has the same form as that encountered earlier in equation (10) and we can immediately write down the two limiting solutions.

For $\Delta\nu_0 T \ll 1$,

$$\overline{\Delta n_1 \Delta n_2} = 2\bar{n}_1 \bar{n}_2 \langle \gamma_{12}^2(0) \rangle \quad \ldots\ldots (22)$$

and for $\Delta\nu_0 T \gg 1$,

$$\overline{\Delta n_1 \Delta n_2} = 2\bar{n}_1 \bar{n}_2 (\kappa/\Delta\nu_0 T) \langle \gamma_{12}^2(0) \rangle. \quad \ldots\ldots (23)$$

Finally, when the two beams are unpolarized, these correlations are halved. Since the fluctuations in two normal planes of polarization are independent, we may associate uncorrelated numbers of photons n_1' and n_1'' with each polarization, such that $\overline{n_1'} = \overline{n_1''} = \frac{1}{2}\bar{n}_1$ and $\overline{\Delta n_1 \Delta n_2} = \overline{2\Delta n_1' \Delta n_2'}$.

The result for $\Delta\nu_0 T \gg 1$ is again equivalent to that obtained by Hanbury Brown and Twiss (1957), although expressed in a slightly different form. The case $\Delta\nu_0 T \ll 1$ was not considered by these authors and is unlikely to be of practical importance. But it is significant that the correlation in this case is independent of the spectral line shape. From (22) and (23) it is clear that $\overline{\Delta n_1 \Delta n_2}$ is directly proportional to the degeneracy of the beams, i.e. to the number of photons occupying the same cell in phase space.

§-6. Superposition of Counts

When the counts n_1 and n_2 of two detectors illuminated by two similar partially coherent plane polarized beams are superposed, we obtained another variate whose degeneracy is in general intermediate between the plane polarized and the unpolarized case.

Thus if

$$n = n_1 + n_2,$$

$$\overline{\Delta n^2} = \overline{\Delta n_1^2} + \overline{\Delta n_2^2} + 2\overline{\Delta n_1 \Delta n_2}$$

and, by using the results of (12), (13), (22) and (23), we obtain

$$\overline{\Delta n^2} = \bar{n}\left\{1 + \bar{n} - \frac{2\bar{n}_1\bar{n}_2}{\bar{n}}[1 - 2\langle \gamma_{12}^2(0) \rangle]\right\} \quad \ldots\ldots (24)$$

for $\Delta\nu_0 T \ll 1$, or

$$\overline{\Delta n^2} = \bar{n}\left\{1 + \bar{n}\left(\frac{\kappa}{\Delta\nu_0 T}\right) - \frac{2\bar{n}_1\bar{n}_2}{\bar{n}}\left(\frac{\kappa}{\Delta\nu_0 T}\right)[1 - 2\langle \gamma_{12}^2(0) \rangle]\right\} \text{ for } \Delta\nu_0 T \gg 1. \quad \ldots\ldots (25)$$

Since $\langle \gamma_{12}^2(0) \rangle \leq \frac{1}{2}$, we see that the degeneracy of the distribution of n is generally less than that of n_1 and n_2. The difference depends on the normalized coherence factor $\langle \gamma_{12}^2(0) \rangle$. It follows at once that this factor also measures the extent to which the two beams share cells in phase space. In particular, for completely coherent beams, the statistical behaviour of n is identical with that of n_1 and n_2, since all the cells are shared. This has already been pointed out by Hanbury Brown and Twiss (1957).

§ 7. Discussion

It has been shown that a full statistical description of the fluctuations in photon beams is possible by associating the photons with Gaussian random waves, which are describable by their spectral density $\phi(\nu)$. The validity of this approach has been questioned by Fellgett (1957), but it is seen to lead to completely consistent results. While the results are all concerned with the number of counts in a definite time interval T, they can immediately be applied to continuous fluctuation and correlation measurements. For all such measurements are limited by a certain resolving time T, which takes the place of the standard interval.

As shown by equation (15), the number n_T in an interval $T \ll 1/\Delta\nu_0$ obeys the pure Bose–Einstein distribution, as we should expect for a boson assembly in a single cell of phase space. In longer intervals the fluctuations of n_T correspond to the boson density fluctuations in a phase space containing $\sim \Delta\nu_0 T$ cells. From the equations (12), (13), (22) and (23) it is clear that the correlation between fluctuations depends essentially on the non-classical fluctuations of the photons, as has already been shown by Purcell (1956) and Hanbury Brown and Twiss (1957) and on the degree of coherence between the beams. These, in turn, depend on the extent of the cells in phase space in real space and time.

The degree of coherence is therefore derivable from correlation measurements (Hanbury Brown and Twiss 1956) as well as from interference experiments (Wolf 1955), although the former always yield the local average $\langle \gamma_{12}^2(\tau) \rangle$. This is of course to be expected, since $\gamma_{12}(\tau)$ contains phase information about the beams which is incompatible with the detection of single photons. Even with a perfect detector having $\alpha = 100\%$ the counts must not be too closely identified with the wave intensity.

There is another important difference between the time dependent correlation effect and the interference effect. Since the correlation depends essentially on two or more photons sharing cells in phase space, it depends on the degeneracy and therefore, unlike the interference effect, varies with the intensity of the beams. This becomes most obvious if we examine the normalized correlation coefficient

$$\rho = \overline{\Delta n_1 \Delta n_2} / (\overline{\Delta n_1^2}\, \overline{\Delta n_2^2})^{1/2}.$$

We then find

$$\rho = \frac{2\bar{n}_1\bar{n}_2(\kappa/\Delta\nu_0 T)\langle \gamma_{12}^2(0) \rangle}{\{\bar{n}_1\bar{n}_2(1+\bar{n}_1\,\kappa/\Delta\nu_0 T)(1+n_2\,\kappa/\Delta\nu_0 T)\}^{1/2}}$$

if $\Delta\nu_0 T \gg 1$, or

$$\rho = \frac{2\bar{n}_1\bar{n}_2\langle \gamma_{12}^2(0) \rangle}{\{\bar{n}_1\bar{n}_2(1+\bar{n}_1)(1+\bar{n}_2)\}^{1/2}}$$

if $\Delta\nu_0 T \ll 1$.

It can be seen that, although $\rho \propto \langle \gamma_{12}^2(0) \rangle$, it is very much less than $\langle \gamma_{12}^2(0) \rangle$ under conditions of low degeneracy, but tends to $2\langle \gamma_{12}^2(0) \rangle$ at high degeneracy. In particular, for highly degenerate completely coherent beams $\rho = 1$. The correlation is therefore appreciable only when the wave properties, as distinct from the particle properties, of the beam become evident. This confirms the view of Hanbury Brown and Twiss (1957) that the effect should be regarded basically as a wave effect and shows that it will be more difficult to detect in an experiment with light than with radio waves (Hanbury Brown and Twiss 1954). It is a considerable credit to these authors that they were able to detect the effect while working with a degeneracy of the order of a few times 10^{-3}.

Acknowledgment

The author is grateful to Dr. R. Fürth for some valuable discussions.

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APPENDIX

The Behaviour of $\langle \gamma_{12}^2(\tau) \rangle$

If $\phi_{12}(\nu)$ is the normalized cross power spectrum of the two beams, then

$$\gamma_{12}(\tau) = \int_{-\infty}^{\infty} \phi_{12}(\nu) \exp(2\pi i \nu \tau)\, d\nu.$$

Since $\phi_{12}(-\nu) = \phi_{12}^*(\nu)$, this can be written in the form

$$\gamma_{12}(\tau) = \int_0^{\infty} [\phi_{12}(\nu) \exp(2\pi i \nu \tau) + \phi_{12}^*(\nu) \exp(-2\pi i \nu \tau)]\, d\nu.$$

If we now introduce the substitution $\nu = \nu' + \nu_0$ and denote

$$\int_{-\nu_0}^{\infty} \phi_{12}(\nu_0 + \nu') \exp(2\pi i \nu' \tau)\, d\nu'$$

by $V(\tau)$, the equation becomes

$$\gamma_{12}(\tau) = V(\tau) \exp(2\pi i \nu_0 \tau) + V^*(\tau) \exp(-2\pi i \nu_0 \tau). \quad \ldots\ldots (\text{A}\,1)$$

Now by hypothesis, $\phi_{12}(\nu_0 + \nu')$ will be appreciably different from zero only for small ν' i.e. for $|\nu'| \lesssim \Delta\nu_0$. It follows from the integral defining $V(\tau)$ that this function will not change significantly in an interval $\Delta\tau \ll 1/\Delta\nu_0$. Thus $V(\tau)$ is a slowly varying function (i.e. compared with $y(t)$). From (A 1)

$$\gamma_{12}^2(\tau) = V^2(\tau) \exp(4\pi i \nu_0 \tau) + V^{*2}(\tau) \exp(-4\pi i \nu_0 \tau) + 2\,|V(\tau)|^2. \quad \ldots\ldots (\text{A}\,2)$$

If we now calculate the local average, it follows at once from the properties of $V(\tau)$ that

$$\langle \gamma_{12}^2(\tau) \rangle = 2\,|V(\tau)|^2, \quad \ldots\ldots (\text{A}\,3)$$

so that $\langle \gamma_{12}^2(\tau) \rangle$ is also a slowly varying function of τ. In particular, for $\tau \ll 1/\Delta\nu_0$,

$$\langle \gamma_{12}^2(\tau) \rangle = \langle \gamma_{12}^2(0) \rangle.$$

If the two beams are completely coherent so that $\phi_{12}(\nu) = \phi(\nu)$,

$$V(0) = \int_0^{\infty} \phi(\nu)\, d\nu = \tfrac{1}{2}.$$

Hence

$$\langle \gamma^2(0) \rangle = \tfrac{1}{2}. \quad \ldots\ldots (\text{A}\,4)$$

Further, from the integral defining $V(\tau)$, we see that, for $\tau \gg 1/\Delta\nu_0$, the integrand contains a rapidly oscillating factor $\exp(2\pi i \nu' \tau)$. It follows that $V(\tau)$—and therefore $\langle \gamma_{12}^2(\tau) \rangle$—is small for $\tau \gg 1/\Delta\nu_0$.

Finally from

$$V(\tau) = \exp(-2\pi i\nu_0\tau) \int_0^\infty \phi_{12}(\nu) \exp(2\pi i\nu\tau)\, d\nu$$

we have,

$$|V(\tau)| = \left| \int_0^\infty \phi_{12}(\nu) \exp(2\pi i\nu\tau)\, d\nu \right| \leqslant \int_0^\infty |\phi_{12}(\nu)|\, d\nu. \quad \ldots\ldots (\text{A }5)$$

Hence, using (A 3) and (A 5) we see that, for a single coherent beam,

$$\langle \gamma^2(\tau) \rangle \leqslant \langle \gamma^2(0) \rangle = \tfrac{1}{2}.$$