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The de Rham Theorem in Algebraic Geometry and in Theory of Motives

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Abstract

The de Rham Theorem in Algebraic Geometry states the existence of a comparison isomorphism between the algebraic de Rham and Betti cohomologies of smooth algebraic varieties. We provide a proof of this result, together with an exposition of the constructions and results involved.

The algebraic de Rham isomorphism naturally produces the period numbers, which are an interesting arithmetic invariant for algebraic varieties. There are some conjectures about polynomial relations between periods, which predict that all such relations should be explained by the geometry of the algebraic variety.

A natural conceptual framework in which these conjectures can be formulated is Theory of Motives. We revisit the de Rham Theorem in this framework and we state a version of the Grothendieck Period Conjecture.

Contents

Introduction

Algebraic Topology is a branch of Mathematics which studies topological spaces by associating to each of them some objects, which should be easier to deal with. Such objects, called *invariants*, are usually linear algebra objects, such as groups or vector spaces. Some of these invariants arise as the sequence of cohomology groups of a complex associated to the given topological space. They are called *cohomological invariants*. The construction of the complex may involve the topological structure alone, or it may use some additional structure considered on the topological space. For example, the *singular cohomology* is a cohomological invariant, which can be defined for any topological space. It is the sequence of the cohomology groups of the *complex of singular cochains*, whose construction is based on continuous maps from the topological standard simplexes to the given topological space. Another one is the *de Rham cohomology*, which can be defined for any differentiable or complex manifold, that is, a topological space with an additional differentiable or complex structure. It is the sequence of the cohomology groups of the *de Rham complex*, whose construction is based on differential forms defined over the given manifold. Although these two cohomological invariants are constructed using tools of very different nature (singular cohomology is of topological nature, while de Rham cohomology is of analytic nature), they turn out to be the same. That is, there exists a canonical isomorphism between each cohomology group of the singular complex and of the de Rham complex associated to a manifold. It is called the *de Rham isomorphism*. This is the content of the *de Rham Theorem*, originally conjectured by Henri Cartan and proved for the first time by Georges de Rham in [deR31]. An explicit isomorphism is given by the integration of closed differential forms along singular cycles over the manifold.

Algebraic Geometry is another branch of Mathematics which deals with a particular kind of topological spaces, which can be studied with purely algebraic methods: the algebraic varieties. We consider the notion of an algebraic variety over a field given by the modern language of theory of schemes. Algebraic Geometry borrows at times from Algebraic Topology the technique of considering cohomological invariants, in order to solve some problems about algebraic varieties. The most famous case is the construction of the étale ℓ-adic cohomologies for algebraic varieties over a field of positive characteristic, for the solution of the Weil Conjectures.

In this thesis we deal with another couple of cohomological invariants for algebraic varieties, which are the analogues of the singular cohomology and of the de Rham cohomology of Algebraic Topology. We consider smooth algebraic varieties over a field k of characteristic zero. For $k = \mathbb{C}$, there exists a universal construction, called the *analytification*, which allows to transform a smooth algebraic variety over $\mathbb{C}$ into a complex manifold. This construction allows to associate, to any smooth algebraic variety over $\mathbb{C}$, the cohomological invariants given by the singular cohomology and the de Rham cohomology of the corresponding complex manifold. They are called, respectively, *Betti cohomology* and *analytic de Rham cohomology* of the given algebraic variety over $\mathbb{C}$. In fact, we can also construct, for any smooth algebraic variety over k, another kind of de Rham cohomology, using the purely algebraic structure of the algebraic variety, called the *algebraic de Rham cohomology*. Compared to the analytic version, its definition involves only the algebraic forms, without considering the transcendental ones. For $k = \mathbb{C}$, the algebraic and the analytic de Rham cohomologies turn out to be canonically isomorphic. The first chapter of this thesis is devoted to give a proof of this result, together with an exposition of the definitions and constructions involved. Composing the isomorphism between the algebraic and the analytic de Rham cohomologies with the de Rham isomorphism, we obtain a canonical isomorphism between the algebraic de Rham and the Betti cohomologies, called the *algebraic de Rham isomorphism*. Explicitly, this isomorphism is given by the integration of algebraic forms along singular cycles over the complex manifold given by the analytification.

Starting from smooth algebraic varieties over $\overline{\mathbb{Q}}$, the field of algebraic numbers, the algebraic de Rham isomorphism naturally produces the *period numbers*. For this reason the algebraic de Rham isomorphism, in this case, is also called the *period isomorphism*. The period numbers are a class of complex numbers that lie between algebraic and transcendental numbers. They are defined as those numbers obtained by integrating closed $\overline{\mathbb{Q}}$-linear algebraic forms along $\mathbb{Q}$-linear singular cycles. Many famous numbers and constants that appear in central conjectures in Number Theory are periods, such as π and special values of some L-functions. Others, like e and the Euler constant γ, are supposed not to be periods. There are several open questions about periods, some of them are considered very hard and out of reach for the present moment. A survey on open problems and connections with arithmetic conjectures is [KZ01]. One of the first questions regards identities between periods. More precisely, we can ask what are the relations between two representations of a single period as an integral. The *Kontsevich and Zagier Period Conjecture* ([KZ01, Conj. 1]) predicts that we can pass from an integral representation to another using only the classical integration rules: linearity, change of variables and Stokes formula. Another questions regards the polynomial relations with coefficients in $\overline{\mathbb{Q}}[\pi^{-1}]$ between periods. Following an initial intuition of Grothendieck, which predicts that all polynomial relations between periods of an algebraic variety over $\overline{\mathbb{Q}}$ should be explained by the geometry (that is, the *algebraic cycles*) of the given algebraic variety and its products, a precise conjecture, usually called the *Grothendieck Period Conjecture*, can be stated inside the conceptual framework of *Theory of Motives*. The conjectures of Grothendieck and Kontsevich-Zagier are also related to each other (see [Ayo14b]).

The *motive* (or *motivic cohomology*) of an algebraic variety is a notion initially envisioned by Grothendieck in the 60's, which can be thought as the essential and deepest cohomological invariant that can be associated to an algebraic variety. All the other cohomological invariants should be concrete manifestations of this master invariant. For many years, Theory of Motives remained a vague ideal project and more precise conjectural pictures grew thanks to further developments in Algebraic Geometry. But nowadays, we have several concrete candidate theories, which also allowed to approach and solve some problems in Arithmetic Algebraic Geometry. The main goal in Theory of Motives is to construct a suitable *category of motives*, which should be thought as the category in which motivic cohomology takes values. Mostly for historical reasons, we usually distinguish between *Theory of Pure Motives* and *Theory of Mixed Motives*: in the first only smooth projective algebraic varieties are considered, while the second includes all (smooth) algebraic varieties. Initially, only pure motives were studied, in order to solve the Weil Conjectures, even though it was clear from the very beginning that they should be included into a wider picture of mixed motives, as we can see in a letter from Grothendieck to Luc Illusie in 1973 (which can be found in the appendix of [Jan]). In the successive years, the scope of mixed motives expanded, until the formulation by Alexander Beilinson in the middle 80's of an ambitious conjectural program, which, besides extending pure motives, should also explain and relate several phenomena across Algebraic Geometry and Number Theory. This made popular Theory of Motives also among arithmeticians.

There are mainly two methods (related to each other) for constructing a category of motives, which give rise to as many versions of the Grothendieck Period Conjecture. One uses the formalism of *tannakian categories*, which allows to talk about the *motivic Galois group* and paves the way for the development of a *Galois Theory of Periods* (see [And08] and [Hub18]). The other method is based on algebraic cycles. This thesis is concerned with the latter approach. In the overview to the second chapter, we outline some facts about Theory of Motives from this cycle-theoretic point of view. In the second chapter we present two concrete constructions of categories of motives: the *category of Chow motives*, which belongs to Theory of Pure Motives, and a version of *Voevodsky's triangulated category of motives*, which is an outcome of Beilinson's program of mixed motives. Besides recovering the algebraic de Rham isomorphism inside these conceptual frameworks, we construct the *cycle class maps* relating algebraic cycles to Betti and algebraic de Rham cohomologies. The cycle class maps and their compatibility under the algebraic de Rham isomorphism are the fundamental tools, which allow to state the cycle-theoretic version of the Grothendieck Period Conjecture.

Chapter 1

The de Rham Theorem in Algebraic Geometry

1.0.1 Overview

The de Rham Theorem is a result in Differential Geometry, which states that:

Theorem 1.0.1 (de Rham Theorem). *For any M differentiable manifold, there exist canonical isomorphisms of $\mathbb{R}$-vector spaces, for each $i \geq 0$,*

$$H^i_{dR}(M/\mathbb{R}) \cong H^i_{Sing}(M;\mathbb{R}),$$

where $H^i_{dR}(M/\mathbb{R})$ is the i^{th}-de Rham cohomology group of M and $H^i_{Sing}(M;\mathbb{R})$ is the i^{th}-singular cohomology group with real coefficients of M.

An analogous result with complex coefficients can be stated for any complex manifold.

In this chapter we want to bring this theorem inside Algebraic Geometry, that is, we want to state an analogous result for algebraic varieties, instead of manifolds. More precisely, we will prove the following:

Theorem 1.0.2 (Algebraic de Rham Theorem). *Let $\sigma : k \hookrightarrow \mathbb{C}$ be a field extension. For any X smooth algebraic variety over k, there exist canonical isomorphisms of $\mathbb{C}$-vector spaces, for each $i \geq 0$,*

$$\varpi^i : H^i_{AdR}(X/k) \otimes_k \mathbb{C} \cong H^i_{Bet}(X_\sigma) \otimes_{\mathbb{Q}} \mathbb{C},$$

where $H^i_{AdR}(X/k)$ is the i^{th}-algebraic de Rham cohomology group of X and $H^i_{Bet}(X_\sigma)$ is the i^{th}-Betti cohomology group of $X_\sigma \coloneqq X \times_k \mathbb{C}$.

To state this theorem we first need to define the *Betti* and the *algebraic de Rham cohomologies.* Given X an algebraic variety over $\mathbb{C}$, the set of $\mathbb{C}$-rational points $X(\mathbb{C})$, called the *analytification* of X, has a canonical structure of *complex analytic space* (a complex manifold with eventually singular points). Given a smooth algebraic variety over $\mathbb{C}$, the analytification is a smooth complex analytic space, that is, a complex manifold. The Betti cohomology of X is defined as the singular cohomology with rational coefficients of $X(\mathbb{C})$

$$H^i_{\text{Bet}}(X) \coloneqq \mathrm{H}^i_{\text{Sing}}(X(\mathbb{C}); \mathbb{Q}).$$

Given X a smooth algebraic variety over any field k, the algebraic de Rham cohomology of X is defined as the sheaf cohomology of $\Omega^\bullet_{X/k}$, a complex of Zariski sheaves over X, called the *algebraic de Rham complex* of X over k,

$$\mathrm{H}^i_{\text{AdR}}(X/k) \coloneqq \mathbb{H}^i(X_{Zar}, \Omega^\bullet_{X/k}).$$

The algebraic de Rham complex is constructed with purely algebraic tools, starting from a sheaf-theoretical version of the module of Kähler differentials. Attention should be paid to the fact that each object $\Omega^p_{X/k}$ is a coherent sheaf of $\mathcal{O}_X$-modules, while the differentials are only k-linear.

The proof of the Algebraic de Rham Theorem consists in showing the following composition of canonical isomorphisms of $\mathbb{C}$-vector spaces, for each $i \geq 0$,

$$\mathrm{H}^i_{\text{AdR}}(X/k) \otimes_k \mathbb{C} \cong \mathrm{H}^i_{\text{AdR}}(X_\sigma/\mathbb{C}) \cong \mathrm{H}^i_{\text{dR}}(X_\sigma(\mathbb{C})) \cong \mathrm{H}^i_{\text{Bet}}(X_\sigma) \otimes_{\mathbb{Q}} \mathbb{C}.$$

The first isomorphism follows from a flat base change result in sheaf cohomology. The last isomorphism is the de Rham Theorem for complex manifolds. So, much of the work consists in proving the middle isomorphism. It compares, for any X algebraic variety over $\mathbb{C}$, the algebraic de Rham cohomology of X with the de Rham cohomology of the analytification, called the *analytic de Rham cohomology* of X. To do this, it's useful to notice that also the analytic de Rham cohomology of X can be computed as the sheaf cohomology of a complex of sheaves. Indeed, it holds that

$$\mathrm{H}^i_{\text{dR}}(X(\mathbb{C})) \cong \mathbb{H}^i(X(\mathbb{C})_{an}, \Omega^\bullet_{X(\mathbb{C})}),$$

where $\Omega^\bullet_{X(\mathbb{C})}$ is the *holomorphic de Rham complex* of $X(\mathbb{C})$, which is a complex of sheaves over $X(\mathbb{C})$, with the classical topology of open covers. In case X is proper, using a spectral sequences argument, the comparison isomorphism between algebraic and analytic de Rham cohomology is a direct consequence of a GAGA Theorem ([Ser56] and [Gro57]), which states that sheaf cohomology of a coherent sheaf of modules over a *proper* algebraic variety over $\mathbb{C}$ is isomorphic to the one of its analytification. In the general case, a first proof was given by Grothendieck in [Gro66]. Another is given by Deligne in [Del70]. The strategy consists in embedding X as an open subset into $\overline{X}$, a smooth proper algebraic variety over $\mathbb{C}$, such that the complementary closed subscheme $D \coloneqq \overline{X} \setminus X$ is a simple normal crossing divisor of $\overline{X}$ (this is possible by Nagata Embedding Theorem [Del10] and Hironaka resolution of singularities [Wlo05]). Then, we consider $\Omega^\bullet_{\overline{X}/\mathbb{C}}(\log D)$, a complex of Zariski sheaves over $\overline{X}$, called the *algebraic de Rham complex with logarithmic poles along D*. We show that its sheaf cohomology computes the algebraic de Rham cohomology of X

$$\mathrm{H}^i_{\mathrm{AdR}}(X/\mathbb{C}) \cong \mathbb{H}^i(\overline{X}_{Zar}, \Omega^\bullet_{\overline{X}/\mathbb{C}}(\log D)).$$

Moreover, analogous constructions and results hold in the analytic context, considering the analytifications. That is, it holds

$$\mathrm{H}^i_{\mathrm{dR}}(X(\mathbb{C})) \cong \mathbb{H}^i(\overline{X}(\mathbb{C})_{an}, \Omega^\bullet_{\overline{X}(\mathbb{C})}(\log D(\mathbb{C}))).$$

We conclude by proving the isomorphism between the sheaf cohomology of the algebraic and the analytic de Rham complexes with logarithmic poles, for which we can use the GAGA Theorem recalled above, since $\overline{X}$ is proper.

We see that the sheaf-theoretic point of view is fundamental in all this work. We refer to Appendix A for notations and results on cohomology of sheaves defined over a general site.

For $k = \overline{\mathbb{Q}}$, with $\sigma : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ given by the inclusion, the Algebraic de Rham Theorem gives the canonical isomorphisms

$$\varpi^i : \mathrm{H}^i_{\mathrm{AdR}}(X/\overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \cong \mathrm{H}^i_{\mathrm{Bet}}(X_\sigma) \otimes_{\mathbb{Q}} \mathbb{C}.$$

These isomorphisms produce the *period numbers*, which are defined as the complex numbers appearing as the entries of a representative matrix of ϖ^i, with respect to a $\overline{\mathbb{Q}}$-basis and a $\mathbb{Q}$-basis, respectively.

1.0.2 Contents of the chapter

In section 1.1, after recalling briefly the definition of the de Rham and singular cohomologies for differentiable manifolds and the statement of the *de Rham Theorem*, we see how these cohomology theories can be described also as sheaf cohomology of some complexes of sheaves. We also give a proof of the de Rham Theorem in this sheaf-theoretic setting, as a direct consequence of the Poincaré Lemma. Then, we discuss the analogous de Rham Theorem for complex manifolds.

In section 1.2, we introduce the *complex analytic spaces* and the *analytification functor*, together with some properties. Then, we talk about sheaves of modules over algebraic varieties over $\mathbb{C}$ and over complex analytic spaces, the analytification of sheaves of modules and their relations, until the statement of *GAGA Theorems*.

In section 1.3, after recalling briefly the constructions of the module of Kähler differentials and the algebraic de Rham complex for morphisms of commutative rings with unit, we describe their sheaf-theoretical analogue using the language of sites. That is, we define the *sheaf of modules of Kähler differentials* and the *algebraic de Rham complex* for morphisms of sheaves of commutative rings with unit over a site. We also prove two general properties: change of sites and functoriality. Then, we describe this construction applied to the data of a morphism of schemes and we discuss some properties.

In section 1.4, we give a definition of *smooth morphisms* of schemes. We choose a suitable definition in order to easily prove that an algebraic variety over $\mathbb{C}$ is smooth if and only if its analytification is a smooth complex analytic space. Then, we prove an important property of smooth morphisms: its sheaf of modules of Kähler differentials is finite locally free. Our definition of smooth morphisms contains, as a particular case, the definition of *étale morphisms*. They allow to construct a new topology (in the sense of Grothendieck topologies) over schemes: the *étale topology*. With this new topology, we obtain a characterization of smooth morphisms, which is analogous to the definition of smooth complex analytic spaces. Moreover, we see that a morphism between smooth algebraic varieties over $\mathbb{C}$ is étale if and only if its analytification is a local isomorphism of smooth complex analytic spaces. This suggests to consider also a new topology over complex analytic spaces, which is defined by local isomorphisms: the *étale-analytic topology*. In fact, this is nothing new, since it is equivalent to the classical topology given by open covers, but it will be useful later in the second chapter. Finally, we see some further properties of smooth schemes over a field.

In section 1.5, we define the *algebraic de Rham cohomology* of smooth algebraic varieties over a field k. We see that it can be also computed by the Čech cohomology relative to an affine open cover. We use this description to compute some examples. Then, we see some properties of the algebraic de Rham cohomology: functoriality, Künneth formula, $\mathbb{A}^1$-invariance and étale descent. Then, taking $k = \mathbb{C}$, we prove the comparison isomorphism between the algebraic and the analytic de Rham cohomology. We first prove it in the case of a proper smooth algebraic variety and then we prove the general case of any smooth algebraic variety. Although the proper case is not necessary for the proof of the general case, we discuss it anyway, because some facts in the general case are nothing more than the logarithmic version of the ones in the proper case, which are easier to discuss.

In section 1.6 we define the *Betti cohomology* of algebraic varieties over $\mathbb{C}$ and we finally prove the *Algebraic de Rham Theorem*. Then, we define the *period numbers*, which are some arithmetic invariants associated to a smooth algebraic variety over $\overline{\mathbb{Q}}$, naturally arising from the corresponding *algebraic de Rham isomorphism*.

1.1 The de Rham Theorem for manifolds

1.1.1 Differentiable manifolds

In Differential Geometry, given a differentiable manifold M, we can associate to M two kinds of cohomological invariants.

- The *de Rham cohomology* of M

$$\mathrm{H}^i_{\mathrm{dR}}(M/\mathbb{R}) := \mathrm{H}^i(\mathcal{A}^\bullet(M)),$$

which is defined as the cohomology groups of the complex of $\mathbb{R}$-vector spaces $\mathcal{A}^\bullet(M)$, the *de Rham complex* of M (see [Bre93, §V,2]). This invariant is of *analytic nature*, since its definition involves differential forms defined over M.

- The *singular cohomology of M with real coefficients*

$$\mathrm{H}^i_{\mathrm{Sing}}(M;\mathbb{R}) := \mathrm{H}^i(\mathcal{C}^\bullet_{\mathrm{Sing}}(M;\mathbb{R})),$$

which is defined as the cohomology groups of the complex of $\mathbb{R}$-vector spaces $\mathcal{C}^\bullet_{Sing}(M,\mathbb{R})$, the *complex of singular cochains* over M with coefficients in $\mathbb{R}$ (see [Bre93, §V,5]). This invariant is of *topological nature*, since it can be defined for any topological space. Starting from a differentiable manifold, we simply forget the differentiable structure.

It's well-known the *de Rham Theorem*, which states that, in fact, de Rham and singular cohomologies with real coefficients provide the same cohomological invariant for M (see [Bre93, §V, Thm. 9.1])

Theorem 1.1.1 (De Rham Theorem)**.** *Let M be a differentiable manifold. There exist isomorphisms of $\mathbb{R}$-vector spaces, for each $i \geq 0$,*

$$H^i_{dR}(M/\mathbb{R}) \cong H^i_{Sing}(M;\mathbb{R}).$$

There exist several proofs of this theorem. There are classical proofs which show explicit isomorphisms, as the one given in [Bre93]. The strategy consists in considering the pairings given by integration of differential forms along singular cycles over M, for any $p \geq 0$,

$$\begin{aligned}\mathcal{A}^p(M) \times \mathcal{C}^{\mathrm{Sing}}_p(M) &\to \mathbb{R}\\ (\omega,\gamma) &\mapsto \int_\gamma \omega,\end{aligned}$$

where $\mathcal{C}^{\mathrm{Sing}}_\bullet(M)$ is the *complex of singular chains* over M (with coefficients in $\mathbb{Z}$). Then, it's proved that these induce the pairings, for any $i \geq 0$,

$$\begin{aligned}\mathrm{H}^i_{\mathrm{dR}}(M/\mathbb{R}) \times \mathrm{H}^{\mathrm{Sing}}_i(M) &\to \mathbb{R}\\ ([\omega],[\gamma]) &\mapsto \int_\gamma \omega,\end{aligned}$$

where the *singular homology* of M (with integral coefficients)

$$\mathrm{H}_i^{\mathrm{Sing}}(M) := \mathrm{H}_i(\mathcal{C}_\bullet^{\mathrm{Sing}}(M))$$

is defined as the homology groups of $\mathcal{C}_\bullet^{\mathrm{Sing}}(M)$. Finally, it's proved that these pairings are perfect, that is, they induce isomorphisms of $\mathbb{R}$-vector spaces

$$\mathrm{H}_{\mathrm{dR}}^i(M/\mathbb{R}) \cong \mathrm{Hom}_{\mathbb{Z}\text{-mod}}(H_i^{\mathrm{Sing}}(M), \mathbb{R}).$$

This proves Theorem 1.1.1. Indeed, the complex of singular cochains over M is defined as

$$\mathcal{C}^\bullet_{\mathrm{Sing}}(M;\mathbb{R}) := \mathrm{Hom}_{\mathbb{Z}\text{-mod}}(\mathcal{C}_\bullet^{\mathrm{Sing}}(M), \mathbb{R})$$

and, since $\mathbb{R}$ is a field, by the Universal Coefficients Theorem, it holds that, for any $i \geq 0$,

$$\mathrm{H}^i(\mathcal{C}^\bullet_{\mathrm{Sing}}(M;\mathbb{R})) \cong \mathrm{Hom}_{\mathbb{Z}\text{-mod}}(\mathrm{H}_i(\mathcal{C}_\bullet^{\mathrm{Sing}}(M)), \mathbb{R}).$$

Another proof uses Homological Algebra methods. A reference is [GH78, p. 44]. The key point is to adopt a sheaf-theoretic point of view, thinking de Rham and singular cohomologies as cohomology groups of M with coefficients in certain complexes of sheaves of $\mathbb{R}$-vector spaces over M. In this sheaf-theoretic language, the de Rham Theorem is reformulated as the existence of a quasi-isomorphism between these complexes of sheaves. For notations and results about sheaf cohomology look at Appendix A. Although this reformulation gives a less explicit proof of the de Rham Theorem, it will be fundamental to move from the setting of Differential Geometry to the one of Algebraic Geometry.

We start describing the de Rham cohomology as sheaf cohomology.

Definition 1.1.2. Given M a differentiable manifold, the *de Rham complex* of M is the complex of presheaves of $\mathbb{R}$-vector spaces over M (presheaves of $\mathbb{R}$-vector spaces over the category of open subsets of M)

$$\mathcal{A}_M^\bullet \ : \ \mathcal{A}_M^0 \xrightarrow{d} \mathcal{A}_M^1 \xrightarrow{d^1} \mathcal{A}_M^2 \to \cdots ,$$

given open-wise by the de Rham complex of differential forms over the open subset. That is, for any $U \subset M$ open subset,

$$\mathcal{A}_M^\bullet(U) := \mathcal{A}^\bullet(U).$$

Restriction morphisms are given by the usual restriction maps of differential forms.

The de Rham complex is a complex of sheaves of $\mathbb{R}$-vector spaces over M (sheaves of $\mathbb{R}$-vector spaces over the classical site of open subsets of M). Notice that $\mathcal{A}_M^0$ is the sheaf of differentiable $\mathbb{R}$-valued functions over M. If M has dimension n, for any point $x \in M$, let $w_{1,x}, \ldots, w_{n,x} \in \mathcal{A}_{M,x}^0$ be the stalks of some local coordinates at x. The ideal generated by $w_{1,x}, \ldots, w_{n,x}$ is the maximal ideal of $\mathcal{A}_{M,x}^0$. Hence, we can think at M as a locally ringed space with structural sheaf $\mathcal{A}^0_M$. The sheaf $\mathcal{A}^1_M$ is a finite locally free sheaf of $\mathcal{A}^0_M$-modules of rank n, with stalks generated by the stalks of differentials of local coordinates

$$\mathcal{A}^1_{M,x} \cong \oplus_{i=1}^n \mathcal{A}^0_{M,x} dw_{i,x}.$$

Also the sheaves $\mathcal{A}^p_M$ are sheaves of $\mathcal{A}^0_M$-modules, such that

$$\mathcal{A}^p_M \cong \bigwedge^p \mathcal{A}^1_M.$$

Hence, they are also finite locally free sheaves of $\mathcal{A}^0_M$-modules. The differentials

$$d^p : \mathcal{A}^p_M \to \mathcal{A}^{p+1}_M$$

are morphisms of sheaves of $\mathbb{R}$-vector spaces (not of $\mathcal{A}^0_M$-modules!), which on stalks are such that

$$\begin{aligned} d^p(f_x dw_{i_1,x} \wedge \cdots \wedge dw_{i_p,x}) &= df_x \wedge dw_{i_1,x} \wedge \cdots \wedge dw_{i_p,x} = \\ &= \sum_{j=1}^n \left(\frac{\partial f}{\partial w_j}\right)_x dw_{j,x} \wedge dw_{i_1,x} \wedge \cdots \wedge dw_{i_p,x}. \end{aligned}$$

Proposition 1.1.3. *Let M be a differentiable manifold. The cohomology groups of M with coefficients in $\mathcal{A}^\bullet_M$ compute the de Rham cohomology of M*

$$\mathbb{H}^i(M, \mathcal{A}^\bullet_M) \cong H^i_{dR}(M/\mathbb{R}).$$

Proof. Since differentiable manifolds admit partitions of unity, then the structural sheaf $\mathcal{A}^0_M$ is a fine sheaf and hence, also all sheaves of $\mathcal{A}^0_M$-modules are fine. So, each $\mathcal{A}^p_M$ is a fine sheaf, hence acyclic. Then, the hyper-cohomology spectral sequence

$$E_1^{p,q} = \mathbb{H}^q(M, \mathcal{A}^p_M) \Rightarrow \mathbb{H}^{p+q}(M, \mathcal{A}^\bullet_M)$$

has page 1 given by

$$\begin{array}{ccccccc} \cdots & & \cdots & & \cdots & & \\ 0 & \to & 0 & \to & 0 & \to \cdots \\ 0 & \to & 0 & \to & 0 & \to \cdots \\ \mathbb{H}^0(M, \mathcal{A}^0_M) & \to & \mathbb{H}^0(M, \mathcal{A}^1_M) & \to & \mathbb{H}^0(M, \mathcal{A}^2_M) & \to \cdots . \end{array}$$

Hence, the spectral sequence degenerates at page 2 and we get that, for any $i \geq 0$,

$$\mathbb{H}^i(M, \mathcal{A}^\bullet_M) \cong \mathrm{H}^i(\mathbb{H}^0(M, \mathcal{A}^\bullet_M)) = \mathrm{H}^i(\Gamma(M, \mathcal{A}^\bullet_M)).$$

By definition, the global sections at M of the de Rham complex is the complex

$\mathcal{A}^\bullet(M)$, which computes de Rham cohomology. We conclude that

$$\mathbb{H}^i(M, \mathcal{A}^\bullet_M) \cong \mathrm{H}^i_{\mathrm{dR}}(M/\mathbb{R}).$$

$\square$

Now we pass to describe singular cohomology as sheaf cohomology. Given an abelian group Λ, we denote by

$$\Lambda_M$$

the constant abelian sheaf over M given by Λ, that is the sheafification of the constant abelian presheaf over M with values Λ.

Proposition 1.1.4. *Let M be a differentiable manifold. The cohomology groups of M with coefficients in Λ_M compute the singular cohomology of M with coefficients in Λ*

$$\mathbb{H}^i(M, \Lambda_M) \cong H^i_{Sing}(M; \Lambda).$$

Proof. A reference is [BT82, chap. III, §15]. This result actually holds for any topological space which admits a good cover. Recall that, given $\mathcal{U} = \{U_i\}_{i \in I}$ an open cover of a topological space, we say that $\mathcal{U}$ is a good cover if all finite intersections

$$U_{i_0 \dots i_p} \coloneqq U_{i_0} \cap \dots \cap U_{i_p}$$

are contractible. Differentiable manifolds always admit a good cover. So, let $\mathcal{U}$ be a good cover of M. Consider the complex of $\mathcal{U}$-small singular chains (with coefficients in $\mathbb{Z}$) of M

$$\mathcal{C}^{\mathcal{U}}_\bullet(M) \ : \ \mathcal{C}^{\mathcal{U}}_0(M) \to \mathcal{C}^{\mathcal{U}}_1(M) \to \mathcal{C}^{\mathcal{U}}_2(M) \to \cdots .$$

The Mayer-Vietoris principle (see [BT82, prop. 15.2]) states that

$$0 \leftarrow \mathcal{C}^{\mathcal{U}}_\bullet(M) \leftarrow \bigoplus_{i_0} \mathcal{C}^{\mathrm{Sing}}_\bullet(U_{i_0}) \leftarrow \bigoplus_{i_0, i_1} \mathcal{C}^{\mathrm{Sing}}_\bullet(U_{i_0 i_1}) \leftarrow \bigoplus_{i_0, i_1, i_2} \mathcal{C}^{\mathrm{Sing}}_\bullet(U_{i_0 i_1 i_2}) \leftarrow \cdots$$

is an exact sequence of complexes of abelian groups. Since each $\mathcal{C}^{\mathcal{U}}_p(M)$ and $\mathcal{C}^{\mathrm{Sing}}_p(U_{i_0 \dots i_p})$ is a free abelian group, then, applying the functor $\mathrm{Hom}_{\mathbb{Z}\text{-mod}}(\ \_\ , \Lambda)$ to the above exact sequence, we get the exact sequence of complexes of abelian groups

$$0 \to \mathcal{C}^\bullet_{\mathcal{U}}(M; \Lambda) \to \prod_{i_0} \mathcal{C}^\bullet_{\mathrm{Sing}}(U_{i_0}; \Lambda) \to \prod_{i_0, i_1} \mathcal{C}^\bullet_{\mathrm{Sing}}(U_{i_0 i_1}; \Lambda) \to \prod_{i_0, i_1, i_2} \mathcal{C}^\bullet_{\mathrm{Sing}}(U_{i_0 i_1 i_2}; \Lambda) \to \cdots . \tag{1.1}$$

Consider the first quadrant double complex

$$\begin{array}{ccccccc}
\cdots & & \cdots & & \cdots & & \\
\uparrow & & \uparrow & & \uparrow & & \\
\prod_{i_0} \mathcal{C}^2_{\mathrm{Sing}}(U_{i_0};\Lambda) & \to & \prod_{i_0,i_1} \mathcal{C}^2_{\mathrm{Sing}}(U_{i_0 i_1};\Lambda) & \to & \prod_{i_0,i_1,i_2} \mathcal{C}^2_{\mathrm{Sing}}(U_{i_0 i_1 i_2};\Lambda) & \to \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
\prod_{i_0} \mathcal{C}^1_{\mathrm{Sing}}(U_{i_0};\Lambda) & \to & \prod_{i_0,i_1} \mathcal{C}^1_{\mathrm{Sing}}(U_{i_0 i_1};\Lambda) & \to & \prod_{i_0,i_1,i_2} \mathcal{C}^1_{\mathrm{Sing}}(U_{i_0 i_1 i_2};\Lambda) & \to \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
\prod_{i_0} \mathcal{C}^0_{\mathrm{Sing}}(U_{i_0};\Lambda) & \to & \prod_{i_0,i_1} \mathcal{C}^0_{\mathrm{Sing}}(U_{i_0 i_1};\Lambda) & \to & \prod_{i_0,i_1,i_2} \mathcal{C}^0_{\mathrm{Sing}}(U_{i_0 i_1 i_2};\Lambda) & \to \cdots .
\end{array}$$

We consider the two spectral sequences associated to it. The first one ${}^{I}E_2^{p,q}$ is obtained by computing first, vertical cohomology and then, the horizontal. Since in Λ-mod cohomology commutes with products, page 1 is given by

$$\begin{array}{ccccccc}
\cdots & & \cdots & & \cdots & & \\
\prod_{i_0} \mathrm{H}^2_{\mathrm{Sing}}(U_{i_0};\Lambda) & \to & \prod_{i_0,i_1} \mathrm{H}^2_{\mathrm{Sing}}(U_{i_0 i_1};\Lambda) & \to & \prod_{i_0,i_1,i_2} \mathrm{H}^2_{\mathrm{Sing}}(U_{i_0 i_1 i_2};\Lambda) & \to \cdots \\
\prod_{i_0} \mathrm{H}^1_{\mathrm{Sing}}(U_{i_0};\Lambda) & \to & \prod_{i_0,i_1} \mathrm{H}^1_{\mathrm{Sing}}(U_{i_0 i_1};\Lambda) & \to & \prod_{i_0,i_1,i_2} \mathrm{H}^1_{\mathrm{Sing}}(U_{i_0 i_1 i_2};\Lambda) & \to \cdots \\
\prod_{i_0} \mathrm{H}^0_{\mathrm{Sing}}(U_{i_0};\Lambda) & \to & \prod_{i_0,i_1} \mathrm{H}^0_{\mathrm{Sing}}(U_{i_0 i_1};\Lambda) & \to & \prod_{i_0,i_1,i_2} \mathrm{H}^0_{\mathrm{Sing}}(U_{i_0 i_1 i_2};\Lambda) & \to \cdots .
\end{array}$$

Since $\mathcal{U}$ is a good cover, by homotopy invariance of singular cohomology, we deduce that

$$\mathrm{H}^q_{\mathrm{Sing}}(U_{i_0\dots i_p};\Lambda) = \begin{cases} \Lambda & \text{if q=0} \\ 0 & \text{else.} \end{cases}$$

Moreover notice that, for any $p \geq 0$

$$\prod_{i_0,\dots,i_p} \mathrm{H}^0_{\mathrm{Sing}}(U_{i_0\dots i_p};\Lambda) \cong \prod_{i_0,\dots,i_p} \Lambda \cong \check{C}^p(\mathcal{U};\Lambda_M).$$

That is, page 1 of the spectral sequence is concentrated in the 0^{th}-row, given by the Čech complex of the constant sheaf Λ_X relative to the cover $\mathcal{U}$. Hence, the spectral sequence degenerates at page 2 and converges to the Čech cohomology of Λ_X relative to $\mathcal{U}$

$${}^{I}E_2^{p,q} \Rightarrow \check{\mathrm{H}}^{p+q}(\mathcal{U};\Lambda_M).$$

The second spectral sequence ${}^{II}E_2^{p,q}$ is obtained by computing first, horizontal cohomology and then, the vertical. By exactness of 1.1, page 1 is given by

$$\begin{array}{ccccccc}
\cdots & & \cdots & & \cdots & & \\
0 & \to & 0 & \to & 0 & \to \cdots \\
0 & \to & 0 & \to & 0 & \to \cdots \\
\mathcal{C}^0_{\mathcal{U}}(M;\Lambda) & \to & \mathcal{C}^1_{\mathcal{U}}(M;\Lambda) & \to & \mathcal{C}^2_{\mathcal{U}}(M;\Lambda) & \to \cdots .
\end{array}$$

Hence ${}^{II}E_2^{p,q}$ degenerates at page 2 and converges to the $\mathcal{U}$-small singular cohomology of M with coefficients in Λ

$$^{II}E_2^{p,q} \Rightarrow \mathrm{H}_{\mathcal{U}}^{p+q}(M;\Lambda).$$

Since both spectral sequences $^{I}E_2^{p,q}$ and $^{II}E_2^{p,q}$ converge to the cohomology of the total complex of the double complex, then we deduce that the limits are isomorphic

$$\check{\mathrm{H}}^i(\mathcal{U};\Lambda_M) \cong \mathrm{H}_{\mathcal{U}}^i(M;\Lambda).$$

Since the complex of singular chains over M is homotopical equivalent to the the complex of $\mathcal{U}$-small singular chains over M and the functor $\mathrm{Hom}_{\mathbb{Z}\text{-mod}}(\_,\Lambda)$ preserves homotopical equivalence of complexes, then the complex of singular cochains over M is homotopical equivalent to the the complex of $\mathcal{U}$-small singular cochains over M. Hence, the $\mathcal{U}$-small singular cohomology is isomorphic to the singular cohomology with coefficients in Λ

$$\mathrm{H}_{\mathcal{U}}^i(M;\Lambda) \cong \mathrm{H}_{\mathrm{Sing}}^i(M;\Lambda).$$

Then, since good covers are cofinal in the category of covers of M, taking direct limit over all good covers of M, we deduce that the Čech cohomology of M with coefficients in Λ_M is isomorphic to the singular cohomology of M with coefficients in Λ

$$\mathrm{H}_{\mathrm{Sing}}^i(M;\Lambda) \cong \varinjlim_{\mathcal{U}\text{ good cover}} \check{\mathrm{H}}^i(\mathcal{U};\Lambda_M) \cong \check{\mathrm{H}}^i(M;\Lambda_M).$$

Since M is paracompact, then Čech cohomology always coincides with sheaf cohomology. Hence, we conclude

$$\mathrm{H}_{\mathrm{Sing}}^i(M;\Lambda) \cong \check{\mathrm{H}}^i(M;\Lambda_M) \cong \mathbb{H}^i(M,\Lambda_M).$$

$\square$

Now, we finally prove the de Rham Theorem. Recall that, from Real Analysis, it's well-known the Poincaré Lemma.

Theorem 1.1.5 (Poincaré Lemma)**.** *For any integer $n \geq 0$, over any open disk of $\mathbb{R}^n$, any closed differential form is also exact.*

Since M at each point has a basis of open neighborhoods diffeomorphic to open disks of $\mathbb{R}^n$, then the Poincaré Lemma is equivalent to say that the de Rham complex $\mathcal{A}_M^\bullet$ is exact on stalks, i.e. exact as a complex of sheaves. Moreover, the kernel of

$$d : \mathcal{A}_M^0 \to \mathcal{A}_M^1$$

is the sheaf of differentiable $\mathbb{R}$-valued functions with differential zero, that is, the constant sheaf $\mathbb{R}_M$. Hence, we have a quasi-isomorphism of complexes of sheaves of $\mathbb{R}$-vector spaces over M

$$\mathbb{R}_M \xrightarrow{\sim} \mathcal{A}_M^\bullet.$$

Since quasi-isomorphisms induce isomorphisms on sheaf cohomology, then, for each $i \geq 0$,

$$\mathrm{H}^i_{\mathrm{Sing}}(M;\mathbb{R}) \cong \mathbb{H}^i(M,\mathbb{R}_M) \cong \mathbb{H}^i(M,\mathcal{A}^\bullet_M) = H^i_{\mathrm{dR}}(M/\mathbb{R}),$$

which is the de Rham Theorem 1.1.1.

1.1.2 Complex manifolds

In Complex Geometry, we can formulate analogous constructions and results, considering complex manifolds instead of differentiable manifolds. We describe what happens in this context because we need it to connect to Algebraic Geometry.

Let M be a complex manifold of dimension n. In particular M is a differentiable manifold of dimension $2n$, so we can consider its de Rham complex $\mathcal{A}^\bullet_M$. Consider $(\mathcal{A}^\bullet_M)_{\mathbb{C}}$ the open-wise complexification of $\mathcal{A}^\bullet_M$. That is, for any $U \subset M$ open subset,

$$(\mathcal{A}^\bullet_M)_{\mathbb{C}}(U) \coloneqq \mathcal{A}^\bullet_M(U) \otimes_{\mathbb{R}} \mathbb{C}.$$

It is a complex of sheaves of $\mathbb{C}$-vector spaces over M (sheaves of $\mathbb{C}$-vector spaces over the classical site of open subsets of M). For any point $x \in M$, let $w_{1,x}, \dots, w_{n,x} \in \mathcal{A}^0_{M,x} \otimes_{\mathbb{R}} \mathbb{C}$ be the stalks of some complex local coordinates at x. We have a canonical decomposition of each $(\mathcal{A}^r_M)_{\mathbb{C}}$

$$(\mathcal{A}^r_M)_{\mathbb{C}} \cong \oplus_{p+q=r} \mathcal{A}^{p,q}_M,$$

such that on stalks

$$\mathcal{A}^{p,q}_{M,x} \cong \oplus_{I,J} (\mathcal{A}^0_{M,x} \otimes_{\mathbb{R}} \mathbb{C}) dw_{I,x} \wedge d\bar{w}_{J,x},$$

where $I = \{1 \leq i_1 < \cdots < i_p \leq n\}$, $J = \{1 \leq j_1 < \cdots < j_q \leq n\}$ and $dw_I \coloneqq dw_{i_1} \wedge \cdots \wedge dw_{i_p}$, $d\bar{w}_J \coloneqq d\bar{w}_{j_1} \wedge \cdots \wedge d\bar{w}_{j_q}$. We also have a canonical decomposition of the differentials

$$d^r_{\mathbb{C}} = \oplus_{p+q=r} d^{p,q} + \bar{d}^{p,q},$$

such that on stalks

$$\begin{aligned} d^r_{\mathbb{C}}(f_x dw_{I,x} \wedge d\bar{w}_{J,x}) &= \sum_{i=1}^n \left(\frac{\partial f}{\partial w_i}\right)_x dw_{i,x} \wedge dw_{I,x} \wedge d\bar{w}_{J,x} + \sum_{j=1}^n \left(\frac{\partial f}{\partial \bar{w}_j}\right)_x d\bar{w}_{j,x} \wedge dw_{I,x} \wedge d\bar{w}_{J,x} \\ &\eqqcolon d^{p,q}(f_x dw_{I,x} \wedge d\bar{w}_{J,x}) + \bar{d}^{p,q}(f_x dw_{I,x} \wedge d\bar{w}_{J,x}). \end{aligned}$$

In other words, we have a first quadrant double complex

$$\begin{tikzcd}
\cdots & \cdots & \cdots & \\
\mathcal{A}_M^{0,2} \arrow[r, "d^{0,2}"] \arrow[u] & \mathcal{A}_M^{1,2} \arrow[r, "d^{1,2}"] \arrow[u] & \mathcal{A}_M^{2,2} \arrow[r] \arrow[u] & \cdots \\
\mathcal{A}_M^{0,1} \arrow[r, "d^{0,1}"] \arrow[u, "\bar{d}^{0,1}"] & \mathcal{A}_M^{1,1} \arrow[r, "d^{1,1}"] \arrow[u, "-\bar{d}^{1,1}"] & \mathcal{A}_M^{2,1} \arrow[r] \arrow[u, "\bar{d}^{2,1}"] & \cdots \\
\mathcal{A}_M^{0,0} \arrow[r, "d^{0,0}"] \arrow[u, "\bar{d}^{0,0}"] & \mathcal{A}_M^{1,0} \arrow[r, "d^{1,0}"] \arrow[u, "-\bar{d}^{1,0}"] & \mathcal{A}_M^{2,0} \arrow[r] \arrow[u, "\bar{d}^{2,0}"] & \cdots ,
\end{tikzcd}$$

whose total complex is $(\mathcal{A}_M^\bullet)_\mathbb{C}$.

Definition 1.1.6. Given M a complex manifolds, the *holomorphic (or analytic) de Rham complex* of M is the complex of sheaves of $\mathbb{C}$-vector spaces over M

$$\Omega_M^\bullet \ : \ \Omega_M^0 \xrightarrow{d} \Omega_M^1 \xrightarrow{d^1} \Omega_M^2 \to \cdots$$

defined as

$$\Omega_M^\bullet \coloneqq ker(\mathcal{A}_M^{\bullet,0} \xrightarrow{\bar{d}^{\bullet,0}} \mathcal{A}_M^{\bullet,1}).$$

Notice that $\Omega_M^0 \cong \mathcal{O}_M$ is the sheaf of holomorphic $\mathbb{C}$-valued functions over M. For any point $x \in M$, let $w_{1,x}, \dots, w_{n,x} \in \mathcal{O}_{M,x}$ be the stalks of some local coordinates at x. The ideal generated by $w_{1,x}, \dots, w_{n,x}$ is the maximal ideal of $\mathcal{O}_{M,x}$. Hence, we can think at M as a locally ringed space with structural sheaf $\mathcal{O}_M$. The sheaf Ω_M^1 is a finite locally free sheaf of $\mathcal{O}_M$-modules of rank n, with stalks generated by the stalks of differentials of local coordinates

$$\Omega_{M,x}^1 \cong \oplus_{i=1}^n \mathcal{O}_{M,x} dw_{i,x}.$$

Also the sheaves Ω_M^p are sheaves of $\mathcal{O}_M$-modules, such that

$$\Omega_M^p \cong \bigwedge^p \Omega_M^1.$$

Hence, they are also finite locally free sheaves of $\mathcal{O}_M$-modules. The differentials

$$d^p : \Omega_M^p \to \Omega_M^{p+1}$$

are morphisms of sheaves of $\mathbb{C}$-vector spaces (not of $\mathcal{O}_M$-modules!), which satisfy the Leibnitz rule: for any ω section of Ω_M^r and η section of Ω_M^{p-r},

$$d^p(\omega \wedge \eta) = d^r\omega \wedge \eta + (-1)^r \omega \wedge d^{p-r}\eta.$$

In analogy with what happens in the real case, we give the following definition.

Definition 1.1.7. Given M a complex manifold, the *holomorphic (or analytic) de Rham cohomology* of M is the cohomology of M with coefficients in the holomorphic de Rham complex

$$\mathrm{H}^i_{\mathrm{dR}}(M/\mathbb{C}) := \mathbb{H}^i(M, \Omega^\bullet_M).$$

Differently from the real case, the sheaves Ω^p_M aren't acyclic in general. Hence, the sheaf cohomology of $\Omega^\bullet_M$ can't be computed by taking the cohomology groups of the complex of its global sections at M, but we need to consider acyclic resolutions of Ω^p_M. The following variant of the Poincaré Lemma (see [GH78, p. 25]) provides such resolutions.

Theorem 1.1.8 ($\bar{d}$-Poincaré Lemma)**.** *For any integer $n \geq 0$, over any open disk of $\mathbb{C}^n$, any $\bar{d}$-closed differential form is also $\bar{d}$-exact.*

Since M at each point has a basis of open neighborhoods biholomorphic to open disks of $\mathbb{C}^n$, then the $\bar{d}$-Poincaré Lemma is equivalent to say that columns of the double complex 1.1.2 are exact on stalks, i.e. exact complexes of sheaves. So we have quasi-isomorphisms, for each $p \geq 0$,

$$\Omega^p_M \xrightarrow{\sim} \mathcal{A}^{p,\bullet}_M$$

and hence, also a quasi-isomorphism

$$\Omega^\bullet_M \xrightarrow{\sim} Tot^{\oplus}(\mathcal{A}^{\bullet,\bullet}_M) \cong (\mathcal{A}^\bullet_M)_{\mathbb{C}}.$$

Notice that, since each $\mathcal{A}^p_M$ is acyclic and $\mathbb{C}$ is flat over $\mathbb{R}$, then also each $(\mathcal{A}^p_M)_{\mathbb{C}}$ is acyclic. Hence, $(\mathcal{A}^\bullet_M)_{\mathbb{C}}$ is an acyclic resolution of $\Omega^\bullet_M$. Then,

$$\begin{gathered}\mathrm{H}^i_{\mathrm{dR}}(M/\mathbb{C}) = \mathbb{H}^i(M, \Omega^\bullet_M) \cong \mathbb{H}^i(M, (\mathcal{A}^\bullet_M)_{\mathbb{C}}) \cong \mathrm{H}^i(\mathcal{A}^\bullet_M(M) \otimes_{\mathbb{R}} \mathbb{C}) \cong \\ \cong \mathrm{H}^i(\mathcal{A}^\bullet(M) \otimes \mathbb{C}) \cong \mathrm{H}^i_{\mathrm{dR}}(M/\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C},\end{gathered}$$

that is, the holomorphic de Rham cohomology of a complex manifold is the complexification of the de Rham cohomology of the underlying differentiable manifold. Moreover, recall that, by Poincaré Lemma, we have a quasi-isomorphism of sheaves of $\mathbb{R}$-modules over M

$$\mathbb{R}_M \xrightarrow{\sim} \mathcal{A}^\bullet_M.$$

Since $\mathbb{C}$ is flat over $\mathbb{R}$, its complexification is still a quasi-isomorphism

$$\mathbb{C}_M \xrightarrow{\sim} (\mathcal{A}^\bullet_M)_{\mathbb{C}}.$$

Since image of $\mathbb{C}_M \to (\mathcal{A}^\bullet_M)_{\mathbb{C}}$ lies inside $\Omega^\bullet_M$, then, we have a commutative diagram of complexes of sheaves over M

$$\begin{tikzcd} \mathbb{C}_M \arrow[r, "\sim"] \arrow[dr] & (\mathcal{A}_M^\bullet)_{\mathbb{C}} \\ & \Omega_M^\bullet \arrow[u, "\wr"] \end{tikzcd}$$

By the 2-out-of-3 property for quasi-isomorphisms, then also

$$c : \mathbb{C}_M \xrightarrow{\sim} \Omega_M^\bullet \tag{1.2}$$

is a quasi-isomorphism. We can think at it as an holomorphic version of Poincaré Lemma. As in the real case, we have an holomorphic version of the de Rham Theorem, which is an immediate consequence of holomorphic Poincaré Lemma.

Theorem 1.1.9 (Holomorphic (or analytic) de Rham Theorem)**.** *Let M be a complex manifold. There exist isomorphisms of $\mathbb{C}$-vector spaces, for each $i \geq 0$,*

$$H^i_{dR}(M/\mathbb{C}) \cong H^i_{Sing}(M; \mathbb{C}).$$

Proof. The quasi-isomorphisms of complexes of abelian sheaves 1.2 induces the isomorphisms on sheaf cohomology, for each $i \geq 0$

$$\mathbb{H}^i(M, \mathbb{C}_M) \cong \mathbb{H}^i(M, \Omega_M^\bullet)$$

By the sheaf-theoretic interpretation of singular cohomology of M (Prop. 1.1.4) and by definition of holomorphic de Rham cohomology, it follows that

$$\mathrm{H}^i_{\mathrm{Sing}}(M; \mathbb{C}) \cong \mathbb{H}^i(M, \mathbb{C}_M) \cong \mathbb{H}^i(M, \Omega_M^\bullet) = \mathrm{H}^i_{\mathrm{dR}}(M/\mathbb{C}).$$

□

Remark 1.1.10. As in the real case, an explicit isomorphism is the one induced by the perfect pairing given by integration of holomorphic forms along singular cycles over M

$$\begin{aligned} H^i_{dR}(M/\mathbb{C}) \times H_i^{Sing}(M) &\to \mathbb{C} \\ ([\omega], [\gamma]) &\mapsto \int_\gamma \omega. \end{aligned}$$

From now on, since we will deal only with complex manifolds, we will denote the analytic de Rham cohomology of a complex manifold M simply by

$$\mathrm{H}^i_{\mathrm{dR}}(M),$$

instead of $\mathrm{H}^i_{\mathrm{dR}}(M/\mathbb{C})$.

1.2 Analytic spaces and GAGA Theorems

1.2.1 Analytic spaces

We want to introduce a kind of locally ringed spaces, which can be thought as complex manifolds with eventually some singular points. These spaces allow to connect to the algebraic varieties over $\mathbb{C}$ in Algebraic Geometry. A reference is [GR84].

We consider $\mathbb{C}^n$ with the euclidean topology and we denote by $\mathcal{O}_{\mathbb{C}^n}$ the sheaf of holomorphic $\mathbb{C}$-valued functions over $\mathbb{C}^n$. It is a sheaf of $\mathbb{C}$-algebras. Stalks of $\mathcal{O}_{\mathbb{C}^n}$ are all isomorphic to $\mathbb{C}\{t_1, \dots, t_n\}$, the ring of power series convergent on some disk, which is a subring of $\mathbb{C}[\![t_1, \dots, t_n]\!]$, the ring of formal power series. It is a local ring because $\mathbb{C}[\![t_1, \dots, t_n]\!]$ is, with maximal ideal generated by $t_1, \dots, t_n$, and an invertible formal power series which is convergent has inverse which is also convergent. Also the maximal ideal of $\mathbb{C}\{t_1, \dots, t_n\}$ is generated by $t_1, \dots, t_n$ and the residue field is isomorphic to $\mathbb{C}$. Hence, we can think at $\mathbb{C}^n$ as a locally ringed space with structural sheaf $\mathcal{O}_{\mathbb{C}^n}$. For any open subset $U \subset \mathbb{C}^n$, we can think at U as a locally ringed space with structural sheaf given by $\mathcal{O}_U \coloneqq \mathcal{O}_{\mathbb{C}^n}\big|_U$.

Definition 1.2.1. A *local model* is a ringed space $(Z, \mathcal{O}_Z)$, such that Z is the zero locus of a finite set of holomorphic functions $S \subset \mathcal{O}_{\mathbb{C}^n}(U)$ on an open subset $U \subset \mathbb{C}^n$

$$Z = Z(S) \coloneqq \{x \in U \mid f(x) = 0 \; \forall f \in S\},$$

with the subspace topology and structural sheaf

$$\mathcal{O}_Z \coloneqq i^{-1}(\mathcal{O}_U / \langle S \rangle),$$

where $i : Z \hookrightarrow U$ denotes the inclusion and $\langle S \rangle \subset \mathcal{O}_U$ is the subsheaf of ideals generated by elements of S.

A *complex analytic space* is a ringed space $(\mathcal{Y}, \mathcal{O}_\mathcal{Y})$, with $\mathcal{Y}$ Hausdorff, which is locally isomorphic to a local model, i.e. such that there exists an open cover $\{\mathcal{W}_i\}_{i \in I}$ of $\mathcal{Y}$, such that, for any $i \in I$,

$$(\mathcal{W}_i, \mathcal{O}_\mathcal{Y}\big|_{\mathcal{W}_i}) \cong (Z, \mathcal{O}_Z)$$

as ringed spaces, for some $(Z, \mathcal{O}_Z)$ local model. The ringed spaces $\mathcal{W}_i$ are called *local charts* of $\mathcal{Y}$.

Example 1.2.2. $\mathbb{C}^n$ and its open subsets are Hausdorff, hence they are complex analytic spaces. In particular $\mathbb{C}^0$ is the complex analytic spaces given by the point with structural sheaf the constant sheaf $\mathbb{C}$. Notice that it is isomorphic to $Spec(\mathbb{C})$ as a ringed space. Since local models have subspace topology of $\mathbb{C}^n$, then they are Hausdorff, hence they are complex analytic spaces. Open subsets of complex analytic spaces are complex analytic spaces.

Remark 1.2.3. Complex analytic spaces are locally ringed spaces. Indeed, it suffices to check that local models are. Let $(Z, \mathcal{O}_Z)$ be a local model. Then, for any $x \in Z$,

$$\mathcal{O}_{Z,x} = i^{-1}(\mathcal{O}_U/\langle S\rangle)_x \cong \mathcal{O}_{\mathbb{C}^n,x}/\langle S\rangle_x$$

is a quotient of a local ring, hence it is local.

Notice that, since $\mathcal{O}_{\mathbb{C}^n}$ is a sheaf of $\mathbb{C}$-algebras, then also the structural sheaf of any complex analytic space is a sheaf of $\mathbb{C}$-algebras. Hence, any complex analytic space is endowed with a unique morphism of ringed spaces into $\mathbb{C}^0$.

Definition 1.2.4. Given $\mathcal{Y}$ and $\mathcal{W}$ complex analytic spaces, a *morphism of complex analytic spaces* $f : \mathcal{Y} \to \mathcal{W}$ is a morphism of locally ringed spaces over $\mathbb{C}^0$. In other words, f is a morphism of locally ringed spaces, such that the morphism on structural sheaves $f^\# : f^{-1}\mathcal{O}_\mathcal{W} \to \mathcal{O}_\mathcal{Y}$ is a morphism of sheaves of $\mathbb{C}$-algebras.

Complex analytic spaces with their morphisms form the *category of complex analytic spaces*, denoted by $An_\mathbb{C}$.

Remark 1.2.5. Let $(Z, \mathcal{O}_Z)$ be a local model. The inclusion

$$i : Z \hookrightarrow U$$

with the morphism of sheaves obtained by applying i^{-1} to the canonical projection on the quotient

$$i^\# : i^{-1}\mathcal{O}_U \twoheadrightarrow i^{-1}(\mathcal{O}_U/\langle S\rangle)$$

defines a closed immersion. Moreover, for any $x \in Z$, the surjective morphism induced on the residue fields at x and $i(x)$ is a morphism of $\mathbb{C}$-algebras

$$\mathbb{C} \twoheadrightarrow k(x).$$

Hence $k(x) \cong \mathbb{C}$. That is, the residue field of any complex analytic spaces at any point is isomorphic to $\mathbb{C}$.

Remark 1.2.6. The translations in $\mathbb{C}^n$

$$\begin{aligned} \tau : \mathbb{C}^n &\to \mathbb{C}^n \\ (y_1, \dots, y_n) &\mapsto (y_1 + a_1, \dots, y_n + a_n), \end{aligned}$$

for some $a_1, \dots, a_n \in \mathbb{C}$, with morphism on structural sheaves given by composition with τ, are isomorphisms of complex analytic spaces, with inverse the opposite translation. Notice that, up to applying a translation, we can assume that, given any complex analytic space $\mathcal{Y}$ and a point $y \in \mathcal{Y}$, a local chart $\mathcal{W} \cong Z$ such that $y \in \mathcal{W}$, maps y into $0 \in Z \subset \mathbb{C}^n$.

Remark 1.2.7. Let $\mathcal{Y}$ be a complex analytic space. Then, we have a canonical map

$$\begin{aligned} \mathrm{Hom}_{An_{\mathbb{C}}}(\mathcal{Y}, \mathbb{C}^n) &\to \mathcal{O}_{\mathcal{Y}}(\mathcal{Y})^n \\ f &\mapsto (s_1, \dots, s_n), \end{aligned}$$

such that, for each $i = 1, \dots, n$,

$$\begin{aligned} f^{\#}_{\mathbb{C}} : \mathcal{O}_{\mathbb{C}^n}(\mathbb{C}) &\to \mathcal{O}_{\mathcal{Y}}(\mathcal{Y}) \\ p_i &\mapsto s_i, \end{aligned}$$

where $p_i : \mathbb{C}^n \to \mathbb{C}$ is the projection to the i^{th}-component. In fact, this defines a bijection (see [GR84, §1.3.1]).

Definition 1.2.8. A complex analytic space $\mathcal{Y}$ is *reduced at a point* $y \in \mathcal{Y}$ if the stalk $\mathcal{O}_{\mathcal{Y},y}$ is a reduced ring. We say that $\mathcal{Y}$ is *reduced* if it is reduced at all its points.

Definition 1.2.9. A complex analytic space $\mathcal{Y}$ is *smooth of dimension* n *at a point* $y \in \mathcal{Y}$, if there exists a local chart containing y, which is isomorphic to an open subset of $\mathbb{C}^n$. We say that $\mathcal{Y}$ is *smooth*, if it is smooth at all its points. We denote by $AnSm_{\mathbb{C}}$ the full subcategory of $An_{\mathbb{C}}$, whose objects are smooth complex analytic spaces.

Remark 1.2.10. It can be checked that the classical definition of a complex manifold via holomorphic atlas is equivalent to the one of smooth complex analytic space. Moreover, morphisms of smooth complex analytic spaces are exactly morphisms of complex manifolds. In this sense, we can think at complex analytic spaces as complex manifolds with eventually some singular points.

There's a useful criterion for smoothness.

Theorem 1.2.11 (Jacobi criterion)**.** *Let $\mathcal{Y}$ be a complex analytic space, $y \in \mathcal{Y}$ a point, $\mathcal{W} \subset \mathcal{Y}$ a local chart containing y, with $\mathcal{W} \cong Z(f_1, \dots, f_{n-r})$ for some $f_1, \dots, f_{n-r} \in \mathcal{O}_{\mathbb{C}^n}(U)$ and $U \subset \mathbb{C}^n$ open subset. Assume that the local chart maps $y \in \mathcal{W}$ into $0 \in Z \subset \mathbb{C}^n$. Then, $\mathcal{Y}$ is smooth of dimension r at y if and only if the Jacobian matrix evaluated at* 0

$$J_{f_1,\dots,f_{n-r}}(0) := \left[\frac{\partial f_i}{\partial t_j}(0)\right]_{\substack{i=1,\dots,n-r \\ j=1,\dots,n}}$$

has maximal rank $n - r$.

Proof. If $\mathcal{Y}$ is smooth of dimension r at y, then we can assume that the local chart $\mathcal{W} \subset \mathcal{Y}$ containing y is sufficiently small such that there exists an isomorphism

$$\varphi : \mathcal{W} \cong V$$

with $V \subset \mathbb{C}^r$ an open subset. Then, we have an isomorphism

$$\mathcal{O}_{\mathbb{C}^r,\varphi(y)} \cong \mathcal{O}_{\mathcal{Y},y}.$$

Hence, the maximal ideal $\mathfrak{m}_y \subset \mathcal{O}_{\mathcal{Y},y}$ is generated by r elements $g_{1,y}, \ldots, g_{r,y} \in \mathcal{O}_{\mathcal{Y},y}$. We can assume that U is sufficiently small, such that there exist $g'_1, \ldots, g'_r \in \mathcal{O}_{\mathbb{C}^n}(U)$, whose stalks $g'_{1,0}, \ldots, g'_{r,0} \in \mathcal{O}_{\mathbb{C}^n,0}$ are some preimages of $g_{1,y}, \ldots, g_{r,y}$ along the surjective morphism

$$\mathcal{O}_{\mathbb{C}^n,0} \twoheadrightarrow \mathcal{O}_{\mathbb{C}^n,0}/(f_{1,0}, \ldots, f_{n-r,0}) \cong \mathcal{O}_{\mathcal{Y},y}.$$

Then it holds that

$$\mathcal{O}_{\mathbb{C}^n,0}/(f_{1,0}, \ldots, f_{n-r,0}, g'_{1,0}, \ldots, g'_{r,0}) \cong \mathcal{O}_{\mathcal{Y},y}/(g_{1,y}, \ldots, g_{r,y}) \cong \mathbb{C},$$

This means that $f_{1,0}, \ldots, f_{n-r,0}, g'_{1,0}, \ldots, g'_{r,0}$ generate the maximal ideal of $\mathcal{O}_{\mathbb{C}^n,0}$, i.e. they are stalks of holomorphic functions defining a local chart of $\mathbb{C}^n$ on an open neighborhood of 0. Hence, the Jacobian matrix of $f_1, \ldots, f_{n-r}, g'_1, \ldots, g'_r$ evaluated at 0 has maximal rank n. Then, also the Jacobian matrix of $f_1, \ldots, f_{n-r}$ evaluated at 0 has maximal rank $n-r$. Conversely, assume that the last $n-r$ columns of the Jacobian matrix evaluated at 0 are linearly independent. By the Holomorphic Implicit Function Theorem, we can assume $\mathcal{W}$ sufficiently small, such that $\mathcal{W}$ is isomorphic to the graph of an holomorphic function $g : V \to \mathbb{C}^{n-r}$, for some $V \subset \mathbb{C}^r$ open subset. Then, $\mathcal{W} \cong V$ as ringed spaces (see [GR84, §1.3.5]). Hence, $\mathcal{Y}$ is smooth at y. □

Now, we see the relation between algebraic varieties over $\mathbb{C}$ and complex analytic spaces. First, we recall the definition of an algebraic variety.

Definition 1.2.12. An *algebraic variety over a field* k is a separated, locally of finite type k-scheme. Algebraic varieties over k with morphisms of k-schemes, i.e. morphisms of locally ringed spaces over $Spec(k)$, form the *category of algebraic varieties over* k, denoted by Var_k.

Recall that a k-scheme X is locally of finite type if, for any $U \subset X$ affine open subset, $\mathcal{O}_X(U)$ is a finitely generated k-algebra.

Both the categories $Var_{\mathbb{C}}$ and $An_{\mathbb{C}}$ fully embed into $Loc_{\mathbb{C}}$, the category of locally ringed spaces over $Spec(\mathbb{C}) \cong \mathbb{C}^0$

$$\begin{tikzcd} & An_{\mathbb{C}} \arrow[d, hook, "\iota_A"] \\ Var_{\mathbb{C}} \arrow[r, hook, "\iota_V"] & Loc_{\mathbb{C}}. \end{tikzcd}$$

There exists a universal construction that transforms an algebraic variety over $\mathbb{C}$ into a complex analytic space.

Proposition 1.2.13 (Analytification)**.** *The full embedding $\iota_V : Var_{\mathbb{C}} \hookrightarrow Loc_{\mathbb{C}}$ admits an absolute right kan lift along the full embedding $\iota_A : An_{\mathbb{C}} \hookrightarrow Loc_{\mathbb{C}}$*

$$\begin{tikzcd} & An_{\mathbb{C}} \arrow[d, hook, "\iota_A"] \\ Var_{\mathbb{C}} \arrow[ur, "an"] \arrow[r, hook, "\iota_V"'] \arrow[ur, phantom, shift left=3, "\Downarrow \alpha"] & Loc_{\mathbb{C}}. \end{tikzcd}$$

Explicitly, there exists a functor

$$\begin{aligned} an : Var_{\mathbb{C}} &\to An_{\mathbb{C}} \\ X &\mapsto an(X) \end{aligned}$$

with a natural transformation

$$\alpha : \iota_A \circ an \Rightarrow \iota_V,$$

satisfying the following universal property: given $X \in Var_{\mathbb{C}}$ and $\mathcal{Y} \in An_{\mathbb{C}}$, for any morphism $f : \mathcal{Y} \to X$ in $Loc_{\mathbb{C}}$, there exists a unique morphism $\mathcal{Y} \to an(X)$ in $An_{\mathbb{C}}$, such that

$$\begin{tikzcd} \mathcal{Y} \arrow[rr, "f"] \arrow[dr] & & X \\ & an(X) \arrow[ur, "\alpha_X"] & \end{tikzcd}$$

is a commutative diagram in $Loc_{\mathbb{C}}$.

Proof. Notice that, equivalently, the universal property in the statement tells that for any $X \in Var_{\mathbb{C}}$ and $\mathcal{Y} \in An_{\mathbb{C}}$ we have a bijection

$$\mathrm{Hom}_{An_{\mathbb{C}}}(\mathcal{Y}, an(X)) \cong \mathrm{Hom}_{Loc_{\mathbb{C}}}(\mathcal{Y}, X),$$

natural in X and $\mathcal{Y}$, which is given by composition with a morphism of locally ringed spaces

$$\alpha_X : an(X) \to X.$$

In other words, for any $X \in Var/\mathbb{C}$, the functor

$$\mathrm{Hom}_{Loc_{\mathbb{C}}}(\_\,, X) : An_{\mathbb{C}}^{op} \to \mathrm{Set}$$

is represented by $an(X)$ together with a canonical morphism $\alpha_X : an(X) \to X$.

We start considering the case $X \cong \mathbb{A}^n_{\mathbb{C}}$ is the affine space of dimension n. We have canonical bijections, for any $\mathcal{Y} \in An_{\mathbb{C}}$,

$$\mathrm{Hom}_{Loc_{\mathbb{C}}}(\mathcal{Y}, \mathbb{A}^n_{\mathbb{C}}) \cong \mathrm{Hom}_{\mathrm{Alg}_{\mathbb{C}}}(\mathbb{C}[t_1, \dots, t_n], \mathcal{O}_{\mathcal{Y}}(\mathcal{Y})) \cong \mathcal{O}_{\mathcal{Y}}(\mathcal{Y})^n \cong \mathrm{Hom}_{An_{\mathbb{C}}}(\mathcal{Y}, \mathbb{C}^n),$$

where the first bijection holds because affine spaces are affine schemes, and the last is remark 1.2.7. So, $an(\mathbb{A}^n_{\mathbb{C}}) \cong \mathbb{C}^n$. Following the bijections, we see that the canonical morphism of locally ringed spaces

$$\alpha_{\mathbb{A}^n_{\mathbb{C}}} : \mathbb{C}^n \to \mathbb{A}^n_{\mathbb{C}}$$

is the one corresponding to the homomorphism of $\mathbb{C}$-algebras

$$\mathbb{C}[t_1, \ldots, t_n] \to \mathcal{O}_{\mathbb{C}^n}(\mathbb{C}^n),$$

which assigns to any polynomial the holomorphic function defined by that polynomial.

Now, consider the case X is an affine algebraic variety. This means that $X \cong Spec(\mathcal{O}_X(X))$ and $\mathcal{O}_X(X)$ is a finitely generated $\mathbb{C}$-algebra

$$\mathcal{O}_X(X) \cong \mathbb{C}[t_1, \ldots, t_n]/I,$$

for some ideal $I \subset \mathbb{C}[t_1, \ldots, t_n]$. By Hilbert's basis theorem, I is a finitely generated ideal. Let $I = (f_1, \ldots, f_m)$ for some $f_1, \ldots, f_m \in \mathbb{C}[t_1, \ldots, t_n]$. So, $X = V(f_1, \ldots, f_m)$ is the closed subscheme of $\mathbb{A}^n_{\mathbb{C}}$ corresponding to the sheaf of ideals

$$\mathcal{I} \coloneqq \langle f_1, \ldots, f_m \rangle \subset \mathcal{O}_{\mathbb{A}^n_{\mathbb{C}}}.$$

Consider the fiber product in $Loc_{\mathbb{C}}$ between X and $\mathbb{C}^n$ over $\mathbb{A}^n_{\mathbb{C}}$

$$\begin{tikzcd}
Z \coloneqq X \times_{\mathbb{A}^n_{\mathbb{C}}} \mathbb{C}^n \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathbb{C}^n \arrow[d, "\alpha_{\mathbb{A}^n_{\mathbb{C}}}"] \\
X \arrow[r, hook] & \mathbb{A}^n_{\mathbb{C}}
\end{tikzcd}$$

Then, $Z \hookrightarrow \mathbb{C}^n$ is the closed immersion corresponding to the sheaf of ideals

$$\alpha^*_{\mathbb{A}^n_{\mathbb{C}}} \mathcal{I} = \langle f_1, \ldots, f_m \rangle \subset \mathcal{O}_{\mathbb{C}^n},$$

where here $f_1, \ldots, f_m \in \mathcal{O}_{\mathbb{C}^n}(\mathbb{C}^n)$ are seen as the holomorphic functions defined by the polynomials. That is, Z is the local model $Z = Z(f_1, \ldots, f_m) \subset \mathbb{C}^n$. We have the canonical bijections, for any $\mathcal{Y} \in An_{\mathbb{C}}$,

$$\begin{aligned}
\mathrm{Hom}_{An_{\mathbb{C}}}(\mathcal{Y}, Z) &\cong \mathrm{Hom}_{Loc_{\mathbb{C}}}(\mathcal{Y}, Z) \cong \\
&\cong \mathrm{Hom}_{Loc_{\mathbb{C}}}(\mathcal{Y}, X) \times_{\mathrm{Hom}_{Loc_{\mathbb{C}}}(\mathcal{Y}, \mathbb{A}^n_C)} \mathrm{Hom}_{Loc_{\mathbb{C}}}(\mathcal{Y}, \mathbb{C}^n) \cong \\
&\cong \mathrm{Hom}_{Loc_{\mathbb{C}}}(\mathcal{Y}, X),
\end{aligned}$$

where the second bijection holds by universal property of fiber product in $Loc_{\mathbb{C}}$, and the last follows from the previous case, since

$$\mathrm{Hom}_{Loc_{\mathbb{C}}}(\mathcal{Y}, \mathbb{A}^n_{\mathbb{C}}) \cong \mathrm{Hom}_{Loc_{\mathbb{C}}}(\mathcal{Y}, \mathbb{C}^n).$$

So, $Z \cong an(X)$ and the morphism $\alpha_X : an(X) \to X$ is the canonical morphism of fiber product.

Finally, consider X a generic algebraic variety over $\mathbb{C}$. Let $\{X_i\}_{i\in I}$ be an affine open cover of X. Each X_i is an affine algebraic variety over $\mathbb{C}$, hence its analytification $an(X_i)$ exists by the previous case. We prove that the $an(X_i)$, for $i \in I$, glue to an analytification of X. First notice that they glue to a locally ringed space. Indeed, notice that, since X is separated, also each intersection $X_{ij} := X_i \cap X_j$ is an affine open subsets of X, hence its analytification $an(X_{ij})$ exists by the previous case. By naturality of analytification, the glueing data $\{X_i, X_{ij}\}_{i,j\in I}$ for X, produces a glueing data of locally ringed spaces $\{an(X_i), an(X_{ij})\}_{i,j\in I}$. Let $\mathcal{X}$ be the glueing locally ringed space. The canonical morphisms of analytification define a morphism between the glueing data of $\mathcal{X}$ and X. They glue to a morphism of locally ringed spaces

$$\alpha : \mathcal{X} \to X.$$

Now, we show that $\mathcal{X}$ is a complex analytic space. Indeed, $\{an(X_i)\}_{i\in I}$ is an open cover of $\mathcal{X}$ and, by the previous case, the $an(X_i)$ are local models. Moreover, since we have the commutative diagram of topological spaces with the diagonal maps

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "\Delta"] \arrow[d, "\alpha"] & \mathcal{X} \times \mathcal{X} \arrow[d, "\alpha\times\alpha"] \\
X \arrow[r, hook, "\Delta"] & X \times X,
\end{tikzcd}$$

and $\Delta\alpha(X) \subset X \times X$ is closed because X is separated, then

$$\Delta(\mathcal{X}) = (\alpha \times \alpha)^{-1}(\Delta\alpha(X))$$

is closed inside $\mathcal{X} \times \mathcal{X}$. Hence, $\mathcal{X}$ is Hausdorff. Finally, we show that $\mathcal{X}$ satisfies the universal property of the analytification of X. Given $\mathcal{Y} \in Loc_{\mathbb{C}}$, for any $f : \mathcal{Y} \to X$ in $Loc_{\mathbb{C}}$, consider $\{\mathcal{Y}_i := f^{-1}(X_i), \mathcal{Y}_{ij} := f^{-1}(X_{ij})\}_{i,j\in I}$, which is a glueing data for $\mathcal{Y}$. For each $i \in I$, by universal property of $an(X_i)$, there exists a unique morphism $\mathcal{Y}_i \to an(X_i)$ such that

$$\begin{tikzcd}
\mathcal{Y}_i \arrow[rr, "f|_{Y_i}"] \arrow[dr] & & X_i \\
& an(X_i) \arrow[ur, "\alpha_{X_i}"] &
\end{tikzcd}$$

commutes in $Loc_{\mathbb{C}}$. By naturality of analytification, these morphisms define a commutative diagram of morphisms between the glueing data of $\mathcal{Y}$, $\mathcal{X}$ and X. Taking colimit, we conclude that there exists a unique morphism $\mathcal{Y} \to \mathcal{X}$ such that

$$\begin{tikzcd} \mathcal{Y} \arrow[rr, "f"] \arrow[dr] & & X \\ & \mathcal{X} \arrow[ur, "\alpha"] & \end{tikzcd}$$

commutes in $Loc_{\mathbb{C}}$. So $\mathcal{X} \cong an(X)$, with canonical morphism $\alpha : \mathcal{X} \to X$. $\square$

Definition 1.2.14. The universal functor

$$an : Var_{\mathbb{C}} \to An_{\mathbb{C}}$$

is called *analytification functor*. Given X an algebraic variety over $\mathbb{C}$, the complex analytic space $an(X)$ is called the *analytification* of X.

Remark 1.2.15. We have an explicit description of what is set-theoretically the analytification of an algebraic variety over $\mathbb{C}$, which however is not so clear form the proof of its existence. Let $X \in Var_{\mathbb{C}}$. On one hand, we have the bijections,

$$\mathrm{Hom}_{Loc_{\mathbb{C}}}(\mathbb{C}^0, X) \cong \mathrm{Hom}_{An_{\mathbb{C}}}(\mathbb{C}^0, an(X)) \cong an(X),$$

where the first is the universal property of analytification and the second holds because the residue field of any complex analytic space is isomorphic to $\mathbb{C}$ at any point, by remark 1.2.5. On the other hand, since $\mathbb{C}^0 \cong Spec(\mathbb{C})$ as locally ringed spaces, then

$$\mathrm{Hom}_{Loc_{\mathbb{C}}}(\mathbb{C}^0, X) \cong \mathrm{Hom}_{Var_{\mathbb{C}}}(Spec(\mathbb{C}), X) = X(\mathbb{C}),$$

where by $X(\mathbb{C})$ we denote the set of $\mathbb{C}$-rational points of X. Hence,

$$an(X) \cong X(\mathbb{C}),$$

that is, set-theoretically $an(X)$ is the set of $\mathbb{C}$-rational points of X. From now on, we will denote by $X(\mathbb{C})$ the analytification of X. Notice that, since $\mathbb{C}$ is algebraically closed, the $\mathbb{C}$-rational points are exactly the closed ones and, since X is locally of finite type over $\mathbb{C}$, they are very dense in X (which means that every locally closed subset contains a point of $X(\mathbb{C})$). Moreover, the canonical map $\alpha_X : X(\mathbb{C}) \hookrightarrow X$ is the inclusion of closed points.

Remark 1.2.16. Let $X \in Var_{\mathbb{C}}$ and $U \subset X$ be an open subset. Then, the analytification of U is isomorphic to the preimage of U along the canonical morphism of analytification

$$U(\mathbb{C}) \cong \alpha_X^{-1}(U).$$

Indeed, by universal properties of analytification and fiber product, we have the bijections, for any $\mathcal{Y} \in An_{\mathbb{C}}$,

$$\begin{aligned}\mathrm{Hom}_{An_{\mathbb{C}}}(\mathcal{Y}, \alpha_X^{-1}(U)) &\cong \mathrm{Hom}_{Loc_{\mathbb{C}}}(\mathcal{Y}, U \times_X X(\mathbb{C})) \cong \\ &\cong \mathrm{Hom}_{Loc_{\mathbb{C}}}(\mathcal{Y}, U) \times_{\mathrm{Hom}_{Loc_{\mathbb{C}}}(\mathcal{Y}, X)} \mathrm{Hom}_{Loc_{\mathbb{C}}}(\mathcal{Y}, X(\mathbb{C})) \cong \\ &\cong \mathrm{Hom}_{Loc_{\mathbb{C}}}(\mathcal{Y}, U) \times_{\mathrm{Hom}_{Loc_{\mathbb{C}}}(\mathcal{Y}, X)} \mathrm{Hom}_{An_{\mathbb{C}}}(\mathcal{Y}, X(\mathbb{C})) \cong \\ &\cong \mathrm{Hom}_{Loc_{\mathbb{C}}}(\mathcal{Y}, U) \cong \mathrm{Hom}_{An_{\mathbb{C}}}(\mathcal{Y}, U(\mathbb{C})).\end{aligned}$$

The claim follows by Yoneda Lemma.

Proposition 1.2.17. *Let X be an algebraic variety over $\mathbb{C}$. The canonical morphism of analytification $\alpha_X : X(\mathbb{C}) \to X$ is such that, for any $x \in X(\mathbb{C})$ closed point of X, the morphism of local rings $\mathcal{O}_{X,x} \to \mathcal{O}_{X(\mathbb{C}),x}$ induces an isomorphism between their adic-completion with respect to their maximal ideal*

$$\widehat{\mathcal{O}}_{X,x} \cong \widehat{\mathcal{O}}_{X(\mathbb{C}),x}.$$

Proof. Since x is a closed point of X, then

$$\mathcal{O}_{X,x} \cong \mathcal{O}_{\mathbb{A}^n_{\mathbb{C}},0}/I,$$

where $0 \in \mathbb{A}^n_{\mathbb{C}}(\mathbb{C}) \cong \mathbb{C}^n$ is the closed point of $\mathbb{A}^n_{\mathbb{C}} = Spec(\mathbb{C}[t_1, \dots, t_n])$ given by the maximal ideal $\mathfrak{m} = (t_1, \dots, t_n)$ and $I \subset \mathcal{O}_{\mathbb{A}^n_{\mathbb{C}},0}$ is an ideal. By construction of the analytification, we see that

$$\mathcal{O}_{X(\mathbb{C}),x} \cong \mathcal{O}_{\mathbb{C}^n,0}/I\mathcal{O}_{\mathbb{C}^n,0}.$$

Since completion commutes with quotients, then it suffices to show that

$$\widehat{\mathcal{O}}_{\mathbb{A}^n_{\mathbb{C}},0} \cong \widehat{\mathcal{O}}_{\mathbb{C}^n,0}.$$

On one hand, we have that

$$\widehat{\mathcal{O}}_{\mathbb{A}^n_{\mathbb{C}},0} \cong \widehat{\mathbb{C}[t_1, \dots, t_n]_{\mathfrak{m}}} \cong \widehat{\mathbb{C}[t_1, \dots, t_n]}_{\widehat{\mathfrak{m}}} \cong \mathbb{C}[\![t_1, \dots, t_n]\!]_{\widehat{\mathfrak{m}}} \cong \mathbb{C}[\![t_1, \dots, t_n]\!],$$

where the second isomorphism holds because, since $\mathfrak{m}$ is maximal, the $\mathfrak{m}$-adic completion commutes with the localization at $\mathfrak{m}$, and the last holds because $\mathbb{C}[\![t_1, \dots, t_n]\!]$ is already local with maximal ideal $\widehat{\mathfrak{m}}$. On the other hand, we have

$$\widehat{\mathcal{O}}_{\mathbb{C}^n,0} = \widehat{\mathbb{C}\{t_1, \dots, t_n\}} \cong \mathbb{C}[\![t_1, \dots, t_n]\!].$$

Hence

$$\widehat{\mathcal{O}}_{\mathbb{A}^n_{\mathbb{C}},0} \cong \mathbb{C}[\![t_1, \dots, t_n]\!] \cong \widehat{\mathcal{O}}_{\mathbb{C}^n,0}.$$

$\square$

1.2.2 GAGA Theorems

Notice that the analytification construction offers an alternative way to study algebraic varieties over $\mathbb{C}$. Indeed, given X an algebraic variety over $\mathbb{C}$, we have two different ways to study it. One is the algebraic way, considering X as a scheme, so dealing with the Zariski topology and the rational functions of the structural sheaf $\mathcal{O}_X$. The other is the analytic way, passing to the analytification $X(\mathbb{C})$, so considering the analytic topology and the transcendental functions of the structural sheaf $\mathcal{O}_{X(\mathbb{C})}$. It turns out that, in case X is a proper algebraic variety, these two ways give rise to analogous results, making the two approaches substantially equivalent. The deep reason for this equivalence is attributable to the equivalence of the categories of coherent sheaves of modules over X and $X(\mathbb{C})$ and to the isomorphism of the sheaf cohomologies of the corresponding coherent sheaves of modules. Below, we state precisely these results, which are called *GAGA (Géométrie Algébrique Géométrie Analytique) Theorems*.

The appropriate language in which to discuss sheaves (of modules) and sheaf cohomology is the one of (ringed) sites. For the moment, we consider the classical sites associated to the underlying topological spaces of X and $X(\mathbb{C})$, that is, the ones given by the category of open subsets, with the Grothendieck topology where the covering families are open covers. Moreover, since X and $X(\mathbb{C})$ are ringed spaces, then the structural sheaves are sheaves of rings with respect to the classical sites, so they define classical ringed sites. The more general notion of sites will be useful later, when we will consider non-classical sites associated to X and $X(\mathbb{C})$ and also big sites. So, explicitly, given X an algebraic variety over $\mathbb{C}$ (or more generally over any field k), we consider the site with underlying category $\mathrm{Op}(X)$, the category of Zariski open subsets of X, and covering families given by open covers. It is called the *small Zariski site* over X and it is denoted by

$$X_{Zar}.$$

Since X is a ringed space, then the structural sheaf $\mathcal{O}_X$ is a sheaf of rings on the site X_{Zar}. Hence, it endows X_{Zar} with a structure of ringed site, denoted by

$$(X_{Zar}, \mathcal{O}_X).$$

Analogously, given $\mathcal{Y}$ a complex analytic space, we consider the site with underlying category $\mathrm{Op}(\mathcal{Y})$, the category of open subsets of $\mathcal{Y}$ as a complex analytic space, and covering families given by open covers. It is called the *small analytic site* over $\mathcal{Y}$ and it is denoted by

$$\mathcal{Y}_{an}.$$

Since $\mathcal{Y}$ is a ringed space, than the structural sheaf $\mathcal{O}_{\mathcal{Y}}$ is a sheaf of rings on the site

$\mathcal{Y}_{an}$. Hence, it endows $\mathcal{Y}_{an}$ with a structure of ringed site, denoted by

$$(\mathcal{Y}_{an}, \mathcal{O}_{\mathcal{Y}}).$$

To relate cohomology of X with the one of $X(\mathbb{C})$, we consider morphisms of (ringed) sites. The continuous map of topological spaces underlying $\alpha_X : X(\mathbb{C}) \to X$, the canonical morphism of analytifcation, induces a morphism of sites

$$\alpha_X : X(\mathbb{C})_{an} \to X_{Zar}.$$

By remark 1.2.16, it is the morphism of sites associated to the continuous functor given by the restriction of the analytification functor

$$\begin{aligned} an : \mathrm{Op}(X) &\to \mathrm{Op}(X(\mathbb{C})) \\ U &\mapsto \alpha_X^{-1}(U) \cong U(\mathbb{C}). \end{aligned}$$

Since $\alpha_X : X(\mathbb{C}) \to X$ is also a morphism of ringed spaces, it induces a morphism of ringed sites

$$\alpha_X : (X(\mathbb{C})_{an}, \mathcal{O}_{X(\mathbb{C})}) \to (X_{Zar}, \mathcal{O}_X).$$

Consider the categories of abelian sheaves and sheaves of $\mathcal{O}_X$-modules over X_{Zar}

$$\mathrm{Ab}(X_{Zar}) \qquad \& \qquad \mathrm{Mod}(\mathcal{O}_X)$$

and, analogously for $X(\mathbb{C})$, the categories

$$\mathrm{Ab}(X(\mathbb{C})_{an}) \qquad \& \qquad \mathrm{Mod}(\mathcal{O}_{X(\mathbb{C})}).$$

By general theory of (ringed) sites, we have the pairs of adjoint functors

$$\alpha_X^{-1} : \mathrm{Ab}(X_{Zar}) \rightleftarrows \mathrm{Ab}(X(\mathbb{C})_{an}) : \alpha_{X*}$$

and

$$\alpha_X^{*} : \mathrm{Mod}(\mathcal{O}_X) \rightleftarrows \mathrm{Mod}(\mathcal{O}_{X(\mathbb{C})}) : \alpha_{X*}.$$

Recall that: α_{X*} is such that, for any $G \in \mathrm{Ab}(X(\mathbb{C})_{an})$ or $\mathrm{Mod}(\mathcal{O}_{X(\mathbb{C})})$,

$$\alpha_{X*}G : U \mapsto G(U(\mathbb{C})),$$

α_X^{-1} is such that, for any $F \in \mathrm{Ab}(X_{Zar})$, $\alpha_X^{-1}F$ is the sheafification of

$$\alpha_X^{p}F : \mathcal{W} \mapsto \varinjlim_{\mathcal{W} \to U(\mathbb{C}) \in \mathrm{Op}(X(\mathbb{C}))} F(U),$$

and α_X^* is such that, for any $F \in \mathrm{Mod}(\mathcal{O}_X)$,

$$\alpha_X^* F := \alpha_X^{-1} F \otimes_{\alpha_X^{-1}\mathcal{O}_X} \mathcal{O}_{X(\mathbb{C})}.$$

Definition 1.2.18. Given X an algebraic variety over $\mathbb{C}$, the functor

$$\begin{aligned} \alpha_X^* : \mathrm{Mod}(\mathcal{O}_X) &\to \mathrm{Mod}(\mathcal{O}_{X(\mathbb{C})}) \\ F &\mapsto F^{an} := \alpha_X^* F \end{aligned}$$

is called the *analytification functor of sheaves of modules* and F^{an} the *analytification* of F.

Proposition 1.2.19. *For any X algebraic variety over $\mathbb{C}$, the morphism of ringed sites*

$$\alpha_X : (X(\mathbb{C})_{an}, \mathcal{O}_{X(\mathbb{C})}) \to (X_{Zar}, \mathcal{O}_X)$$

is faithfully-flat. Hence, the analytification functor of sheaves of modules α_X^* is exact.

Proof. We have to prove that

$$\alpha_X^{\#} : \alpha_X^{-1}\mathcal{O}_X \to \mathcal{O}_{X(\mathbb{C})}$$

is a faithfully-flat morphism of abelian sheaves over the site $X(\mathbb{C})_{an}$. Since $X(\mathbb{C})_{an}$ is the classical site given by open covers, this is equivalent to prove that for any $x \in X(\mathbb{C})$

$$\mathcal{O}_{X,x} \to \mathcal{O}_{X(\mathbb{C}),x}$$

is a faithfully-flat morphism of local rings. Taking the induced morphism on adic-completions with respect to the maximal ideals, we obtain the commutative diagram of rings

$$\begin{array}{ccc} \mathcal{O}_{X,x} & \longrightarrow & \mathcal{O}_{X(\mathbb{C}),x} \\ \downarrow & & \downarrow \\ \widehat{\mathcal{O}}_{X,x} & \longrightarrow & \widehat{\mathcal{O}}_{X(\mathbb{C}),x}. \end{array}$$

By proposition 1.2.17, the lower horizontal morphism is an isomorphism. Since $\mathcal{O}_{X,x}$ and $\mathcal{O}_{X(\mathbb{C}),x}$ are noetherian local rings (because are isomorphic to quotients and localizations of $\mathbb{C}[t_1, \dots, t_n]$ and $\mathbb{C}\{t_1, \dots, t_n\}$, which are noetherian rings), then the canonical morphisms into their adic-completion with respect to their maximal ideal, i.e. vertical morphisms in the diagram, are faithfully-flat. By commutativity of the diagram, it follows that also $\mathcal{O}_{X,x} \to \mathcal{O}_{X(\mathbb{C}),x}$ is faithfully-flat. $\square$

Before stating GAGA Theorems, we see some properties of coherent sheaves of modules over the ringed sites X_{Zar} and $X(\mathbb{C})_{an}$.

Proposition 1.2.20. *For any X algebraic variety over a field k, the structural sheaf $\mathcal{O}_X$ is a coherent sheaf of $\mathcal{O}_X$-modules. Analogously, for any $\mathcal{Y}$ complex analytic space, the structural sheaf $\mathcal{O}_\mathcal{Y}$ is a coherent sheaf of $\mathcal{O}_\mathcal{Y}$-modules.*

Proof. Let X be an algebraic variety over k. Since X is a locally noetherian scheme, then coherent sheaves of $\mathcal{O}_X$-modules are exactly the quasi-coherent and locally of finite type ones (see [GW20, Prop. 7.46]). Since the structural sheaf $\mathcal{O}_X$ is a quasi-coherent and locally of finite type sheaf of $\mathcal{O}_X$-modules, then it is a coherent sheaf of $\mathcal{O}_X$-modules.

Let $\mathcal{Y}$ be a complex analytic space. In case $\mathcal{Y} \cong \mathbb{C}^n$, the statement is Oka's Theorem (see [GR84, §2.5.2]). Since coherence is a local property, then also $\mathcal{O}_U$ is a coherent sheaf $\mathcal{O}_U$-modules for any $U \subset \mathbb{C}^n$ open subset. For a general $\mathcal{Y}$, since coherence is a local property, we can assume that $\mathcal{Y}$ is a local model. Let $\mathcal{Y} \cong Z = Z(S)$, zero locus of a finite set of holomorphic functions S over an open subset $U \subset \mathbb{C}^n$. Let $i : \mathcal{Y} \hookrightarrow U$ be the corresponding closed immersion. Since S is finite, then $\mathcal{O}_U/\langle S\rangle$ is a locally of finite presentation sheaf of $\mathcal{O}_U$-modules. Since $\mathcal{O}_U$ is a coherent sheaf of $\mathcal{O}_U$-modules and $\mathcal{O}_U/\langle S\rangle$ is a locally of finite presentation sheaf of $\mathcal{O}_U$-modules, then $\mathcal{O}_U/\langle S\rangle$ is also a coherent sheaf of $\mathcal{O}_U$-modules. Since a sheaf of $\mathcal{O}_U/\langle S\rangle$-modules that is coherent as a sheaf of $\mathcal{O}_U$-modules is also a coherent sheaf of $\mathcal{O}_U/\langle S\rangle$-modules, then $\mathcal{O}_U/\langle S\rangle$ is a coherent sheaf of $\mathcal{O}_U/\langle S\rangle$-modules. Using exactness of i^{-1}, we conclude that $\mathcal{O}_\mathcal{Y} \cong i^{-1}(\mathcal{O}_U/\langle S\rangle)$ is a coherent sheaf of $i^{-1}(\mathcal{O}_U/\langle S\rangle)$-modules. □

Proposition 1.2.21. *Let X be an algebraic variety over $\mathbb{C}$. If F is a coherent sheaf of $\mathcal{O}_X$-modules, then F^{an} is a coherent sheaf of $\mathcal{O}_{X(\mathbb{C})}$-modules.*

Proof. Since, by proposition 1.2.20, for both algebraic varieties over $\mathbb{C}$ and complex analytic spaces the structural sheaf is coherent, then the statement is equivalent to: if F is a locally of finite presentation sheaf of $\mathcal{O}_X$-modules, then F^{an} is a locally of finite presentation sheaf of $\mathcal{O}_{X(\mathbb{C})}$-modules. So, let F be a locally of finite presentation sheaf of $\mathcal{O}_X$-modules. For any $x \in X(\mathbb{C})$ closed point of X, consider a local finite presentation of F over an open subset $U \subset X$ containing x, that is an exact sequence, for some $n, m \geq 0$,

$$\mathcal{O}_U^m \to \mathcal{O}_U^n \to F\big|_U \to 0.$$

Since α_X^* is exact by proposition 1.2.19, then it induces the exact sequence

$$\mathcal{O}_{U(\mathbb{C})}^m \to \mathcal{O}_{U(\mathbb{C})}^n \to F^{an}\big|_{U(\mathbb{C})} \to 0,$$

which is a local finite presentation of F^{an} over the open subset $U(\mathbb{C}) \subset X(\mathbb{C})$ containing x. Hence, F^{an} is locally of finite presentation. □

In other words, proposition 1.2.21 tells that, for any X algebraic variety over $\mathbb{C}$, the analytification functor of sheaves of modules restricts to the categories of coherent sheaves of modules

$$\alpha_X^* : \mathrm{Coh}(\mathcal{O}_X) \to \mathrm{Coh}(\mathcal{O}_{X(\mathbb{C})}).$$

Now we state GAGA Theorems. They were first formulated and proved by Serre for projective algebraic varieties in [Ser56] and then generalized by Grothendieck for proper algebraic varieties in [Gro57]. Moreover, Grothendieck adopts a relative approach. That is, instead of considering only algebraic varieties, he deals more generally with morphisms of algebraic varieties. Hence, statements about sheaf cohomology of algebraic varieties become more generally statements about the right-derived pushforward of morphisms of algebraic varieties. However, we are interested only in Serre's non-relative version, which is recovered by applying Grothendieck's relative version to the structural morphism of algebraic varieties into $Spec(\mathbb{C})$. We also remark that, while Grothendieck considers sheaf cohomology, Serre considers Čech cohomology. However, notice that it makes no difference, since, in both algebraic and analytic contexts, they coincide for coherent sheaves of modules. Indeed, in the algebraic context, recall that, given U an affine algebraic variety and $F \in \mathrm{Coh}(\mathcal{O}_U)$, it holds that (see [Har77, §III, Thm. 3.5])

$$\mathbb{H}^q(U_{Zar}, F) = 0 \qquad \text{for any } q > 0.$$

So, given $X \in Var_{\mathbb{C}}$, consider $\mathcal{U} = \{U_i\}_{i \in I}$ an affine open cover of X. Since X is separated, then also finite intersections $U_{i_0 \dots i_n} \coloneqq U_{i_0} \cap \dots \cap U_{i_n}$ are affine. Hence, it holds that, for any $F \in \mathrm{Coh}(\mathcal{O}_X)$,

$$\mathbb{H}^q(U_{i_0 \dots i_n}, F) = 0 \qquad \text{for each } n \geq 0 \text{ and } q > 0.$$

By Leray's Theorem, it follows that, for each $i \geq 0$,

$$\mathbb{H}^i(X_{Zar}, F) \cong \check{\mathrm{H}}^i(\mathcal{U}; F).$$

In the analytic context, there is the notion of a *Stein space*, a kind of complex analytic space, which plays the analogous role of an affine algebraic variety (see [GR84, §1.4.4-6]). Indeed, Cartan's Theorem B states that, given V a Stein space and $G \in \mathrm{Coh}(\mathcal{O}_V)$, it holds that

$$\mathbb{H}^q(V_{an}, G) = 0 \qquad \text{for any } q > 0.$$

Moreover, any $\mathcal{Y} \in An_{\mathbb{C}}$ admits an open cover of Stein spaces $\mathcal{V} = \{V_i\}_{i \in I}$ and finite intersections $V_{i_0 \dots i_n} \coloneqq V_{i_0} \cap \dots \cap V_{i_n}$ are again Stein. Hence, it holds that, for any $G \in \mathrm{Coh}(\mathcal{O}_{\mathcal{Y}})$,

$$\mathbb{H}^q(V_{i_0 \dots i_n}, G) = 0 \qquad \text{for each } n \geq 0 \text{ and } q > 0.$$

Then, by Leray's Theorem, it follows that

$$\mathbb{H}^i(\mathcal{Y}_{an}, G) \cong \check{\mathrm{H}}^i(\mathcal{V}; G).$$

We will state GAGA theorems considering sheaf cohomology.

Consider the morphism of ringed sites

$$\alpha_X : (X(\mathbb{C})_{an}, \mathcal{O}_{X(\mathbb{C})}) \to (X_{Zar}, \mathcal{O}_X).$$

Recall that, by functoriality of sheaf cohomology with respect to morphisms of ringed sites, we have canonical morphisms of $\mathbb{C}$-vector spaces, for any $F \in \mathrm{Mod}(\mathcal{O}_X)$ and $i \geq 0$,

$$\mathbb{H}^i(X_{Zar}, F) \to \mathbb{H}^i(X(\mathbb{C})_{an}, \alpha_X^* F) = \mathbb{H}^i(X(\mathbb{C})_{an}, F^{an}),$$

natural in F.

Theorem 1.2.22 (GAGA Theorem I)**.** *Let X be a proper algebraic variety over $\mathbb{C}$. Then, for any $F \in Coh(\mathcal{O}_X)$, the canonical morphisms of $\mathbb{C}$-vector spaces, for each $i \geq 0$,*

$$\mathbb{H}^i(X_{Zar}, F) \to \mathbb{H}^i(X(\mathbb{C})_{an}, F^{an})$$

are isomorphisms, natural in F.

Proof. See [Ser56, §12-13, Thm. 1], [Gro57, §6, Thm. 5]. □

Theorem 1.2.23 (GAGA Theorem II)**.** *Let X be a proper algebraic variety over $\mathbb{C}$. Then, the analytification functor of sheaves of modules restricted to the categories of coherent sheaves of modules*

$$\alpha_X^* : Coh(\mathcal{O}_X) \to Coh(\mathcal{O}_{X(\mathbb{C})})$$

is an equivalence of categories.

Proof. See [Ser56, §14-17, Thm. 2,3], [Gro57, §8, Thm. 6]. □

1.3 The algebraic de Rham complex

1.3.1 Sheaf-theoretic Kähler differentials

Recall that, given a morphism of commutative rings with unit $\varphi : A \to B$, the *functor of A-derivations* from B

$$\mathrm{Der}_A(B, \_\,) : B\text{-mod} \to Set$$

is such that, for any $M \in B$-mod, $\mathrm{Der}_A(B, M)$ is the set of morphisms of A-modules $D : B \to M$ satisfying Leibniz rule

$$D(bb') = bD(b') + b'D(b) \qquad \text{for any } b, b' \in B.$$

It is a representable functor and we denote a representation by $(\Omega_{B/A}, d)$. The B-module $\Omega_{B/A}$ is called the *module of Kähler differentials* of B over A, and the universal morphism

$$d : B \to \Omega_{B/A}$$

is called the *universal derivation*. We have an explicit description of $\Omega_{B/A}$: it is the free B-module generated by formal elements db, for $b \in B$, modulo the relations given by A-linearity and Leibniz rule. For each $p \geq 0$, we denote by

$$\Omega^p_{B/A} \coloneqq \bigwedge^p \Omega_{B/A},$$

where the wedge product is taken as B-modules. In particular, $\Omega^0_{A/B} \cong B$ and $\Omega^1_{A/B} \cong \Omega_{A/B}$. $\Omega^p_{B/A}$ is generated as a B-module by the elements of the kind $db_1 \wedge \cdots \wedge db_p$, for $b_1, \ldots, b_p \in B$. Moreover, for any $p \geq 0$, there exists a unique morphism of A-modules (not of B-modules!)

$$d^p : \Omega^p_{B/A} \to \Omega^{p+1}_{B/A},$$

such that

$$d^p(b_0 db_1 \wedge \cdots \wedge db_p) = db_0 \wedge db_1 \wedge \cdots \wedge db_p.$$

These morphisms are characterized by the following properties:

- $d^0 = d$,
- $d^{p+1} \circ d^p = 0$ for any $p \geq 0$,
- $d^p(\omega \wedge \eta) = d^r\omega \wedge \eta + (-1)^r \omega \wedge d^{p-r}\eta$ for any $\omega \in \Omega^r_{B/A}$ and $\eta \in \Omega^{p-r}_{B/A}$.

The second property tells that we have a complex of A-modules

$$\Omega^\bullet_{B/A} \ : \ B \xrightarrow{d} \Omega^1_{B/A} \xrightarrow{d^1} \Omega^2_{B/A} \to \cdots ,$$

called the *algebraic de Rham complex* of B over A. For more details and properties (such as functoriality and exact sequences), we refer to [Eis95, §16].

We have sheaf-theoretical analogues of all these constructions in the general language of sites.

Definition 1.3.1. Given $\mathcal{C}$ a site, $f : \mathcal{O}_1 \to \mathcal{O}_2$ a morphism of sheaves of commutative rings with unit over $\mathcal{C}$ and $\mathcal{M}$ a sheaf of $\mathcal{O}_2$-modules, we define an $\mathcal{O}_1$-*derivation* from $\mathcal{O}_2$ to $\mathcal{M}$ a morphism of sheaves of $\mathcal{O}_1$-modules

$$D : \mathcal{O}_2 \to \mathcal{M},$$

such that, for any open subset $U \subset X$,

$$D_U : \mathcal{O}_2(U) \to \mathcal{M}(U)$$

is an $\mathcal{O}_1(U)$-derivation from $\mathcal{O}_2(U)$ to $\mathcal{M}(U)$. We denote by $\mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{M})$ the set of $\mathcal{O}_1$-derivations from $\mathcal{O}_2$ to $\mathcal{M}$. This gives rise to the functor

$$\mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \_\,) : \mathrm{Mod}(\mathcal{O}_2) \to \mathit{Set},$$

called the *functor of $\mathcal{O}_1$-derivations from $\mathcal{O}_2$*

Proposition 1.3.2. *Let $\mathcal{C}$ be a site and $f : \mathcal{O}_1 \to \mathcal{O}_2$ a morphism of sheaves of commutative rings with unit over $\mathcal{C}$. Then, the functor*

$$\mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \_\,) : \mathit{Mod}(\mathcal{O}_2) \to \mathit{Set}$$

is representable.

Proof. We define the sheaf of $\mathcal{O}_2$-modules

$$\Omega_{\mathcal{O}_2/\mathcal{O}_1}$$

obtained by sheafifying the presheaf of $\mathcal{O}_2$-modules

$$\Omega^{\mathscr{P}}_{\mathcal{O}_2/\mathcal{O}_1} : U \mapsto \Omega_{\mathcal{O}_2(U)/\mathcal{O}_1(U)},$$

where $\Omega_{\mathcal{O}_2(U)/\mathcal{O}_1(U)}$ is the module of Kähler differentials of $\mathcal{O}_2(U)$ over $\mathcal{O}_1(U)$. Moreover, we define the morphism of sheaves of $\mathcal{O}_1$-modules

$$d : \mathcal{O}_2 \to \Omega_{\mathcal{O}_2/\mathcal{O}_1}$$

obtained by sheafifying the morphism of presehaves of $\mathcal{O}_1$-modules

$$U \mapsto (d_U : \mathcal{O}_2(U) \to \Omega_{\mathcal{O}_2(U)/\mathcal{O}_1(U)}),$$

where d_U is the universal derivation for $\Omega_{\mathcal{O}_2(U)/\mathcal{O}_1(U)}$. We have the bijections, for any $\mathcal{M} \in \mathrm{Mod}(\mathcal{O}_2)$,

$$\mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{M}) \cong \mathrm{Hom}_{\mathrm{PMod}(\mathcal{O}_2)}(\Omega^{\mathscr{P}}_{\mathcal{O}_2/\mathcal{O}_1}, \mathcal{M}) \cong \mathrm{Hom}_{\mathrm{Mod}(\mathcal{O}_2)}(\Omega_{\mathcal{O}_2/\mathcal{O}_1}, \mathcal{M}),$$

where the first follow from open-wise representability of $\mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \_)$ and the second is universal property of sheafification. Then $\mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \_\,)$ is representable, with a representation given by $(\Omega_{\mathcal{O}_2/\mathcal{O}_1}, d)$. $\square$

Definition 1.3.3. Given $\mathcal{C}$ a site and $f : \mathcal{O}_1 \to \mathcal{O}_2$ a morphism of sheaves of commutative rings with unit over $\mathcal{C}$, the sheaf of $\mathcal{O}_2$-modules $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$ is called the *sheaf of modules of Kähler differentials of* $\mathcal{O}_2$ *over* $\mathcal{O}_1$, and the morphism of sheaves of $\mathcal{O}_1$-modules $d : \mathcal{O}_2 \to \Omega_{\mathcal{O}_2/\mathcal{O}_1}$ is called the *universal derivation.* For each $p \geq 0$, we define

$$\Omega^p_{\mathcal{O}_2/\mathcal{O}_1} := \bigwedge^p \Omega_{\mathcal{O}_2/\mathcal{O}_1},$$

where the wedge product is taken as sheaves of $\mathcal{O}_2$-modules. It is the sheafification of the presheaf of $\mathcal{O}_2$-modules

$$U \mapsto \Omega^p_{\mathcal{O}_2(U)/\mathcal{O}_1(U)}.$$

In particular, $\Omega^0_{\mathcal{O}_2/\mathcal{O}_1} \cong \mathcal{O}_2$ and $\Omega^1_{\mathcal{O}_2/\mathcal{O}_1} \cong \Omega_{\mathcal{O}_2/\mathcal{O}_1}$. Moreover we define the morphisms of sheaves of $\mathcal{O}_1$-modules (not of $\mathcal{O}_2$-modules!), for any $p \geq 0$,

$$d^p : \Omega^p_{\mathcal{O}_2/\mathcal{O}_1} \to \Omega^{p+1}_{\mathcal{O}_2/\mathcal{O}_1},$$

the sheafification of the morphism of presheves of $\mathcal{O}_1$-modules

$$U \mapsto (d^p_U : \Omega^p_{\mathcal{O}_2(U)/\mathcal{O}_1(U)} \to \Omega^{p+1}_{\mathcal{O}_2(U)/\mathcal{O}_1(U)}).$$

In particular, $d^0 = d$. By the corresponding property for rings, we have that, for any $p \geq 0$

$$d^{p+1} \circ d^p = 0.$$

Hence, we have a complex of sheaves of $\mathcal{O}_1$-modules

$$\Omega^\bullet_{\mathcal{O}_2/\mathcal{O}_1} : \mathcal{O}_2 \xrightarrow{d} \Omega^1_{\mathcal{O}_2/\mathcal{O}_1} \xrightarrow{d^1} \Omega^2_{\mathcal{O}_2/\mathcal{O}_1} \to \cdots,$$

called *algebraic de Rham complex* of $\mathcal{O}_2$ over $\mathcal{O}_1$.

Given a morphism of ringed sites $f : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$, we can apply this construction to the morphism of sheaves of rings over $\mathcal{C}$

$$f^\# : f^{-1}\mathcal{O}_\mathcal{D} \to \mathcal{O}_\mathcal{C}.$$

We obtain a complex of sheaves of $f^{-1}\mathcal{O}_\mathcal{D}$-modules, denoted by

$$\Omega^\bullet_{\mathcal{C}/\mathcal{D}} := \Omega^\bullet_{\mathcal{O}_\mathcal{C}/f^{-1}\mathcal{O}_\mathcal{D}},$$

called the *algebraic de Rham complex* of $\mathcal{C}$ over $\mathcal{D}$.

We see some properties of the sheaf of modules of Kähler differentials and the algebraic de Rham complex.

Proposition 1.3.4. *Let $f : \mathcal{C} \to \mathcal{D}$ be a morphism of sites and $\mathcal{O}_1 \to \mathcal{O}_2$ a morphism of sheaves of commutative rings with unit over $\mathcal{D}$. Then, we have a* *canonical isomorphism of complexes of abelian sheaves*

$$f^{-1}\Omega^{\bullet}_{\mathcal{O}_2/\mathcal{O}_1} \cong \Omega^{\bullet}_{f^{-1}\mathcal{O}_2/f^{-1}\mathcal{O}_1}.$$

Proof. By universal property of the sheaf of modules of Kähler differentials and by the adjunctions

$$f^{-1} : \mathrm{Mod}(\mathcal{O}_2) \rightleftarrows \mathrm{Mod}(f^{-1}\mathcal{O}_2) : f_*$$

and

$$f^{-1} : \mathrm{Mod}(\mathcal{O}_1) \rightleftarrows \mathrm{Mod}(f^{-1}\mathcal{O}_1) : f_*,$$

(which are both restrictions of the adjunction (f^{-1}, f_*) on categories of abelian sheaves over $\mathcal{C}$ and $\mathcal{D}$), we have the bijections, for any $\mathcal{M}$ sheaf of $f^{-1}\mathcal{O}_2$-modules,

$$\begin{aligned}\mathrm{Hom}_{f^{-1}\mathcal{O}_2}(f^{-1}\Omega^1_{\mathcal{O}_2/\mathcal{O}_1}, \mathcal{M}) &\cong \mathrm{Hom}_{\mathcal{O}_2}(\Omega^1_{\mathcal{O}_2/\mathcal{O}_1}, f_*\mathcal{M}) \cong \mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, f_*\mathcal{M}) \cong \\ &\cong \mathrm{Der}_{f^{-1}\mathcal{O}_1}(f^{-1}\mathcal{O}_2, \mathcal{M}) \cong \mathrm{Hom}_{f^{-1}\mathcal{O}_2}(\Omega^1_{f^{-1}\mathcal{O}_2/f^{-1}\mathcal{O}_1}, \mathcal{M}).\end{aligned}$$

By Yoneda lemma, it follows the isomorphism of sheaves of $f^{-1}\mathcal{O}_2$-modules

$$f^{-1}\Omega^1_{\mathcal{O}_2/\mathcal{O}_1} \cong \Omega^1_{f^{-1}\mathcal{O}_2/f^{-1}\mathcal{O}_1}.$$

It is compatible with universal derivations, that is, we have the commutative diagram of morphisms of sheaves of $f^{-1}\mathcal{O}_1$-modules

$$\begin{tikzcd} f^{-1}\mathcal{O}_2 \arrow[r, "f^{-1}d"] \arrow[d, equal] & f^{-1}\Omega^1_{\mathcal{O}_2/\mathcal{O}_1} \arrow[d, "\wr"] \\ f^{-1}\mathcal{O}_2 \arrow[r, "d"] & \Omega^1_{f^{-1}\mathcal{O}_2/f^{-1}\mathcal{O}_1}. \end{tikzcd}$$

Since f^{-1} commutes with exterior powers, we also have the canonical isomorphisms of sheaves of $f^{-1}\mathcal{O}_2$-modules, for each $p \geq 0$,

$$f^{-1}\Omega^p_{\mathcal{O}_2/\mathcal{O}_1} \cong f^{-1}\left(\bigwedge^p \Omega^1_{\mathcal{O}_2/\mathcal{O}_1}\right) \cong \bigwedge^p f^{-1}\Omega^1_{\mathcal{O}_2/\mathcal{O}_1} \cong \bigwedge^p \Omega^1_{f^{-1}\mathcal{O}_2/f^{-1}\mathcal{O}_1} \cong \Omega^p_{f^{-1}\mathcal{O}_2/f^{-1}\mathcal{O}_1},$$

which are compatible with differentials of the algebraic de Rham complexes. That is, they define an isomorphism of complexes of abelian sheaves

$$f^{-1}\Omega^{\bullet}_{\mathcal{O}_2/\mathcal{O}_1} \cong \Omega^{\bullet}_{f^{-1}\mathcal{O}_2/f^{-1}\mathcal{O}_1}.$$

□

The following is the sheafified versions of the analogous functoriality property for the module of Kähler differentials for rings.

Proposition 1.3.5 (Functoriality)**.** *Let $\mathcal{C}$ be a site and*

$$\begin{tikzcd} \mathcal{O}_1 \arrow[r] \arrow[d] & \mathcal{O}_2 \arrow[d] \\ \mathcal{O}'_1 \arrow[r] & \mathcal{O}'_2 \end{tikzcd}$$

a commutative square of sheaves of commutative rings with unit over $\mathcal{C}$. Then, we have a canonical morphism of complexes of abelian sheaves

$$\Omega^{\bullet}_{\mathcal{O}_2/\mathcal{O}_1} \to \Omega^{\bullet}_{\mathcal{O}'_2/\mathcal{O}'_1}$$

and canonical morphisms of sheaves of $\mathcal{O}'_2$-modules, for each $p \geq 0$,

$$\Omega^p_{\mathcal{O}_2/\mathcal{O}_1} \otimes_{\mathcal{O}_2} \mathcal{O}'_2 \to \Omega^p_{\mathcal{O}'_2/\mathcal{O}'_1}.$$

Moreover, if the square is object-wise cocartesian, that is, for any $U \in \mathcal{C}$, $\mathcal{O}'_2(U) \cong \mathcal{O}'_1(U) \otimes_{\mathcal{O}_1(U)} \mathcal{O}_2(U)$, then

$$\Omega^p_{\mathcal{O}_2/\mathcal{O}_1} \otimes_{\mathcal{O}_2} \mathcal{O}'_2 \cong \Omega^p_{\mathcal{O}'_2/\mathcal{O}'_1}.$$

Proof. For any $U \in \mathcal{C}$, the commutative square of rings

$$\begin{tikzcd} \mathcal{O}_1(U) \arrow[r] \arrow[d] & \mathcal{O}_2(U) \arrow[d] \\ \mathcal{O}'_1(U) \arrow[r] & \mathcal{O}'_2(U) \end{tikzcd}$$

induces the canonical morphism of $\mathcal{O}_2(U)$-modules (see [Eis95, pg.386])

$$\Omega^1_{\mathcal{O}_2(U)/\mathcal{O}_1(U)} \to \Omega^1_{\mathcal{O}'_2(U)/\mathcal{O}'_1(U)}$$

and hence, the canonical morphism of $\mathcal{O}'_2(U)$-modules

$$\Omega^1_{\mathcal{O}_2(U)/\mathcal{O}_1(U)} \otimes_{\mathcal{O}_2(U)} \mathcal{O}'_2(U) \to \Omega^1_{\mathcal{O}'_2(U)/\mathcal{O}'_1(U)},$$

which is an isomorphism in case the square of sheaves of rings is object-wise cocartesian. These define morphisms of presheaves, whose sheafification are the morphism of sheaves of $\mathcal{O}_2$-modules

$$\Omega^1_{\mathcal{O}_2/\mathcal{O}_1} \to \Omega^1_{\mathcal{O}'_2/\mathcal{O}'_1}$$

and the morphisms of sheaves of $\mathcal{O}'_2$-modules

$$\Omega^1_{\mathcal{O}_2/\mathcal{O}_1} \otimes_{\mathcal{O}_2} \mathcal{O}'_2 \to \Omega^1_{\mathcal{O}'_2/\mathcal{O}'_1},$$

which is an isomorphism in case the square of sheaves of rings is object-wise cocartesian. Taking exterior powers for each $p \geq 0$, we get the morphisms of sheaves of $\mathcal{O}_2$-modules

$$\Omega^p_{\mathcal{O}_2/\mathcal{O}_1} \to \Omega^p_{\mathcal{O}'_2/\mathcal{O}'_1}$$

and, since pullback commutes with exterior power, the morphism of sheaves of $\mathcal{O}'_2$-modules

$$\Omega^p_{\mathcal{O}_2/\mathcal{O}_1} \otimes_{\mathcal{O}_2} \mathcal{O}'_2 \to \Omega^p_{\mathcal{O}'_2/\mathcal{O}'_1},$$

which is an isomorphism in case the square of sheaves of rings is object-wise cocartesian. Moreover, the morphisms $\Omega^p_{\mathcal{O}_2/\mathcal{O}_1} \to \Omega^p_{\mathcal{O}'_2/\mathcal{O}'_1}$ are compatible with differentials, hence they define a morphism of complexes of abelian sheaves between the algebraic de Rham complexes

$$\Omega^\bullet_{\mathcal{O}_2/\mathcal{O}_1} \to \Omega^\bullet_{\mathcal{O}'_2/\mathcal{O}'_1}.$$

□

1.3.2 Kähler differentials for schemes

We are interested in this construction applied to the classical morphism of ringed sites given by a morphism of schemes $f : X \to S$, that is, applied to the morphism of ringed sites

$$f : (X_{Zar}, \mathcal{O}_X) \to (S_{Zar}, \mathcal{O}_S).$$

We denote by

$$\Omega^\bullet_{X/S} \coloneqq \Omega^\bullet_{\mathcal{O}_X/f^{-1}\mathcal{O}_S}.$$

We discuss some properties in this case.

Proposition 1.3.6. *Let $f : X \to S$ be a morphism of schemes. For each $p \geq 0$, $\Omega^p_{X/S}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules, such that, for any affine open subsets $Spec(B) \cong U \subset X$ and $Spec(A) \cong V \subset S$, with $f(U) \subset V$,*

$$\Omega^p_{X/S}\big|_U \cong \Omega^p_{B/A}{}^{\sim}.$$

Moreover, for any $x \in X$,

$$(\Omega^p_{X/S})_x \cong \Omega^p_{\mathcal{O}_{X,x}/\mathcal{O}_{S,f(x)}}$$

Proof. Let $Spec(B) \cong U \subset X$ and $Spec(A) \cong V \subset S$ be affine open subsets, such that $f(U) \subset V$. We denote by

$$f\big|_U : U \to V$$

the restriction of f. By universal property of the module of Kähler differentials, we have the bijections, for any $\mathcal{M}$ sheaf of $\mathcal{O}_U$-modules,

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Mod}(\mathcal{O}_U)}(\Omega^1_{U/V}, \mathcal{M}) \cong \mathrm{Der}_{f|_U{}^{-1}\mathcal{O}_V}(\mathcal{O}_U, \mathcal{M}) \cong \mathrm{Der}_{f^{-1}\mathcal{O}_S\big|_U}(\mathcal{O}_X\big|_U, \mathcal{M}) \cong \\ \cong \mathrm{Hom}_{\mathrm{Mod}(\mathcal{O}_X|_U)}(\Omega^1_{X/S}\big|_U, \mathcal{M}) \cong \mathrm{Hom}_{\mathrm{Mod}(\mathcal{O}_U)}(\Omega^1_{X/S}\big|_U, \mathcal{M}).\end{aligned}$$

By Yoneda Lemma, it follows the isomorphism of sheaves of $\mathcal{O}_U$-modules

$$\Omega^1_{X/S}\big|_U \cong \Omega^1_{U/V}.$$

By universal property of the module of Kähler differentials and the equivalence of categories between B-modules and quasi-coherent sheaves of $\mathcal{O}_U$-modules, we have the bijections, for any B-module M,

$$\begin{aligned}\mathrm{Hom}_{\mathrm{QCoh}(\mathcal{O}_U)}(\Omega^1_{B/A}{}^{\sim}, M^{\sim}) &\cong \mathrm{Hom}_{B\text{-mod}}(\Omega^1_{B/A}, M) \cong \mathrm{Der}_A(B, M) \cong \\ &\cong \mathrm{Der}_{A^{\sim}}(B^{\sim}, M^{\sim}) \cong \mathrm{Der}_{f|_U{}^{-1}\mathcal{O}_V}(\mathcal{O}_U, M^{\sim}) \cong \\ &\cong \mathrm{Hom}_{\mathrm{QCoh}(\mathcal{O}_U)}(\Omega^1_{U/V}, M^{\sim}).\end{aligned}$$

By Yoneda Lemma, it follows the isomorphism of sheaves of $\mathcal{O}_U$-modules

$$\Omega^1_{U/V} \cong \Omega^1_{B/A}{}^{\sim}.$$

Hence,

$$\Omega^1_{X/S}\big|_U \cong \Omega^1_{B/A}{}^{\sim}$$

as sheaves of $\mathcal{O}_U$-modules. Then, for any $p \geq 0$,

$$\Omega^p_{X/S}\big|_U \cong \bigwedge^p \Omega^1_{X/S}\big|_U \cong \bigwedge^p (\Omega^1_{B/A}{}^{\sim}) \cong \left(\bigwedge^p \Omega^1_{B/A}\right)^{\sim} \cong \Omega^p_{B/A}{}^{\sim}.$$

Moreover, for any $x \in X$, take some affine open subsets $Spec(B) \cong U \subset X$ and $Spec(A) \cong V \subset S$, such that $f(U) \subset V$ and $x \in U$. Let $\mathfrak{q} \in Spec(B)$ and $\mathfrak{p} \in Spec(A)$ be the prime ideals corresponding to $x \in U$ and $f(x) \in V$ respectively. Then, we have the isomorphisms of $\mathcal{O}_{X,x}$-modules

$$(\Omega^1_{X/S})_x \cong (\Omega^1_{U/V})_x \cong (\Omega^1_{B/A})_{\mathfrak{q}} \cong \Omega^1_{B_{\mathfrak{q}}/A_{\mathfrak{p}}} \cong \Omega^1_{\mathcal{O}_{X,x}/\mathcal{O}_{S,s}}.$$

Then, for any $p \geq 0$,

$$(\Omega^p_{X/S})_x \cong (\bigwedge^p \Omega^1_{X/S})_x \cong \bigwedge^p (\Omega^1_{X/S})_x \cong \bigwedge^p \Omega^1_{\mathcal{O}_{X,x}/\mathcal{O}_{S,s}} \cong \Omega^p_{\mathcal{O}_{X,x}/\mathcal{O}_{S,s}}.$$

$\square$

Example 1.3.7. Given a commutative ring with unit R and an R-scheme X, we denote by

$$\Omega^1_{X/R} \coloneqq \Omega^1_{X/Spec(R)}$$

the sheaf of modules of Kähler differentials of the structural morphism $X \to Spec(R)$. Consider the R-scheme $\mathbb{A}^n_R = Spec(R[t_1, \ldots, t_n])$, the n-dimensional affine space over R. Recall that

$$\Omega^1_{R[t_1,\ldots,t_n]/R} \cong \oplus_{i=1}^n R[t_1, \ldots, t_n]dt_i.$$

Then,

$$\Omega^1_{\mathbb{A}^n_R/R} \cong \Omega^1_{R[t_1,\ldots,t_n]/R}{}^{\sim} \cong \oplus_{i=1}^n \mathcal{O}_{\mathbb{A}^n_R} dt_i$$

is a free $\mathcal{O}_{\mathbb{A}^n_R}$-module of rank n. Moreover, given $f_1, \ldots, f_m \in R[t_1, \ldots, t_n]$, consider the closed subscheme

$$X \coloneqq V(f_1, \ldots, f_m) \subset \mathbb{A}^n_R,$$

which is such that $X \cong Spec(B)$, with $B \cong R[t_1, \ldots, t_n]/(f_1, \ldots, f_m)$. Recall that

$$\Omega^1_{B/R} \cong \Omega^1_{R[t_1,\ldots,t_n]/R}/\langle df_1, \ldots, df_m\rangle \cong \oplus_{i=1}^n R[t_1, \ldots, t_n]dt_i/\langle df_1, \ldots, df_m\rangle,$$

hence it is a finitely presented $R[t_1, \ldots, t_n]$-module. Then

$$\Omega^1_{X/R} \cong \Omega^1_{B/R}{}^{\sim}$$

is a finitely presented sheaf of $\mathcal{O}_{\mathbb{A}^n_R}$-modules, hence also a finitely presented sheaf of $\mathcal{O}_X$-modules.

Remark 1.3.8. Let $g : X \to S$ be morphism of schemes locally of finite presentation, that is, for any $x \in X$, there exist affine open subsets $Spec(B) \cong U \subset X$ and $Spec(A) \cong V \subset S$, such that $x \in U$, $f(U) \subset V$ and

$$B \cong A[t_1, \ldots, t_n]/(f_1, \ldots, f_m),$$

for some $f_1, \ldots, f_m \in A[t_1, \ldots, t_n]$. Since, by proposition 1.3.6,

$$\Omega^1_{X/S}\big|_U \cong \Omega^1_{B/A}{}^{\sim},$$

by example 1.3.7, it follows that $\Omega^1_{X/S}$ is a locally of finite presentation sheaf of $\mathcal{O}_X$-modules.

Proposition 1.3.9 (Functoriality)**.** *Let*

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle q} \\ S' & \xrightarrow{g} & S \end{array}$$

be a commutative square of schemes. Then, we have a canonical morphism of *complexes of abelian sheaves*

$$f^{-1}\Omega^{\bullet}_{X/S} \to \Omega^{\bullet}_{X'/S'}$$

and canonical morphisms of sheaves of $\mathcal{O}'_X$-modules, for each $p \geq 0$,

$$f^*\Omega^p_{X/S} \to \Omega^p_{X'/S'}.$$

Moreover, if the square is cartesian, that is $X' \cong X \times_S S'$, then

$$f^*\Omega^p_{X/S} \cong \Omega^p_{X'/S'}.$$

Proof. We apply proposition 1.3.5 to the site X'_{Zar} and the commutative square of sheaves of rings

$$\begin{tikzcd} f^{-1}q^{-1}\mathcal{O}_S \arrow[r] \arrow[d] & f^{-1}\mathcal{O}_X \arrow[d] \\ p^{-1}\mathcal{O}_{S'} \arrow[r] & \mathcal{O}_{X'}. \end{tikzcd}$$

We deduce the canonical morphism of complexes of abelian sheaves

$$\Omega^{\bullet}_{f^{-1}\mathcal{O}_X/f^{-1}q^{-1}\mathcal{O}_S} \to \Omega^{\bullet}_{\mathcal{O}_{X'}/p^{-1}\mathcal{O}_{S'}} = \Omega^{\bullet}_{X'/S'}$$

and the canonical morphisms of sheaves of $\mathcal{O}_X$-modules, for each $p \geq 0$,

$$\Omega^p_{f^{-1}\mathcal{O}_X/f^{-1}q^{-1}\mathcal{O}_S} \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_{X'} \to \Omega^p_{\mathcal{O}_{X'}/p^{-1}\mathcal{O}_{S'}} = \Omega^p_{X'/S'},$$

which are isomorphisms if the square of schemes is cartesian. [1] By proposition 1.3.4 applied to the morphism of sites $f : X'_{Zar} \to X_{Zar}$ and the morphism of sheaves of rings $q^{-1}\mathcal{O}_S \to \mathcal{O}_X$ over X_{Zar}, we get that

$$f^{-1}\Omega^{\bullet}_{X/S} = f^{-1}\Omega^{\bullet}_{\mathcal{O}_X/q^{-1}\mathcal{O}_S} \cong \Omega^{\bullet}_{f^{-1}\mathcal{O}_X/f^{-1}q^{-1}\mathcal{O}_S}.$$

By composition, we get the morphism of complexes of abelian sheaves

$$f^{-1}\Omega^{\bullet}_{X/S} \to \Omega^{\bullet}_{X'/S'}.$$

Moreover, we have the morphisms of $\mathcal{O}_X$-modules, for each $p \geq 0$,

$$f^*\Omega^p_{X/S} \cong f^{-1}\Omega^p_{X/S} \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_{X'} \to \Omega^p_{X'/S'},$$

[1] The fact that the square of schemes is cartesian doesn't imply that the square of sheaves of rings is object-wise cocartesian. However, it is on an affine open cover of X'. By quasi-coherence of the sheaf of modules of Kähler differentials, we see that this is indeed sufficient to conclude the isomorphism.

which are isomorphisms if the square of schemes is cartesian. □

1.4 Smooth and étale morphisms of schemes

In this section we want to give a suitable notion of smoothness for algebraic varieties over $\mathbb{C}$, such that it corresponds exactly to the notion of smoothness for complex analytic spaces via analytification. That is, we would like that, for any X algebraic variety over $\mathbb{C}$,

$$X \text{ is smooth} \iff X(\mathbb{C}) \text{ is smooth.}$$

Since smooth complex analytic spaces of dimension r are those complex analytic spaces that are *locally* isomorphic to $\mathbb{C}^r$, which is the analytification of the r-dimensional affine space $\mathbb{A}^r_{\mathbb{C}}$, then, we might be pushed to define smooth algebraic varieties over $\mathbb{C}$ as those algebraic varieties over $\mathbb{C}$ which are *locally* isomorphic to $\mathbb{A}^r_{\mathbb{C}}$. However, this definition turns out to be too restrictive for our purpose, since there exist algebraic varieties over $\mathbb{C}$, which are not smooth in the above naive sense, but whose analytification is a smooth complex analytic space. The reason why this naive definition doesn't work is that, when we say said *locally* for algebraic varieties over $\mathbb{C}$, we meant locally with respect to the Zariski topology, which is too coarse. In fact, the same definition is indeed the correct one for our purpose, once we replace Zariski topology with another suitable topology (in the sense of Grothendieck topologies).

1.4.1 Smooth morphisms of schemes

For the moment, we leave aside the idea of introducing a new Grothendieck topology and we take another reasonable way to define smoothness for algebraic varieties over $\mathbb{C}$. Notice that another possible approach can be imitate the Jacobi criterion 1.2.11. Following this idea, we give more generally a definition of smoothness for morphisms of schemes, which will also allow to define the wanted new Grothendieck topology. A reference is [Bos12, §8.5].

Recall that, given a polynomial $f \in R[t_1, \dots, t_n]$, we denote by $\frac{\partial f}{\partial t_j}$ the polynomial given by the formal derivative of f with respect to the variable t_j. Given a point $x \in \mathbb{A}^n_R = Spec(R[t_1, \dots, t_n])$, the evaluation of f at x is the image of f along the canonical morphism into the the residue field at x

$$\begin{aligned} R[t_1, \dots, t_n] &\to k(x) \\ f &\mapsto f(x). \end{aligned}$$

Definition 1.4.1. Given a morphism of schemes $g : X \to S$ locally of finite presentation and a point $x \in X$, we say that g is *smooth at x* of *relative dimension* r, if there exist affine open subsets $U \subset X$ and $Spec(R) \cong V \subset S$, with $x \in U$ and

$f(U) \subset V$, and a commutative diagram

$$\begin{tikzcd} U \arrow[rr, hook] \arrow[dr, "g"'] & & \mathbb{A}^n_R \arrow[dl] \\ & V, & \end{tikzcd}$$

such that

$$U \cong V(f_1, \dots, f_{n-r}) = Spec(R[t_1, \dots, t_n]/(f_1, \dots, f_{n-r})),$$

for some $f_1, \dots, f_{n-r} \in R[t_1, \dots, t_n]$, such that the *Jacobian matrix* evaluated at x

$$J_{f_1,\dots,f_{n-r}}(x) := \left[\frac{\partial f_i}{\partial t_j}(x)\right]_{\substack{i=1,\dots,n-r \\ j=1,\dots,n}}$$

has maximal rank $n-r$.

We say that f is *smooth*, if it is smooth at all its points.

A smooth morphism of relative dimension 0 at each point is called an *étale morphism*.

Given a scheme S, we say that an S-scheme X is *smooth* if its structural morphism $X \to S$ is smooth. We denote by Sm_S the full subcategory of S-schemes given by smooth S-schemes.

Remark 1.4.2. Since having maximal rank is a local property, then also being smooth is, i.e. if $f : X \to S$ is smooth at $x \in X$ of relative dimension r, then there exists and open subset $U \subset X$, with $x \in U$, such that $f\big|_U : U \to S$ is smooth of relative dimension r. In particular, being étale is a local property.

Example 1.4.3. For any scheme S and any $n \in \mathbb{N}$, the n-dimensional affine space $\mathbb{A}^n_S$ is a smooth S-scheme of relative dimension n.

There exists also other equivalent definitions of smooth morphisms of schemes. With this one, it's immediate to prove that it gives the definition of smooth algebraic varieties over $\mathbb{C}$ we were looking for.

Proposition 1.4.4. *Let X be an algebraic variety over $\mathbb{C}$, $x \in X(\mathbb{C})$ a closed point of X. Then, X is smooth of relative dimension r at x if and only if $X(\mathbb{C})$ is smooth of dimension r at x . Moreover, X is smooth if and only if $X(\mathbb{C})$ is.*

Proof. Assume that X is smooth at $x \in X(\mathbb{C})$ of relative dimension r. Take affine open subsets $U \subset X$ containing x and $V = Spec(\mathbb{C})$, as in definition 1.4.1. By construction of analytification (proposition 1.2.13), we see that $U(\mathbb{C})$ is the local model $Z(f_1, \dots, f_{n-r}) \subset \mathbb{C}^n$, where $f_1, \dots, f_{n-r}$ are seen as the holomorphic functions on $\mathbb{C}^n$ defined by the corresponding polynomials. Since x is a closed point of X, the Jacobian matrix of polynomials $f_1, \dots, f_{n-r}$ evaluated at x in the definition of smooth morphisms of schemes coincides with the Jacobian matrix of the corresponding holomorphic functions evaluated at x. Hence, one Jacobian matrix has maximal rank if and only if also the other does. By definition of smoothness for X as a $\mathbb{C}$-scheme and by Jacobi criterion 1.2.11 for $X(\mathbb{C})$, then X is smooth at x if and only if $X(\mathbb{C})$ is smooth at x. Moreover, since being smooth is a local property and, by remark 1.2.15, $X(\mathbb{C})$ is very dense in X, then smoothness for X can be checked at points in $X(\mathbb{C})$. So, X is smooth if and only if $X(\mathbb{C})$ is. □

Remark 1.4.5. Let's see more explicitly which are local charts of $X(\mathbb{C})$ analytification of X, a smooth algebraic variety over $\mathbb{C}$. Given $x \in X(\mathbb{C})$, let $U \subset X$ be an affine open subset containing x, as in the proof of the previous proposition. By construction of analytification, we have the naturality square of α

$$\begin{tikzcd} U(\mathbb{C}) \arrow[r, "\alpha_U"] \arrow[d, hook] & U \arrow[d, hook] \\ \mathbb{C}^n \arrow[r, "\alpha_{\mathbb{A}^n_{\mathbb{C}}}"] & \mathbb{A}^n_{\mathbb{C}}. \end{tikzcd}$$

Consider the corresponding commutative diagram of global sections of the structural sheaves

$$\begin{tikzcd} \mathcal{O}_{X(\mathbb{C})}(U(\mathbb{C})) & \mathcal{O}_X(U) \arrow[l] \\ \mathcal{O}(\mathbb{C}^n) \arrow[u] & \mathbb{C}[t_1, \dots, t_n]. \arrow[l] \arrow[u] \end{tikzcd}$$

We denote by $u_1, \dots, u_n \in \mathcal{O}_X(U)$ and $w_1, \dots, w_n \in \mathcal{O}_{X(\mathbb{C})}(U(\mathbb{C}))$ the images of $t_1, \dots, t_n \in \mathbb{C}[t_1, \dots, t_n]$. Assuming that the Jacobian matrix of $f_1, \dots, f_{n-r}$ evaluated at x has the last $n-r$ columns which are linearly independent, by the Holomorphic Implicit Function Theorem, $w_{1,x}, \dots, w_{r,x}$ are stalks of local coordinates of $X(\mathbb{C})$ at x.

The definition of smooth morphism of schemes given above is also convenient to prove the following result, which will be useful later.

Proposition 1.4.6. *Let $g : X \to S$ be a smooth morphism of schemes. Then, the sheaf of modules of Kähler differentials $\Omega^1_{X/S}$ is a finite locally free sheaf of $\mathcal{O}_X$-modules, with rank at a point x equal to the relative dimension of g at x.*

Proof. Since g is locally of finite presentation, then, by remark 1.3.8, $\Omega^1_{X/S}$ is a locally of finite presentation sheaf of $\mathcal{O}_X$-modules. Then, to prove that $\Omega^1_{X/S}$ is a finite locally free sheaf of $\mathcal{O}_X$-modules, it's sufficient to prove that $\Omega^1_{X/S}$ is free on stalks (see [GW20, prop. 7.41]). Let $x \in X$ be a point and r be the relative dimension of g at x. We prove that $(\Omega^1_{X/S})_x$ is a free $\mathcal{O}_{X,x}$-module of rank r. Take affine open subsets $U \subset X$ containing x and $V \subset S$, as in definition 1.4.1. We denote by

$$A := \mathcal{O}_{S,g(x)}, \qquad B := \mathcal{O}_{\mathbb{A}^n_R,x} \qquad \text{and} \qquad C := \mathcal{O}_{X,x}.$$

By proposition 1.3.6, we have that

$$(\Omega^1_{X/S})_x \cong \Omega^1_{C/A}.$$

Hence, we have to prove that $\Omega^1_{C/A}$ is a free C-module of rank r. Notice that

$$B \cong A[t_{1,x}, \dots, t_{n,x}] \qquad \text{and} \qquad C \cong B/I_x,$$

where I_x is the ideal generated by the stalks $f_{1,x}, \dots, f_{n-r,x} \in B$. Then, $\Omega^1_{B/A}$ is the free B-module generated by $dt_{1,x}, \dots, dt_{n,x}$

$$\Omega^1_{B/A} \cong \oplus_{i=1}^n B dt_{i,x}.$$

Let $k(x)$ be the residue field of B. Consider the $k(x)$-vector space

$$\Omega^1_{B/A} \otimes_B k(x) \cong \oplus_{i=1}^n k(x) dt_i(x).$$

Let $df_1(x), \dots, df_{n-r}(x) \in \Omega^1_{B/A} \otimes_B k(x)$ be the images of $df_{1,x}, \dots, df_{n-r,x} \in \Omega^1_{B/A}$ along the canonical morphism $\Omega^1_{B/A} \to \Omega^1_{B/A} \otimes_B k(x)$. They are such that, for each $i = 1, \dots, n$,

$$df_i(x) = \sum_{j=1}^n \frac{\partial f_i}{\partial t_j}(x) dt_j(x).$$

Since the Jacobian matrix evaluated at x has maximal rank $n-r$, then the elements $df_1(x), \dots, df_{n-r}(x) \in \Omega^1_{B/A} \otimes_B k(x)$ are $k(x)$-linearly independent. Assuming that the last $n-r$ columns of the Jacobian matrix of $f_1, \dots, f_{n-r}$ evaluated at x

$$J_{f_1,\dots,f_{n-r}}(x) := \left[\frac{\partial f_i}{\partial t_j}(x)\right]_{\substack{i=1,\dots,n-r \\ j=1,\dots,n}}$$

are $k(x)$-linearly independent, then we can complete the $k(x)$-linearly independent set $\{df_1(x), \dots, df_{n-r}(x)\}$ with $dt_1(x), \dots, dt_r(x)$ to obtain a $k(x)$-basis of $\Omega^1_{B/A} \otimes_B k(x)$. Since $\Omega^1_{B/A}$ is a finitely generated B-module, then, by Nakayama lemma, $\Omega^1_{B/A}$ is generated by $df_{1,x}, \dots, df_{n-r,x}, dt_{1,x}, \dots, dt_{r,x}$ as a B-module. Since $\Omega^1_{B/A}$ is a free B-module of rank n, then, they are also free generators

$$\Omega^1_{B/A} \cong (\oplus_{i=1}^{n-r} B df_{i,x}) \oplus (\oplus_{i=1}^r B dt_{i,x}).$$

Consider the conormal sequence for $A \to B \twoheadrightarrow C$ (see [Eis95, Prop. 16.3]). It is the exact sequence of C-modules

$$I_x/I_x^2 \to \Omega^1_{B/A} \otimes_B C \to \Omega^1_{C/A} \to 0.$$

Since I_x/I_x^2 is generated by classes of $f_{1,x}, \dots, f_{n-r,x}$, which are sent into $df_{1,x}, \dots, df_{n-r,x}$ in $\Omega_{B/A} \otimes_B C$, and

$$\Omega^1_{B/A} \otimes_B C \cong (\oplus_{i=1}^{n-r} Cdf_{i,x}) \oplus (\oplus_{i=1}^{r} Cdt_{i,x}),$$

then, the coker of $I_x/I_x^2 \to \Omega^1_{B/A} \otimes_B C$ is

$$\Omega^1_{C/A} \cong \oplus_{i=1}^{r} Cdt_{i,x}.$$

Hence, $\Omega^1_{C/A}$ is a free C-module of rank r. □

Definition 1.4.7. Given $g : X \to S$ a smooth morphism of relative dimension r at a point $x \in X$, by proposition 1.4.6, there exist $u_{1,x}, \dots u_{r,x} \in \mathcal{O}_{X,x}$, such that their images $du_{1,x}, \dots du_{r,x}$ along $d_x : \mathcal{O}_{X,x} \to (\Omega^1_{X/S})_x$, the stalk of the universal derivation, are free generators of $(\Omega^1_{X/S})_x$ as an $\mathcal{O}_{X,x}$-module

$$(\Omega^1_{X/S})_x \cong \oplus_{i=1}^{r} \mathcal{O}_{X,x} du_{i,x},$$

Such a set $\{u_{1,x}, \dots u_{r,x}\}$ is called a *system of local parameters* of g at x.

Remark 1.4.8. Let's see more explicitly how to describe a system of local parameters at a point $x \in X$. Take affine open subsets $U \subset X$ containing x and $V \subset S$, as in definition 1.4.1. Then, we have a closed immersion

$$U \hookrightarrow \mathbb{A}^n_R.$$

Consider the corresponding morphism on global sections of the structural sheaves

$$\mathcal{O}_X(U) \leftarrow R[t_1, \dots, t_n].$$

We denote by $u_1, \dots, u_n \in \mathcal{O}_X(U)$ the images of $t_1, \dots, t_n \in R[t_1, \dots, t_n]$. Assuming that the Jacobian matrix of $f_1, \dots, f_{n-r}$ evaluated at x has the last $n - r$ columns which are linearly independent, by the proof of the previous proposition, we see that $(\Omega^1_{X/S})_x$ is the free $\mathcal{O}_{X,x}$-module generated by the stalks $du_{1,x}, \dots, du_{r,x}$. Hence, $\{u_{1,x} \dots u_{r,x}\}$ is a system of local parameters.

An immediate consequence of proposition 1.4.6 is the following characterization of étale morphisms. Recall that a morphism of schemes $g : X \to S$ locally of finite presentation is *unramified at a point* $x \in X$, if $(\Omega^1_{X/S})_x = 0$ and it is *unramified*, if it is unramified at all its points.

Corollary 1.4.9. *Let $g : X \to S$ be a morphism of schemes locally of finite presentation. Then, g is an étale morphism if and only if it is smooth and unramified.*

Proof. It follows from proposition 1.4.6 and definition of étale morphisms. □

1.4.2 The étale topology

Now, we want to define a new Grothendieck topology on schemes and obtain a characterization of smooth morphisms of schemes, following the idea described at the beginning of this section.

In the following proposition we collect some properties of smooth and étale morphisms of schemes.

Proposition 1.4.10. *The following facts hold true:*

(1) Let $f : X \to Y$ and $g : Y \to Z$ be smooth morphisms of relative dimension r and s respectively. Then, $gf : X \to Z$ is smooth of relative dimension $r + s$. In particular, composition of étale morphisms is étale.

(2) Smooth (étale) morphisms are stable under base change.

(3) Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of finite presentation, with g unramified. If gf is smooth (étale), then also f is.

(4) Open immersions are étale morphisms.

Proof. For (1) and (2), see [Bos12, §8.5, Prop. 2]. For (3), see [Bos12, §8.5, Lemma 11]. (4) is true because open immersions trivially satisfy the definition of étale morphisms of schemes. □

Definition 1.4.11. Given X a scheme, we define an *étale cover* of X any family of étale morphisms of schemes

$$\{f_i : X_i \to X\}_{i \in I},$$

such that $X = \bigcup_{i \in I} f_i(X_i)$. We define the *small étale site* over X

$$X_{ét}$$

the site with underlying category $\acute{E}t/X$, the category of étale morphisms over X, and covering families of objects given by étale covers. Properties (1), (2) and (3) in proposition 1.4.10 and corollary 1.4.10 assure that étale covers define a Grothendieck topology on this site, called *étale topology.* The étale sheaf of rings

$$\mathcal{O}_X^{ét} : U \mapsto \mathcal{O}_U(U)$$

endows $X_{ét}$ with a structure of ringed site, called *small étale ringed site*

$$(X_{ét}, \mathcal{O}_X^{ét}).$$

Remark 1.4.12. For any $f : X \to Y$ morphism of schemes, we have an induced morphism on small étale ringed sites

$$f^{ét} : (X_{ét}, \mathcal{O}_X^{ét}) \to (Y_{ét}, \mathcal{O}_Y^{ét}).$$

Indeed, since étale morphisms are stable under base change, we have a functor

$$\begin{aligned} \acute{E}t/Y &\to \acute{E}t/X \\ V &\mapsto V \times_Y X. \end{aligned}$$

By definition of étale covers, this functor is continuous. Hence, it gives rise to a morphism of sites $f : X_{ét} \to Y_{ét}$. Moreover, we have a morphism of sheaves of rings over $Y_{ét}$

$$\mathcal{O}_Y^{ét} \to f_*^{ét} \mathcal{O}_X^{ét},$$

given by, for any $V \in \acute{E}t/Y$,

$$\mathcal{O}_V(V) \to \mathcal{O}_{V \times_Y X}(V \times_Y X),$$

the global sections of the morphism on the structural sheaves of the canonical projection of the fiber product $V \times_Y X \to V$. The adjoint morphism of sheaves of rings

$$f^{ét^{-1}} \mathcal{O}_Y^{ét} \to \mathcal{O}_X^{ét}$$

defines the morphism on small étale ringed sites.

Remark 1.4.13. Given X a scheme, property (4) in proposition 1.4.10 tells that we have the inclusion functor

$$\mathrm{Op}(X) \hookrightarrow \acute{E}t/X.$$

By definition of Zariski and étale covers, this inclusion functor is continuous. Hence, it defines a morphism of sites

$$\pi_X : X_{ét} \to X_{Zar}.$$

Then, we have the pair of adjoint functors

$$\pi_X^{-1} : \mathrm{Ab}(X_{Zar}) \rightleftarrows \mathrm{Ab}(X_{ét}) : \pi_{X*}.$$

Moreover, notice that $\pi_{X*}\mathcal{O}_X^{ét}$ is the restriction of $\mathcal{O}_X^{ét}$ to $\mathrm{Op}(X)$, hence it coincides with the structural sheaf of X

$$\mathcal{O}_X \cong \pi_{X*}\mathcal{O}_X^{ét}.$$

The adjoint morphism of étale sheaves of rings

$$\pi_X^{-1}\mathcal{O}_X \to \mathcal{O}_X^{\acute{e}t}$$

defines a morphism of ringed sites

$$\pi_X : (X_{\acute{e}t}, \mathcal{O}_X^{\acute{e}t}) \to (X_{Zar}, \mathcal{O}_X).$$

It's useful the following characterization of étale morphisms between smooth schemes over a base.

Proposition 1.4.14. *Let $f : X \to Y$ be a morphism of S-schemes locally of finite presentation and $x \in X$ a point. Let $s \in S$ be the image of x along $X \to S$. If X is smooth at x and Y is smooth at $f(x)$, then the following are equivalent:*

(i) f is étale at x.

(ii) The canonical functoriality morphism of sheaves of $\mathcal{O}_X$-modules (proposition 1.3.9)

$$f^*\Omega^1_{Y/S} \to \Omega^1_{X/S},$$

is an isomorphism on the stalk at x

Proof. See [Bos12, §8.5, cor. 12]. □

The following result tells that smooth S-schemes are those *étale-locally* isomorphic to an affine space over S.

Proposition 1.4.15. *Let $g : X \to S$ be a morphisms of schemes locally of finite presentation and $x \in X$ a point. Then, f is smooth at x of relative dimension r if and only if there exist affine open subsets $U \subset X$ and $Spec(R) \cong V \subset Y$ with $x \in U$ and $f(U) \subset V$, such that there exists an étale morphism of R-schemes*

$$h : U \to \mathbb{A}^r_R.$$

Proof. If h exists, then g is smooth at x of relative dimension r becuse $\mathbb{A}^r_S \to S$ is smooth of relative dimension r, and by property (1) in proposition 1.4.10. Conversely, assume that g is smooth at x of relative dimension r. Let $u_{1,x}, \ldots, u_{r,x} \in \mathcal{O}_{X,x}$ be a system of local parameters of X at x. Then, we can take affine open subsets $U \subset X$, $Spec(R) \cong V \subset S$, with $x \in U$ and $f(U) \subset V$, and sections $u_1, \ldots, u_r \in \mathcal{O}_X(U)$, whose stalks at x are the given system of local parameters, such that

$$\Omega^1_{X/S}\big|_U \cong \oplus_{i=1}^r \mathcal{O}_X\big|_U du_i$$

The sections $u_1, \ldots, u_r \in \mathcal{O}_X(U)$ define a morphism of R-schemes

$$h : U \to \mathbb{A}^r_R \cong Spec(R[t_1, \ldots, t_r]),$$

corresponding to the morphism of R-algebras

$$\begin{aligned} R[t_1, \dots, t_r] &\to \mathcal{O}_X(U) \\ t_i &\mapsto u_i. \end{aligned}$$

Hence, for any $y \in U$, the canonical morphism of $\mathcal{O}_{X,y}$-modules

$$(h^*\Omega^1_{\mathbb{A}^r_R/R})_y = (\Omega^1_{\mathbb{A}^r_R/R})_{h(y)} \otimes_{\mathcal{O}_{\mathbb{A}^r_R,h(y)}} \mathcal{O}_{X,y} \to (\Omega^1_{X/S})_y \tag{1.3}$$

maps $dt_{i,y}$ into $du_{i,y}$, for $i = 1, \dots, r$. Since

$$(\Omega^1_{\mathbb{A}^r_R/R})_{h(y)} \cong \oplus_{i=1}^r \mathcal{O}_{\mathbb{A}^r_R,h(y)} dt_{i,y}$$

and

$$(\Omega^1_{X/S})_y \cong \oplus_{i=1}^r \mathcal{O}_{X,y} du_{i,y},$$

then, the canonical morphism 1.3 is an isomorphism. By proposition 1.4.14, h is étale at y. Since this holds for any $y \in U$, then h is étale. □

1.4.3 The étale-analytic topology

Another consequence of proposition 1.4.14 is the following result, which gives a geometric interpretation of étale morphisms, in case of smooth algebraic varieties over $\mathbb{C}$, via the corresponding property of its analytification.

Corollary 1.4.16. *Let*

$$f : X \to Y$$

be a morphism of smooth algebraic varieties over $\mathbb{C}$*. Then,* f *is étale if and only if its analytification*

$$f^{an} : X(\mathbb{C}) \to Y(\mathbb{C})$$

is a local isomorphism of smooth complex analytic spaces.

Proof. Let $x \in X(\mathbb{C})$ be a closed point of X. By the Inverse Function Theorem, f^{an} is a local isomorphism at x if and only if the canonical functoriality morphism of sheaves of $\mathcal{O}_{X(\mathbb{C})}$-modules between the sheaves of holomorphic 1-forms

$$f^{an*}\Omega^1_{Y(\mathbb{C})} \to \Omega^1_{X(\mathbb{C})} \tag{1.4}$$

is an isomorphism on the stalk at x. We will see in proposition 1.5.10 that the analytification of the sheaf of modules of Kähler differentials of X over $\mathbb{C}$ is canonically isomorphic to the sheaf of holomorphic 1-forms of $X(\mathbb{C})$, and the same holds for Y. Moreover, these canonical isomorphisms are compatible with the canonical functoriality morphisms. That is, the morphism 1.4 is the analytification of the canonical functoriality morphism of sheaves of $\mathcal{O}_X$-modules between the sheaves of modules of Kähler differentials (proposition 1.3.9)

$$f^*\Omega^1_{Y/\mathbb{C}} \to \Omega^1_{X/\mathbb{C}}. \tag{1.5}$$

By proposition 1.2.19, the analytification functor of sheaves of $\mathcal{O}_X$-modules is faithfully-flat, hence it preserves and reflects isomorphisms. So, 1.5 is an isomorphism on stalk at x if and only if 1.4 is an isomorphism on stalk at x. By proposition 1.4.14, this means that f is étale at x if and only if f^{an} is a local isomorphism at x. Since being étale is a local property and $X(\mathbb{C})$ is very dense in X, then being étale for f can be checked at closed points of X. Hence, f is étale if and only if f^{an} is a local isomorphism. □

Remark 1.4.17. Given X a smooth algebraic variety over $\mathbb{C}$, by proposition 1.4.4, we know that the analytification $X(\mathbb{C})$ is a smooth complex analytic space. We want to describe explicitly local charts. Let $x \in X(\mathbb{C})$, such that X has relative dimension n over $\mathbb{C}$ at x. By proposition 1.4.15, there exists an affine open subset $U \subset X$ containing x, such that there exists an étale morphism of $\mathbb{C}$-schemes

$$h : U \to \mathbb{A}^n_{\mathbb{C}}.$$

By corollary 1.4.16, its analytification

$$h^{an} : U(\mathbb{C}) \to \mathbb{C}^n$$

is a local isomorphism. Then, we can choose an open subset $\mathcal{W} \subset U(\mathbb{C})$ containing x, such that h^{an} restricts on $\mathcal{W}$ to an isomorphism into an open subset of $\mathbb{C}^n$. Such $\mathcal{W}$ is a local chart of $X(\mathbb{C})$ at x, with local coordinates given by $w_1, \dots, w_n \in \mathcal{O}_{X(\mathbb{C})}(\mathcal{W})$ the images of $t_1, \dots, t_n$ along

$$\mathbb{C}[t_1, \dots, t_n] \xrightarrow{h^\#} \mathcal{O}_X(U) \xrightarrow{\alpha_X^\#} \mathcal{O}_{X(\mathbb{C})}(U(\mathbb{C})) \to \mathcal{O}_{X(\mathbb{C})}(\mathcal{W}).$$

The last corollary suggests to consider also another topology for complex analytic spaces, that will be useful later.

Definition 1.4.18. Given $\mathcal{Y}$ a complex analytic space, we define an *étale-analytic cover* of $\mathcal{Y}$ any family of local isomorphisms (or *étale-analytic morphisms*) of complex analytic spaces

$$\{f_i : \mathcal{Y}_i \to \mathcal{Y}\}_{i \in I},$$

such that $\mathcal{Y} = \bigcup_{i \in I} f_i(\mathcal{Y}_i)$. We define the *small étale-analytic site* of $\mathcal{Y}$

$$\mathcal{Y}_{\textit{ét-an}}$$

the site with underlying category *Ét-An*$/\mathcal{Y}$, the category of local isomorphisms over $\mathcal{Y}$, and covering families of objects given by étale-analytic covers.

Remark 1.4.19. Let $\mathcal{Y}$ be a complex analytic space. Since open immersions are local isomorphisms, we have the inclusion functor

$$\mathrm{Op}(\mathcal{Y}) \hookrightarrow \acute{E}t\text{-}An/\mathcal{Y}.$$

By definition of analytic open and étale-analytic covers, this inclusion functor is continuous. Hence, it defines a morphism of sites

$$\pi^{an}_{\mathcal{Y}} : \mathcal{Y}_{\acute{e}t\text{-}an} \to \mathcal{Y}_{an}.$$

Notice that, since any étale-analytic cover is refined by an analytic open cover and viceversa, then the induced adjunction

$$\pi^{an-1}_{\mathcal{Y}} : Ab(\mathcal{Y}_{an}) \rightleftarrows \mathrm{Ab}(\mathcal{Y}_{\acute{e}t\text{-}an}) : \pi^{an}_{\mathcal{Y}*}$$

defines an equivalence of categories and sheaf cohomologies of corresponding sheaves are isomorphic.

Remark 1.4.20. Let X be a smooth algebraic variety over $\mathbb{C}$. Corollary 1.4.16 tells that the analytification functor restricts to

$$an : \acute{E}t/X \to \acute{E}t\text{-}An/X(\mathbb{C}).$$

By definition of étale and étale-analytic covers, this inclusion functor is continuous. Hence, it defines a morphism of sites

$$\alpha^{\acute{e}t}_X : X(\mathbb{C})_{\acute{e}t\text{-}an} \to X_{\acute{e}t}.$$

1.4.4 Smooth schemes over a field

To conclude, we see some properties of smooth schemes over a field. In this case, the relative dimension is simply called *dimension*. We see that smoothness is related to regularity property.

Recall that a locally noetherian scheme X is *regular at a point* $x \in X$, if

$$dim\mathcal{O}_{X,x} = dim_{k(x)}\mathfrak{m}_x/\mathfrak{m}_x^2,$$

where $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ is the maximal ideal. By Nakayama lemma, there exists a set of generators $u_{1,x}, \dots, u_{r,x}$ of $\mathfrak{m}_x$, with $r = dim\mathcal{O}_{X,x}$, such that their classes $u_1(x), \dots, u_r(x) \in \mathfrak{m}_x/\mathfrak{m}_x^2$ are a $k(x)$-basis. The set $\{u_{1,x}, \dots, u_{r,x}\}$ is called a *system of regular parameters* of X at x. X is *regular*, if it is regular at all its points.

Proposition 1.4.21. *Let X be a scheme locally of finite type over a field k, $x \in X$ a point. If X is smooth at x of dimension r, then X is regular at x with $r = dim\mathcal{O}_{X,x}$. Moreover, a system of regular parameters at x is also a system of local parameters at*

x. If k is perfect, also the converse holds true: if X is regular at x with $r = dim\mathcal{O}_{X,x}$, then X is smooth at x of dimension r.

Proof. See [Bos12, §8.5, prop. 15]. □

Remark 1.4.22. Notice that, given X a scheme locally of finite type over a field k and $x \in X$ a closed point, is not true that a system of local parameters at x is also a system of regular parameters at x! For example, take $X = \mathbb{A}^2_k \cong Spec(k[t_1, t_2])$ and x the point corresponding to the prime ideal $(t_1 - 1, t_2 - 1)$. Since

$$(\Omega^1_{\mathbb{A}^2_k})_x \cong \oplus_{i=1,2}\mathcal{O}_{\mathbb{A}^2_k,x} dt_{i,x},$$

then $\{t_{1,x}, t_{2,x}\}$ is system of local parameters of $\mathbb{A}^2_k$ at x, but it is not a system of regular parameters, since they are invertible elements in $\mathcal{O}_{\mathbb{A}^2_k,x}$, so they don't generate the maximal ideal. Viceversa, notice that a system of regular parameters is given by $\{u_{1,x}, u_{2,x}\}$, where $u_i \coloneqq t_i - 1$, for $i = 1, 2$. In agreement with proposition 1.4.21, it is also a system of local parameters, since $du_{i,x} = dt_{i,x}$, for $i = 1, 2$.

Corollary 1.4.23. *Let X be a scheme locally of finite type over a field k. If X is smooth, then X is reduced and its irreducible components coincide with connected components.*

Proof. By proposition 1.4.21, X is regular. This means that $\mathcal{O}_{X,x}$ is a regular local ring for any $x \in X$. By Auslander-Buchsbaum Theorem, $\mathcal{O}_{X,x}$ is an UFD, in particular is reduced, for any $x \in X$. So X is reduced. Moreover, in particular $\mathcal{O}_{X,x}$ is a domain, hence x belongs to exactly one irreducible component of X. So, irreducible components are disjoint and they coincide with connected components. □

1.5 The algebraic de Rham cohomology

Given X an algebraic variety over a field k, we defined its algebraic de Rham complex (subsection 1.3.2)

$$\Omega^\bullet_{X/k} \; : \; \mathcal{O}_X \to \Omega^1_{X/k} \to \Omega^2_{X/k} \to \cdots .$$

If X is smooth, it gives rise to a good cohomology theory for X, which, in case $k = \mathbb{C}$, coincides with the analytic de Rham cohomology of the analytification $X(\mathbb{C})$. The aim of this section is to prove this fact.

1.5.1 The algebraic de Rham cohomology

Definition 1.5.1. Let X be a smooth algebraic variety over a field k. The cohomology of X with coefficients in the algebraic de Rham complex

$$\mathrm{H}^i_{\mathrm{AdR}}(X/k) \coloneqq \mathbb{H}^i(X_{Zar}, \Omega^\bullet_{X/k})$$

is called the *algebraic de Rham cohomology of X*.

Remark 1.5.2. Algebraic de Rham cohomology is defined as sheaf cohomology. However, it can also be computed as Čech cohomology relative to an affine open cover. Indeed, given X an algebraic variety over $\mathbb{C}$, let $\mathcal{U} = \{U_i\}_{i \in I}$ be an affine open cover of X. Since X is separated, then also the finite intersections $U_{i_0 \dots i_n} \coloneqq U_{i_0} \cap \cdots \cap U_{i_n}$ are affine. Since by proposition 1.3.6, for any $p \geq 0$, $\Omega^p_{X/k}$ is a coherent sheaf of $\mathcal{O}_X$-modules, it holds that (see [Har77, §III, Thm. 3.5])

$$\mathbb{H}^q(U_{i_0 \dots i_n}, \Omega^p_{X/k}) = 0 \qquad \text{for each } n \geq 0 \text{ and } q > 0.$$

By Leray's Theorem, it follows that, for each $i \geq 0$,

$$\mathbb{H}^i(X_{Zar}, \Omega^\bullet_{X/k}) \cong \check{\mathrm{H}}^i(\mathcal{U}; \Omega^\bullet_{X/k}).$$

This fact is useful to compute some examples of algebraic de Rham cohomology.

Example 1.5.3. 1) $X = Spec(k)$. Since $Spec(k)$ is affine, we choose the affine open cover $\mathcal{U} = \{Spec(k)\}$. The Čech complex of $\Omega^\bullet_{Spec(k)/k}$ relative to $\mathcal{U}$ is

$$0 \to k \to 0.$$

Hence, we get

$$\mathrm{H}^i_{\mathrm{AdR}}(Spec(k)/k) \cong \begin{cases} k & \text{for } i = 0 \\ 0 & \text{else.} \end{cases}$$

2) $X = \mathbb{A}^1_k$. Since $\mathbb{A}^1_k \cong Spec(k[t])$ is affine, we choose the affine open cover $\mathcal{U} = \{\mathbb{A}^1_k\}$. The Čech complex of $\Omega^\bullet_{\mathbb{A}^1_k/k}$ relative to $\mathcal{U}$ is

$$0 \to k[t] \to k[t]dt \to 0.$$

If $char(k) = 0$, then polynomials with differential zero are the constant ones and every polynomial is differential of some other polynomial. Hence, we get

$$\mathrm{H}^i_{\mathrm{AdR}}(\mathbb{A}^1_k/k) \cong \begin{cases} k & \text{for } i = 0 \\ 0 & \text{else.} \end{cases}$$

3) $X = \mathbb{G}_m$. Since $\mathbb{G}_m \cong Spec(k[t,u]/(tu-1))$ is affine, we choose the affine open cover $\mathcal{U} = \{\mathbb{G}_m\}$. The Čech complex of $\Omega^\bullet_{\mathbb{G}_m/k}$ relative to $\mathcal{U}$ is

$$0 \to \frac{k[t,u]}{(tu-1)} \to \left(\frac{k[t,u]}{(tu-1)} dt \oplus \frac{k[t,u]}{(tu-1)} du \right) / \langle udt + tdu \rangle \to 0.$$

If $char(k) = 0$, polynomials with differential zero are again the constant ones and $udt = dt/t = -du/u = -tdu$ is not differential of any polynomial, hence generates cohomology in degree 1. Hence, we get

$$\mathrm{H}^i_{\mathrm{AdR}}(\mathbb{G}_m/k) \cong \begin{cases} k & \text{for } i = 0, 1 \\ 0 & \text{else.} \end{cases}$$

4) $X = \mathbb{P}^1_k = Proj(k[z_0, z_1])$. Consider the affine open cover $\mathcal{U} = \{U_0, U_1\}$, where

$$\begin{aligned} U_0 &:= D_+(z_0) \cong Spec(k[z_1/z_0]) \cong Spec(k[t]) \cong \mathbb{A}^1_k \\ U_1 &:= D_+(z_1) \cong Spec(k[z_0/z_1]) \cong Spec(k[u]) \cong \mathbb{A}^1_k \\ U_0 \cap U_1 &\cong D(t) \cong D(u) \cong Spec(k[t, u]/(tu - 1)) \cong \mathbb{G}_m. \end{aligned}$$

The Čech complex of $\Omega^\bullet_{\mathbb{P}^1_k/k}$ relative to $\mathcal{U}$ is the total complex of the double complex

$$\begin{array}{ccccc} 0 & & 0 & & \\ \uparrow & & \uparrow & & \\ \frac{k[t,u]}{(tu-1)} & \to & (\frac{k[t,u]}{(tu-1)}dt \oplus \frac{k[t,u]}{(tu-1)}du)/\langle udt + tdu \rangle & \to & 0 \\ \uparrow & & \uparrow & & \\ k[t] \oplus k[u] & \to & k[t]dt \oplus k[u]du & \to & 0. \end{array}$$

Assume $char(k) = 0$. In degree 0, a cycle is a pair of constant polynomials $(c, -c)$. In degree 1, a cycle is a triple $(F(t, u), f(t)dt, g(u)du)$, such that

$$\frac{\partial F(t,u)}{\partial t} = f(t) + cu \qquad \& \qquad \frac{\partial F(t,u)}{\partial u} = g(u) + ct,$$

for some $c \in k$. This implies that $F(t, u)$ is of the kind $F_1(t) + F_2(u)$ in $k[t, u]/(tu - 1)$, with

$$\frac{\partial F_1(t)}{\partial t} = f(t) \qquad \& \qquad \frac{\partial F_2(u)}{\partial u} = g(u).$$

But then, the element $(F(t, u), f(t)dt, g(u)du)$ is the boundary of the pair of polynomials $(F_1(t), F_2(u))$ in degree 0. In degree 2, the element $\omega := dt/t = -du/u$ is not a boundary. So, it generates the cohomology in degree 2. Hence, we get

$$\mathrm{H}^i_{\mathrm{AdR}}(\mathbb{P}^1_k/k) \cong \begin{cases} k & \text{for } i = 0, 2 \\ 0 & \text{else.} \end{cases}$$

Now, we see some properties of the algebraic de Rham cohomology.

Proposition 1.5.4 (Functoriality). *Let $f : X \to Y$ be a morphism of smooth algebraic varieties over a field k. Then, we have canonical morphisms of k-vector* *spaces, for each $i \geq 0$,*

$$H^i_{AdR}(Y/k) \to H^i_{AdR}(X/k).$$

They define a contravariant functor

$$H^i_{AdR}(\_/k) : Sm_k \to Vect_k.$$

Proof. By proposition 1.3.9, we have the canonical morphism of complexes of abelian sheaves over X_{Zar}

$$f^{-1}\Omega^\bullet_{Y/k} \to \Omega^\bullet_{X/k}.$$

It induces the morphisms on sheaf cohomology

$$\mathbb{H}^i(X_{Zar}, f^{-1}\Omega^\bullet_{Y/k}) \to \mathbb{H}^i(X_{Zar}, \Omega^\bullet_{X/k}) = \mathrm{H}^i_{\mathrm{AdR}}(X/k).$$

Composing with the functoriality morphisms of sheaf cohomology

$$\mathrm{H}^i_{\mathrm{AdR}}(Y/k) = \mathbb{H}^i(Y_{Zar}, \Omega^\bullet_{Y/k}) \to \mathbb{H}^i(X_{Zar}, f^{-1}\Omega^\bullet_{Y/k}),$$

we get the canonical morphisms of abelian groups

$$\mathrm{H}^i_{\mathrm{AdR}}(Y/k) \to \mathrm{H}^i_{\mathrm{AdR}}(X/k).$$

Since all the considered morphisms of complexes of abelian sheaves are also k-linear, then these are also morphisms of k-vector spaces. Functorial properties follow from functoriality of inverse image and naturality of the functoriality morphisms of sheaf cohomology. □

Proposition 1.5.5 (Künneth formula)**.** *Let X and Y be smooth algebraic varieties over a field k. Then, we have canonical isomorphisms of k-vector spaces, for each $i \geq 0$,*

$$H^i_{AdR}(X \times_k Y/k) \cong \oplus_{p+q=i} H^p_{AdR}(X/k) \otimes_k H^q_{AdR}(Y/k)$$

Proof. See [Stacks, Tag 0FM9]

□

Proposition 1.5.6 ($\mathbb{A}^1$-invariance)**.** *Let X be a smooth algebraic variety over a field k. Then, we have canonical isomorphisms of k-vector spaces, for each $i \geq 0$,*

$$H^i_{AdR}(X \times_k \mathbb{A}^1_k/k) \cong H^i_{AdR}(X/k).$$

Proof. Recall from example 1.5.3 that

$$H^i_{AdR}(\mathbb{A}^1_k/k) \cong \begin{cases} k & \text{for } i = 0 \\ 0 & \text{else.} \end{cases}$$

By Künneth formula (proposition 1.5.5), we obtain that, for each $i \geq 0$,

$$H^i_{AdR}(X \times_k \mathbb{A}^1_k) \cong \oplus_{p+q=i} H^p_{AdR}(X) \otimes_k H^q_{AdR}(\mathbb{A}^1_k) \cong$$
$$\cong H^i_{AdR}(X) \otimes_k k \cong H^i_{AdR}(X).$$

□

Another property tells that the algebraic de Rham cohomology can be computed also as a sheaf cohomology of a complex of étale sheaves. This complex is obtained by applying the general construction of the algebraic de Rham complex to the morphism on small étale ringed sites. More precisely, recall that, given a morphism of schemes $g : X \to S$, we have the morphism of ringed sites (remark 1.4.12)

$$g^{ét} : (X_{ét}, \mathcal{O}^{ét}_X) \to (S_{ét}, \mathcal{O}^{ét}_S).$$

Applying the general construction of algebraic de Rham complex to this morphism of ringed sites, we get a complex of abelian sheaves over $X_{ét}$

$$(\Omega^\bullet_{X/S})^{ét} := \Omega^\bullet_{\mathcal{O}^{ét}_X / f^{ét-1}\mathcal{O}^{et}_S}.$$

Remark 1.5.7. Recall that we have the morphism of ringed sites (remark 1.4.13)

$$\pi_X : (X_{ét}, \mathcal{O}^{ét}_X) \to (X_{Zar}, \mathcal{O}_X).$$

Notice that, for each $p \geq 0$, we have a canonical isomorphism of $\mathcal{O}^{ét}_X$-modules

$$\pi^*_X \Omega^p_{X/S} \cong (\Omega^p_{X/S})^{ét}.$$

Indeed, we have the commutative diagram of ringed sites

$$\begin{tikzcd} (X_{ét}, \mathcal{O}^{ét}_X) \arrow[r, "g^{ét}"] \arrow[d, "\pi_X"] & (S_{ét}, \mathcal{O}^{ét}_S) \arrow[d, "\pi_S"] \\ (X_{Zar}, \mathcal{O}_X) \arrow[r, "g"] & (S_{Zar}, \mathcal{O}_S), \end{tikzcd}$$

which induces the commutative square of sheaves of rings over $X_{ét}$

$$\begin{tikzcd} \pi^{-1}_X g^{-1} \mathcal{O}_S \arrow[r] \arrow[d] & \pi^{-1}_X \mathcal{O}_X \arrow[d] \\ g^{ét^{-1}} \mathcal{O}^{ét}_S \arrow[r] & \mathcal{O}^{ét}_X. \end{tikzcd}$$

On one hand, by proposition 1.3.4, we have the canonical isomorphism of sheaves of

$\mathcal{O}_X^{ét}$-modules

$$\Omega^p_{\pi_X^{-1}\mathcal{O}_X/\pi_X^{-1}g^{-1}\mathcal{O}_S} \otimes_{\pi_X^{-1}\mathcal{O}_X} \mathcal{O}_X^{ét} \cong \pi_X^{-1}\Omega^p_{\mathcal{O}_X/g^{-1}\mathcal{O}_S} \otimes_{\pi_X^{-1}\mathcal{O}_X} \mathcal{O}_X^{ét} = \pi_X^*\Omega^p_{X/S}.$$

On the other hand, by proposition 1.3.5, we have the canonical isomorphism of sheaves of $\mathcal{O}_X^{ét}$-modules [2]

$$\Omega^p_{\pi_X^{-1}\mathcal{O}_X/\pi_X^{-1}g^{-1}\mathcal{O}_S} \otimes_{\pi_X^{-1}\mathcal{O}_X} \mathcal{O}_X^{ét} \cong \Omega^p_{\mathcal{O}_X^{ét}/g^{ét^{-1}}\mathcal{O}_S^{ét}} = (\Omega^p_{X/S})^{ét}.$$

Remark 1.5.8. For each $p \geq 0$, since $\Omega^p_{X/S}$ is a quasi-coherent $\mathcal{O}_X$-module, then we have the following explicit description of $\pi_X^*\Omega^p_{X/S}$ (see [Stacks, Tag 070S] and [Stacks, Tag 03DV]): for any $f : U \to X$ étale morphism

$$\pi_X^*\Omega^p_{X/S}(U) \cong \Gamma(U, f^*\Omega^p_{X/S}).$$

If moreover X is smooth over S, by proposition 1.4.14, we have that

$$f^*\Omega^p_{X/S} \cong \Omega^p_{U/S}.$$

Hence, in this case we have that $(\Omega^p_{X/S})^{ét} \cong \pi_X^*\Omega^p_{X/S}$ is the étale sheaf

$$(\Omega^p_{X/S})^{ét} \ : \ U \mapsto \Gamma(U, \Omega^p_{U/S}).$$

Proposition 1.5.9 (Étale descent)**.** *Let X be a smooth algebraic variety over a field k. The complex of abelian sheaves over $X_{ét}$*

$$(\Omega^\bullet_{X/k})^{ét} := \Omega^\bullet_{\mathcal{O}_X^{ét}/f^{ét^{-1}}\mathcal{O}^{et}_{Spec(k)}}$$

computes the algebraic de Rham cohomology of X. That is, we have canonical isomorphisms of k-vector spaces, for each $i \geq 0$,

$$H^i_{AdR}(X/k) \cong \mathbb{H}^i(X_{ét}, (\Omega^\bullet_{X/k})^{ét}).$$

Proof. By propositions 1.3.4 and 1.3.5 applied to the commutative square of sheaves of rings in remark 1.5.7, we have a canonical morphism of complexes of abelian sheaves over $X_{ét}$

$$\pi_X^{-1}\Omega^\bullet_{X/k} \cong \Omega^\bullet_{\pi_X^{-1}\mathcal{O}_X/\pi_X^{-1}g^{-1}\mathcal{O}_{Spec(k)}} \to \Omega^\bullet_{\mathcal{O}_X^{ét}/g^{ét^{-1}}\mathcal{O}^{ét}_{Spec(k)}} = (\Omega^\bullet_{X/k})^{ét}.$$

[2]The square of sheaves of rings is not object-wise cocartesian. However, it is on an affine open cover of X and this is indeed sufficient to conclude the isomorphism.

It induces the morphisms on sheaf cohomology

$$\mathbb{H}^i(X_{\acute{e}t}, \pi_X^{-1}\Omega^\bullet_{X/k}) \to \mathbb{H}^i(X_{\acute{e}t}, (\Omega^\bullet_{X/k})^{\acute{e}t}).$$

Composing with the functoriality morphisms of sheaf cohomology

$$\mathrm{H}^i_{\mathrm{AdR}}(X/k) = \mathbb{H}^i(X_{Zar}, \Omega^\bullet_{X/k}) \to \mathbb{H}^i(X_{\acute{e}t}, \pi_X^{-1}\Omega^\bullet_{X/k}),$$

we get the canonical morphisms of abelian groups

$$\mathrm{H}^i_{\mathrm{AdR}}(X/k) \to \mathbb{H}^i(X_{\acute{e}t}, (\Omega^\bullet_{X/k})^{\acute{e}t}). \tag{1.6}$$

Since all the considered morphisms of complexes of abelian sheaves are also k-linear, then these are also morphisms of k-vector spaces. We prove that these canonical morphisms of k-vector spaces are isomorphisms. For each $p \geq 0$, since $\Omega^p_{X/k}$ is a quasi-coherent $\mathcal{O}_X$-module and $(\Omega^p_{X/k})^{\acute{e}t} \cong \pi_X^*\Omega^p_{X/k}$ (remark 1.5.7), then we have canonical isomorphisms (see [Stacks, Tag 03DW]), for each $q \geq 0$,

$$\mathbb{H}^q(X_{Zar}, \Omega^p_{X/k}) \cong \mathbb{H}^q(X_{\acute{e}t}, (\Omega^p_{X/k})^{\acute{e}t}).$$

They define a canonical isomorphism between the hyper-cohomology spectral sequences

$$E_1^{p,q} = \mathbb{H}^q(X_{Zar}, \Omega^p_{X/S}) \Rightarrow \mathbb{H}^{p+q}(X_{Zar}, \Omega^\bullet_{X/k}) = \mathrm{H}^{p+q}_{\mathrm{AdR}}(X/k)$$

and

$$E_1^{p,q} = \mathbb{H}^q(X_{\acute{e}t}, (\Omega^p_{X/k})^{\acute{e}t}) \Rightarrow \mathbb{H}^{p+q}(X_{\acute{e}t}, (\Omega^\bullet_{X/k})^{\acute{e}t}).$$

Hence, we conclude that the morphisms induced on the limits of the spectral sequences, which are the canonical morphisms 1.6, are isomorphisms. □

1.5.2 Comparison of algebraic and analytic de Rham cohomology

Now, we consider $k = \mathbb{C}$. As explained in the subsection 1.2.2 on GAGA Theorems, we have an algebraic and an analytic way to study algebraic varieties in this case. For what concerns de Rham cohomology, this means that, given X a smooth algebraic variety over $\mathbb{C}$, we can either consider its algebraic de Rham cohomology

$$\mathrm{H}^i_{\mathrm{AdR}}(X/\mathbb{C}) := \mathbb{H}^i(X_{Zar}, \Omega^\bullet_{X/\mathbb{C}}),$$

or pass to the analytification $X(\mathbb{C})$, which is a smooth analytic space (i.e. a complex manifold, by remark 1.2.10), and consider its analytic de Rham cohomology

$$\mathrm{H}^i_{\mathrm{dR}}(X(\mathbb{C})) := \mathbb{H}^i(X_{an}, \Omega^\bullet_{X(\mathbb{C})}).$$

The aim of the rest of this section is to show that these two approaches give indeed the same objects, that is, we have canonical isomorphisms of $\mathbb{C}$-vector spaces, for each $i \geq 0$,

$$\mathrm{H}^i_{\mathrm{AdR}}(X/\mathbb{C}) \cong \mathrm{H}^i_{\mathrm{dR}}(X(\mathbb{C})).$$

We first prove it in case X is proper, where we can use GAGA Theorem I. Then, we prove the general case, where the idea is to reduce to the proper case.

We start constructing the canonical morphisms of $\mathbb{C}$-vector spaces

$$\mathrm{H}^i_{\mathrm{AdR}}(X/\mathbb{C}) \to \mathrm{H}^i_{\mathrm{dR}}(X(\mathbb{C})).$$

Let $g : X \to \mathit{Spec}(\mathbb{C})$ be the structural morphism. Since the analytic differential

$$d : \mathcal{O}_{X(\mathbb{C})} \to \Omega^1_{X(\mathbb{C})}$$

is $\mathbb{C}$-linear and satisfies the Leibnitz rule, then the composition

$$\mathcal{O}_X \to \alpha_{X*}\mathcal{O}_{X(\mathbb{C})} \xrightarrow{\alpha_{X*}d} \alpha_{X*}\Omega^1_{X(\mathbb{C})}$$

is a $g^{-1}\mathcal{O}_{\mathit{Spec}(\mathbb{C})}$-derivation. By universal property of $\Omega^1_{X/\mathbb{C}}$, there exists a canonical morphism of sheaves of $\mathcal{O}_X$-modules

$$\Omega^1_{X/\mathbb{C}} \to \alpha_{X*}\Omega^1_{X(\mathbb{C})},$$

such that the following diagram commutes

$$\begin{tikzcd} \mathcal{O}_X \arrow[rr, "d"] \arrow[d] & & \Omega^1_{X/\mathbb{C}} \arrow[d] \\ \alpha_{X*}\mathcal{O}_{X(\mathbb{C})} \arrow[rr, "\alpha_{X*}d"] & & \alpha_{X*}\Omega^1_{X(\mathbb{C})}. \end{tikzcd} \tag{1.7}$$

Consider the adjoint morphism of sheaves of $\mathcal{O}_{X(\mathbb{C})}$-modules via the adjunction $(\alpha_X^*, \alpha_{X*})$

$$\alpha_X^*\Omega^1_{X/\mathbb{C}} \to \Omega^1_{X(\mathbb{C})}.$$

Since pullback commutes with exterior powers, we get the morphisms of sheaves of $\mathcal{O}_{X(\mathbb{C})}$-modules, for each $p \geq 0$,

$$\alpha_X^*\Omega^p_{X/\mathbb{C}} \cong \alpha_X^*\left(\bigwedge^p \Omega^1_{X/\mathbb{C}}\right) \cong \bigwedge^p \alpha_X^*\Omega^1_{X/\mathbb{C}} \to \bigwedge^p \Omega^1_{X(\mathbb{C})} \cong \Omega^p_{X(\mathbb{C})}.$$

Taking back the adjoint morphisms of sheaves of $\mathcal{O}_X$-modules via the adjunction

$(\alpha_X^*, \alpha_{X*})$ we get the morphisms of sheaves of $\mathcal{O}_X$-modules

$$\Omega^p_{X/\mathbb{C}} \to \alpha_{X*}\Omega^p_{X(\mathbb{C})}.$$

They are compatible with differentials, that is, we have commutative diagrams

$$\begin{tikzcd}
\Omega^p_{X/\mathbb{C}} \arrow[r, "d^p"] \arrow[d] & \Omega^{p+1}_{X/\mathbb{C}} \arrow[d] \\
\alpha_{X*}\Omega^p_{X(\mathbb{C})} \arrow[r, "\alpha_{X*}d^p"] & \alpha_{X*}\Omega^{p+1}_{X(\mathbb{C})}.
\end{tikzcd}$$

Indeed, since $d^1 = d$, the commutative diagram 1.7 tells that it's true for $p = 1$. Moreover, the analytic differentials are such that $d^{p+1} \circ d^p = 0$, are $\mathbb{C}$-linear and satisfy the Leibnitz rule. This is sufficient to prove the commutativity of the diagram, since these properties characterize the algebraic differentials d^p. So, we have a morphism of complexes of abelian sheaves over X_{Zar}

$$\Omega^\bullet_{X/\mathbb{C}} \to \alpha_{X*}\Omega^\bullet_{X(\mathbb{C})}.$$

Taking the adjoint morphism via the adjunction $(\alpha_X^{-1}, \alpha_{X*})$, we get the canonical morphism of complexes of abelian sheaves over $X(\mathbb{C})_{an}$

$$\alpha_X^{-1}\Omega^\bullet_{X/\mathbb{C}} \to \Omega^\bullet_{X(\mathbb{C})}.$$

It induces the morphisms on sheaf cohomology

$$\mathbb{H}^i(X(\mathbb{C})_{an}, \alpha_X^{-1}\Omega^\bullet_{X/\mathbb{C}}) \to \mathbb{H}^i(X(\mathbb{C})_{an}, \Omega^\bullet_{X(\mathbb{C})}) = \mathrm{H}^i_{\mathrm{dR}}(X(\mathbb{C})).$$

Composing with the functoriality morphism of sheaf cohomology

$$\mathrm{H}^i_{\mathrm{AdR}}(X/\mathbb{C}) = \mathbb{H}^i(X_{Zar}, \Omega^\bullet_{X/\mathbb{C}}) \to \mathbb{H}^i(X(\mathbb{C})_{an}, \alpha_X^{-1}\Omega^\bullet_{X/\mathbb{C}}),$$

we get the canonical morphisms of abelian groups, for each $i \geq 0$,

$$\mathrm{H}^i_{\mathrm{AdR}}(X/\mathbb{C}) \to \mathrm{H}^i_{\mathrm{dR}}(X(\mathbb{C})). \tag{1.8}$$

Since all the considered morphisms of complexes of abelian sheaves are also $\mathbb{C}$-linear, then these are also morphisms of $\mathbb{C}$-vector spaces.

It is fundamental the following fact, which tells that the analytification of the sheaves of modules of Kähler differentials are the sheaves of modules of holomorphic forms.

Proposition 1.5.10. *Let X be a smooth algebraic variety over $\mathbb{C}$. Then, we have* *canonical isomorphisms of sheaves of $\mathcal{O}_{X(\mathbb{C})}$-modules, for each $p \geq 0$,*

$$(\Omega^p_{X/\mathbb{C}})^{an} \cong \Omega^p_{X(\mathbb{C})}.$$

Proof. Consider the canonical morphisms of sheaves of $\mathcal{O}_{X(\mathbb{C})}$-modules constructed above

$$(\Omega^p_{X/\mathbb{C}})^{an} = \alpha_X^* \Omega^p_{X/\mathbb{C}} \to \Omega^p_{X(\mathbb{C})}.$$

We show that these are isomorphisms. It suffices to prove it for $p = 1$, since all others isomorphisms are obtained by taking its exterior powers. We show that we have isomorphisms on the stalk at each point $x \in X(\mathbb{C})$. Let $u_{1,x}, \dots, u_{n,x} \in \mathcal{O}_{X,x}$ be a system of local parameters of X at x. By definition of local parameters, $(\Omega^1_{X/\mathbb{C}})_x$ is freely generated as an $\mathcal{O}_{X,x}$-module by their differentials

$$(\Omega^1_{X/\mathbb{C}})_x \cong \oplus_{i=1}^n \mathcal{O}_{X,x} du_{i,x}.$$

We denote by $w_{1,x}, \dots, w_{n,x} \in \mathcal{O}_{X(\mathbb{C}),x}$ the images of $u_{1,x}, \dots, u_{n,x} \in \mathcal{O}_{X,x}$ along the morphism $\alpha^{\#}_{X,x} : \mathcal{O}_{X,x} \to \mathcal{O}_{X(\mathbb{C}),x}$. By remark 1.4.17, $w_{1,x}, \dots, w_{n,x} \in \mathcal{O}_{X(\mathbb{C}),x}$ are stalks of local coordinates of a local chart of $X(\mathbb{C})$ at x. Hence, $(\Omega^1_{X(\mathbb{C})})_x$ is freely generated as an $\mathcal{O}_{X(\mathbb{C}),x}$-module by their analytic differentials

$$(\Omega^1_{X(\mathbb{C})})_x \cong \oplus_{i=1}^n \mathcal{O}_{X(\mathbb{C}),x} dw_{i,x}.$$

By commutativity of the diagram 1.7, we have that the morphism

$$(\Omega^1_{X/\mathbb{C}})_x \to (\alpha_{X*}\Omega^1_{X(\mathbb{C})})_x \to (\Omega^1_{X(\mathbb{C})})_x$$

maps $du_{i,x}$ into $dw_{i,x}$, for $i = 1, \dots, n$. Hence,

$$(\alpha_X^* \Omega^1_{X/\mathbb{C}})_x \cong (\oplus_{i=1}^n \mathcal{O}_{X,x} du_{i,x}) \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X(\mathbb{C}),x} \to \oplus_{i=1}^n \mathcal{O}_{X(\mathbb{C}),x} dw_{i,x} \cong (\Omega^1_{X(\mathbb{C})})_x$$

is an isomorphism. □

Remark 1.5.11. Since we have the morphism of small analytic ringed sites

$$(X(\mathbb{C})_{an}, \mathcal{O}_{X(\mathbb{C})}) \to (\mathbb{C}^0_{an}, \mathcal{O}_{\mathbb{C}^0})$$

we could also apply the general construction of the algebraic de Rham complex to it. However, this is not a relevant object. It doesn't coincide with $\Omega^\bullet_{X(\mathbb{C})}$, the analytic de Rham complex of $X(\mathbb{C})$, which is indeed the right object to consider in the analytic context. Intuitively, a reason why they don't coincide, is that the analytic differential can be linear with respect to infinite sums, in the sense that, for example, we have

$$d(e^z) = d(1 + z + z^2/2 + \dots) = dz + zdz + z^2/2dz + \dots = e^z dz.$$

The algebraic differential, instead, is only linear with respect to finite sums. For example, it can be proved that

$$d(e^z) \neq e^z dz.$$

Now, we prove the comparison isomorphism between the algebraic and the analytic de Rham cohomology.

The proper case

Theorem 1.5.12. *Let X be a smooth proper algebraic variety over $\mathbb{C}$. Then, the canonical morphisms of $\mathbb{C}$-vector spaces constructed in 1.8, for each $i \geq 0$,*

$$H^i_{AdR}(X/\mathbb{C}) \to H^i_{dR}(X(\mathbb{C})),$$

are isomorphisms.

Proof. For each $p \geq 0$, the sheaf of $\mathcal{O}_X$-modules $\Omega^p_{X/k}$ is finite locally free (proposition 1.4.6). Hence, it is a quasi-coherent and locally of finite type sheaf of $\mathcal{O}_X$-modules. Since X is a locally noetherian scheme and $\mathcal{O}_X$ is a coherent sheaf of $\mathcal{O}_X$-modules (proposition 1.2.20), then $\Omega^p_{X/k}$ is a coherent sheaf of $\mathcal{O}_X$-modules (see [GW20, Prop. 7.46]). Since X is proper, by GAGA Theorem I 1.2.22, we have canonical isomorphisms of $\mathbb{C}$-vector spaces, for each $q \geq 0$,

$$\mathbb{H}^q(X_{Zar}, \Omega^p_{X/\mathbb{C}}) \cong \mathbb{H}^q(X(\mathbb{C})_{an}, (\Omega^p_{X/\mathbb{C}})^{an}).$$

Composing with the canonical isomorphism

$$\mathbb{H}^q(X(\mathbb{C})_{an}, (\Omega^p_{X/\mathbb{C}})^{an}) \cong \mathbb{H}^q(X(\mathbb{C})_{an}, \Omega^p_{X(\mathbb{C})})$$

induced by the canonical isomorphism of $\mathcal{O}_{X(\mathbb{C})}$-modules $(\Omega^p_{X/\mathbb{C}})^{an} \cong \Omega^p_{X(\mathbb{C})}$ of proposition 1.5.10, we get canonical isomorphisms of $\mathbb{C}$-vector spaces

$$\mathbb{H}^q(X_{Zar}, \Omega^p_{X/\mathbb{C}}) \cong \mathbb{H}^q(X(\mathbb{C})_{an}, \Omega^p_{X(\mathbb{C})}).$$

They define a canonical isomorphism between the hyper-cohomology spectral sequences

$$E_1^{p,q} = \mathbb{H}^q(X_{Zar}, \Omega^p_{X/\mathbb{C}}) \Rightarrow \mathbb{H}^{p+q}(X_{Zar}, \Omega^\bullet_{X/\mathbb{C}}) = \mathrm{H}^{p+q}_{\mathrm{AdR}}(X/\mathbb{C})$$

and

$$E_1^{p,q} = \mathbb{H}^q(X(\mathbb{C})_{an}, \Omega^p_{X(\mathbb{C})}) \Rightarrow \mathbb{H}^{p+q}(X(\mathbb{C})_{an}, \Omega^\bullet_{X(\mathbb{C})}) = \mathrm{H}^{p+q}_{\mathrm{dR}}(X(\mathbb{C}))$$

Hence, we conclude that the morphisms induced on the limits of the spectral sequences, which are the canonical morphism constructed in 1.8, are isomorphisms. $\square$

The general case

The strategy to prove the general case consists of the following steps.

1) Given X a smooth algebraic variety over $\mathbb{C}$, we embed X as an open subset inside $\overline{X}$, a smooth proper algebraic variety over $\mathbb{C}$.

2) We define a complex of abelian sheaves over $\overline{X}_{Zar}$, whose sheaf cohomology computes the algebraic de Rham cohomology of X.

3) We do analogous constructions in the analytic context: we define a complex of abelian sheaves over $\overline{X}(\mathbb{C})_{an}$, whose sheaf cohomology computes the analytic de Rham cohomology of $X(\mathbb{C})$.

4) We prove, using GAGA Theorem I 1.2.22, that the cohomology of $\overline{X}$ and $\overline{X}(\mathbb{C})$ with coefficients in the complexes of abelian sheaves defined in 2) and 3) are isomorphic. Since they compute the algebraic and the analytic de Rham cohomology of X and $X(\mathbb{C})$ respectively, we conclude the isomorphisms
$$\mathrm{H}^i_{\mathrm{AdR}}(X/\mathbb{C}) \cong \mathrm{H}^i_{\mathrm{dR}}(X(\mathbb{C})).$$

We start with step 1). Given X a smooth algebraic variety over $\mathbb{C}$, by Nagata Embedding Theorem for schemes (see [Del10, thm. 1.6]), we can be embedded X as an open subscheme into $\tilde{X}$, a proper algebraic variety over $\mathbb{C}$,
$$X \hookrightarrow \tilde{X}.$$
Since $\tilde{X}$ might not be smooth, we apply Hironaka Resolution of Singularities in characteristic zero (see [Wlo05, thm. 1.0.3]) to obtain a proper morphism
$$\rho : \overline{X} \to \tilde{X},$$
with $\overline{X}$ a smooth algebraic variety over $\mathbb{C}$, such that the restriction to the smooth locus $\tilde{X}^{sm} \subset \tilde{X}$, the open subset of smooth points,
$$\rho|_{\rho^{-1}(\tilde{X}^{sm})} : \rho^{-1}(\tilde{X}^{sm}) \to \tilde{X}^{sm}$$
is an isomorphism. Since $\tilde{X}$ and ρ are proper, then also $\overline{X}$ is. Since X is a smooth open subscheme of $\tilde{X}$, then $X \subset \tilde{X}^{sm}$. Hence, $X \cong \rho^{-1}(X)$ can be identified with an open subscheme of $\overline{X}$. Moreover, by Hironaka Embedded Desingularization Theorem (see [Wlo05, thm. 1.0.2]), up to applying blow-ups, we can assume that the complementary closed subscheme, with the reduced scheme structure,
$$D := \overline{X} \setminus X$$
is a simple normal crossing divisor.

Now, we address step 2). For the moment we leave aside the setting obtained in step 1) and we consider the more generally the setting given by Y a smooth algebraic variety over a field k with $D \subset Y$ a normal crossing divisor. This is indeed sufficient for all constructions and results, that we are about to face in this step.

First, we recall the definition of normal crossing divisor.

Definition 1.5.13. Let Y be a smooth algebraic variety over a field k. Recall from proposition 1.4.21, that Y is regular. A closed subvariety $D \subset Y$ is a *normal crossing divisor* (NCD) of Y if, for any $y \in D$, there exists $u_{1,y}, \ldots, u_{n,y} \in O_{Y,y}$ a system of regular parameters of Y at y, such that

$$\mathcal{O}_{D,y} \cong \mathcal{O}_{Y,y}/(u_{1,y} \cdots u_{m,y}),$$

for some $m \leq n$. Moreover, we say that D is *simple*, if its irreducible components are smooth.

Remark 1.5.14. Let Y be a smooth algebraic variety over a field k, $y \in Y$ a point and $u_{1,y}, \ldots, u_{n,y} \in \mathcal{O}_{Y,y}$ a system of local parameters of Y at y. As seen in the proof of proposition 1.4.15, we can take an affine open subset $U \subset Y$ containing y and sections $u_1, \ldots u_n \in \mathcal{O}_Y(U)$, whose stalks at y are the system of local parameters, which define an étale morphism $h : U \to \mathbb{A}^n_k$. This means that

$$\Omega^1_{Y/k}(U) \cong \oplus_{i=1}^n \mathcal{O}_Y(U) du_i.$$

Recall that, by proposition 1.4.21, a system of regular parameters is also a system of local parameters of Y at y. Then, the definition of $D \subset Y$ NCD implies that, for any $y \in Y$, there exists such an affine open subset $U \subset Y$, such that

$$D \cap U \cong V(u_1 \cdots u_m) = V(u_1) \cup \cdots \cup V(u_m)$$

for some $m \leq n$ (eventually $m = 0$, meaning that $y \notin D$). So, we can think at a NCD as a closed subscheme, which étale-locally is a union of coordinate hyperplanes. Moreover, if we denote by $X \coloneqq Y \setminus D$ the complementary open subscheme, then

$$X \cap U \cong D(u_1 \cdots u_m) = D(u_1) \cap \cdots \cap D(u_m).$$

Hence, the restrictions $u_1, \ldots, u_m \in \mathcal{O}_Y(X \cap U)$ are invertible elements.

Whenever we have a NCD, we can define the following logarithmic variant of the sheaf of Kähler differentials.

Definition 1.5.15. Let Y be a smooth algebraic variety over a field k and $D \subset Y$ a NCD. Let $X \coloneqq Y \setminus D \subset Y$ be the complementary open subscheme and denote by $j : X \hookrightarrow Y$ the open immersion. Consider the sheaf of $\mathcal{O}_Y$-modules

$$j_* \Omega^1_{X/k}.$$

We define the *sheaf of modules of Kähler differentials of* Y *with logarithmic poles along* D the $\mathcal{O}_Y$-submodule

$$\Omega_{Y/k}(\log D) \subset j_* \Omega^1_{X/k},$$

such that, for any $y \in Y$, if $U \subset Y$ is an affine open subset as in remark 1.5.14, then

$$\Omega_{Y/k}(\log D)(U) \subset j_* \Omega^1_{X/k}(U) \cong \Omega^1_{Y/k}(X \cap U) \cong \oplus_{i=1}^n \mathcal{O}_Y(X \cap U) du_i,$$

is the $\mathcal{O}_Y(U)$-submodule generated by the elements [3]

$$\frac{du_1}{u_1}, \dots, \frac{du_m}{u_m}, du_{m+1}, \dots, du_n.$$

Remark 1.5.16. By corollary 1.4.23, Y is reduced and has irreducible components that coincide with connected components. Hence, connected components of Y are integral. This implies that, for any open subset $U \subset Y$, the restriction morphism

$$\mathcal{O}_Y(U) \hookrightarrow \mathcal{O}_Y(X \cap U)$$

is injective, i.e. the canonical morphism $\mathcal{O}_Y \hookrightarrow j_* \mathcal{O}_X$ is injective. Moreover, if $U \subset Y$ is an affine open subset as in remark 1.5.14, since the elements

$$du_1, \dots, du_m, du_{m+1}, \dots, du_n \in \Omega^1_{Y/k}(X \cap U)$$

are $\mathcal{O}_Y(X \cap U)$-linearly independent, then also the elements

$$\frac{du_1}{u_1}, \dots, \frac{du_m}{u_m}, du_{m+1}, \dots, du_n \in \Omega^1_{Y/k}(X \cap U)$$

are $\mathcal{O}_Y(X \cap U)$-linearly independent and hence also $\mathcal{O}_Y(U)$-linearly independent. So,

$$\Omega_{Y/k}(\log D)(U) \cong \left(\oplus_{i=1}^m \mathcal{O}_Y(U) \frac{du_i}{u_i} \right) \oplus \left(\oplus_{i=m+1}^n \mathcal{O}_Y(U) du_i \right).$$

That is, $\Omega_{Y/k}(\log D)$ is a finite locally free sheaf of $\mathcal{O}_Y$-modules, with stalks at $y \in Y$

$$(\Omega_{Y/k}(\log D))_y \cong \left(\oplus_{i=1}^m \mathcal{O}_{Y,y} \frac{du_{i,y}}{u_{i,y}} \right) \oplus \left(\oplus_{i=m+1}^n \mathcal{O}_{Y,y} du_{i,y} \right).$$

As for the algebraic de Rham complex, we can construct a complex of abelian sheaves out of the sheaf of modules of Kähler differentials with logarithmic poles

[3] We define $\Omega_{Y/k}(\log D)$ only on an affine open cover of Y. It can be proved that these definitions don't depend on the choice of a system of local parameters, hence they are compatible over intersections of affine open subsets. So, $\Omega_{Y/k}(\log D)$ is indeed a sheaf of $\mathcal{O}_Y$-modules.

along D. For each $p \geq 0$, we define

$$\Omega^p_{Y/k}(\log D) := \bigwedge^p \Omega^1_{Y/k}(\log D),$$

where the wedge product is taken as sheaves of $\mathcal{O}_Y$-modules. In particular, $\Omega^0_{Y/k}(\log D) \cong \mathcal{O}_Y$ and $\Omega^1_{Y/k}(\log D) \cong \Omega_{Y/k}(\log D)$. It's immediate to check that we have inclusions of sheaves of $\mathcal{O}_Y$-modules, for each $p \geq 0$,

$$\Omega^p_{Y/k}(\log D) \subset j_*\Omega^p_{X/k}. \tag{1.9}$$

Moreover, notice that the first differential of $j_*\Omega^\bullet_{X/k}$ is such that, on an affine open subset $U \subset Y$ as in remark 1.5.16,

$$d\left(\frac{du_i}{u_i}\right) = d\left(\frac{1}{u_i}\right) \wedge du_i = -\frac{1}{u_i^2} du_i \wedge du_i = 0.$$

So, we see that the differentials of $j_*\Omega^\bullet_{X/k}$ restrict to the inclusions 1.9. Hence they define a complex of abelian sheaves

$$\Omega^\bullet_{Y/k}(\log D),$$

which is also k-linear. By construction, we have the inclusion of complexes of abelian sheaves over Y_{Zar}

$$\Omega^\bullet_{Y/k}(\log D) \hookrightarrow j_*\Omega^\bullet_{X/k}, \tag{1.10}$$

which is also k-linear.

Definition 1.5.17. Let Y be a smooth algebraic variety over a field k and $D \subset Y$ a NCD. The complex of sheaves of k-vector spaces over Y_{Zar}

$$\Omega^\bullet_{Y/k}(\log D)$$

is called the *algebraic de Rham complex of Y with logarithmic poles along D*.

Now, we want to prove that the sheaf cohomology of the algebraic de Rham complex with logarithmic poles computes the algebraic de Rham cohomology of the complementary open subscheme, which is smooth. We start constructing a canonical morphism from one to the other. The inclusion of complexes of abelian sheaves over Y_{Zar} 1.10 induces canonical morphisms on sheaf cohomology

$$\mathbb{H}^i(Y_{Zar}, \Omega^\bullet_{Y/k}(\log D)) \to \mathbb{H}^i(Y_{Zar}, j_*\Omega^\bullet_{X/k}).$$

Composing with the canonical edge morphisms of the Leray spectral sequence for j and $\Omega^\bullet_{X/k}$

$$\mathbb{H}^i(Y_{Zar}, j_*\Omega^\bullet_{X/k}) \to \mathbb{H}^i(X_{Zar}, \Omega^\bullet_{X/k}) = \mathrm{H}^i_{\mathrm{AdR}}(X/k),$$

we get the canonical morphisms of abelian groups, for each $i \geq 0$,

$$\mathbb{H}^i(Y_{Zar}, \Omega^\bullet_{Y/k}(\log D)) \to \mathrm{H}^i_{\mathrm{AdR}}(X/k).$$

Since all the considered morphisms of complexes of abelian sheaves are also k-linear, then these are also morphisms of k-vector spaces.

Proposition 1.5.18. *Let k be a field of characteristic 0. Given Y a smooth algebraic variety over k and $D \subset Y$ a NCD, let $X := Y \setminus D \subset Y$ be the complementary open subscheme and denote by $j : X \hookrightarrow Y$ the open immersion. Then, the canonical morphisms of k-vector spaces, for each $i \geq 0$,*

$$\mathbb{H}^i(Y_{Zar}, \Omega^\bullet_{Y/k}(\log D)) \to H^i_{AdR}(X/k)$$

are isomorphisms.

Proof. We prove that each of the morphisms in the composition

$$\mathbb{H}^i(Y_{Zar}, \Omega^\bullet_{Y/k}(\log D)) \to \mathbb{H}^i(Y_{Zar}, j_*\Omega^\bullet_{X/k}) \to \mathrm{H}^i_{\mathrm{AdR}}(X/k)$$

is an isomorphism. Consider the second morphism in the composition. Notice that it is the limit morphism of the morphism between the hyper-cohomology spectral sequences

$$E_1^{p,q} = \mathbb{H}^q(Y_{Zar}, j_*\Omega^p_{X/k}) \Rightarrow \mathbb{H}^{p+q}(Y_{Zar}, j_*\Omega^\bullet_{X/k})$$

and

$$E_1^{p,q} = \mathbb{H}^q(X_{Zar}, \Omega^p_{X/k}) \Rightarrow \mathbb{H}^{p+q}(X_{Zar}, \Omega^\bullet_{X/k}) = \mathrm{H}^{p+q}_{\mathrm{dR}}(X),$$

such that, for each $p \geq 0$, the canonical morphisms

$$\mathbb{H}^q(Y_{Zar}, j_*\Omega^p_{X/k}) \to \mathbb{H}^q(X_{Zar}, \Omega^p_{X/k})$$

are the edge morphisms of the Leray spectral sequence for j and $\Omega^p_{X/k}$. So, it suffices to prove that these are isomorphisms. Recall that the Leray spectral sequence for j and $\Omega^p_{X/k}$ is

$$E_2^{r,s} = \mathbb{H}^r(Y_{Zar}, \mathbf{R}^s j_*\Omega^p_{X/k}) \Rightarrow \mathbb{H}^{r+s}(X_{Zar}, \Omega^p_{X/k}).$$

Recall that the higher derived pushforward $\mathbf{R}^s j_*\Omega^p_{X/k}$ is the abelian sheaf over Y_{Zar} given by the sheafification of the abelian presheaf

$$U \mapsto \mathbb{H}^s(X \cap U, \Omega^p_{X/k}).$$

If we take $U \subset Y$ an affine open subsets as in remark 1.5.14, then $X \cap U$ is an affine scheme. Since $\Omega^p_{X/k}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules, then (see [Har77, §3, Thm. 3.5])

$$\mathbb{H}^s(X \cap U, \Omega^p_{X/k}) = 0 \qquad \text{for each } s > 0.$$

Since such open subsets $U \subset Y$ form an open cover of Y, then we conclude that

$$\mathbf{R}^q j_* \Omega^p_{X/k} = 0 \qquad \text{for each } q > 0.$$

Hence, the Leray spectral sequence for j and $\Omega^p_{X/k}$ degenerates at page 2 and the edge morphisms are isomorphisms. Now, consider the first morphism in the composition. Recall that it is induced by the inclusion of complexes of abelian sheaves over Y_{Zar}

$$\Omega^\bullet_{Y/k}(\log D) \hookrightarrow j_* \Omega^\bullet_{X/k}. \tag{1.11}$$

We prove that it is a quasi-isomorphism. A reference is [HM17, Prop. 3.1.16]. We prove that the cokernel of 1.11 is an exact complex of abelian sheaves over Y_{Zar}. Since being exact for a complex of sheaves is a local property, this is equivalent to check exactness on stalks at each point $y \in Y$. In fact, using the étale descent property of algebraic de Rham cohomology, we see that we can equivalently consider the étale topology, so we can equivalently check exactness at étale stalks (see [Mil13, p. 47]). Hence, we can assume that $Y \cong \mathbb{A}^n_k \cong Spec(k[t_1, \dots, t_n])$, $y \in Y$ is the point given by the prime ideal $(t_1, \dots, t_n)$ and $D = V(t_1, \dots, t_m)$ for some $m \leq n$. Reasoning by induction, it suffices to prove the case $n = 1$. If $m = 0$, that is D is empty, there is nothing to prove. So, assume that $D = V(t)$. Recall that the étale stalk of $\mathcal{O}_Y = \mathcal{O}_{\mathbb{A}^1_k}$ at y is the Henselization of $k[t]_{(t)}$ (see [Mil13, Cor. 4.14]), which is the ring of algebraic formal power series (see [Mil13, Cor. 4.17])

$$\mathcal{O} := k[\![t]\!] \cap k(t)^{alg}.$$

The étale stalk at y of the cokernel of 1.11 is the complex

$$0 \to \frac{\mathcal{O}[t^{-1}]}{\mathcal{O}} \to \frac{\mathcal{O}[t^{-1}]}{t^{-1}\mathcal{O}} dt \to 0. \tag{1.12}$$

The only non-trivial differential is such that, for any $f \in \mathcal{O}$ and $i > 0$,

$$\frac{f}{t^i} \mapsto \begin{cases} \frac{f'}{t^i} dt - i \frac{f}{t^{i+1}} dt & i > 1 \\ -\frac{f}{t^2} dt & i = 1. \end{cases}$$

Since, by hypothesis, $char(k) = 0$, then the differential is injective. Reasoning by induction, we see that it is also surjective. Hence, the complex 1.12 is exact.

□

Now, we address step 3). In the analytic context we can do exactly the analogous constructions of the algebraic context of step 2). Everything can be literally rewritten, replacing:

- Y a smooth algebraic variety over a field k, with M a smooth complex analytic space.

- $u_{1,y}, \dots, u_{n,y} \in \mathcal{O}_{Y,y}$ a system of local parameters of N at y, with $w_{1,y}, \dots, w_{n,y} \in \mathcal{O}_{M,y}$ stalks of local coordinates of a local chart of N at y. Notice that $D \subset N$ a NCD is a closed complex analytic subspace, which locally, i.e. on a local chart, is the union of some coordinate hyperplanes.

- $U \subset Y$ an affine open subset endowed with an étale morphism $h : U \to \mathbb{A}^n_k$, with $U \subset N$ an open subset of a local chart.

- The corresponding sections $u_1, \dots, u_n \in \mathcal{O}_Y(U)$, with coordinates of the local chart $w_1, \dots, w_n \in \mathcal{O}_N(U)$.

- The sheaf of Kähler differentials and the algebraic de Rham compex, with the sheaf of holomorphic 1-forms and the analytic de Rham complex.

So, given N a smooth complex analytic space with $D \subset N$ a NCD, if we denote by $M := N \setminus D$ the complementary open subspace, which is a smooth complex analytic space, and $j : M \hookrightarrow N$ the open immersion, we obtain a finite locally free sheaf of $\mathcal{O}_N$-modules

$$\Omega_N(\log D) \subset j_*\Omega^1_M,$$

called the *sheaf of modules of holomorphic 1-forms of N with logarithmic poles along D*. It has stalks at $y \in N$

$$(\Omega_N(\log D))_y \cong \left(\oplus_{i=1}^m \mathcal{O}_{N,y} \frac{dw_{i,y}}{w_{i,y}} \right) \oplus \left(\oplus_{i=m+1}^n \mathcal{O}_{N,y} dw_{i,y} \right),$$

where $w_1, \dots, w_n$ are local coordinates of a local chart $U \subset N$ containing y and $D \cap U = Z(w_1 \cdots w_m)$. Its exterior powers form the *analytic de Rham complex of N with logarithmic poles along D*

$$\Omega^\bullet_N(\log D) \hookrightarrow j_*\Omega^\bullet_M,$$

which is also $\mathbb{C}$-linear. Notice that, for smooth complex analytic spaces, the fact that, for any $U \subset N$ open subset, the restriction morphisms

$$\mathcal{O}_N(U) \hookrightarrow \mathcal{O}_N(M \cap U)$$

are injective, i.e. $\mathcal{O}_N \hookrightarrow j_*\mathcal{O}_M$ is injective, is due to the Identity Principle of holomorphic functions. We also have the canonical morphism of $\mathbb{C}$-vector spaces, for each $i \geq 0$,

$$\mathbb{H}^i(N_{an}, \Omega^\bullet_N(\log D)) \to \mathrm{H}^i_{\mathrm{dR}}(M).$$

Remark 1.5.19. In the analytic context, we can define the morphism of abelian sheaves

$$\log : \mathcal{O}^\times_M \to \mathcal{O}_M,$$

given by the complex logarithm. Its composition with the first differential of the analytic de Rham complex gives the *logarithmic derivation*, that is the morphism of abelian sheaves

$$d\log : \mathcal{O}_M^\times \to \Omega_M^1,$$

which assigns to a section f of $\mathcal{O}_M^\times$ the section $\frac{df}{f}$ of Ω_M^1. The logarithmic derivation makes sense also in the algebraic context: we have the morphism of abelian sheaves

$$d\log : \mathcal{O}_X^\times \to \Omega_{X/k}^1,$$

which assigns to a section f of $\mathcal{O}_X^\times$ the section $\frac{df}{f}$ of $\Omega_{X/k}^1$. The sheaves $\Omega_Y^1(\log D)$ and $\Omega_N^1(\log D)$ contain the sections given by the logarithmic derivations of the rational functions defining D. For this reason they are called the algebraic/analytic de Rham complexes with logarithmic poles along D.

It holds the analogous of the algebraic proposition 1.5.18.

Proposition 1.5.20. *Let N be a smooth complex analytic space and $D \subset Y$ be a NCD. Let $M \coloneqq N \setminus D \subset N$ be the complementary open subspace and denote by $j : M \hookrightarrow N$ the open immersion. Then, the canonical morphisms of $\mathbb{C}$-vector spaces, for each $i \geq 0$,*

$$\mathbb{H}^i(N_{an}, \Omega_N^\bullet(\log D)) \to H_{dR}^i(M)$$

are isomorphisms.

Proof. As for the algebraic analogue, we prove that each of the morphisms in the composition

$$\mathbb{H}^i(N_{an}, \Omega_N^\bullet(\log D)) \to \mathbb{H}^i(N_{an}, j_*\Omega_M^\bullet) \to \mathrm{H}_{\mathrm{dR}}^i(M)$$

is an isomorphism. Consider the second morphism in the composition. The same proof of the algebraic version works literally with the replacements pointed above, because, given $U \subset N$ a local chart, where D is the union of some coordinate hyperplanes, then $M \cap U$ is a Stein space. Since, for each $p \geq 0$, Ω_M^p is a finite locally free sheaf of $\mathcal{O}_M$-modules and $\mathcal{O}_M$ is a coherent sheaf of $\mathcal{O}_M$-modules, then Ω_M^p is a coherent sheaf of $\mathcal{O}_M$-modules. Hence, by Cartan's Theorem B, we have that

$$\mathbb{H}^s(M \cap U, \Omega_M^p) = 0 \qquad \text{for each } s > 0.$$

We conclude as in the algebraic case. Now, consider the first morphism in the composition. As in the algebraic case, we prove that the inclusion of complexes of abelian sheaves over N_{an}

$$\Omega_N^\bullet(\log D) \hookrightarrow j_*\Omega_M^\bullet \tag{1.13}$$

is a quasi-isomorphism. A reference is [HM17, Prop. 4.1.16]. We prove that the cokernel of 1.13 is an exact complex of abelian sheaves over N_{an}. Since being exact for a complex of sheaves is a local property, this is equivalent to check exactness on stalks at each point $y \in N$. Hence, we can assume that N is isomorphic to a disk in $\mathbb{C}^n$, $y = 0 \in \mathbb{C}^n$ and $D = Z(t_1, \dots, t_m)$ for some $m \leq n$. Reasoning by induction, it suffices to prove the case $n = 1$. If $m = 0$, that is D is empty, there is nothing to prove. So, assume that $D = Z(t)$. We denote by $\mathcal{O}$ the ring of germs at 0 of holomorphic functions and by $\mathcal{K}$ the ring of germs at 0 of meromorphic functions with an isolated singularity at 0. Then, $\mathcal{O} \cong \mathbb{C}\{t\}$ is the ring of power series convergent on some disk and $\mathcal{K}$ is the ring of Laurent series convergent on some annulus. The stalk at 0 of the cokernel of 1.13 is the complex

$$0 \to \frac{\mathcal{K}}{\mathcal{O}} \to \frac{\mathcal{K}}{t^{-1}\mathcal{O}} dt \to 0. \tag{1.14}$$

The only non-trivial differential is such that

$$\sum_{i>0} a_i t^{-i} \mapsto \sum_{i>0} (-i) a_i t^{-i-1}.$$

It is bijective, with inverse given by

$$\sum_{j>1} b_i t^{-j} \mapsto \sum_{j>1} \frac{b_j}{-j+1} t^{-j+1}.$$

Hence, the complex 1.14 is exact.

□

We conclude with step 4). Consider the setting of step 2) with $k = \mathbb{C}$. So, let Y be a smooth algebraic variety over $\mathbb{C}$ and $D \subset Y$ be a NCD. We denote by $X \coloneqq Y \setminus D$ the complementary open subscheme and $j : X \hookrightarrow Y$ the open immersion. So, we have the inclusion of complexes of abelian sheaves over Y_{Zar}

$$\Omega^\bullet_{Y/\mathbb{C}}(\log D) \hookrightarrow j_* \Omega^\bullet_{X/\mathbb{C}}.$$

We can also consider the associated complex analytic spaces given by the analytifications: $Y(\mathbb{C})$, $X(\mathbb{C})$ and $D(\mathbb{C})$. $Y(\mathbb{C})$ is a smooth complex analytic space, $D(\mathbb{C}) \subset Y(\mathbb{C})$ is a closed subspace and $X(\mathbb{C}) = Y(\mathbb{C}) \setminus D(\mathbb{C})$ is the complementary open subspace, with open immersion $j^{an} : X(\mathbb{C}) \hookrightarrow Y(\mathbb{C})$. By remark 1.4.17, which describes explicitly the local charts of the analytification, we see that $D(\mathbb{C}) \subset Y(\mathbb{C})$ is a NCD in the sense of a smooth complex analytic spaces. Hence, $Y(\mathbb{C})$, $D(\mathbb{C})$ and $X(\mathbb{C})$ give the setting of step 3). So, we have the inclusion of complexes of abelian sheaves over $Y(\mathbb{C})_{an}$

$$\Omega^\bullet_{Y(\mathbb{C})}(\log D(\mathbb{C})) \hookrightarrow j^{an}_* \Omega^\bullet_{X(\mathbb{C})}.$$

We want to compare the sheaf cohomology of $\Omega^\bullet_{Y/\mathbb{C}}(\log D)$ with the one of $\Omega^\bullet_{Y(\mathbb{C})}(\log D(\mathbb{C}))$. Recall that we have a morphism of complexes of abelian sheaves

over X_{Zar}
$$\Omega^\bullet_{X/\mathbb{C}} \to \alpha_{X*}\Omega^\bullet_{X(\mathbb{C})}.$$
Applying the functor j_*, we get the morphism of complexes of abelian sheaves over $X(\mathbb{C})_{an}$
$$j_*\Omega^\bullet_{X/\mathbb{C}} \to j_*\alpha_{X*}\Omega^\bullet_{X(\mathbb{C})} \cong \alpha_{Y*}j^{an}_*\Omega^\bullet_{X(\mathbb{C})}.$$
Following definitions, we see that it restricts to a morphism between the de Rham complexes with logarithmic poles

$$\Omega^\bullet_{Y/\mathbb{C}}(\log D) \to \alpha_{Y*}\Omega^\bullet_{Y(\mathbb{C})}(\log D(\mathbb{C})).$$

Taking the adjoint morphism via the adjunction $(\alpha_Y^{-1}, \alpha_{Y*})$, we get the canonical morphism of complexes of abelian sheaves over $Y(\mathbb{C})_{an}$

$$\alpha_Y^{-1}\Omega^\bullet_{Y/\mathbb{C}}(\log D) \to \Omega^\bullet_{Y(\mathbb{C})}(\log D(\mathbb{C})).$$

It induce the morphism on sheaf cohomology

$$\mathbb{H}^i(Y(\mathbb{C})_{an}, \alpha_Y^{-1}\Omega^\bullet_{Y/\mathbb{C}}(\log D)) \to \mathbb{H}^i(Y(\mathbb{C})_{an}, \Omega^\bullet_{Y(\mathbb{C})}(\log D(\mathbb{C}))).$$

Composing with the functoriality morphism of sheaf cohomology

$$\mathbb{H}^i(Y_{Zar}, \Omega^\bullet_{Y/\mathbb{C}}(\log D)) \to \mathbb{H}^i(Y(\mathbb{C})_{an}, \alpha_Y^{-1}\Omega^\bullet_{Y/\mathbb{C}}(\log D)),$$

we get the canonical morphisms of abelian groups, for each $i \geq 0$,

$$\mathbb{H}^i(Y_{Zar}, \Omega^\bullet_{Y/\mathbb{C}}(\log D)) \to \mathbb{H}^i(Y(\mathbb{C})_{an}, \Omega^\bullet_{Y(\mathbb{C})}(\log D(\mathbb{C}))),$$

which are also $\mathbb{C}$-linear.

It holds the logarithmic version of proposition 1.5.10, which tells that analytification of sheaves of modules of Kähler differentials with logarithmic poles are the sheaves of holomorphic forms with logarithmic poles.

Proposition 1.5.21. *Let Y be a smooth algebraic variety over $\mathbb{C}$, $D \subset Y$ a NCD. Denote by $X := Y \setminus D$ the complementary open subscheme and $j : X \hookrightarrow Y$ the open immersion. Then, we have canonical isomorphisms of sheaves of $\mathcal{O}_{Y(\mathbb{C})}$-modules, for each $p \geq 0$,*
$$(\Omega^p_{Y/\mathbb{C}}(\log D))^{an} \cong \Omega^p_{Y(\mathbb{C})}(\log D(\mathbb{C})).$$
Proof. We observed above that we have morphisms of sheaves of $\mathcal{O}_Y$-modules

$$\Omega^p_{Y/\mathbb{C}}(\log D) \to \alpha_{Y*}\Omega^p_{Y(\mathbb{C})}(\log D(\mathbb{C})).$$

The adjoint morphisms of $\mathcal{O}_{Y(\mathbb{C})}$-modules via the adjunction $(\alpha_Y^*, \alpha_{Y*})$ are

$$(\Omega^p_{Y/\mathbb{C}}(\log D))^{an} = \alpha_Y^*\Omega^p_{Y/\mathbb{C}}(\log D) \to \Omega^p_{Y(\mathbb{C})}(\log D(\mathbb{C})).$$

We show that these are isomorphisms. It suffices to prove it for $p = 1$, since all other isomorphisms are obtained by taking exterior powers. We show that we have isomorphisms on each stalk at $y \in Y(\mathbb{C})$. Let $u_{1,y}, \ldots, u_{n,y} \in \mathcal{O}_{Y,y}$ be a system of local parameters of Y at y and $w_{1,y}, \ldots, w_{n,y} \in \mathcal{O}_{Y(\mathbb{C}),y}$ the images along the morphism $\alpha^{\#}_{Y,y} : \mathcal{O}_{Y,y} \to \mathcal{O}_{Y(\mathbb{C}),y}$. Recall that the stalks of the sheaf of modules of Kähler differentials of Y with logarithmic poles along D are

$$(\Omega^1_{Y/\mathbb{C}}(\log D))_y \cong \left(\oplus_{i=1}^m \mathcal{O}_{Y,y} \frac{du_{i,y}}{u_{i,y}}\right) \oplus \left(\oplus_{i=m+1}^n \mathcal{O}_{Y,y} du_{i,y}\right),$$

and the stalks of the sheaf of modules of the holomorphic 1-forms of $Y(\mathbb{C})$ with logarithmic poles along $D(\mathbb{C})$ are

$$(\Omega^1_{Y(\mathbb{C})}(\log D(\mathbb{C})))_y \cong \left(\oplus_{i=1}^m \mathcal{O}_{Y(\mathbb{C}),y} \frac{dw_{i,y}}{w_{i,y}}\right) \oplus \left(\oplus_{i=m+1}^n \mathcal{O}_{Y(\mathbb{C}),y} dw_{i,y}\right).$$

Since

$$(\Omega^1_{Y/\mathbb{C}}(\log D))_y \to (\alpha_{Y*}\Omega^1_{Y(\mathbb{C})}(\log D(\mathbb{C})))_y \to (\Omega^1_{Y(\mathbb{C})}(\log D(\mathbb{C})))_y$$

maps $u_{i,y}$ into $w_{i,y}$ and $du_{i,y}$ into $dw_{i,y}$, for $i = 1, \ldots, n$, then

$$(\alpha_Y^*\Omega^1_{Y/\mathbb{C}}(\log D))_y \cong (\Omega^1_{Y/\mathbb{C}}(\log D))_y \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{Y(\mathbb{C}),y} \to (\Omega^1_{Y(\mathbb{C})}(\log D(\mathbb{C})))_y$$

is an isomorphism. □

Remark 1.5.22. Notice that, instead,

$$(j_*\Omega^p_{X/\mathbb{C}})^{an} \ncong j_*^{an}\Omega^p_{X(\mathbb{C})}.$$

Indeed, the first one is the sheaf of meromorphic p-forms over $Y(\mathbb{C})$ with poles along $D(\mathbb{C})$, while the second is the sheaf of p-forms over $Y(\mathbb{C})$ with eventually some essential singularity along $D(\mathbb{C})$.

In case Y is proper, we have the logarithmic version of theorem 1.5.12.

Theorem 1.5.23. *Let Y be a smooth proper algebraic variety over $\mathbb{C}$, $D \subset Y$ a NCD. Denote by $X := Y \setminus D$ the complementary open subscheme and $j : X \hookrightarrow Y$ the open immersion. Then, the canonical morphisms of $\mathbb{C}$-vector spaces, for each $i \geq 0$,*

$$\mathbb{H}^i(Y_{Zar}, \Omega^p_{Y/\mathbb{C}}(\log D)) \cong \mathbb{H}^i(Y(\mathbb{C})_{an}, \Omega^p_{Y(\mathbb{C})}(\log D(\mathbb{C})))$$

are isomoprhisms.

Proof. For each $p \geq 0$, the sheaf of $\mathcal{O}_Y$-modules $\Omega^p_{Y/\mathbb{C}}(\log D)$ is finite locally free (remark 1.5.16). Hence, it is a quasi-coherent and locally of finite type sheaf of $\mathcal{O}_Y$-modules. Since Y is a locally noetherian scheme and $\mathcal{O}_Y$ is a coherent sheaf of $\mathcal{O}_Y$-modules (proposition 1.2.20), then $\Omega^p_{Y/\mathbb{C}}(\log D)$ is a coherent sheaf of $\mathcal{O}_Y$-modules (see [GW20, Prop. 7.46]). Since Y is proper, arguing as in theorem 1.5.12, using proposition 1.5.21, GAGA Theorem I 1.2.22 and the hyper-cohomology spectral sequences for $\Omega^\bullet_{X/\mathbb{C}}(\log D)$, we get that the canonical morphisms of $\mathbb{C}$-vector spaces

$$\mathbb{H}^i(Y_{Zar}, \Omega^p_{Y/\mathbb{C}}(\log D)) \cong \mathbb{H}^i(Y(\mathbb{C})_{an}, \Omega^p_{Y(\mathbb{C})}(\log D(\mathbb{C})))$$

are isomorphisms.

$\square$

This allows to conclude the generalization of theorem 1.5.12 to any smooth algebraic variety over $\mathbb{C}$.

Theorem 1.5.24. *Let X be a smooth algebraic variety over $\mathbb{C}$. Then, the canonical morphisms of $\mathbb{C}$-vector spaces constructed in 1.8, for each $i \geq 0$,*

$$H^i_{AdR}(X/\mathbb{C}) \to H^i_{dR}(X(\mathbb{C})),$$

are isomorphisms.

Proof. By step 1), there exists $\overline{X}$ a smooth projective algebraic variety over $\mathbb{C}$ with an open immersion $j : X \hookrightarrow \overline{X}$, such that $D := \overline{X} \setminus X$ the complementary closed subset with the reduced scheme structure is a NCD of $\overline{X}$. By above constructions applied to $Y = \overline{X}$, we have a commutative square of canonical morphisms of $\mathbb{C}$-vector spaces

$$\begin{array}{ccc} \mathbb{H}^i(\overline{X}_{Zar}, \Omega^\bullet_{\overline{X}/\mathbb{C}}(\log D)) & \longrightarrow & \mathbb{H}^i(\overline{X}(\mathbb{C})_{an}, \Omega^\bullet_{\overline{X}(\mathbb{C})}(\log D(\mathbb{C}))) \\ \downarrow & & \downarrow \\ \mathrm{H}^i_{\mathrm{AdR}}(X/\mathbb{C}) & \longrightarrow & \mathrm{H}^i_{\mathrm{dR}}(X(\mathbb{C})) \end{array}$$

By propositions 1.5.18 and 1.5.20, the vertical morphisms are isomorphisms. Since $\overline{X}$ is proper, by theorem 1.5.23, the upper horizontal morphism is an isomorphism. Hence, also the lower horizontal morphism is an isomorphism. $\square$

1.6 The algebraic de Rham Theorem

1.6.1 The algebraic de Rham isomorphism

Let $\sigma : k \hookrightarrow \mathbb{C}$ be a field extension. Given X an algebraic variety over k we can associate to X two different cohomology theories. One is the algebraic de Rham cohomology defined in section 1.5, that is, the family of k-vector spaces

$$\mathrm{H}^i_{\mathrm{AdR}}(X/k) := \mathbb{H}^i(X_{Zar}, \Omega^\bullet_{X/k}).$$

The other is obtained by considering the base change of X along σ

$$X_\sigma := X \times_k \mathbb{C},$$

which is a smooth algebraic variety over $\mathbb{C}$, since smooth morphisms are stable under base change, and taking the singular cohomology with coefficients in $\mathbb{Q}$ of the analytification. This cohomology theory takes the following name in Algebraic Geometry.

Definition 1.6.1. Given X an algebraic variety over $\mathbb{C}$, the family of $\mathbb{Q}$-vector spaces

$$\mathrm{H}^i_{\mathrm{Bet}}(X) := \mathrm{H}^i_{\mathrm{Sing}}(X(\mathbb{C}); \mathbb{Q})$$

is called the *Betti cohomology* of X.

The *algebraic de Rham Theorem* states that these cohomology theories provide the same invariants extending scalars in $\mathbb{C}$.

Theorem 1.6.2 (Algebraic de Rham Theorem)**.** *Let $\sigma : k \hookrightarrow \mathbb{C}$ be a field extension. For any X smooth algebraic variety over k, there exists a canonical isomorphism of $\mathbb{C}$-vector spaces, for each $i \geq 0$,*

$$H^i_{AdR}(X/k) \otimes_k \mathbb{C} \cong H^i_{Bet}(X_\sigma) \otimes_{\mathbb{Q}} \mathbb{C},$$

natural in X.

Proof. The canonical isomorphisms are obtained by composition of the following canonical isomorphisms, which are also all natural.

- The canonical isomorphisms of the analytic de Rham Theorem 1.1.9 for $X_\sigma(\mathbb{C})$

$$\mathrm{H}^i_{\mathrm{dR}}(X_\sigma(\mathbb{C})) \cong \mathrm{H}^i_{\mathrm{Sing}}(X_\sigma(\mathbb{C}); \mathbb{C}) \cong \mathrm{H}^i_{\mathrm{Sing}}(X_\sigma(\mathbb{C}); \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \cong \mathrm{H}^i_{\mathrm{Bet}}(X_\sigma) \otimes_{\mathbb{Q}} \mathbb{C}.$$

- The canonical comparison isomorphism between the algebraic and the analytic de Rham cohomology 1.5.24 for X_σ

$$\mathrm{H}^i_{\mathrm{AdR}}(X_\sigma/\mathbb{C}) \cong \mathrm{H}^i_{\mathrm{dR}}(X_\sigma(\mathbb{C})).$$

- The canonical isomorphisms

$$\mathrm{H}^i_{\mathrm{AdR}}(X/k) \otimes_k \mathbb{C} \cong \mathrm{H}^i_{\mathrm{AdR}}(X_\sigma/\mathbb{C}),$$

  which are constructed as follows. By proposition 1.3.9, we have the canonical morphism of complexes of abelian sheaves over $X_{\sigma Zar}$

$$\tilde{\sigma}^{-1}\Omega^\bullet_{X/k} \to \Omega^\bullet_{X_\sigma/\mathbb{C}},$$

where $\tilde{\sigma} : X_\sigma \to X$ is the canonical morphism of the fiber product. This induces the morphisms on sheaf cohomology

$$\mathbb{H}^i(X_{\sigma Zar}, \tilde{\sigma}^{-1}\Omega^\bullet_{X/k}) \to \mathbb{H}^i(X_{\sigma Zar}, \Omega^\bullet_{X_\sigma/\mathbb{C}}) = \mathrm{H}^i_{\mathrm{AdR}}(X_\sigma/\mathbb{C}).$$

Composing with the canonical functoriality morphisms

$$\mathrm{H}^i_{\mathrm{AdR}}(X/k) = \mathbb{H}^i(X_{Zar}, \Omega^\bullet_{X/k}) \to \mathbb{H}^i(X_{\sigma Zar}, \tilde{\sigma}^{-1}\Omega^\bullet_{X/k}),$$

we get the canonical morphisms of abelian groups

$$\mathrm{H}^i_{\mathrm{AdR}}(X/k) \to \mathrm{H}^i_{\mathrm{AdR}}(X_\sigma/\mathbb{C}),$$

which are also k-linear. Since $\mathrm{H}^i_{\mathrm{AdR}}(X_\sigma/\mathbb{C})$ are also $\mathbb{C}$-vector spaces, we get the canonical morphisms of $\mathbb{C}$-vector spaces

$$\mathrm{H}^i_{\mathrm{AdR}}(X/k) \otimes_k \mathbb{C} \to \mathrm{H}^i_{\mathrm{AdR}}(X_\sigma/\mathbb{C}). \tag{1.15}$$

Consider the hyper-cohomology spectral sequences

$$E_1^{p,q} = \mathbb{H}^q(X_{Zar}, \Omega^p_{X/k}) \Rightarrow \mathbb{H}^{p+q}(X_{Zar}, \Omega^\bullet_{X/k}) = \mathrm{H}^{p+q}_{\mathrm{AdR}}(X/k),$$

which induces the spectral sequence

$$E_1^{p,q} = \mathbb{H}^q(X_{Zar}, \Omega^p_{X/k}) \otimes_k \mathbb{C} \Rightarrow \mathbb{H}^{p+q}(X_{Zar}, \Omega^\bullet_{X/k}) \otimes_k \mathbb{C} = \mathrm{H}^{p+q}_{\mathrm{AdR}}(X/k) \otimes_k \mathbb{C},$$

and

$$E_1^{p,q} = \mathbb{H}^q(X_{\sigma Zar}, \Omega^p_{X_\sigma/\mathbb{C}}) \Rightarrow \mathbb{H}^{p+q}(X_{\sigma Zar}, \Omega^\bullet_{X_\sigma/\mathbb{C}}) = \mathrm{H}^{p+q}_{\mathrm{AdR}}(X_\sigma/\mathbb{C}).$$

Since $\Omega^p_{X/k}$ are quasi-coherent $\mathcal{O}_X$-modules (proposition 1.3.6), by flat base change (see [Har77, §III, Prop. 9.3]) and proposition 1.3.9, we get the canonical isomorphisms

$$\mathbb{H}^q(X_{Zar}, \Omega^p_{X/k}) \otimes_k \mathbb{C} \cong \mathbb{H}^q(X_{\sigma Zar}, \tilde{\sigma}^*\Omega^p_{X/k}) \cong \mathbb{H}^q(X_{\sigma Zar}, \Omega^p_{X_\sigma/\mathbb{C}}),$$

which define an isomorphism between the last two spectral sequences. Hence, the morphism induced on the limits of the spectral sequences, which is the canonical morphism 1.15, is an isomorphism.

□

Definition 1.6.3. Given X a smooth algebraic variety over a field k with a field extension $\sigma : k \hookrightarrow \mathbb{C}$, the canonical morphism of $\mathbb{C}$-vector spaces, for each $i \geq 0$,

$$\varpi^i_X : \mathrm{H}^i_{\mathrm{AdR}}(X/k) \otimes_k \mathbb{C} \cong \mathrm{H}^i_{\mathrm{Bet}}(X_\sigma) \otimes_{\mathbb{Q}} \mathbb{C}$$

is called the *algebraic de Rham isomorphism.*

Remark 1.6.4. The algebraic de Rham isomorphism is *canonical* once we fix the field extension σ, in the sense that for different field extensions we get different isomorphisms.

1.6.2 Period numbers

For $k = \overline{\mathbb{Q}}$, with the inclusion into $\mathbb{C}$, given X an algebraic variety over $\overline{\mathbb{Q}}$, we have the associated algebraic de Rham isomorphisms

$$\varpi_X^i : \mathrm{H}^i_{\mathrm{AdR}}(X/\overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \cong \mathrm{H}^i_{\mathrm{Bet}}(X_\sigma) \otimes_{\mathbb{Q}} \mathbb{C}.$$

These canonical isomorphisms of $\mathbb{C}$-vector spaces produce some arithmetic invariants associated to the algebraic variety X. Notice that a $\overline{\mathbb{Q}}$-basis of $\mathrm{H}^i_{\mathrm{AdR}}(X/\overline{\mathbb{Q}})$ induces a $\mathbb{C}$-basis of $\mathrm{H}^i_{\mathrm{AdR}}(X/\overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$, which we still call a $\overline{\mathbb{Q}}$-basis. Analogously, we can consider a $\mathbb{Q}$-basis of $\mathrm{H}^i_{\mathrm{Bet}}(X_\sigma) \otimes_{\mathbb{Q}} \mathbb{C}$.

Definition 1.6.5. Let X be a smooth algebraic variety over $\overline{\mathbb{Q}}$. The complex numbers arising as entries some representative matrix of the canonical isomorphisms ϖ_X^i, with respect to a $\overline{\mathbb{Q}}$-basis and a $\mathbb{Q}$-basis, are called *period numbers.*

For this reason, in this case, the canonical isomorphism ϖ_X^i is also called *period isomorphisms.* Concretely, following the isomorphisms which give rise to the period isomorphisms (see the proof of theorem 1.6.2) and taking account of the concrete description of the analytic de Rham theorem in remark 1.1.10, we see that period numbers are obtained by integrating closed $\overline{\mathbb{Q}}$-linear algebraic form along $\mathbb{Q}$-linear singular cycles over $X_\sigma(\mathbb{C})$.

Example 1.6.6. Take $X = \mathbb{G}_m \cong \mathit{Spec}(\overline{\mathbb{Q}}[t,u](tu-1))$. Recall from example 1.5.3, 3) that its algebraic de Rham cohomology is given by

$$\mathrm{H}^i_{\mathrm{AdR}}(\mathbb{G}_m/\overline{\mathbb{Q}}) \cong \begin{cases} \overline{\mathbb{Q}} & \text{for } i = 0,1 \\ 0 & \text{else.} \end{cases}$$

Consider the period isomorphism

$$\varpi^1_{\mathbb{G}_m} : \mathrm{H}^1_{\mathrm{AdR}}(\mathbb{G}_m/\overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \cong \mathrm{H}^1_{\mathrm{Bet}}(\mathbb{G}_m) \otimes_{\mathbb{Q}} \mathbb{C}.$$

Notice that the analytification of $\mathbb{G}_m$ is $\mathbb{C}$ without the origin

$$\mathbb{G}_m(\mathbb{C}) \cong \mathbb{C}^\times.$$

A $\overline{\mathbb{Q}}$-basis of $\mathrm{H}^i_{\mathrm{AdR}}(\mathbb{G}_m/\overline{\mathbb{Q}})$ corresponds to the class of the algebraic form $\frac{dz}{z}$ over $\mathbb{C}^\times$. A $\mathbb{Q}$-basis of $\mathrm{H}^1_{\mathrm{Bet}}(\mathbb{G}_m)$ corresponds to the class of the singular cycle $\gamma : t \mapsto \exp(2\pi i t)$

over $\mathbb{C}^\times$. Then,

$$\int_\gamma \frac{dz}{z} = 2\pi i$$

is a period number of $\mathbb{G}_m$.

Chapter 2

The de Rham Theorem in Theory of Motives

2.0.1 Overview

In this chapter we revisit the Algebraic de Rham Theorem inside *Theory of Motives*.

Theory of Motives is a vast research area with many different approaches, conjectural programs and candidate concrete theories. The whole picture remains to be understood.

Broadly speaking, one of the ultimate aims of Theory of Motives is that of providing a unifying framework in which to study both *arithmetic-analytic invariants* and *algebro-geometric invariants* of algebraic varieties, as well as relations between them. Examples of the former are algebraic de Rham and Betti cohomologies. Examples of the latter are Chow groups and algebraic K-theory, which are some invariants constructed out, respectively, algebraic cycles (linear combinations of integral closed subvarieties) and algebraic vector bundles over the given algebraic variety.

The relations between these two kinds of invariants are the content of some deep conjectures in Algebraic Geometry, such as the *Grothendieck-Hodge Conjecture*, the *Tate Conjecture* and the *Grothendieck Period Conjecture*. The final aim of this chapter is to state a version of the latter, as presented in [And+20, §1.3].

To do it, we have to, not only recover the algebraic de Rham isomorphism inside the conceptual framework given by Theory of Motives, but also construct maps relating algebraic cycles to de Rham and Betti cohomologies, in a compatible way under the algebraic de Rham isomorphism. So, revisiting the algebraic de Rham Theorem in Theory of Motives, means to obtain more than the only algebraic de Rham isomorphism of the previous chapter. In order to understand this revisitation, we outline some facts about Theory of Motives.

The idea for a Theory of Motives was introduced for the first time by Grothendieck in the 60's, while studying the Weil Conjectures. The matter is that there exist many different cohomology theories for smooth projective algebraic varieties over a field (the kind of spaces involved in the Weil Conjectures). Examples are the algebraic de Rham and Betti cohomologies

$$\begin{aligned} \mathrm{H}^*_{\mathrm{Bet}} = \oplus_{i\geq 0}\mathrm{H}^i_{\mathrm{Bet}} :& SmProj^{op}_{\mathbb{C}} \to grVect_{\mathbb{Q}} \\ \mathrm{H}^*_{\mathrm{AdR}} = \oplus_{i\geq 0}\mathrm{H}^i_{\mathrm{AdR}} :& SmProj^{op}_{k} \to grVect_{k}, \end{aligned}$$

which are defined over the fields $\mathbb{C}$ and k of characteristic 0. Others, defined over finite fields $\mathbb{F}_p$ of positive characteristic, the *étale ℓ-adic cohomologies* (see [Mil13, §19])

$$\mathrm{H}^*_{\ell} = \oplus_{i\geq 0}\mathrm{H}^i_{\ell} : SmProj^{op}_{\mathbb{F}_p} \to grVect_{\mathbb{Q}_\ell},$$

were introduced in that years, in order to solve the Weil Conjectures. Although these cohomology theories are defined using different tools (for example, tools of topological nature for the Betti cohomology, of algebraic nature for the algebraic de Rham cohomology and of arithmetic nature for the ℓ-adic cohomologies), they turn out to share some common properties. Moreover, under suitable conditions, they are also related by comparison isomorphisms. The algebraic de Rham isomorphism proved in the previous chapter is an example. The common properties of these cohomology theories are encoded in the abstract notion of a *Weil cohomology theory* (see subsection 2.1.1). Briefly, a Weil cohomology over a field k, with coefficients in a field K of characteristic 0, is the data of a contravariant functor from smooth projective varieties over k into finite dimensional graded K-vector spaces

$$\mathrm{H}^* \coloneqq \oplus_i \mathrm{H}^i : SmProj^{op}_k \to grVect_K,$$

satisfying some axioms. The main axioms are: *Künneth formula*, *Poincaré duality* and the existence of *cycle class maps*, for any $i \geq 0$,

$$cl^i_X : \mathrm{CH}^i(X) \to \mathrm{H}^{2i}(X)(i),$$

where $\mathrm{H}^{2i}(X)(i) \coloneqq H^{2i}(X) \otimes K(1)^{\otimes i}$, with $K(1) \in grVect_K$ a given object of dimension 1 concentrated in degree -2, called the *Tate module*. Its inverse $K(-1) \coloneqq K(1)^{\otimes -1}$ is canonically isomorphic to $\mathrm{H}^2(\mathbb{P}^1_k)$, called the *Lefschetz module*. Recall that $\mathrm{CH}^i(X)$, the *Chow group* of codimension i, is the group $Z^i(X)$ of algebraic cycles of codimension i (the free abelian group generated by integral subvarieties of X of codimension i), modulo the *rational equivalence* $\sim_{rat}$ (see [Ful98, §1.3])

$$\mathrm{CH}^i(X) \coloneqq Z^i(X)/\sim_{rat}.$$

The cycle class map is the important tool that relates algebraic cycles to the given Weil cohomology. The algebraic de Rham, Betti and ℓ-adic cohomologies are all examples of Weil cohomologies. A natural question is whether, for any fixed base field k, they all come from some universal Weil cohomology with coefficients in $\mathbb{Q}$, in the sense that any other Weil cohomology with coefficients in K is canonically isomorphic to it, after extending coefficients. This is proved not to be the case (there exists a counterexample of Serre in positive characteristic, see [Mil12, §3], and another argument in characteristic 0, given in subsection 2.1.3, is related to the existence of transcendental periods). Then, in order to find a deep reason and explain a common origin to the various concrete cohomology theories of smooth projective algebraic varieties over k, Grothendieck's idea was to look for a cohomology theory with values, not in the category of finite dimensional graded $\mathbb{Q}$-vector spaces, but more generally in some category $\mathcal{M}(k)$, called the *category of motives*. Accounts of this conjectural program can be found for example in [Ser91]. The category of motives $\mathcal{M}(k)$ is expected to have properties similar to the ones of finite dimensional $\mathbb{Q}$-vector spaces: it should be rigid, tensor, $\mathbb{Q}$-linear, abelian, semi-simple and with finite dimensional hom-sets. The cohomology theory with values in $\mathcal{M}(k)$ should be a monoidal contravariant functor

$$h : SmProj_k^{op} \to \mathcal{M}(k),$$

called *motivic cohomology*, such that any Weil cohomology H^* over k with coefficients in K factors uniquely through it, with a faithful exact tensor functor R_{H}, called *realization functor*,

$$\begin{tikzcd} SmProj_k^{op} \arrow[rr, "\mathrm{H}^*"] \arrow[dr, "h"'] & & grVect_K. \\ & \mathcal{M}(k) \arrow[ur, "R_{\mathrm{H}}"'] & \end{tikzcd}$$

For example, we should have the *Betti, algebraic de Rham* and *ℓ-adic realization functors*

$$\begin{aligned} R_{\mathrm{Bet}} :& \mathcal{M}(\mathbb{C}) \to grVect_{\mathbb{Q}} \\ R_{\mathrm{AdR}} :& \mathcal{M}(k) \to grVect_k \\ R_\ell :& \mathcal{M}(\mathbb{F}_p) \to grVect_{\mathbb{Q}_\ell}. \end{aligned}$$

Moreover, the functor h should have a decomposition

$$h = \oplus_{i\geq 0} h^i,$$

which induces on the motivic cohomology of any $X \in SmProj_k$ a graduation

$$h^*(X) \cong \oplus_{i\geq 0} h^i(X),$$

called *graduation by weights*. The realization functors should be such that

$$R_{\mathrm{H}}(h^i(X)) = \mathrm{H}^i(X),$$

that is, each $h^i(X)$ should *realize* into $\mathrm{H}^i(X)$. The object $\mathbb{1}(-1) := h^2(\mathbb{P}^1_k)$ is called the *Lefschetz object*. Its dual should coincide with its inverse with respect to tensor product $\mathbb{1}(1) := \mathbb{1}(-1)^{\otimes -1}$, called the *Tate object*, which realizes into the Tate module. For any $j \in \mathbb{Z}$, we denote by $h^i(X)(j) := h^i(X) \otimes \mathbb{1}(1)^{\otimes j}$.

So, we can think at the category of motives $\mathcal{M}(k)$ as a linearization of $SmProj_k$, whose objects, the *motives*, are the "cohomological essence" of smooth projective algebraic varieties over k, and which is endowed with realization functors into concrete cohomology theories.

The main concrete outcome of this conjectural program is a construction proposed by Grothendieck: the motivic cohomology into the *category of Chow motives*

$$h : SmProj_k^{op} \to \mathrm{CHM}(k; \mathbb{Q}).$$

It is a very formal construction based on algebraic cycles modulo rational equivalence (see subsection 2.1.3). However, this category doesn't fulfill all the expected requirements for $\mathcal{M}(k)$. For example, it is not even an abelian category. A slight variation of the category of Chow motives, the *category of numerical motives*, obtained by considering the *numerical equivalence* instead of the rational equivalence, gives rise to an abelian category (see [Jan92]). The category of numerical motives is conjecturally the expected abelian category of motives $\mathcal{M}(k)$, provided that some conjectures hold true. These conjectures are part of the *Standard Conjectures*, some statements about the existence of some algebraic cycles, formulated by Grothendieck in order to prove the Weil Conjectures (see [And04, §5] or [Kle]). However, nowadays Standard Conjectures are still unsolved and we still don't have the abelian category of motives $\mathcal{M}(k)$ predicted by Grothendieck.

Progress in Theory of Motives occurred by trying to extend the conjectural picture from the context of smooth projective algebraic varieties to the one of general algebraic varieties. Indeed, concrete cohomology theories can be defined also for general algebraic varieties and some properties and comparison isomorphisms still hold, so it is reasonable to expect such an extension. In this passage was fundamental the work of Deligne on Hodge Theory ([Del71b] and [Del74b], or see also [Ste]). Deligne proved that the Betti cohomology of algebraic varieties over $\mathbb{C}$ has a *mixed Hodge structure*, generalizing the Hodge decomposition, or equivalently the *pure Hodge structure*, of the Betti cohomology of smooth projective algebraic varieties over $\mathbb{C}$. Mixed Hodge structures form a rigid tensor abelian category $\mathrm{MHS}_{\mathbb{Q}}$, whose objects are endowed with an increasing filtration, with successive quotients which are pure Hodge structures. Analogous results hold also for the étale ℓ-adic cohomologies, where $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_k)$, the category of finite $\mathbb{Q}_\ell$-linear continuous representations of G_k, the absolute Galois group of k, plays the analogous role of $\mathrm{MHS}_{\mathbb{Q}}$ (see [Del74a] and [Del71a]). Assuming that these structures should have a motivic origin, Deligne conjectured the existence of a rigid tensor $\mathbb{Q}$-linear abelian *category of mixed motives*

$\mathcal{MM}(k)$ with a monoidal contravariant functor from algebraic varieties over k

$$h : Var_k^{op} \to \mathcal{MM}(k),$$

which should contain $\mathcal{M}(k)$, now renamed the *category of pure motives*, as the full subcategory of semi-simple objects. Accounts of this conjectural program can be found for example in [Dela]. Each mixed motive $M \in \mathcal{MM}(k)$ is expected to be endowed with an increasing filtration $W_\bullet$, called the *weight filtration*, such that the successive quotients are pure motives, that is

$$gr_i^W M := W_i M / W_{i-1} M \in \mathcal{M}(k).$$

Moreover, we expect that realization functors from $\mathcal{M}(k)$ can be extended to $\mathcal{MM}(k)$ and also factor through the category $\mathrm{MHS}_{\mathbb{Q}}$, for the Betti realization, and $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_k)$, for the étale ℓ-adic realizations. That is, there exist the *enriched realization functors*

$$\begin{aligned} R_{\mathrm{Bet}} :& \mathcal{MM}(k) \to \mathrm{MHS}_{\mathbb{Q}} \\ R_\ell :& \mathcal{MM}(k) \to \mathrm{Rep}_{\mathbb{Q}_\ell}(G_k), \end{aligned}$$

called *Hodge* and *Tate realizations* respectively. The Grothendieck-Hodge and the Tate conjectures asserts that these enriched realization functors are full on $\mathcal{M}(k)$ (see [And04, Prop. 7.2.1.3, Prop. 7.3.1.3]). Analogously, the version of the Grothendieck Period Conjecture we will describe is about fullness of an enriched realization functor.

Another crucial point for the development of Theory of Motives was progress in the study of algebraic cycles. Algebraic cycles are usually studied modulo an *adequate equivalence*, which allows to define a ring structure over the quotient (and which also allows to define composition in Grothendieck's categories of pure motives). The coarser adequate equivalence is the rational equivalence, which gives rise to the *Chow ring* (and which is the one used in the construction of the category of Chow motives)

$$\mathrm{CH}^*(X) := \oplus_{i\geq 0} \mathrm{CH}^i(X) = \oplus_{i\geq 0} Z^i(X)/\sim_{rat} .$$

The problem is that the groups $\mathrm{CH}^i(X)$ in general are very large, so they are difficult to study. Indeed, taking rational coefficients, $\mathrm{CH}^i(X)_{\mathbb{Q}} := \mathrm{CH}^i(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ are infinite dimensional $\mathbb{Q}$-vector spaces. Instead, the choice of other finer adequate equivalences gives rise to finite dimensional $\mathbb{Q}$-vector spaces. An example is the *homological equivalence* $\sim_{hom}$, which is induced by the kernel of the cycle class map of some Weil cohomology H^* (one of the Standard Conjectures states that it is equivalent to the numerical equivalence, hence it is independent from H^*). Indeed, by definition, the quotients by homological equivalence are isomorphic to a sub-vector space of the cohomology groups

$$Z^i(X)_{\mathbb{Q}}/\sim_{hom} \cong \mathrm{CH}^i(X)_{\mathbb{Q}}/ker(cl_X^i) \hookrightarrow \mathrm{H}^{2i}(X)(i),$$

which are finite dimensional $\mathbb{Q}$-vector spaces. Moreover (see [Jan, §1]), for both for Betti and étale ℓ-adic cohomologies, there exist the following $\mathbb{Q}$-linear maps

$$\begin{aligned} cl^{(1),i}_{\mathrm{Bet},X} :& ker(cl^i_{\mathrm{Bet},X}) \to \mathrm{Ext}^1_{\mathrm{MHS}_{\mathbb{Q}}}(\mathbb{Q}, \mathrm{H}^{2i-1}_{\mathrm{Bet}}(X)(i)) \\ cl^{(1),i}_{\ell,X} :& ker(cl^i_{\ell,X}) \to \mathrm{Ext}^1_{\mathrm{Rep}_{\mathbb{Q}_\ell}(G_k)}(\mathbb{Q}_\ell, \mathrm{H}^{2i-1}_{\ell}(X)(i)), \end{aligned}$$

whose images are finite dimensional $\mathbb{Q}$-vector spaces. The first are usually called the *Abel-Jacobi maps*, while the second are deduced from the *Hochschild-Serre spectral sequence*. From the latter, we can also obtain the $\mathbb{Q}$-linear maps, for any $\nu > 1$,

$$cl^{(\nu),i}_{\ell,X} : ker(cl^{(\nu-1)i}_{\ell,X}) \to \mathrm{Ext}^\nu_{\mathrm{Rep}_{\mathbb{Q}_\ell}(G_k)}(\mathbb{Q}_\ell, \mathrm{H}^{2i-\nu}_{\ell}(X)(i)),$$

whose images are finite dimensional $\mathbb{Q}$-vector spaces. This induces a descending filtration $F^\bullet_\ell$ on rational Chow groups $\mathrm{CH}^i(X)_{\mathbb{Q}}$, given by

$$\begin{aligned} &F^0_\ell \mathrm{CH}^i(X)_{\mathbb{Q}} \coloneqq \mathrm{CH}^i(X)_{\mathbb{Q}}, \\ &F^1_\ell \mathrm{CH}^i(X)_{\mathbb{Q}} \coloneqq ker(cl^i_{\ell,X}), \\ &F^\nu_\ell \mathrm{CH}^i(X)_{\mathbb{Q}} \coloneqq ker(cl^{(\nu-1),i}_{\ell,X}) \qquad \text{for } \nu > 1. \end{aligned}$$

The successive quotients

$$gr^\nu_{F_\ell} \mathrm{CH}^i(X)_{\mathbb{Q}} \coloneqq F^\nu_\ell \mathrm{CH}^i(X)_{\mathbb{Q}} / F^{\nu+1}_\ell \mathrm{CH}^i(X)_{\mathbb{Q}} \cong Im(cl^{(\nu),i}_{\ell,X})$$

are finite dimensional $\mathbb{Q}$-vector spaces, contained into $\mathrm{Ext}^\nu_{\mathrm{Rep}_{\mathbb{Q}_\ell}(G_k)}(\mathbb{Q}_\ell, \mathrm{H}^{2i-\nu}_{\ell}(X)(i))$. Assuming that this should have a motivic origin, Beilinson conjectured (see [Jan, Conj. 2.3]) the existence of a descending filtration $F^\bullet$ on rational Chow groups $\mathrm{CH}^i(X)_{\mathbb{Q}}$, for any $X \in SmProj_k$, which starts with

$$F^0 \mathrm{CH}^i(X)_{\mathbb{Q}} \coloneqq \mathrm{CH}^i(X)_{\mathbb{Q}} \qquad \& \qquad F^1 \mathrm{CH}^i(X)_{\mathbb{Q}} \coloneqq ker(cl^i_X),$$

and whose successive quotients are such that

$$gr^\nu_F \mathrm{CH}^i(X)_{\mathbb{Q}} \coloneqq F^\nu \mathrm{CH}^i(X)_{\mathbb{Q}} / F^{\nu+1} \mathrm{CH}^i(X)_{\mathbb{Q}} \cong \mathrm{Ext}^\nu_{\mathcal{MM}(k)}(\mathbb{1}, h^{2i-\nu}(X)(i)),$$

where $\mathbb{1} \in \mathcal{MM}(k)$ is the unit object.

Further, there is the fact that Betti and étale ℓ-adic cohomologies arise as cohomologies of functorial complexes in the (bounded) derived categories of $\mathrm{MHS}_{\mathbb{Q}}$ and $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_k)$ respectively. That is, we have monoidal contravariant functors, given by the total right-derived of global section functors,

$$\begin{aligned} \mathbf{R}_{\mathrm{Bet}} : Var^{op}_{\mathbb{C}} &\to \mathrm{D}^b(\mathrm{MHS}_{\mathbb{Q}}) \\ X &\mapsto \mathbf{R}\Gamma(X(\mathbb{C}), \mathbb{Q}), \end{aligned}$$

and

$$\begin{aligned}\mathbf{R}_\ell : Var_{\mathbb{F}_p}^{op} &\to \mathrm{D}^b(\mathrm{Rep}_{\mathbb{Q}_\ell}(G_k))\\ X &\mapsto \mathbf{R}\Gamma(X \times_k \bar{k}, \mathbb{Q}_\ell),\end{aligned}$$

such that

$$H^*_{\mathrm{Bet}}(X) \cong \oplus_{i\geq 0}\mathrm{H}^i(\mathbf{R}\Gamma(X(\mathbb{C}),\mathbb{Q})) \qquad \& \qquad \mathrm{H}^*_\ell(X) \cong \oplus_{i\geq 0}\mathrm{H}^i(\mathbf{R}\Gamma(X\times_k \bar{k},\mathbb{Q}_\ell)).$$

This suggests to formulate a derived version of the conjectural program of mixed motives (see [Jan, Conj. 4.1]). We can expect that there exists a monoidal contravariant functor

$$\mathbf{R} : Var_k^{op} \to \mathrm{D}^b(\mathcal{MM}(k)),$$

such that $\mathbf{R}_{\mathrm{Bet}}$ and $\mathbf{R}_\ell$ factor with exact faithful monoidal triangulated realization functors

$$\begin{aligned}R_{\mathrm{Bet}} :&\mathrm{D}^b(\mathcal{MM}(k)) \to \mathrm{D}^b(\mathrm{MHS}_{\mathbb{Q}})\\ R_\ell :&\mathrm{D}^b(\mathcal{MM}(k)) \to \mathrm{D}^b(\mathrm{Rep}_{\mathbb{Q}_\ell}(G_k)).\end{aligned}$$

The expected relation with the abelian category of pure motives is that, for any $X \in SmProj_k$,

$$\mathrm{H}^i(\mathbf{R}(X)) \cong h^i(X) \in \mathcal{M}(k)$$

and we have a quasi-isomorphism of complexes in $\mathcal{MM}(k)$ (see [Jan, Lemma 4.3] and [Dela, §3.3])

$$\mathbf{R}(X) \simeq \oplus_{i\geq 0} h^i(X)[-i].$$

So, we have $\mathbb{1}(-1) \cong \mathrm{H}^2(\mathbf{R}(\mathbb{P}^1_k))$. For any $j \in \mathbb{Z}$, we denote by $\mathbf{R}(X)(j) \coloneqq \mathbf{R}(X) \otimes \mathbb{1}(1)^{\otimes j}$. Moreover, for any $X \in SmProj_k$ and $i \geq 0$, we expect the existence of a canonical isomorphism

$$\mathrm{CH}^i(X)_{\mathbb{Q}} \cong \mathrm{Hom}_{\mathrm{D}^b(\mathcal{MM}(k))}(\mathbb{1}, \mathbf{R}(X)(i)[2i]). \tag{2.1}$$

This requirement can be thought as a derived version of the conjectural descending filtration on Chow groups described above. Indeed, recalling that the Ext-groups in an abelian category $\mathring{A}$ are computed by hom-sets in its derived category as, for any $M, N \in \mathcal{A}$ and $p \geq 0$,

$$\mathrm{Hom}_{\mathrm{D}^b(\mathcal{A})}(M, N[p]) \cong \mathrm{Ext}^p_{\mathcal{A}}(M,N),$$

we have the Ext spectral sequence in $\mathcal{MM}(k)$

$$E_2^{p,q} \coloneqq \mathrm{Hom}_{\mathrm{D}^b(\mathcal{MM}(k))}(\mathbb{1}, \mathrm{H}^q(\mathbf{R}(X)(i))[p]) \Rightarrow \mathrm{Hom}_{\mathrm{D}^b(\mathcal{MM}(k))}(\mathbb{1}, \mathbf{R}(X)(i)[p+q]).$$

This spectral sequence induces a descending filtration $F^\bullet$ on the target. In particular, for $p+q=2i$, by 2.1, we get a descending filtration on $\mathrm{CH}^i(X)_{\mathbb{Q}}$. By the degeneracy of the spectral sequence at page 2 (see [Jan, Prop. 4.4]), it follows that the filtration is such that

$$\begin{aligned} gr_F^\nu \mathrm{CH}^i(X)_{\mathbb{Q}} \cong E_2^{\nu,2i-\nu} &\cong \mathrm{Hom}_{\mathrm{D}^b(\mathcal{MM}(k))}(\mathbb{1}, \mathrm{H}^{2i-\nu}(\mathbf{R}(X)(i))[\nu]) \cong \\ &\cong \mathrm{Ext}^\nu_{\mathcal{MM}(k)}(\mathbb{1}, h^{2i-\nu}(X)(i)). \end{aligned}$$

Moreover, it follows from the expected requirements that it should exist a fully-faithful embedding of $\mathbb{Q}$-linear tensor categories (see [Jan, Lemma 4.6])

$$\mathrm{CHM}(k;\mathbb{Q}) \hookrightarrow \mathrm{D}^b(\mathcal{MM}(k)), \tag{2.2}$$

which maps $h(X)$ into $\mathbf{R}(X)$. In general, the groups

$$\mathrm{H}^i(X, \mathbb{Q}(j)) \coloneqq \mathrm{Hom}_{\mathrm{D}^b(\mathcal{MM}(k))}(\mathbb{1}, \mathbf{R}(X)(j)[i])$$

are called *motivic cohomology groups* of X (in [Dela] are also called *absolute cohomology*, to avoid confusion with motivic cohomology of X, which is the motive in $\mathcal{MM}(k)$ associated to X).

There are several candidate *triangulated categories of mixed motives* which try to fulfill the requirements expected from $\mathrm{D}^b(\mathcal{MM}(k))$. However, the abelian category of mixed motives $\mathcal{MM}(k)$ is still missing (or, less restrictively, the existence of a *t-structure* on the candidate triangulated categories of mixed motives, whose heart would be $\mathcal{MM}(k)$, see [Dela, §3.1] and [Jan, §4.7]). One of these proposals is *Voevodsky's triangulated category of mixed motives*, described in [Voe] and [MVW06]. It is considered a very promising candidate triangulated category of mixed motives, since it satisfies many of the expected properties, including 2.1 and 2.2. More generally, the motivic cohomology groups in Voevodsky's category are isomorphic to *Bloch's higher Chow groups* (see [Lev94]), whose definition is still based on algebraic cycles. In subsection 2.2.1 we consider a variant of this category, described in [Ayo13, §2.1]. It is equivalent to Voevodsky's category in the case we are interested in, that is k of characteristic zero and rational coefficients (see [Ayo13, Thm. B1]). This version is in fact more similar to the construction of the *Morel and Voevodsky's* $\mathbb{A}^1$*-homotopy category* (see [MV99]), except that we consider chain complexes instead of simplicial sets.

There are many other important aspects and properties expected from a category of mixed motives, that we won't consider in this thesis. One is the expected relation between motivic cohomology groups with higher algebraic K-theory (see [Dela, §3.7], [Jan, Conj. 4.1 (v)], [BMS87, §0.2]). Another aspect is a further extension of the conjectural picture of mixed motives (see [Jan, Conj. 4.8]), by which it is expected the existence of rigid tensor $\mathbb{Q}$-linear abelian categories $\mathcal{MM}(S)$, for any base scheme S, whose derived categories give rise to a *Grothendieck's six functor formalism*, similarly to what happens for étale ℓ-adic cohomology. This has been accomplished for the Voevodsky's category in [CD19]. Moreover, since both categories $\mathrm{MHS}_{\mathbb{Q}}$ and $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_k)$ are neutral tannakian categories, this suggests that also the abelian category of mixed motives should be a $\mathbb{Q}$-linear *neutral tannakian category* (see [DM82]), with fibre functor given by the Betti realization (see [And04, §6.2]). In particular, this last aspect allows to discuss a stronger formulation of the Grothendieck Period Conjecture, which involves the *motivic Galois group* (see [BC14, §2], [And04, Prop. 23.1.4.1], [Hub18, Conj. 5.14], [And08, Conj. 4.1.1]).

It may seems that the development of Theory of Motives took a different direction from the original Grothendieck's purpose of constructing a universal cohomology theory for algebraic varieties. However, for example Dugger's work (see [Dug00]), whose original motivation was indeed that of "explaining" the construction of the $\mathbb{A}^1$-homotopy category, describes a framework for a *universal homotopy theory* for a general category of spaces. Morel and Voevodsky's construction turns out to fit into this framework, applied to the category of smooth algebraic varieties (see [Dug00, §8]). An analogous point of view for cohomology is taken in [Bar23] and [Bar24], where it is described a framework for a *universal cohomology theory* for a general category of spaces. Applied to the category of smooth projective algebraic varieties, this tells that a universal Weil cohomology with values in an abelian category exists, and that we can use it to define a theory of pure motives.

2.0.2 Contents of the chapter

In section 2.1 we revisit the algebraic de Rham Theorem inside the conceptual framework of *Theory of Pure Motives.* In subsection 2.1.1, we start giving a precise definition of the central concept in Theory of Pure Motives: the *Weil cohomology theories.* The data and axioms in the definition of a Weil cohomology theory introduce the fundamental concepts of *Tate twist*, which can be thought as a technical device to express duals and the property of *Poincaré duality*, and the *cycle class map.* We see some consequences and constructions following directly from this definition. The most important ones are the construction of the *pushforward map* and the result that cohomology of $\mathbb{P}^1_k$ is completely determined by the axioms. In subsection 2.1.2 we consider the concrete examples of algebraic de Rham and Betti cohomologies. We state the result telling that they give rise to Weil cohomologies. In particular, we are interested in the construction of their Tate module and cycle class map. We do it using *Grothendieck's Theory of Chern classes*, for which we refer to Appendix B for the main constructions and results. Chern classes allow to reduce the construction of the cycle class map to the one of the *first Chern class*, which is a natural transformation from the *Picard group* to the 1-twisted second cohomology group. To define them, we also have to establish suitable Tate modules for algebraic de Rham and Betti cohomology. We also see that the canonical algebraic de Rham isomorphism proved in the previous chapter (theorem 1.6.2) generalizes to a version including Tate twists, obtaining a canonical *twisted algebraic de Rham isomorphism*. Then, we prove the main result of this section (proposition 2.1.15): the compatibility of the algebraic de Rham and the Betti cycle class maps under the twisted algebraic de Rham isomorphism. By properties in Grothendieck's Theory of Chern classes, we see that we can reduce to prove the compatibility of the first Chern class under the twisted algebraic de Rham isomorphism. In subsection 2.1.3 we describe the construction of $\mathrm{CHM}(k;\mathbb{Q})$, the *category of Chow motives*. This construction uses Chow groups and the cycle class map to define a category with realization functors. At the end of the construction, we remark that, viceversa, Chow groups and the cycle class map can be recovered from the category of Chow motives respectively as hom-sets and maps between hom-sets induced by realization functors.

In section 2.2 we pass to the conceptual framework of *Theory of Mixed Motives*. In subsection 2.2.1 we start describing the triangulated category of mixed motives we will work with. We define $\mathbf{DA}^{\mathrm{eff}}_{ét}(k;\Lambda)$, the *category of effective étale motivic sheaves*. The idea is to enlarge the category Sm_k, considering the category of complexes of presheaves over it, and formally impose the properties of *étale descent* and $\mathbb{A}^1$-*homotopy invariance*, by inverting some morphisms corresponding to these properties. The technical tool used to invert morphisms is that of Bousfield localization in model categories. We describe also an alternative equivalent construction, which uses the technical tool of Verdier localization in triangulated categories. This alternative description and the related notion of $\mathbb{A}^1$-*local objects* are useful to compute hom-sets in $\mathbf{DA}^{\mathrm{eff}}_{ét}(k;\Lambda)$. We also define the concepts of *motives associated* to a smooth algebraic variety, *Tate motives* and *étale motivic cohomology groups*. The latter arise as hom-sets in $\mathbf{DA}^{\mathrm{eff}}_{ét}(k;\Lambda)$ between the motive associated to a smooth algebraic variety and the Tate motives, that is, étale motivic cohomology groups are represented by the Tate motives. We conclude stating a result, which relates $\mathbf{DA}^{\mathrm{eff}}_{ét}(k;\Lambda)$ and its étale motivic cohomology groups to $\mathrm{CHM}^{\mathrm{eff}}(k,\Lambda)$ and Chow groups. In subsection 2.2.2 the aim is to construct a triangulated functor from $\mathbf{DA}^{\mathrm{eff}}_{ét}(\mathbb{C};\mathbb{Q})$ to $\mathrm{D}(\mathbb{Q})$, the derived category of $\mathbb{Q}$-vector spaces which assign to any motive associated to a smooth algebraic variety over $\mathbb{C}$ the complex of singular chains of the analytification of the variety. This functor is called the *Betti realization functor*. The strategy consists in introducing the category $\mathbf{AnDA}^{\mathrm{eff}}(\Lambda)$, which is the analogous of the category $\mathbf{DA}^{\mathrm{eff}}_{ét}(\mathbb{C};\Lambda)$, but replacing algebraic objects with analytic ones. These two categories are related by an adjunction, which is induced by the morphism of sites given by the analytification functor. Moreover, we prove that the category $\mathbf{AnDA}^{\mathrm{eff}}(\Lambda)$ is equivalent to $\mathrm{D}(\Lambda)$, the derived category of Λ-modules. The Betti realization functor is defined as the composition of the left adjoint with the equivalence of categories. It allows to define an object in the category $\mathbf{DA}^{\mathrm{eff}}_{ét}(\mathbb{C};\mathbb{Q})$, which represents the Betti cohomology. In subsection 2.2.3, in analogy with Betti cohomology, we define an object in $\mathbf{DA}^{\mathrm{eff}}_{ét}(k;k)$, which represents the algebraic de Rham cohomology. The aim it to prove that these two objects are canonically isomorphic, taking complex coefficients. This isomorphism can be thought as the formulation of the Algebraic de

Rham Theorem with the language of the considered triangulated category of mixed motives. The proof is obtained by putting together three canonical isomorphisms, which correspond exactly to the three canonical isomorphisms of the proof of the Algebraic de Rham Theorem of the previous chapter.

In section 2.3 we state a version of the *Grothendieck Period Conjecture*. We start in subsection 2.3.1 by considering only smooth projective algebraic varieties. The commutative squares expressing the compatibility of the algebraic de Rham and Betti cycle class maps under the twisted algebraic de Rham isomorphism (proposition 2.1.15) tells that algebraic cycles on an algebraic variety and its powers determine polynomial relations between periods of the algebraic variety. This observation leads to the formulation of the cycle-theoretic version of the Grothendieck Period Conjecture. This conjecture can be rephrased by stating that an enriched realization functor from the category of Chow motives, the *de Rham-Betti realization functor*, is full. In subsection 2.3.1 we pass to consider all smooth algebraic varieties. The existence of a commutative square with representatives of étale motivic cohomology groups, Betti cohomology and algebraic de Rham cohomology, induces commutative squares which generalize the ones in the projective case. This leads to formulate a natural generalization of the Grothendieck Period Conjecture for étale motivic cohomology groups.

2.1 Pure Motives

2.1.1 Weil cohomology theories

We give a possible precise definition of a Weil cohomology theory. There is not a univocal axiomatization. We refer to the one in [And04, §3.3]. We denote by $SmProj_k$ the full subcategory of Sm_k given by smooth projective algebraic varieties over a field k.

Definition 2.1.1. Given k, K fields, with K of characteristic 0, a *Weil cohomology theory* over k with coefficients in K is given by the following data and axioms.

D1) A contravariant functor from the category of smooth projective algebraic varieties over k into the category of finite dimensional $\mathbb{Z}$-graded K-vector spaces

$$\mathrm{H}^* = \oplus_{i\in\mathbb{Z}}\mathrm{H}^i : SmProj_k^{op} \to grVect_K,$$

called *cohomology*. For any morphism $f : X \to Y$ in $SmProj_k$

$$f^* \coloneqq \mathrm{H}^*(f) : \mathrm{H}^*(Y) \to \mathrm{H}^*(X)$$

is called the *pullback map* of f.

D2) An object $K(1) \in grVect_K$ concentrated in degree -2 and of dimension 1 as a K-vector space, hence (not canonically!) isomorphic to K, called the *Tate module*.

Given $V^* \in grVect_K$, its grading is denoted by $V^* = \oplus_i V^i$, where V^i is the homogeneous component of degree i. Recall that $grVect_K$ is a rigid tensor abelian category with tensor product of $V^*, W^* \in grVect_k$

$$V^* \otimes W^* := \oplus_k(\oplus_{i+j=k} V^i \otimes_K W^j).$$

We consider the commutativity constraint $V^* \otimes W^* \cong W^* \otimes V^*$ given by Koszul sign rule: for any $v \in V^i$ and $w \in W^j$, $v \otimes w \mapsto (-)^{ij} w \otimes v$. The unit object is K concentrated in degree 0. It is such that $\mathrm{End}(K) \cong K$, hence $grVect_K$ is K-linear. The internal-hom is

$$\underline{\mathrm{Hom}}(V^*, W^*) := \oplus_{j-i} \mathrm{Hom}_K(V^i, W^j).$$

In particular we have dual objects

$$V^{*\vee} := \underline{\mathrm{Hom}}(V^*, K) \cong \oplus_{-i} \mathrm{Hom}_K(V^i, K).$$

We denote by $K(-1) := K(1)^\vee$, called the *Lefschetz module*. It is concentrated in degree 2 and it is also the inverse of $K(1)$ with respect to tensor product, i.e. $K(-1) \cong K(1)^{\otimes -1}$. For any $r \in \mathbb{Z}$, we denote by

$$K(r) := K(1)^{\otimes r},$$

called the r^{th}*-twisted Tate module*, and, for any $V^* \in grVect_K$,

$$V^*(r) := V^* \otimes K(1)^{\otimes r} \cong \oplus_{i-2r} V^i(r),$$

where $V^i(r) := V^i \otimes_K K(r)$. $V^*(r)$ is called the r^{th}*-Tate twist* of V^*. Since $K(1)$ is invertible with respect to tensor product, we have the autoequivalence

$$(1) := \_ \otimes K(1) : grVect_K \to grVect_K,$$

called *Tate twist*, with inverse $(-1) := \_ \otimes K(-1)$. Notice that, for any $V^*, W^* \in grVect_K$ and $r, s \in \mathbb{Z}$,

$$V^*(r) \otimes W^*(s) \cong V^* \otimes W^*(r+s).$$

D3) A morphism of K-vector spaces, for any $X \in SmProj_k$ irreducible of dimension d,

$$\mathrm{Tr}_X : \mathrm{H}^{2d}(X)(d) \to K,$$

called the *trace map* of X.

D4) A group homomorphism, for any $X \in SmProj_k$ and $i \geq 0$,

$$cl_X^i : \mathrm{CH}^i(X) \to \mathrm{H}^{2i}(X)(i),$$

called the i^{th}*-cycle class map*, where $\mathrm{CH}^i(X)$ is the Chow group of codimension i. Recall that

$$\mathrm{CH}^i(X) \coloneqq Z^i(X)/\sim_{rat}.$$

is the group of algebraic cycles $Z^i(X)$, the free abelian group generated by integral closed subschemes) of X of codimension i, modulo the rational equivalence $\sim_{rat}$. For more details, we refer to [Ful98, §1]. It is equivalent to give a morphism in $grVect_{\mathbb{Q}}$ from the Chow group, graded by codimension, with rational coefficients

$$cl_X : \mathrm{CH}^*(X)_{\mathbb{Q}} \coloneqq \oplus_i \mathrm{CH}^i(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to \oplus_i \mathrm{H}^{2i}(X)(i).$$

These data satisfy the following axioms.

A1) (*Coproducts*) H^* preserves finite coproducts, i.e. we have canonical isomorphisms in $grVect_K$, for any $X, Y \in SmProj_k$,

$$\mathrm{H}^*(X \coprod Y) \cong \mathrm{H}^*(X) \oplus \mathrm{H}^*(Y).$$

A2) (*Dimension*) For any $X \in SmProj_k$ of dimension d,

$$\mathrm{H}^i(X) = 0 \qquad \text{for } i < 0 \text{ and } i > 2d.$$

Notice that, since $\mathrm{H}^*(X)$ is a finite dimensional graded K-vector space, then each $\mathrm{H}^i(X)$ is a finite dimensional K-vector space.

A3) (*Künneth formula*) H^* is a tensor functor, i.e. we have isomorphisms in $grVect_K$

$$\mathrm{H}^*(X \times_k Y) \cong \mathrm{H}^*(X) \otimes \mathrm{H}^*(Y),$$

natural in X and Y. Moreover, they are compatible with trace maps, i.e. if X and Y are of dimension d and e respectively, then $X \times_k Y$ is of dimension $d + e$ and its trace map $\mathrm{Tr}_{X \times_k Y}$ is given by the composition

$$\mathrm{H}^{2(d+e)}(X \times_k Y)(d+e) \cong \mathrm{H}^{2d}(X)(d) \otimes_K H^{2e}(Y)(e) \xrightarrow{\mathrm{Tr}_X \otimes \mathrm{Tr}_Y} K \otimes_K K \cong K.$$

It follows that $\mathrm{H}^*(X)$ is an anti-commutative [1] graded K-algebra with multiplication,

[1]Anti-commutative means that, for any $x \in \mathrm{H}^i(X)$ and $x' \in \mathrm{H}^j(X)$, $x \cup x' = (-)^{ij} x' \cup x$. This comes from the Koszul sign rule on the commutativity constraint of tensor product in $grVect_K$.

called *cup product*, given by the composition

$$\cup : \mathrm{H}^*(X) \otimes \mathrm{H}^*(X) \cong \mathrm{H}^*(X \times_k X) \xrightarrow{\Delta^*} \mathrm{H}^*(X),$$

where the latter morphism is the pullback map of the diagonal $\Delta : X \to X \times_k X$.

A4) (*Poincaré duality*) For any $X \in SmProj_k$ irreducible of dimension d, the trace map $\mathrm{Tr}_X : \mathrm{H}^{2d}(X)(d) \to K$ is an isomorphism, and, for any $i \geq 0$, the composition

$$\mathrm{H}^i(X) \otimes_K \mathrm{H}^{2d-i}(X)(d) \xrightarrow{\cup} \mathrm{H}^{2d}(X)(d) \xrightarrow{\mathrm{Tr}_X} K$$

defines a perfect pairing of K-vector spaces. [2]

A5) The cycle class map satisfies the following compatibility conditions:

- (*Naturality*) For any $f : Y \to X$ morphism in $SmProj_k$ and $i \in \mathbb{Z}$, we have the naturality square of the i^{th} cycle class map

$$\begin{tikzcd} \mathrm{CH}^i(X) \arrow[r, "cl^i_X"] \arrow[d, "f^*"] & \mathrm{H}^{2i}(X)(i) \arrow[d, "f^*(i)"] \\ \mathrm{CH}^i(Y) \arrow[r, "cl^i_Y"] & \mathrm{H}^{2i}(Y)(i), \end{tikzcd}$$

  where $f^* : \mathrm{CH}^i(X) \to \mathrm{CH}^i(Y)$ is the pullback map on the Chow groups.
- (*Exterior product*) For any $X, Y \in SmProj_k$, we have the commutative diagram

$$\begin{tikzcd} \mathrm{CH}^i(X) \otimes \mathrm{CH}^j(Y) \arrow[r, "\times"] \arrow[d, "cl^i_X \otimes cl^j_Y"] & \mathrm{CH}^{i+j}(X \times_k Y) \arrow[d, "cl^{i+j}_{X \times_k Y}"] \\ \mathrm{H}^{2i}(X)(i) \otimes_K \mathrm{H}^{2j}(Y)(j) \arrow[r] & \mathrm{H}^{2(i+j)}(X \times_k Y)(i+j), \end{tikzcd}$$

  where the upper arrow is the exterior product on Chow groups (see [Ful98, §1.10]) and the lower arrow is a component of the Künneth isomorphism.
- (*Normalization*) For any $X \in SmProj_k$ irreducible of dimension d, the composition

$$\mathrm{CH}^d(X) \xrightarrow{cl^d_X} \mathrm{H}^{2d}(X)(d) \xrightarrow{\mathrm{Tr}_X} K$$

[2] Notice the following abuse of notation: here $\cup$ denotes the 0-degree component of the d^{th}-twisted cup product.

is the *degree map*, i.e. it is such that $[x] \mapsto [k(x) : k]$, for any rational class of an algebraic cycle of codimension d of X, that is, a closed point $x : Spec(k(x)) \to X$, where $k(x)$ is the residue field at $x \in X$.

We see some direct consequences of the definition of a Weil cohomology theory.

Remark 2.1.2. Notice that

$$\mathrm{H}^0(Spec(k)) \cong \mathrm{H}^0(Spec(k) \times_k Spec(k)) \cong \mathrm{H}^0(Spec(k)) \otimes_K \mathrm{H}^0(Spec(k)).$$

Hence,

$$\mathrm{H}^0(Spec(k)) \cong K.$$

For any $X \in SmProj_k$, let $g : X \to Spec(k)$ be the structural morphism, and $pr_X : X \times Spec(k) \to X$ the canonical projection on X, which is an isomorphism. We have the commutative diagram, for any $i \in \mathbb{Z}$,

$$\begin{tikzcd}
\mathrm{H}^i(X) \otimes_K \mathrm{H}^0(Spec(k)) \arrow[r, "\simeq"] \arrow[d, "id_X^* \otimes g^*"] & \mathrm{H}^i(X \times_k Spec(k)) \arrow[r, "pr_X^{*-1}"] \arrow[d, "(id_X \times g)^*"] & \mathrm{H}^i(X) \arrow[d, equal] \\
\mathrm{H}^i(X) \otimes_K \mathrm{H}^0(X) \arrow[r, "\simeq"] & \mathrm{H}^i(X \times_k X) \arrow[r, "\Delta^*"] & \mathrm{H}^i(X),
\end{tikzcd}$$

where the first is a naturality square of Künneth isomorphism and the second commutes because $pr_X^{-1} = (id_X \times g) \circ \Delta$. It follows that $1 \in K \cong \mathrm{H}^0(Spec(k))$ is such that $1_X \coloneqq g^*(1) \in \mathrm{H}^0(X)$ is the unit of $\mathrm{H}^*(X)$. Moreover, for any $f : X \to Y$ morphism in $SmProj_k$, by naturality of Künneth isomorphism, we have the commutative diagram

$$\begin{tikzcd}
\mathrm{H}^*(X) \otimes \mathrm{H}^*(X) \arrow[r, "\simeq"] \arrow[d, "f^* \otimes f^*"] & \mathrm{H}^*(X \times_k X) \arrow[r, "\Delta_X^*"] \arrow[d, "(f \times f)^*"] & \mathrm{H}^*(X) \arrow[d, "f^*"] \\
\mathrm{H}^*(Y) \otimes \mathrm{H}^*(Y) \arrow[r, "\simeq"] & \mathrm{H}^*(Y \times_k Y) \arrow[r, "\Delta_Y^*"] & \mathrm{H}^*(Y),
\end{tikzcd}$$

which tells that the pullback map f^* is a morphism of graded K-algebras with unit. Notice that, viceversa, we can express the Künneth isomorphism by means of cup product and pullback maps

$$\begin{aligned}
\mathrm{H}^*(X) \otimes \mathrm{H}^*(Y) &\cong \mathrm{H}^*(X \times_k Y) \\
x \otimes y &\mapsto pr_X^*(x) \cup pr_Y^*(y),
\end{aligned}$$

where pr_X and pr_Y are canonical projections of the fiber product $X \times_k Y$.

Remark 2.1.3. Taking $X = Y$ in the exterior product axiom for the cycle class map and composing with the naturality square for the cycle class map for the diagonal

$\Delta : X \to X \times X$, we get that the internal product of the Chow group is compatible with cup product, i.e.

$$cl_X : \mathrm{CH}^*(X)_{\mathbb{Q}} \to \oplus_{i\geq 0}\mathrm{H}^{2i}(X)(i)$$

is a morphism of commutative graded rings.

Remark 2.1.4. Let $X \in SmProj_k$ and let $g : X \to Spec(k)$ be the structural morphism. Consider the rational class $[Spec(k)] \in \mathrm{CH}^0(Spec(k))$. By normalization axiom of the cycle class map,

$$\mathrm{Tr}_{Spec(k)}(cl^0_{Spec(k)}([Spec(k)])) = [k : k] = 1$$

and $\mathrm{Tr}_{Spec(k)}$ is an isomorphism, then

$$cl^0_{Spec(k)}([Spec(k)]) = 1 \in \mathrm{H}^0(Spec(k))$$

is the unit of $\mathrm{H}^*(Spec(k))$. Now, consider the rational class $[X] = g^*[Spec(k)] \in \mathrm{CH}^0(X)$. By naturality axiom of the cycle class map, we have that

$$cl^0_X([X]) = g^*cl^0_{Spec(k)}([Spec(k)]) = g^*(1) \in \mathrm{H}^0(X),$$

which is the unit of $\mathrm{H}^*(X)$, by remark 2.1.2.

Remark 2.1.5. The notion of Tate twist can be thought as a technical device, useful to express appropriately Poincaré duality and duals. Notice that, given $X \in SmProj_k$ irreducible of dimension d, Poincaré duality tells that, more generally, for any $r \in \mathbb{Z}$, we have the perfect pairing of K-vector spaces

$$\mathrm{H}^i(X)(r) \otimes_K \mathrm{H}^{2d-i}(X)(d-r) \xrightarrow{\cup} \mathrm{H}^{2d}(X)(d) \xrightarrow{\mathrm{Tr}_X} K,$$

i.e. we have the isomorphism of K-vector spaces

$$\mathrm{H}^{2d-i}(X)(d-r) \cong \mathrm{Hom}_K(\mathrm{H}^i(X)(r), K).$$

We deduce the isomorphisms in $grVect_K$

$$\begin{aligned}\mathrm{H}^*(X)(r)^\vee &\cong (\oplus_{i-2r}\mathrm{H}^i(X)(r))^\vee \cong \oplus_{2r-i}\,\mathrm{Hom}_K(H^i(X)(r), K) \cong \\ &\cong \oplus_{2r-i}\mathrm{H}^{2d-i}(X)(d-r) = \oplus_{j-2(d-r)}\mathrm{H}^j(X)(d-r) = \mathrm{H}^*(X)(d-r).\end{aligned}$$

In particular, this allows to rewrite the dual object of $\mathrm{H}^*(X)$ as

$$\mathrm{H}^*(X)^\vee \cong \mathrm{H}^*(X)(d).$$

Definition 2.1.6. Given $f : X \to Y$ a morphism in $SmProj_k$, with X and Y of dimension d and e respectively, the morphism in $grVect_K$ given by the composition

$$f_* : \mathrm{H}^*(X)(d) \cong \mathrm{H}^*(X)^\vee \xrightarrow{f^{*\vee}} \mathrm{H}^*(Y)^\vee \cong \mathrm{H}^*(Y)(e),$$

is called the *pushforward map*

Remark 2.1.7. The pushforward map can be thought as the dual of the pullback map via Poincaré duality. Following definitions, we see that, for any $i \in \mathbb{Z}$ and $x \in \mathrm{H}^{2d-i}(X)(d)$, the image $f_*x \in \mathrm{H}^{2e-i}(Y)(e)$ is characterized by the formula, for any $y \in \mathrm{H}^i(Y)$,

$$\mathrm{Tr}_X(f^*y \cup x) = \mathrm{Tr}_Y(y \cup f_*x). \tag{2.3}$$

Let $g_X : X \to Spec(k)$ and $g_Y : Y \to Spec(k)$ be the structural morphisms. Taking $i = 0$ and $y = 1 = g_Y^*(1) \in \mathrm{H}^0(Y)$ the unit, since $f^*(1) = g_X^*(1) \in \mathrm{H}^0(X)$ is the unit, we obtain that trace maps are compatible with the pushforward map, i.e. we have the commutative diagram

$$\begin{tikzcd} \mathrm{H}^{2d}(X)(d) \arrow[rr, "f_*"] \arrow[dr, "\mathrm{Tr}_X"'] & & \mathrm{H}^{2e}(Y)(e). \arrow[dl, "\mathrm{Tr}_Y"] \\ & K & \end{tikzcd}$$

In particular, taking $f = g_X$ the structural morphism of X, we get that the trace map Tr_X is given by the composition

$$\mathrm{H}^{2d}(X)(d) \xrightarrow{g_{X*}} \mathrm{H}^0(Spec(k)) \xrightarrow{\mathrm{Tr}_{Spec(k)}} K.$$

By the formula 2.3, since the pullback map defines a morphisms of graded K-algebras with respect to cup product, it follows that, for any $i, j \in \mathbb{Z}$, $x \in H^{2d-i}(X)(d)$, $y \in H^j(Y)$ and $z \in H^{i-j}(Y)$

$$\mathrm{Tr}_Y(z \cup y \cup f_*x) = \mathrm{Tr}_X(f^*(z \cup y) \cup x) = \mathrm{Tr}_X(f^*z \cup f^*y \cup x) = \mathrm{Tr}_Y(z \cup f_*(f^*y \cup x)).$$

Since, by the perfect pairing of Poincaré duality,

$$\mathrm{Tr}_Y(z \cup \_\,) : \mathrm{H}^{2e-i+j}(X)(e) \to K$$

is an isomorphism, it follows that

$$y \cup f_*x = f_*(f^*y \cup x),$$

called the *projection formula*. Since pullback maps and duals are functorial, then also the pushforward map is functorial, i.e. for any $f : X \to Y$ and $g : Y \to Z$ morphisms in $SmProj_k$,

$$(gf)_* = g_*f_*.$$

It can be proved that the cycle class map is compatible with pushforward, i.e. we have commutative diagrams, for any $i \in \mathbb{Z}$,

$$\begin{tikzcd} \mathrm{CH}^{d-i}(X) \arrow[r, "cl_X^{d-i}"] \arrow[d, "f_*"] & \mathrm{H}^{2(d-i)}(X)(d-i) \arrow[d, "f_*(-i)"] \\ \mathrm{CH}^{e-i}(Y) \arrow[r, "cl_Y^{e-i}"] & \mathrm{H}^{2(e-i)}(Y)(e-i), \end{tikzcd}$$

where $f_* : \mathrm{CH}^{d-i}(X) \to \mathrm{CH}^{e-i}(Y)$ is the pushforward map on the Chow groups. Moreover, if $i : Z \hookrightarrow X$ is a closed immersion in $SmProj_k$, where Z has codimension c in X, then the associated rational class $[i(Z)] \in \mathrm{CH}^c(X)$ is such that

$$[i(Z)] = i_*[Z],$$

where $[Z] \in \mathrm{CH}^0(Z)$. By compatibility of the cycle class map with the pushforward map, we get

$$cl_X^c([i(Z)]) = cl_X^c(i_*[Z]) = i_* cl_Z^0([Z]) = i_*(1_Z) \in \mathrm{H}^{2c}(X)(c),$$

where the las equality holds because, by remark 2.1.4, $cl_Z^0([Z]) = 1_Z \in \mathrm{H}^0(Z)$ is the unit of $\mathrm{H}^*(Z)$.

Remark 2.1.8. Given $X, Y \in SmProj_k$, with X irreducible of dimension d, using Künneth formula and Poincaré duality axioms, we have the following canonical isomorphisms of K-vector spaces

$$\begin{aligned} \mathrm{H}^{2d}(X \times_k Y)(d) &\cong \oplus_i \mathrm{H}^{2d-i}(X)(d) \otimes_K \mathrm{H}^i(Y) \cong \\ &\cong \oplus_i \mathrm{Hom}_K(\mathrm{H}^i(X), K) \otimes_K \mathrm{H}^i(Y) \cong \\ &\cong \oplus_i \mathrm{Hom}_K(\mathrm{H}^i(X), \mathrm{H}^i(Y)) \cong \\ &\cong \mathrm{Hom}_{grVect_K}(\mathrm{H}^*(X), \mathrm{H}^*(Y)). \end{aligned}$$

If we denote by $pr_X : X \times_k Y \to X$ and $pr_Y : X \times_k Y \to Y$ the canonical projections of the fiber product, following the isomorphisms, we see that they are such that

$$\begin{aligned} \mathrm{H}^{2d}(X \times_k Y)(d) &\cong \mathrm{Hom}_{grVect_K}(\mathrm{H}^*(X), \mathrm{H}^*(Y)) \\ u &\mapsto \underline{u} := (x \mapsto pr_{Y*}(pr_X^*(x) \cup u)). \end{aligned}$$

By composition with the d^{th}-component of the cycle class map of $X \times_k Y$, we obtain the morphism of $\mathbb{Q}$-vector spaces

$$r_{\mathrm{H}} : \mathrm{CH}^d(X \times_k Y)_{\mathbb{Q}} \xrightarrow{cl^d_{X \times_k Y}} \mathrm{H}^{2d}(X \times_k Y)(d) \cong \mathrm{Hom}_{grVect_K}(\mathrm{H}^*(X), \mathrm{H}^*(Y)),$$

For any morphism $f : Y \to X$ in $SmProj_k$, we can consider the closed subscheme of the fiber product $\Gamma_f^t \subset X \times_k Y$ given by the transpose of the graph of f. Since Γ_f is isomorphic to Y via the canonical projection pr_Y, then it is an algebraic cycle of codimension d of $X \times_k Y$. Its rational class determines an element $[\Gamma_f^t] \in \mathrm{CH}^d(X \times_k Y)$. Following the definitions, we see that its image along the morphism r_H is exactly the pullback map of the Weil cohomology

$$r_{\mathrm{H}}([\Gamma_f^t]) = \underline{cl^d_{X \times_k Y}([\Gamma_f^t])} = f^* : \mathrm{H}^*(X) \to \mathrm{H}^*(Y).$$

Proposition 2.1.9. *For any Weil cohomology theory H^* over k with coefficients in K, there exists a canonical isomorphism of graded K-vector spaces*

$$H^*(\mathbb{P}^1_k) \cong K \oplus 0 \oplus K(-1).$$

That is, cohomology of $\mathbb{P}^1_k$ is completely determined by the axioms.

Proof. Since $\mathbb{P}^1_k$ is irreducible of dimension 1, its trace map defines a canonical isomorphism $\mathrm{H}^2(\mathbb{P}^1_k)(1) \cong K$, hence

$$\mathrm{H}^2(\mathbb{P}^1_k) \cong K(-1).$$

By Poincaré duality, we get the canonical isomorphism

$$\mathrm{H}^0(\mathbb{P}^1_k) \cong \mathrm{Hom}_K(\mathrm{H}^2(\mathbb{P}^1_k)(1), K) \cong K.$$

It remains to prove that $\mathrm{H}^1(\mathbb{P}^1_k) = 0$. Consider $[\Delta] \in \mathrm{CH}^1(\mathbb{P}^1_k \times_k \mathbb{P}^1_k)$ the rational class associated to the closed immersion $\Delta : \mathbb{P}^1_k \hookrightarrow \mathbb{P}^1_k \times_k \mathbb{P}^1_k$, the diagonal. Since the diagonal is the transpose of the graph of the identity map $id : \mathbb{P}^1_k \to \mathbb{P}^1_k$

$$\Delta = \Gamma^t_{id},$$

then, by remark 2.1.8, we have that

$$r_{\mathrm{H}}([\Delta]) = id^* : \mathrm{H}^*(\mathbb{P}^1_k) \to \mathrm{H}^*(\mathbb{P}^1_k)$$

is the identity of $\mathrm{H}^*(\mathbb{P}^1_k)$. Let $x : Spec(k) \to \mathbb{P}^1_k$ be a k-rational point, hence a closed point. It is an algebraic cycle of codimension 1 of $\mathbb{P}^1_k$. Consider its rational class $[x] \in \mathrm{CH}^1(\mathbb{P}^1_k)$. Moreover consider the class $[\mathbb{P}^1_k] \in \mathrm{CH}^0(\mathbb{P}^1_k)$. Recall that in $\mathrm{CH}^1(\mathbb{P}^1_k \times_k \mathbb{P}^1_k)$ the diagonal decomposes as (see [And04, Ex. 3.2.2.2 (1)])

$$[\Delta] = [x] \times [\mathbb{P}^1_k] + [\mathbb{P}^1_k] \times [x].$$

Following the definitions and by the compatibility axiom of the cycle class maps with the exterior product of Chow groups, we see that this decomposition induces, via r_{H}, the decomposition of the identity on $\mathrm{H}^*(\mathbb{P}^1_k)$

$$id = p_0 + p_2,$$

where p_i is the identity on $\mathrm{H}^i(X)$, for $i = 0, 2$. It follows that p_1, the identity on $\mathrm{H}^1(\mathbb{P}^1_k)$ is zero. Hence, $\mathrm{H}^1(\mathbb{P}^1_k) = 0$.

□

2.1.2 The algebraic de Rham and Betti cycle class maps

Recall that in the previous chapter we defined two cohomology theories for smooth algebraic varieties: the algebraic de Rham cohomology and the Betti cohomology. Restricting to smooth projective algebraic varieties, it turns out that these cohomology theories are examples of Weil cohomologies.

Proposition 2.1.10. *Given k a field of characteristic 0, the algebraic de Rham cohomology*

$$\begin{aligned} H^*_{AdR} : SmProj_k^{op} &\to grVect_k \\ X &\mapsto H^*_{AdR}(X/k) := \oplus_{i\geq 0} H^i_{AdR}(X/k) \end{aligned}$$

defines a Weil cohomology theory over k with coefficients in k.

The Betti cohomology

$$\begin{aligned} H^*_{Bet} : SmProj_{\mathbb{C}}^{op} &\to grVect_{\mathbb{Q}} \\ X &\mapsto H^*_{Bet}(X) := \oplus_{i\geq 0} H^i_{Bet}(X) \end{aligned}$$

defines a Weil cohomology theory over $\mathbb{C}$ with coefficients in $\mathbb{Q}$.

It's not our interest to check that all the axioms are satisfied. For Betti cohomology, most of the axioms are well-known from classical results in Algebraic Topology. For algebraic de Rham cohomology, some axioms (for example functoriality and Künneth formula) have already been proved in the previous chapter, while others (for example Poincaré duality) are more laborious. The only axiom we are interested is the one of cycle class map. We want to define cycle class maps for these cohomology theories and prove that they are compatible under a twisted version of the algebraic de Rham isomorphism. Following [Delb, §1], we define cycle class maps using Grothendieck's Theory of Chern classes. See Appendix B for the main constructions and results.

The idea is to define, for $\mathrm{H}^* = \mathrm{H}^*_{\mathrm{Bet}}, \mathrm{H}^*_{\mathrm{AdR}}$, a suitable Tate twist and a natural transformation of contravariant functors $SmProj_k^{op} \to \mathrm{Ab}$

$$p^1 : \mathrm{Pic} \to \mathrm{H}^2(\_)(1),$$

such that the contravariant functor

$$\begin{aligned} \mathrm{H}^{2*}(\_)(*) : SmProj_k^{op} &\to grRing \\ X &\mapsto \oplus_{i\geq 0}\mathrm{H}^{2i}(X)(i), \end{aligned}$$

together with the natural transformation, satisfies the axioms of Grothendieck's Theory of Chern classes. Then, as explained in Appendix B, we obtain a morphisms of graded rings

$$cl_X : \mathrm{CH}^*(X)_{\mathbb{Q}} \to \oplus_{i\geq 0}\mathrm{H}^{2i}(X)(i).$$

This will be the definition of the cycle class map for H^*.

We start with the algebraic de Rham cohomology over k, with $char(k) = 0$. Recall that, given a ringed site $(\mathcal{C}, \mathcal{O})$, we have a canonical isomorphism of abelian groups

$$\mathbb{H}^1(\mathcal{C}, \mathcal{O}^\times) \cong \mathrm{Pic}(\mathcal{O}), \tag{2.4}$$

where $\mathcal{O}^\times$ is the abelian sheaf over $\mathcal{C}$ of invertible sections of $\mathcal{O}$, and $\mathrm{Pic}(\mathcal{O})$ is the group of isomorphism classes of invertible $\mathcal{O}$-modules. Moreover, it is natural in $\mathcal{C}$. The isomorphism 2.4, applied to the ringed site X_{Zar}, for any $X \in SmProj_k$, gives the canonical isomorphism

$$\mathbb{H}^1(X_{Zar}, \mathcal{O}_X^\times) \cong \mathrm{Pic}(X),$$

natural in X. Consider the morphism of complexes of abelian sheaves over X_{Zar}

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{O}_X^\times & \to & 0 & \to & 0 & \to \cdots \\ \downarrow & & \downarrow d\log & & \downarrow & & \downarrow & \\ \mathcal{O}_X & \to & \Omega^1_{X/k} & \to & \Omega^2_{X/k} & \to & \Omega^3_{X/k} & \to \cdots, \end{array}$$

where the morphism $d\log$ assigns to a section f of $\mathcal{O}_X^\times$ the section $\frac{df}{f}$ of $\Omega^1_{X/k}$. It induces the morphism on sheaf cohomology

$$\mathbb{H}^2(X_{Zar}, \mathcal{O}_X^\times[-1]) \to \mathbb{H}^2(X_{Zar}, \Omega^\bullet_{X/k}) = \mathrm{H}^2_{\mathrm{AdR}}(X/k).$$

Since

$$\mathbb{H}^2(X_{Zar}, \mathcal{O}_X^\times[-1]) \cong \mathbb{H}^1(X_{Zar}, \mathcal{O}_X^\times) \cong \mathrm{Pic}(X),$$

then, we obtained the morphism of abelian groups

$$p^1_{\mathrm{AdR},X} : \mathrm{Pic}(X) \to \mathrm{H}^2_{\mathrm{AdR}}(X/k).$$

Given a morphism $f : X \to Y$ in $SmProj_k$, we have the commutative diagram of abelian sheaves over X_{Zar}

$$\begin{array}{ccc} f^{-1}\mathcal{O}_Y^\times[-1] & \xrightarrow{f^{-1}d\log} & f^{-1}\Omega^\bullet_{Y/k} \\ \downarrow & & \downarrow \\ \mathcal{O}_X^\times[-1] & \xrightarrow{d\log} & \Omega^\bullet_{X/k}. \end{array}$$

It induces a commutative square on sheaf cohomology, which, composed with the natural square of functoriality morphism of sheaf cohomology for the morphism $d\log : \mathcal{O}_Y^\times[-1] \to \Omega^\bullet_{Y/k}$, gives the commutative diagram of abelian groups

$$\begin{array}{ccc} \mathrm{Pic}(Y) & \xrightarrow{p^1_{\mathrm{AdR},Y}} & \mathrm{H}^2_{\mathrm{AdR}}(Y/k) \\ \downarrow & & \downarrow \\ \mathbb{H}^2(X_{Zar}, f^{-1}\mathcal{O}_Y^\times[-1]) & \longrightarrow & \mathbb{H}^2(X_{Zar}, f^{-1}\Omega^\bullet_{Y/k}) \\ \downarrow & & \downarrow \\ \mathrm{Pic}(X) & \xrightarrow{p^1_{\mathrm{AdR},X}} & \mathrm{H}^2_{\mathrm{AdR}}(X/k), \end{array}$$

where vertical morphisms are pullback maps. This suggests to define the Tate module for algebraic de Rham cohomology

$$k(1) := k.$$

So, in this case, the Tate module $k(1)$ is canonically isomorphic to k and we have a canonical isomorphism

$$\mathrm{H}^2_{\mathrm{AdR}}(\_/k)(1) \cong \mathrm{H}^2_{\mathrm{AdR}}(\_/k)$$

Then, the above construction defines a natural transformation of contravarinat functors $SmProj_k^{op} \to \mathrm{Ab}$

$$p^1_{\mathrm{AdR}} : \mathrm{Pic} \to \mathrm{H}^2_{\mathrm{AdR}}(\_/k) \cong \mathrm{H}^2_{\mathrm{AdR}}(\_/k)(1).$$

Now, we pass to Betti cohomology. For any $X \in SmProj_{\mathbb{C}}$, consider the morphism of abelian sheaves over $X(\mathbb{C})_{an}$ given by the exponential map

$$\exp : \mathcal{O}_{X(\mathbb{C})} \to \mathcal{O}^\times_{X(\mathbb{C})}.$$

We denote by

$$\mathbb{Z}(1)_{X(\mathbb{C})} := \ker(\mathcal{O}_{X(\mathbb{C})} \xrightarrow{\exp} \mathcal{O}^\times_{X(\mathbb{C})}).$$

It is isomorphic to the constant abelian sheaf over $X(\mathbb{C})_{an}$ associated to the abelian group

$$\mathbb{Z}(1) \coloneqq 2\pi i\mathbb{Z} = \{2\pi i z \mid z \in \mathbb{Z}\},$$

and the canonical morphism of the kernel

$$c : \mathbb{Z}(1)_{X(\mathbb{C})} \to \mathcal{O}_{X(\mathbb{C})}$$

associates to any section, i.e. to any element of $\mathbb{Z}(1)$, the constant function of the corresponding value. Notice that $\mathbb{Z}(1)_{X(\mathbb{C})}$ is an abelian sheaf isomorphic to $\mathbb{Z}_{X(\mathbb{C})}$, the constant abelian sheaf of value $\mathbb{Z}$ over $X(\mathbb{C})_{an}$, but the isomorphism is not canonical, since it depends on the choice of a square root if -1 in $\mathbb{C}$. We have the exact sequence of abelian sheaves over $X(\mathbb{C})_{an}$

$$0 \to \mathbb{Z}(1)_{X(\mathbb{C})} \xrightarrow{c} \mathcal{O}_{X(\mathbb{C})} \xrightarrow{\exp} \mathcal{O}^{\times}_{X(\mathbb{C})} \to 0,$$

called the *exponential sequence*. Consider the connecting homomorphism of the long exact sequence induced on sheaf cohomology

$$\mathbb{H}^1(X(\mathbb{C})_{an}, \mathcal{O}^{\times}_{X(\mathbb{C})}) \to \mathbb{H}^2(X(\mathbb{C})_{an}, \mathbb{Z}(1)_{X(\mathbb{C})}).$$

Consider the canonical inclusion of abelian sheaves over $X(\mathbb{C})_{an}$ given by the extension to rational scalars

$$\mathbb{Z}(1)_{X(\mathbb{C})} \hookrightarrow \mathbb{Z}(1)_{X(\mathbb{C})} \otimes \mathbb{Q}_{X(\mathbb{C})} \eqqcolon \mathbb{Q}(1)_{X(\mathbb{C})},$$

where $\mathbb{Q}_{X(\mathbb{C})}$ denotes the constant abelian sheaf of value $\mathbb{Q}$ over $X(\mathbb{C})$, which is also a sheaf of rings, and $\otimes$ denotes the tensor product of abelian sheaves. Then, $\mathbb{Q}(1)_{X(\mathbb{C})}$ is the constant abelian sheaf over $X(\mathbb{C})_{an}$ associated to the abelian group $\mathbb{Q}(1) \coloneqq 2\pi i\mathbb{Q}$ It induces the morphism on sheaf cohomology

$$\mathbb{H}^2(X(\mathbb{C})_{an}, \mathbb{Z}(1)_{X(\mathbb{C})}) \to \mathbb{H}^2(X(\mathbb{C})_{an}, \mathbb{Q}(1)_{X(\mathbb{C})}).$$

The isomorphism 2.4, applied to the ringed site $X(\mathbb{C})_{an}$, gives the canonical isomorphism

$$\mathbb{H}^1(X(\mathbb{C})_{an}, \mathcal{O}^{\times}_{X(\mathbb{C})}) \cong \mathrm{Pic}(X(\mathbb{C})),$$

natural in X. Since the pullback of sheaves of modules commutes with tensor product, then the analytification functor of sheaves of $\mathcal{O}_X$-modules induces a morphism of abelian groups (which is, in fact, an isomorphism by GAGA Theorem II 1.2.23, since pullback of sheaves of modules preserve the property of being finite locally free and also the rank)

$$\begin{aligned} \mathrm{Pic}(X) &\to \mathrm{Pic}(X(\mathbb{C})) \\ \mathcal{L} &\mapsto \mathcal{L}^{an}. \end{aligned}$$

Following definitions, we see that we have the commutative diagram

$$\begin{array}{ccccc}
\mathrm{Pic}(X) & & \longrightarrow & & \mathrm{Pic}(X(\mathbb{C})) \\
\wr\| & & & & \wr\| \\
\mathbb{H}^1(X_{Zar}, \mathcal{O}_X^\times) & \longrightarrow & \mathbb{H}^1(X(\mathbb{C})_{an}, \alpha_X^{-1}\mathcal{O}_X^\times) & \longrightarrow & \mathbb{H}^1(X(\mathbb{C})_{an}, \mathcal{O}_{X(\mathbb{C})}^\times).
\end{array}$$

Composing the morphisms constructed above, we obtain the morphism of abelian groups

$$p^1_{\mathrm{Bet},X} : \mathrm{Pic}(X) \cong \mathrm{Pic}(X(\mathbb{C})) \to \mathbb{H}^2(X(\mathbb{C})_{an}, \mathbb{Z}(1)_{X(\mathbb{C})}) \to \mathbb{H}^2(X(\mathbb{C})_{an}, \mathbb{Q}(1)_{X(\mathbb{C})}).$$

Given a morphism $f : X \to Y$ in $SmProj_{\mathbb{C}}$, let $f^{an} : X(\mathbb{C}) \to Y(\mathbb{C})$ be the analytification. We have the commutative diagram of abelian groups

$$\begin{array}{ccccc}
\mathbb{H}^1(Y_{Zar}, \mathcal{O}_Y^\times) & \to & \mathbb{H}^1(Y(\mathbb{C})_{an}, \alpha_Y^{-1}\mathcal{O}_Y^\times) & \to & \mathbb{H}^1(Y(\mathbb{C})_{an}, \mathcal{O}_{Y(\mathbb{C})}^\times) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{H}^1(X_{Zar}, f^{-1}\mathcal{O}_Y^\times) & \to & \mathbb{H}^1(X(\mathbb{C})_{an}, (f\alpha_X)^{-1}\mathcal{O}_Y^\times) & \to & \mathbb{H}^1(X(\mathbb{C})_{an}, (f^{an})^{-1}\mathcal{O}_{Y(\mathbb{C})}^\times) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{H}^1(X_{Zar}, \mathcal{O}_X^\times) & \to & \mathbb{H}^1(X(\mathbb{C})_{an}, \alpha_X^{-1}\mathcal{O}_{X(\mathbb{C})}^\times) & \to & \mathbb{H}^1(X(\mathbb{C})_{an}, \mathcal{O}_{X(\mathbb{C})}^\times)),
\end{array}$$

where the first is a square of functoriality morphisms on sheaf cohomology for $\mathcal{O}_Y$, the second and the third are naturality squares of functoriality morphisms on sheaf cohomology for the morphisms $\alpha_Y^{-1}\mathcal{O}_Y^\times \to \mathcal{O}_{Y(\mathbb{C})}^\times$ and $f^{-1}\mathcal{O}_Y^\times \to \mathcal{O}_X^\times$ respectively and the fourth is induced by the commuative diagram of abelian sheaves over $X(\mathbb{C})_{an}$

$$\begin{array}{ccc}
(f\alpha_X)^{-1}\mathcal{O}_Y^\times & \longrightarrow & (f^{an})^{-1}\mathcal{O}_{Y(\mathbb{C})}^\times \\
\Big\downarrow & & \Big\downarrow \\
\alpha_X^{-1}\mathcal{O}_{X(\mathbb{C})}^\times & \longrightarrow & \mathcal{O}_{X(\mathbb{C})}^\times,
\end{array}$$

which follows from a naturality square of analytification. As noticed above, the horizontal morphisms are the isomorphisms $\mathrm{Pic}(Y) \cong \mathrm{Pic}(Y(\mathbb{C}))$ and $\mathrm{Pic}(X) \cong \mathrm{Pic}(X(\mathbb{C}))$. Moreover, consider the morphism of short exact sequences of abelian sheaves over $X(\mathbb{C})_{an}$

$$\begin{array}{ccccccccc}
0 \to & (f^{an})^{-1}\mathbb{Z}(1)_{Y(\mathbb{C})} & \to & (f^{an})^{-1}\mathcal{O}_{Y(\mathbb{C})} & \xrightarrow{(f^{an})^{-1}\exp} & (f^{an})^{-1}\mathcal{O}_{Y(\mathbb{C})}^\times & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \to & \mathbb{Z}(1)_{X(\mathbb{C})} & \to & \mathcal{O}_{X(\mathbb{C})} & \xrightarrow{\exp} & \mathcal{O}_{X(\mathbb{C})}^\times & \to 0\ .
\end{array}$$

Then, we have the commutative diagram of abelian groups

$$\begin{array}{ccccc}
\mathbb{H}^1(Y(\mathbb{C})_{an}, \mathcal{O}^{\times}_{Y(\mathbb{C})}) & \to & \mathbb{H}^2(Y(\mathbb{C})_{an}, \mathbb{Z}(1)_{Y(\mathbb{C})}) & \to & \mathbb{H}^2(Y(\mathbb{C})_{an}, \mathbb{Q}(1)_{Y(\mathbb{C})}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{H}^1(X(\mathbb{C})_{an}, (f^{an})^{-1}\mathcal{O}^{\times}_{Y(\mathbb{C})}) & \to & \mathbb{H}^2(X(\mathbb{C})_{an}, (f^{an})^{-1}\mathbb{Z}(1)_{Y(\mathbb{C})}) & \to & \mathbb{H}^2(X(\mathbb{C})_{an}, (f^{an})^{-1}\mathbb{Q}(1)_{Y(\mathbb{C})}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{H}^1(X(\mathbb{C})_{an}, \mathcal{O}^{\times}_{X(\mathbb{C})}) & \to & \mathbb{H}^2(X(\mathbb{C})_{an}, \mathbb{Z}(1)_{X(\mathbb{C})}) & \to & \mathbb{H}^2(X(\mathbb{C})_{an}, \mathbb{Q}(1)_{X(\mathbb{C})}),
\end{array}$$

where the first and the third are squares of morphisms of long exact sequences induced on sheaf cohomology, the second is a naturality square of functoriality morphism on sheaf cohomology for the inclusion $\mathbb{Z}(1)_{Y(\mathbb{C})} \hookrightarrow \mathbb{Q}(1)_{Y(\mathbb{C})}$ and the fourth is induced by the commutative diagram of abelian sheaves over $X(\mathbb{C})_{an}$

$$\begin{tikzcd}
(f^{an})^{-1}\mathbb{Z}(1)_{Y(\mathbb{C})} \arrow[r] \arrow[d] & (f^{an})^{-1}\mathbb{Q}(1)_{Y(\mathbb{C})} \arrow[d] \\
\mathbb{Z}(1)_{X(\mathbb{C})} \arrow[r] & \mathbb{Q}(1)_{X(\mathbb{C})}.
\end{tikzcd}$$

So, we obtained the commutative square

$$\begin{tikzcd}
\mathrm{Pic}(Y) \arrow[r, "p^1_{\mathrm{Bet},Y}"] \arrow[d, "f^*"] & \mathbb{H}^2(Y(\mathbb{C})_{an}, \mathbb{Q}(1)_{Y(\mathbb{C})}) \arrow[d] \\
\mathrm{Pic}(X) \arrow[r, "p^1_{\mathrm{Bet},X}"] & \mathbb{H}^2(X(\mathbb{C})_{an}, \mathbb{Q}(1)_{X(\mathbb{C})}).
\end{tikzcd}$$

This suggests to define the Tate module for the Betti cohomology

$$\mathbb{Q}(1) \coloneqq 2\pi i\mathbb{Q} = \{2\pi i r \mid r \in \mathbb{Q}\}.$$

Notice that, as for the associated constant abelian sheaf over $X(\mathbb{C})_{an}$, $\mathbb{Q}(1)$ is a $\mathbb{Q}$-vector space isomorphic to $\mathbb{Q}$, but the isomorphism is not canonical, since it depends on the choice of a square root of -1 in $\mathbb{C}$. Since, given Λ a group, the cohomology of $\Lambda_{X(\mathbb{C})}$, the associated constant abelian sheaf over $X(\mathbb{C})$, computes singular cohomology with coefficients in Λ (proposition 1.1.4), then we have the canonical isomorphisms

$$\mathbb{H}^2(X(\mathbb{C})_{an}, \mathbb{Q}(1)_{X(\mathbb{C})}) \cong \mathrm{H}^2_{\mathrm{Sing}}(X(\mathbb{C}); \mathbb{Q}(1)) \cong \mathrm{H}^2_{\mathrm{Sing}}(X(\mathbb{C}); \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}(1) = \mathrm{H}^2_{\mathrm{Bet}}(X)(1).$$

So, we obtained a natural transformation of contravariant functors $SmProj^{op}_{\mathbb{C}} \to \mathrm{Ab}$

$$p^1_{\mathrm{Bet}} : \mathrm{Pic} \to \mathrm{H}^2_{\mathrm{Bet}}(\_)(1).$$

Definition 2.1.11. The natural transformations

$$p^1_{\mathrm{AdR}} : \mathrm{Pic} \to \mathrm{H}^2_{adr}(\_/k)(1) \qquad \& \qquad p^1_{\mathrm{Bet}} : \mathrm{Pic} \to \mathrm{H}^2_{\mathrm{Bet}}(\_)(1)$$

are called the *algebraic de Rham* and *Betti first Chern class*, respectively.

The contravariant functor

$$\begin{aligned} \mathrm{H}^{2*}_{\mathrm{AdR}}(\_/k)(*) : SmProj_k^{op} &\to grRing \\ X &\mapsto \oplus_{i\geq 0}\mathrm{H}^{2i}_{\mathrm{AdR}}(X/k)(i), \end{aligned}$$

with the natural transformation c^1_{AdR}, satisfies the axioms of Grothendieck's Theory of Chern classes (see [Gro58, §2, ex. 2]). Moreover, since $char(k) = 0$, then, for any $X \in SmProj_k$, $\oplus_{i\geq 0}\mathrm{H}^{2i}_{\mathrm{AdR}}(X/k)(i)$ is a graded $\mathbb{Q}$-algebra. Hence, as explained in Appendix B, we can construct a morphism of commutative graded rings

$$cl_{\mathrm{AdR},X} : \mathrm{CH}^*(X)_{\mathbb{Q}} \to \oplus_{i\geq 0}\mathrm{H}^{2i}_{\mathrm{AdR}}(X/k)(i).$$

Analogously for Betti cohomology, the contravariant functor

$$\begin{aligned} \mathrm{H}^{2*}_{\mathrm{Bet}}(\_)(*) : SmProj_{\mathbb{C}}^{op} &\to grRing \\ X &\mapsto \oplus_{i\geq 0}\mathrm{H}^{2i}_{\mathrm{Bet}}(X)(i), \end{aligned}$$

with the natural transformation p^1_{Bet}, satisfies the axioms of Grothendieck's Theory of Chern classes (see [Gro58, §2, ex. 3]). Moreover, for any $X \in SmProj_{\mathbb{C}}$, $\oplus_{i\geq 0}\mathrm{H}^{2i}_{\mathrm{Bet}}(X)(i)$ is a graded $\mathbb{Q}$-algebra. Hence, as explained in Appendix B, we can construct a morphism of commutative graded $\mathbb{Q}$-algebras

$$cl_{\mathrm{Bet},X} : \mathrm{CH}^*(X)_{\mathbb{Q}} \to \oplus_{i\geq 0}\mathrm{H}^{2i}_{\mathrm{Bet}}(X)(i).$$

The maps $cl_{\mathrm{AdR},X}$ and $cl_{\mathrm{Bet},X}$ satisfy the axioms of the cycle class maps of a Weil cohomology theory. The naturality and the exterior product axioms of the cycle class map immediately follow from naturality of the Chern character and the fact that it is a ring homomorphism.

Definition 2.1.12. Given $X \in SmProj_k$, with $char(k) = 0$, the morphism of commutative graded $\mathbb{Q}$-algebras

$$cl_{\mathrm{AdR},X} : \mathrm{CH}^*(X)_{\mathbb{Q}} \to \oplus_{i\geq 0}\mathrm{H}^{2i}_{\mathrm{AdR}}(X/k)(i)$$

is called the *algebraic de Rham cycle class map* of X.

Given $X \in SmProj_{\mathbb{C}}$, the morphism of commutative graded $\mathbb{Q}$-algebras

$$cl_{\mathrm{Bet},X} : \mathrm{CH}^*(X)_{\mathbb{Q}} \to \oplus_{i\geq 0}\mathrm{H}^{2i}_{\mathrm{Bet}}(X)(i)$$

is called the *Betti cycle class map* of X.

Now, recall that, fixed a field extension $\sigma : k \hookrightarrow \mathbb{C}$, we have the algebraic de Rham isomorphism (theorem 1.6.2), for any $X \in Sm_k$ and each $i \geq 0$,

$$\varpi_X^i : \mathrm{H}^i_{\mathrm{AdR}}(X/k) \otimes_k \mathbb{C} \cong \mathrm{H}^i_{\mathrm{Bet}}(X_\sigma) \otimes_{\mathbb{Q}} \mathbb{C}.$$

We want to consider a twisted version of this canonical isomorphism. Since, by defintion, $k(1) = k$ as a k-vector space, then, for any $q \in \mathbb{Z}$, we have canonical isomorphisms of k-vector spaces

$$k(q) = k(1)^{\otimes q} \cong k^{\otimes q} \cong k.$$

Then, for any $p \geq 0$, we have canonical isomorphisms of k-vector spaces

$$\mathrm{H}^p_{\mathrm{AdR}}(X/k)(q) = \mathrm{H}^p_{\mathrm{AdR}}(X/k) \otimes_k k(q) \cong \mathrm{H}^p_{\mathrm{AdR}}(X/k) \otimes_k k \cong \mathrm{H}^p_{\mathrm{AdR}}(X/k).$$

Hence, we have canonical isomorphisms of $\mathbb{C}$-vector spaces

$$\mathrm{H}^p_{\mathrm{AdR}}(X/k)(q) \otimes_k \mathbb{C} \cong \mathrm{H}^p_{\mathrm{AdR}}(X/k) \otimes_k \mathbb{C}.$$

Notice that, for any $q \in \mathbb{Z}$, we have canonical isomorphisms of $\mathbb{Q}$-vector spaces

$$\mathbb{Q}(q) = \mathbb{Q}(1)^{\otimes q} \cong (2\pi i)^q \mathbb{Q} \cong \{(2\pi i)^q r \mid r \in \mathbb{Q}\}.$$

Then, we have canonical isomorphisms of $\mathbb{Q}$-vector spaces given by multiplication

$$\mathbb{Q}(q) \otimes_{\mathbb{Q}} \mathbb{C} \cong \mathbb{C}.$$

Hence, we have the canonical isomorphisms of $\mathbb{C}$-vector spaces

$$\mathrm{H}^p_{\mathrm{Bet}}(X_\sigma)(q) \otimes_{\mathbb{Q}} \mathbb{C} = \mathrm{H}^p_{\mathrm{Bet}}(X_\sigma) \otimes_{\mathbb{Q}} \mathbb{Q}(q) \otimes_{\mathbb{Q}} \mathbb{C} \cong \mathrm{H}^p_{\mathrm{Bet}}(X_\sigma) \otimes_{\mathbb{Q}} \mathbb{C}.$$

Notice that we are not saying that there's a canonical isomorphism of $\mathbb{Q}$-vector spaces between $\mathrm{H}^p_{\mathrm{Bet}}(X_\sigma)(q)$ and $\mathrm{H}^p_{\mathrm{Bet}}(X_\sigma)$, which indeed doesn't exists (an isomorphism exists, but it's not canonical!). However, it exists taking coefficients in $\mathbb{C}$. Composing with the canonical algebraic de Rham isomorphism ϖ_X^p, we get the canonical isomorphisms of $\mathbb{C}$-vector spaces

$$\mathrm{H}^p_{\mathrm{AdR}}(X/k)(q) \otimes_k \mathbb{C} \cong \mathrm{H}^p_{\mathrm{AdR}}(X/k) \otimes_k \mathbb{C} \cong \mathrm{H}^p_{\mathrm{Bet}}(X_\sigma) \otimes_{\mathbb{Q}} \mathbb{C} \cong \mathrm{H}^p_{\mathrm{Bet}}(X_\sigma)(q) \otimes_{\mathbb{Q}} \mathbb{C}.$$

Definition 2.1.13. Given X a smooth algebraic variety over a field k with a field extension $\sigma : k \hookrightarrow \mathbb{C}$, the canonical morphism of $\mathbb{C}$-vector spaces, for each $p \geq 0$ and $q \in \mathbb{Z}$,

$$\varpi_X^{p,q} : \mathrm{H}^p_{\mathrm{AdR}}(X/k)(q) \otimes_k \mathbb{C} \cong \mathrm{H}^p_{\mathrm{Bet}}(X_\sigma)(q) \otimes_{\mathbb{Q}} \mathbb{C}$$

is called the q^{th}*-twisted algebraic de Rham isomorphism.*

Remark 2.1.14. As we can see in the proof of the algebraic de Rham Theorem 1.6.2, the algebraic de Rham isomorphism, factors through the cohomology of $X_\sigma(\mathbb{C})_{an}$ with coefficients in the constant abelian sheaf $\mathbb{C}_{X_\sigma(\mathbb{C})}$

$$\mathrm{H}^i_{\mathrm{AdR}}(X/k) \otimes_k \mathbb{C} \cong \mathbb{H}^i(X_\sigma(\mathbb{C})_{an}, \mathbb{C}_{X_\sigma(\mathbb{C})}) \cong \mathrm{H}^i_{\mathrm{Bet}}(X_\sigma) \otimes_{\mathbb{Q}} \mathbb{C}.$$

Since, by construction, the twisted algebraic de Rham isomorphism factors through the untwisted version, then this holds also for the twisted version

$$\mathrm{H}^p_{\mathrm{AdR}}(X/k)(q) \otimes_k \mathbb{C} \cong \mathbb{H}^p(X_\sigma(\mathbb{C})_{an}, \mathbb{C}_{X_\sigma(\mathbb{C})}) \cong \mathrm{H}^p_{\mathrm{Bet}}(X_\sigma)(q) \otimes_{\mathbb{Q}} \mathbb{C}.$$

This isomorphism allows to compare elements coming from algebraic de Rham cohomology with the ones from Betti cohomology. More precisely, given $X \in SmProj_k$ we have the canonical injections, for any $p \geq 0$ and $q \in \mathbb{Z}$,

$$\mathrm{H}^p_{\mathrm{AdR}}(X/k)(q) \hookrightarrow \mathrm{H}^p_{\mathrm{AdR}}(X/k)(q) \otimes_k \mathbb{C} \cong \mathbb{H}^p(X_\sigma(\mathbb{C})_{an}, \mathbb{C}_{X_\sigma(\mathbb{C})})$$

and

$$\mathrm{H}^p_{\mathrm{Bet}}(X_\sigma)(q) \hookrightarrow \mathrm{H}^p_{\mathrm{Bet}}(X_\sigma)(q) \otimes_{\mathbb{Q}} \mathbb{C} \cong \mathbb{H}^p(X_\sigma(\mathbb{C})_{an}, \mathbb{C}_{X_\sigma(\mathbb{C})}).$$

If we denote by

$$\tilde{\sigma}^* : \mathrm{CH}^i(X)_{\mathbb{Q}} \to \mathrm{CH}^i(X_\sigma)_{\mathbb{Q}}$$

the pullback map on the Chow group along the canonical projection of the fiber product $\tilde{\sigma} : X_\sigma \to X$, then, we can consider the the following diagram, for any $i \geq 0$,

$$\begin{tikzcd}
\mathrm{CH}^i(X)_{\mathbb{Q}} \arrow[r, "cl^i_{\mathrm{Bet},X} \circ \tilde{\sigma}^*"] \arrow[d, "cl^i_{\mathrm{AdR},X}"] & \mathrm{H}^{2i}_{\mathrm{Bet}}(X_\sigma)(i) \arrow[d, hook] \\
\mathrm{H}^{2i}_{\mathrm{AdR}}(X/k)(i) \arrow[r, hook] & \mathbb{H}^{2i}(X_\sigma(\mathbb{C})_{an}, \mathbb{C}_{X_\sigma(\mathbb{C})}).
\end{tikzcd} \tag{2.5}$$

Proposition 2.1.15. *Let $\sigma : k \hookrightarrow \mathbb{C}$ be a field extension. Then, for any $X \in SmProj_k$ and $i \geq 0$, the digram 2.5 commutes. That is, the algebraic de Rham and Betti cycle class maps are compatible under the twisted algebraic de Rham isomorphism $\varpi^{2i,i}$.*

Proof. Notice that the diagram 2.5 is the same of

$$\begin{tikzcd}
\mathrm{CH}^i(X) \arrow[r, "\tilde{\sigma}^*"] \arrow[d, "cl^i_{\mathrm{AdR},X}"] & \mathrm{CH}^i(X_\sigma) \arrow[r, "cl^i_{\mathrm{Bet},X_\sigma}"] \arrow[d, "cl^i_{\mathrm{AdR},X_\sigma}"] & \mathrm{H}^{2i}_{\mathrm{Bet}}(X_\sigma)(i) \arrow[d, "cl^i_{\mathrm{Bet},X_\sigma}"] \\
\mathrm{H}^{2i}_{\mathrm{AdR}}(X/k)(i) \arrow[r, "\tilde{\sigma}^*"] & \mathrm{H}^{2i}_{\mathrm{AdR}}(X_\sigma/\mathbb{C})(i) \arrow[r] & \mathbb{H}^{2i}(X_\sigma(\mathbb{C})_{an}, \mathbb{C}_{X_\sigma(\mathbb{C})}).
\end{tikzcd}$$

The first square commutes because it is a naturality square of the algebraic de Rham cycle class map. Then, the theorem is equivalent to prove the commutativity of the second square. In other words, we can assume that X is an algebraic variety over $\mathbb{C}$. Recall form Appendix B, that the cycle class maps are such that, in each degree $i \geq 0$, for any $[Z] \in \mathrm{CH}^i(X)$

$$cl_X^i([Z]) = \frac{1}{(i-1)!} c_X^i([\mathcal{O}_Z]),$$

where c_X^i are the Chern classes. So, it suffices to prove that the algebraic de Rham and Betti Chern classes are compatible under the twisted algebraic de Rham isomorphism, that is, for any $i \geq 0$,

$$\begin{tikzcd} \mathrm{K}^0(X) \arrow[r, "c^i_{\mathrm{Bet},X}"] \arrow[d, "c^i_{\mathrm{AdR},X}"] & \mathrm{H}^{2i}_{\mathrm{Bet}}(X)(i) \arrow[d, hook] \\ \mathrm{H}^{2i}_{\mathrm{AdR}}(X/\mathbb{C})(i) \arrow[r, hook] & \mathbb{H}^{2i}(X(\mathbb{C})_{an}, \mathbb{C}_{X(\mathbb{C})}) \end{tikzcd}$$

commutes. Let $\mathcal{E}$ be an algebraic vector bundle over X and $\pi : \mathbb{P}(\mathcal{E}) \to X$ is its projectivization. Recall also that Chern classes $c^i_{\mathrm{AdR},X}(\mathcal{E}) \in \mathrm{H}^{2i}_{\mathrm{AdR}}(X/\mathbb{C})(i)$ and $c^i_{\mathrm{Bet},X}(\mathcal{E}) \in \mathrm{H}^{2i}_{\mathrm{Bet}}(X)(i)$ are the unique elements such that

$$\sum_{i \geq 0} (-1)^{i+1} \pi^* c^i_{\mathrm{AdR},X}(\mathcal{E}) \cup p^1_{\mathrm{AdR},\mathbb{P}(\mathcal{E})}(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) = 0 \quad \text{in } \mathrm{H}^{2i}_{\mathrm{AdR}}(\mathbb{P}(\mathcal{E})/\mathbb{C})(i)$$

and

$$\sum_{i \geq 0} (-1)^{i+1} \pi^* c^i_{\mathrm{Bet},X}(\mathcal{E}) \cup p^1_{\mathrm{Bet},\mathbb{P}(\mathcal{E})}(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) = 0 \quad \text{in } \mathrm{H}^{2i}_{\mathrm{Bet}}(\mathbb{P}(\mathcal{E}))(i).$$

Notice that, since the algebraic de Rham isomorphism is natural in X, then also the twisted version is. So, we have the naturality square of the 1-twisted algebraic de Rham isomorphism

$$\begin{tikzcd} \mathrm{H}^{2i}_{\mathrm{AdR}}(X/\mathbb{C})(i) \arrow[r, "\varpi_X^{2i,i}"] \arrow[d, hook, "\pi^*"] & \mathrm{H}^{2i}_{\mathrm{Bet}}(X)(i) \otimes_{\mathbb{Q}} \mathbb{C} \arrow[d, hook, "(\pi^*)_{\mathbb{C}}"] \\ \mathrm{H}^{2i}_{\mathrm{AdR}}(\mathbb{P}(\mathcal{E})/\mathbb{C})(i) \arrow[r, "\varpi_X^{2i,i}"] & \mathrm{H}^{2i}_{\mathrm{Bet}}(\mathbb{P}(\mathcal{E}))(i) \otimes_{\mathbb{Q}} \mathbb{C}. \end{tikzcd}$$

Then, we see that it suffices to show that the algebraic de Rham and Betti first Chern classes are compatible under the twisted algebraic de Rham isomorphism, that is

$$\begin{tikzcd}
\mathrm{Pic}(X) \arrow[r, "p^1_{\mathrm{Bet},X}"] \arrow[d, "p^1_{\mathrm{AdR},X}"] & \mathrm{H}^2_{\mathrm{Bet}}(X)(1) \arrow[d, hook] \\
\mathrm{H}^2_{\mathrm{AdR}}(X/\mathbb{C})(1) \arrow[r] & \mathbb{H}^2(X(\mathbb{C})_{an}, \mathbb{C}_{X(\mathbb{C})})
\end{tikzcd}$$

commutes. A reference for this is [Del71b, §2.2.5]. Consider the following diagram of complexes of abelian sheaves over $X(\mathbb{C})_{an}$

$$\begin{tikzcd}
\mathcal{O}^{\times}_{X(\mathbb{C})}[-1] \arrow[d, "d\log"] & {[\mathbb{Z}(1)_{X(\mathbb{C})} \xrightarrow{c} \mathcal{O}_{X(\mathbb{C})}]} \arrow[l, "\exp"'] \arrow[r] \arrow[d, hook] & \mathbb{Z}(1)_{X(\mathbb{C})} \arrow[r, hook] \arrow[d, hook] & \mathbb{Q}(1)_{X(\mathbb{C})} \arrow[d, hook] \\
\sigma^{\geq 1}\Omega^{\bullet}_{X(\mathbb{C})} \arrow[d] & {[\mathbb{C}_{X(\mathbb{C})} \xrightarrow{c} \mathcal{O}_{X(\mathbb{C})}]} \arrow[l, "d"'] \arrow[r] & \mathbb{C}_{X(\mathbb{C})} \arrow[r, equal] & \mathbb{C}_{X(\mathbb{C})} \arrow[d, equal] \\
\Omega^{\bullet}_{X(\mathbb{C})} & & & \mathbb{C}_{X(\mathbb{C})}, \arrow[lll, "c"']
\end{tikzcd} \tag{2.6}$$

where $\sigma^{\geq 1}\Omega^{\bullet}_{X(\mathbb{C})}$ denotes the left dumb truncation of $\Omega^{\bullet}_{X(\mathbb{C})}$ at level 1. Notice that the first square commutes because the compositions

$$\begin{array}{ccccccccc}
\mathbb{Z}(1)_{X(\mathbb{C})} & \xrightarrow{c} & \mathcal{O}_{X(\mathbb{C})} & \to & 0 & \to & 0 & \to \cdots \\
\downarrow & & \downarrow \exp & & \downarrow & & \downarrow & \\
0 & \to & \mathcal{O}^{\times}_{X(\mathbb{C})} & \to & 0 & \to & 0 & \to \cdots \\
\downarrow & & \downarrow d\log & & \downarrow & & \downarrow & \\
0 & \to & \Omega^1_{X(\mathbb{C})} & \to & \Omega^2_{X(\mathbb{C})} & \to & \Omega^3_{X(\mathbb{C})} & \to \cdots
\end{array}$$

and

$$\begin{array}{ccccccccc}
\mathbb{Z}(1)_{X(\mathbb{C})} & \xrightarrow{c} & \mathcal{O}_{X(\mathbb{C})} & \to & 0 & \to & 0 & \to \cdots \\
\downarrow & & \| & & \downarrow & & \downarrow & \\
\mathbb{C}_{X(\mathbb{C})} & \xrightarrow{c} & \mathcal{O}_{X(\mathbb{C})} & \to & 0 & \to & 0 & \to \cdots \\
\downarrow & & \downarrow d & & \downarrow & & \downarrow & \\
0 & \to & \Omega^1_{X(\mathbb{C})} & \to & \Omega^2_{X(\mathbb{C})} & \to & \Omega^3_{X(\mathbb{C})} & \to \cdots
\end{array}$$

are the same. The second square clearly commutes. The third square is a commutative diagram of inclusions. The fourth is anti-commutative up to an equivalence. Indeed, the compositions are

$$\begin{array}{ccccccccc}
\mathbb{C}_{X(\mathbb{C})} & \xrightarrow{c} & \mathcal{O}_{X(\mathbb{C})} & \to & 0 & \to & 0 & \to \cdots \\
\downarrow & & \downarrow d & & \downarrow & & \downarrow & \\
0 & \to & \Omega^1_{X(\mathbb{C})} & \to & \Omega^2_{X(\mathbb{C})} & \to & \Omega^3_{X(\mathbb{C})} & \to \cdots \\
\downarrow & & \| & & \| & & \| & \\
\mathcal{O}_{X(\mathbb{C})} & \to & \Omega^1_{X(\mathbb{C})} & \to & \Omega^2_{X(\mathbb{C})} & \to & \Omega^3_{X(\mathbb{C})} & \to \cdots
\end{array}$$

and

$$
\begin{array}{ccccccccc}
\mathbb{C}_{X(\mathbb{C})} & \xrightarrow{c} & \mathcal{O}_{X(\mathbb{C})} & \to & 0 & \to & 0 & \to \cdots \\
\parallel & & \downarrow & & \downarrow & & \downarrow & \\
\mathbb{C}_{X(\mathbb{C})} & \to & 0 & \to & 0 & \to & 0 & \to \cdots \\
\downarrow c & & \downarrow & & \downarrow & & \downarrow & \\
\mathcal{O}_{X(\mathbb{C})} & \to & \Omega^1_{X(\mathbb{C})} & \to & \Omega^2_{X(\mathbb{C})} & \to & \Omega^3_{X(\mathbb{C})} & \to \cdots ,
\end{array}
$$

and we have the homotopy equivalence of morphisms of complexes

$$
\begin{tikzcd}
\mathbb{C}_{X(\mathbb{C})} \arrow[r, "c"] \arrow[d, shift right, "0"'] \arrow[d, shift left, "-c"] & \mathcal{O}_{X(\mathbb{C})} \arrow[r] \arrow[dl, "id"'] \arrow[d, shift right, "d"'] \arrow[d, shift left, "0"] & 0 \arrow[r] \arrow[dl] \arrow[d, shift right] \arrow[d, shift left] & 0 \arrow[r] \arrow[dl] \arrow[d, shift right] \arrow[d, shift left] & \cdots \\
\mathcal{O}_{X(\mathbb{C})} \arrow[r, "d"] & \Omega^1_{X(\mathbb{C})} \arrow[r] & \Omega^2_{X(\mathbb{C})} \arrow[r] & \Omega^3_{X(\mathbb{C})} \arrow[r] & \cdots .
\end{tikzcd}
$$

Taking the second sheaf cohomology of diagram 2.6, we obtain the anti-commutative diagram

$$
\begin{tikzcd}
\mathbb{H}^2(X(\mathbb{C})_{an}, \mathcal{O}^{\times}_{X(\mathbb{C})}[-1]) \arrow[r] \arrow[d] & \mathbb{H}^2(X(\mathbb{C})_{an}, \mathbb{Z}(1)_{X(\mathbb{C})}) \arrow[r] & \mathbb{H}^2(X(\mathbb{C})_{an}, \mathbb{Q}(1)_{X(\mathbb{C})}) \arrow[d, hook] \\
\mathbb{H}^2(X(\mathbb{C})_{an}, \Omega^{\bullet}_{X(\mathbb{C})}) \arrow[rr, hook] & & \mathbb{H}^2(X(\mathbb{C})_{an}, \mathbb{C}_{X(\mathbb{C})}),
\end{tikzcd}
$$

where the first arrow in the upper horizontal composition is the opposite [3] of the connecting homomorphism of the long exact sequence induced on sheaf cohomology by the exponential sequence and the lower horizontal arrow is the analytic de Rham isomorphism. Hence, we get a commutative diagram, if we replace the first arrow in the upper horizontal composition with the connecting homomorphism. We also have

[3] Indeed, the exponential sequence induces the exact triangle in the derived category $\mathrm{D}^+(\mathrm{Ab}(X(\mathbb{C})_{an}))$

$$\mathbb{Z}(1)_{X(\mathbb{C})} \xrightarrow{c} \mathcal{O}_{X(\mathbb{C})} \xrightarrow{\exp} \mathcal{O}^{\times}_{X(\mathbb{C})} \xrightarrow{\delta} \mathbb{Z}(1)_{X(\mathbb{C})}[1],$$

where δ is the morphism in $\mathrm{D}^+(\mathrm{Ab}(X(\mathbb{C})_{an}))$

$$\mathcal{O}^{\times}_{X(\mathbb{C})} \xleftarrow{\exp} [\mathbb{Z}(1)_{X(\mathbb{C})} \xrightarrow{c} \mathcal{O}_{X(\mathbb{C})}][1] \to \mathbb{Z}(1)_{X(\mathbb{C})}[1],$$

which induces the connecting homomorphism in the long exact sequence on sheaf cohomology. Its shifted $\delta[-1]$ is the composition appearing in diagram 2.6. Since

$$\mathcal{O}^{\times}_{X(\mathbb{C})}[-1] \xrightarrow{-\delta[-1]} \mathbb{Z}(1)_{X(\mathbb{C})} \xrightarrow{c} \mathcal{O}_{X(\mathbb{C})} \xrightarrow{\exp} \mathcal{O}^{\times}_{X(\mathbb{C})}$$

is an exact triangle, then $\delta[-1]$ induces the opposite of the connecting homomorphism.

the commutative diagram

$$\begin{tikzcd}
\mathbb{H}^2(X_{Zar}, \mathcal{O}_X^\times[-1]) \arrow[r] \arrow[d] & \mathbb{H}^2(X(\mathbb{C})_{an}, \alpha_X^{-1}\mathcal{O}_X^\times[-1]) \arrow[r] \arrow[d] & \mathbb{H}^2(X(\mathbb{C})_{an}, \mathcal{O}_{X(\mathbb{C})}^\times[-1]) \arrow[d] \\
\mathbb{H}^2(X_{Zar}, \Omega^\bullet_{X/\mathbb{C}}) \arrow[r] & \mathbb{H}^2(X(\mathbb{C})_{an}, \alpha_X^{-1}\Omega^\bullet_{X/\mathbb{C}}) \arrow[r] & \mathbb{H}^2(X(\mathbb{C})_{an}, \Omega^\bullet_{X(\mathbb{C})}),
\end{tikzcd}$$

where the first is a naturality square of functoriality morphism on sheaf cohomology for the morphism $d\log : \mathcal{O}_X^\times[-1] \to \Omega^\bullet_{X/\mathbb{C}}$ and the second is induced by the commutative diagram of abelian sheaves over $X(\mathbb{C})_{an}$

$$\begin{tikzcd}
\alpha_X^{-1}\mathcal{O}_X^\times[-1] \arrow[r] \arrow[d, "\alpha_X^{-1}(d\log)"] & \mathcal{O}_{X(\mathbb{C})}^\times[-1] \arrow[d, "d\log"] \\
\alpha_X^{-1}\Omega^\bullet_{X/\mathbb{C}} \arrow[r] & \Omega^\bullet_{X(\mathbb{C})}.
\end{tikzcd}$$

We already noticed that the upper horizontal composition is the morphism of abelian groups

$$\mathrm{Pic}(X) \to \mathrm{Pic}(X(\mathbb{C})).$$

The lower horizontal composition is the one that gives the comparison isomorphism between algebraic and analytic de Rham cohomology, constructed in 1.8. Putting together the diagrams above, we obtain the commutative diagram

$$\begin{tikzcd}
\mathrm{Pic}(X) \arrow[r] \arrow[d] & \mathrm{Pic}(X(\mathbb{C})) \arrow[r] \arrow[d] & \mathbb{H}^2(X(\mathbb{C})_{an}, \mathbb{Q}(1)_{X(\mathbb{C})}) \arrow[d] \\
\mathbb{H}^2(X_{Zar}, \Omega^\bullet_{X/\mathbb{C}}) \arrow[r] & \mathbb{H}^2(X(\mathbb{C})_{an}, \Omega^\bullet_{X(\mathbb{C})}) \arrow[r] & \mathbb{H}^2(X(\mathbb{C})_{an}, \mathbb{C}_{X(\mathbb{C})}).
\end{tikzcd}$$

By definition of algebraic de Rham and Betti first Chern classes, this is exactly the commutative diagram 2.5 we wanted to prove. □

Example 2.1.16. Take $X = \mathbb{P}^1_{\overline{\mathbb{Q}}}$. Recall, from example 1.5.3, that we have

$$\mathrm{H}^2_{\mathrm{AdR}}(\mathbb{P}^1_{\overline{\mathbb{Q}}}/\overline{\mathbb{Q}}) \cong \overline{\mathbb{Q}}.$$

Take $x \in \mathbb{P}^1_{\overline{\mathbb{Q}}}$ a $\overline{\mathbb{Q}}$-rational point. Consider $[x] \in \mathrm{CH}^1(\mathbb{P}^1_{\overline{\mathbb{Q}}})$ its rational class. Consider the cycle classes

$$\begin{aligned}
&cl^1_{\mathrm{AdR},\mathbb{P}^1_{\overline{\mathbb{Q}}}}([x]) \in \mathrm{H}^2_{\mathrm{AdR}}(\mathbb{P}^1_{\overline{\mathbb{Q}}})(1) \cong \mathrm{H}^2_{\mathrm{AdR}}(\mathbb{P}^1_{\overline{\mathbb{Q}}}) \\
&cl^1_{\mathrm{Bet},\mathbb{P}^1_{\mathbb{C}}}([x]) \in \mathrm{H}^2_{\mathrm{Bet}}(\mathbb{P}^1_{\mathbb{C}})(1).
\end{aligned}$$

They are generators because, by the normalization axiom of cycle class maps, we have in both cases

$$\mathrm{Tr}_{\mathbb{P}^1_{\overline{\mathbb{Q}}}}(cl^1_{\mathbb{P}^1_{\overline{\mathbb{Q}}}}([x])) = [k(x) : \overline{\mathbb{Q}}] = [\overline{\mathbb{Q}} : \overline{\mathbb{Q}}] = 1 \neq 0$$

and the trace map $\mathrm{Tr}_{\mathbb{P}^1_{\overline{\mathbb{Q}}}}$ is an isomorphism, since $\mathbb{P}^1_{\overline{\mathbb{Q}}}$ is irreducible. Consider some $\overline{\mathbb{Q}}$-linear and $\mathbb{Q}$-linear generators

$$\omega \in \mathrm{H}^2_{\mathrm{AdR}}(\mathbb{P}^1_{\overline{\mathbb{Q}}}/\overline{\mathbb{Q}}) \qquad \& \qquad \tilde{\gamma} \in \mathrm{H}^2_{\mathrm{Bet}}(\mathbb{P}^1_{\mathbb{C}}),$$

where $\gamma \in \mathrm{H}_2^{\mathrm{Sing}}(\mathbb{P}^1_{\mathbb{C}}(\mathbb{C}); \mathbb{Q})$ is a $\mathbb{Q}$-linear generator and $\tilde{\gamma} \in \mathrm{H}^2_{\mathrm{Sing}}(\mathbb{P}^1_{\mathbb{C}}(\mathbb{C}); \mathbb{Q}) \cong \mathrm{H}^2_{\mathrm{Bet}}(\mathbb{P}^1_{\mathbb{C}})$ is the dual element. Then, we write

$$cl^1_{\mathrm{AdR},\mathbb{P}^1_{\overline{\mathbb{Q}}}}([x]) = a\omega \qquad \& \qquad cl^1_{\mathrm{Bet},\mathbb{P}^1_{\mathbb{C}}}([x]) = 2\pi i b\tilde{\gamma},$$

with $a \in \overline{\mathbb{Q}} \setminus \{0\}$ and $b \in \mathbb{Q} \setminus \{0\}$. The compatibility of the cycle class maps with the twisted period isomorphism

$$\mathrm{H}^2_{\mathrm{AdR}}(\mathbb{P}^1_{\overline{\mathbb{Q}}}/\overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \cong \mathrm{H}^2_{\mathrm{Bet}}(\mathbb{P}^1_{\mathbb{C}})(1) \otimes_{\mathbb{Q}} \mathbb{C}$$

implies that

$$a \int_\gamma \omega = 2\pi i b.$$

Hence, $2\pi i$ is a period of $\mathbb{P}^1_{\overline{\mathbb{Q}}}$.

2.1.3 The category of Chow motives

In the previous sections we defined the abstract notion of a Weil cohomology theory and we saw the two examples given by algebraic de Rham and Betti cohomology. As already explained in the overview, a natural question is whether, for any field k, it is possible to construct a universal Weil cohomology $\mathrm{H}^*(\_\,; \mathbb{Q})$ over k with rational coefficients, in the sense that any other Weil cohomology $\mathrm{H}^*(\_\,; K)$ over k with coefficients in some other field K of characteristic 0 (hence, containing $\mathbb{Q}$) can be obtained from it changing coefficients, i.e. such that there exist a canonical isomorphism

$$\mathrm{H}^*(\_\,; K) \cong \mathrm{H}^*(\_\,; \mathbb{Q}) \otimes_{\mathbb{Q}} K.$$

However, this is not the case. One explanation is due to the existence of transcendental periods. Indeed, recall that, given $X \in \mathit{SmProj}_{\overline{\mathbb{Q}}}$, a period of X is defined as a complex number appearing in a representative matrix of the period isomorphism

$$\varpi^i_X : \mathrm{H}^i_{\mathrm{AdR}}(X/\overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \cong \mathrm{H}^i_{\mathrm{Bet}}(X_\sigma) \otimes_{\mathbb{Q}} \mathbb{C}$$

with respect to a $\overline{\mathbb{Q}}$-basis of $\mathrm{H}^i_{\mathrm{AdR}}(X/\overline{\mathbb{Q}})$ and a $\mathbb{Q}$-basis of $\mathrm{H}^i_{\mathrm{Bet}}(X_\sigma)$. If such a universal Weil cohomology theory $\mathrm{H}^*(\_\,;\mathbb{Q})$ existed, then we would have canonical isomorphisms of $\overline{\mathbb{Q}}$ and $\mathbb{Q}$-vector spaces

$$\mathrm{H}^i(X;\mathbb{Q})\otimes_{\mathbb{Q}}\overline{\mathbb{Q}}\cong \mathrm{H}^i_{\mathrm{AdR}}(X/\overline{\mathbb{Q}}) \qquad \& \qquad \mathrm{H}^i(X;\mathbb{Q})\cong \mathrm{H}^i_{\mathrm{Bet}}(X/\mathbb{Q}).$$

Since also the period isomorphism is canonical, this implies that the period isomorphism factors through the canonical isomorphisms obtained by extending coefficients into $\mathbb{C}$

$$\mathrm{H}^i_{\mathrm{AdR}}(X/\overline{\mathbb{Q}})\otimes_{\overline{\mathbb{Q}}}\mathbb{C}\cong \mathrm{H}^*(X;\mathbb{Q})\otimes_{\mathbb{Q}}\overline{\mathbb{Q}}\otimes_{\overline{\mathbb{Q}}}\mathbb{C}\cong \mathrm{H}^*(X;\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{C}\cong \mathrm{H}^i_{\mathrm{Bet}}(X_\sigma)\otimes_{\mathbb{Q}}\mathbb{C}.$$

But then, the period isomorphism should map any $\overline{\mathbb{Q}}$-basis of $\mathrm{H}^i_{\mathrm{AdR}}(X/\overline{\mathbb{Q}})$ into a $\mathbb{Q}$-basis of $\mathrm{H}^i_{\mathrm{Bet}}(X/\mathbb{Q})$, hence periods should be all algebraic numbers, which is false (for example, as we saw in 2.1.16, $2\pi i$ is a period of $\mathbb{P}^1_{\overline{\mathbb{Q}}}$, which is transcendental).

Then, recall that the idea to obtain a notion of universal cohomology theory, is to look for a $\mathbb{Q}$-linear category $\mathcal{M}(k)$ with a monoidal contravariant functor, called the *motivic cohomology*,

$$h : SmProj_k^{op} \to \mathcal{M}(k),$$

such that any Weil cohomology H^* over k with coefficients in K factors uniquely through it, with a tensor functor R_{H}

$$\begin{tikzcd} SmProj_k^{op} \arrow[rr, "H^*"] \arrow[dr, "h"'] & & grVect_K \\ & \mathcal{M}(k). \arrow[ur, "R_{\mathrm{H}}"'] & \end{tikzcd}$$

Moreover, to obtain a *category of pure motives*, we ask some further properties, as explained in the overview. We describe Grothendieck's construction for such a category $\mathcal{M}(k)$, which is based on algebraic cycles modulo rational equivalence, called the *category of Chow motives*. References are [Sch, §1] or [And04, §4]. The construction consists of three steps.

First step: the category of correspondences

To have such a category $\mathcal{M}(k)$, we should have at least an object $h(X)$ for each $X \in SmProj_k$, which will be the motivic cohomology of X. Then, we want to give a suitable notion of morphisms between such objects. Recall remark 2.1.8, where we observed that, for any Weil cohomology H^* over k with coefficients in K and $X, Y \in SmProj_k$ with X irreducible of dimension d, we have the following canonical isomorphism of K-vector spaces

$$\mathrm{H}^{2d}(X\times_k Y)(d)\cong \oplus_i \mathrm{H}^{2d-i}(X)(d)\otimes_K \mathrm{H}^i(Y)\cong \mathrm{Hom}_{grVect_K}(\mathrm{H}^*(X),\mathrm{H}^*(Y))$$

Composing with the d^{th}-component of the cycle class map we obtain the morphism of $\mathbb{Q}$-vector spaces

$$r_{\mathrm{H}} : \mathrm{CH}^d(X \times_k Y)_{\mathbb{Q}} \xrightarrow{cl^d_{X\times_k Y}} \mathrm{H}^{2d}(X \times_k Y)(d) \cong \mathrm{Hom}_{grVect_K}(\mathrm{H}^*(X), \mathrm{H}^*(Y)),$$

which is such that, for any morphism $f : Y \to X$ in $SmProj_k$, the transpose of the graph of f is mapped into its pullback map of the Weil cohomology

$$r_{\mathrm{H}}([\Gamma^t_f]) = \underline{cl^d_{X\times_k Y}([\Gamma^t_f])} = f^* : \mathrm{H}^*(X) \to \mathrm{H}^*(Y),$$

The idea is that the morphism r_{H} should be exactly the map induced on hom-sets by the realization functor. This leads to the following definition.

Definition 2.1.17. We define the *category of correspondences* modulo rational equivalence and with coefficients in $\mathbb{Q}$

$$\mathrm{Cor}_{rat}(k; \mathbb{Q}),$$

the category with objects

$$h(X),$$

for each $X \in SmProj_k$, and with morphisms

$$\mathrm{Hom}_{\mathrm{Cor}_{rat}(k;\mathbb{Q})}(h(X), h(Y)) := \mathrm{CH}^d(X \times_k Y)_{\mathbb{Q}},$$

if X is irreducible of dimension d. For general X, we take direct sums over the irreducible components of X.

It is called the category of correspondences because the elements of the $\mathbb{Q}$-vector space, for any $r \in \mathbb{Z}$ and $X \in SmProj_k$ irreducible of dimension d,

$$\mathrm{CH}^{d+r}(X \times_k Y)_{\mathbb{Q}}$$

are called *algebraic correspondences* of *degree* r from X to Y, modulo rational equivalence and with coefficients in $\mathbb{Q}$. So, morphisms in the category of correspondences are given by the algebraic correspondences of degree 0. Since rational equivalence is an *adequate equivalence* (see [Sam58]), then $\mathrm{Cor}_{rat}(k; \mathbb{Q})$ is a $\mathbb{Q}$-linear category, with $\mathbb{Q}$-bilinear composition given by, for any $X, Y, Z \in SmProj_k$ with X and Y irreducible of dimension d and e respectively,

$$\begin{aligned} \mathrm{CH}^d(X \times_k Y)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathrm{CH}^e(Y \times_k Z)_{\mathbb{Q}} &\to \mathrm{CH}^d(X \times_k Z)_{\mathbb{Q}} \\ \alpha \otimes \beta &\mapsto \beta \circ \alpha := pr_{XZ*}(\alpha \times \beta), \end{aligned}$$

where $\alpha \times \beta := pr^*_{XY}\alpha \cdot pr^*_{YZ}\beta$ is the exterior product of Chow groups and $pr_{XY}, pr_{YZ}, pr_{XZ}$ are canonical projections of the fiber product $X \times_k Y \times_k Z$. For general X and Y, we extend by $\mathbb{Q}$-bilinearity. The rational class of the diagonal $\Delta_X : X \to X \times X$ is the identity on $h(X)$, for any $X \in SmProj_k$. $\mathrm{Cor}_{rat}(k;\mathbb{Q})$ is an additive category with biproducts, for any $X, Y \in SmProj_k$,

$$h(X) \oplus h(Y) \coloneqq h(X \coprod Y).$$

The zero object is $h(\emptyset)$. The biproduct of morphisms is given by the sum in Chow groups. Moreover, $\mathrm{Cor}_{rat}(k;\mathbb{Q})$ is a tensor category with tensor product

$$h(X) \otimes h(Y) \coloneqq h(X \times_k Y).$$

The unit object is $\mathbb{1} \coloneqq h(Spec(k))$. The tensor product of morphisms is given by the exterior product of the Chow groups. Associativity and commutativity constraints are those inherited by the product in $SmProj_k$, the fiber product over k.

Consider on $SmProj_k$ the monoidal structure given by the fiber product over k and unit object $Spec(k)$. We have a contravariant monoidal functor

$$\begin{aligned} h : SmProj_k &\to \mathrm{Cor}_{rat}(k;\mathbb{Q}) \\ X &\mapsto h(X) \\ f \uparrow \qquad & \qquad \downarrow [\Gamma_f^t] \\ Y &\mapsto h(Y). \end{aligned}$$

By the above discussion, we see that any Weil cohomology H^* over k with coefficients in K uniquely factors through $\mathrm{Cor}_{rat}(k;\mathbb{Q})$

$$\begin{tikzcd} SmProj_k^{op} \arrow[rr, "\mathrm{H}^*"] \arrow[dr, "h"'] & & grVect_K. \\ & \mathrm{Cor}_{rat}(k;\mathbb{Q}) \arrow[ur, "R_{\mathrm{H}}^C"'] & \end{tikzcd}$$

The functor R_{H}^C, called the *realization functor*, is such that, for any $X, Y \in SmProj_k$, with X irreducible of dimension d,

$$\begin{aligned} R_{\mathrm{H}}^C : \mathrm{Cor}_{rat}(k;\mathbb{Q}) &\to grVect_K \\ h(X) &\mapsto H^*(X) \\ \alpha \downarrow \qquad & \qquad \downarrow \underline{cl^d_{X \times_k Y}(\alpha)} = r_{\mathrm{H}}(\alpha) \\ h(Y) &\mapsto H^*(Y). \end{aligned}$$

For general X, we extend by $\mathbb{Q}$-linearity. By exterior product axiom of the cycle class map, it follows that R_{H}^C is a tensor functor. We see that, as we wanted, the morphism r_{H} is exactly the map induced on hom-sets by the realization functor R_{H}^C. For a general X, we still denote by r_{H} the map induced on hom-sets by the realization functor

$$r_{\mathrm{H}} : \mathrm{Hom}_{\mathrm{Cor}_{rat}(k;\mathbb{Q})}(h(X), h(Y)) \to \mathrm{Hom}_{grVect_K}(\mathrm{H}^*(X), \mathrm{H}^*(Y))$$
$$\alpha \mapsto R^C_{\mathrm{H}}(\alpha).$$

The category of correspondence $\mathrm{Cor}_{rat}(k;\mathbb{Q})$ constitutes the bulk of Grothendieck's construction. Recall that the original aim was to construct a category of pure motives, which is expected to be rigid abelian. The category of correspondences $\mathrm{Cor}_{rat}(k;\mathbb{Q})$ has none of these properties. In the following steps, we apply two formal categorical constructions to $\mathrm{Cor}_{rat}(k;\mathbb{Q})$, trying to force them.

Second step: pseudo-abelian completion

The category of correspondences $\mathrm{Cor}_{rat}(k;\mathbb{Q})$ constructed in the previous step is an additive $\mathbb{Q}$-linear category, but it is not abelian. In fact, it is not even pseudo-abelian. [4] In this step we force this latter property. In order to get a pseudo-abelian category starting from any category, there exists a universal construction: the *pseudo-abelian completion.* This construction consists in formally adding an image for each idempotent morphism. Applied to the category $\mathrm{Cor}_{rat}(k;\mathbb{Q})$, it produces the following category.

Definition 2.1.18. We define the *category of effective Chow motives* with coefficients in $\mathbb{Q}$

$$\mathrm{CHM}^{\mathrm{eff}}(k;\mathbb{Q}),$$

the pseudo-abelian completion of $\mathrm{Cor}_{rat}(k;\mathbb{Q})$. That is, the category with objects

$$ph(X),$$

for each $X \in SmProj_k$ and $p \in \mathrm{End}_{\mathrm{Cor}_{rat}(k;\mathbb{Q})}(h(X))$ an idempotent morphism, called a *projector*, and with morphisms

$$\mathrm{Hom}_{\mathrm{CHM}^{\mathrm{eff}}(k;\mathbb{Q})}(ph(X), qh(Y)) \coloneqq q \circ \mathrm{Hom}_{\mathrm{Cor}_{rat}(k;\mathbb{Q})}(h(X), h(Y)) \circ p.$$

The pseudo-abelian completion is such that an object $ph(X)$ is canonically isomorphic to the categorical image of the idempotent morphism $p : h(X) \to h(X)$

$$ph(X) \cong Im(p).$$

[4]Pseudo-abelian means that any idempotent morphism p has an image (or equivalently a kernel, since $Ker(p) \cong Im(id - p)$ and $id - p$ is also idempotent). Equivalently, any idempotent morphism $p : C \to C$ splits, i.e. there exist morphisms $C \xrightarrow{f} K \xrightarrow{g} C$ such that $gf = p$ and $fg = id$. Moroever, the splitting is unique and $K \cong Im(p)$. Hence, $C \cong Im(p) \oplus Im(id - p)$. More generally, any decomposition of the identity into orthogonal idempotents $id = \sum_i p_i$ induces the decomposition $C \cong \oplus_i Im(p_i)$.

By construction, $\mathrm{CHM}^{\mathrm{eff}}(k;\mathbb{Q})$ is a pseudo-abelian category with biproducts

$$ph(X) \oplus qh(Y) \cong (p \oplus q)h(X \coprod Y).$$

The zero object is $h(\emptyset)$. The biproduct of morphisms is naturally induced by the one in $\mathrm{Cor}_{rat}(k;\mathbb{Q})$. Moreover, $\mathrm{CHM}^{\mathrm{eff}}(k;\mathbb{Q})$ is a tensor category with tensor product

$$ph(X) \otimes qh(Y) \cong (p \otimes q)h(X \times_k Y).$$

The unit object is $\mathbb{1} := h(Spec(k))$. The tensor product of morphisms, associativity and commutativity constraints are naturally induced by the ones in $\mathrm{Cor}_{rat}(k;\mathbb{Q})$.

We have a canonical embedding functor

$$\begin{aligned} \mathrm{Cor}_{rat}(k;\mathbb{Q}) &\hookrightarrow \mathrm{CHM}^{\mathrm{eff}}(k;\mathbb{Q}) \\ h(X) &\mapsto id_X h(X) =: h(X). \end{aligned}$$

It is a $\mathbb{Q}$-linear tensor functor. By composition with the contravariant monoidal functor $h : SmProj_k^{op} \to \mathrm{Cor}_{rat}(k;\mathbb{Q})$ described above, we obtain the contravariant monoidal functor, still denoted by h,

$$h : SmProj_k^{op} \xrightarrow{h} \mathrm{Cor}_{rat}(k;\mathbb{Q}) \hookrightarrow \mathrm{CHM}^{\mathrm{eff}}(k;\mathbb{Q}).$$

Since any Weil cohomology H^* over k with coefficients in K uniquely factors through $\mathrm{Cor}_{rat}(k;\mathbb{Q})$ and since $grVect_K$ is a pseudo-abelian category, then, by universal property of pseudo-abelian completion, we have that H^* also uniquely factors through $\mathrm{CHM}^{\mathrm{eff}}(k;\mathbb{Q})$

$$\begin{tikzcd} SmProj_k \arrow[rrr, "\mathrm{H}^*"] \arrow[dr, "h"'] & & & grVect_k. \\ & \mathrm{Cor}_{rat}(k;\mathbb{Q}) \arrow[rru, "R^C_{\mathrm{H}}"] \arrow[r, hook] & \mathrm{CHM}^{\mathrm{eff}}(k;\mathbb{Q}) \arrow[ru, "R^{\mathrm{eff}}_{\mathrm{H}}"'] & \end{tikzcd}$$

The functor $R^{\mathrm{eff}}_{\mathrm{H}}$, called the *effective realization functor*, is a $\mathbb{Q}$-linear tensor functor. Explicitly, for any $X, Y \in SmProj_k$, given a morphism

$$q \circ \alpha \circ p : ph(X) \to qh(Y),$$

where $\alpha \in \mathrm{Hom}_{\mathrm{Cor}_{rat}(k;\mathbb{Q})}(h(X), h(Y))$ and $p \in \mathrm{End}_{\mathrm{Cor}_{rat}(k;\mathbb{Q})}(h(X))$, $q \in \mathrm{End}_{\mathrm{Cor}_{rat}(k;\mathbb{Q})}(h(Y))$ are projectors, consider the composition

$$\mathrm{H}^*(X) \xrightarrow{r_{\mathrm{H}}(p)} \mathrm{H}^*(X) \xrightarrow{r_{\mathrm{H}}(\alpha)} \mathrm{H}^*(Y) \xrightarrow{r_{\mathrm{H}}(q)} \mathrm{H}^*(Y).$$

The realization of $q \circ \alpha \circ p$ is the restriction to the images of the realization of the projectors

$$Im(r_{\mathrm{H}}(p)) \xrightarrow{r_{\mathrm{H}}(\alpha)} Im(r_{\mathrm{H}}(q)).$$

Remark 2.1.19. The property of being pseudo-abelian, allows to decompose objects in $\mathrm{CHM}^{\mathrm{eff}}(k;\mathbb{Q})$. For example, consider $h(\mathbb{P}^1_k) \in \mathrm{CHM}^{\mathrm{eff}}(k;\mathbb{Q})$. Let $g : \mathbb{P}^1_k \to Spec(k)$ be the structural morphism and $x : Spec(k) \to \mathbb{P}^1_k$ a k-rational point. The composition in $SmProj_k$

$$\mathbb{P}^1_k \xrightarrow{g} Spec(k) \xrightarrow{x} \mathbb{P}^1_k$$

is an idempotent morphism and this factorization provides a splitting. Hence also its image in $\mathrm{CHM}^{\mathrm{eff}}(k;\mathbb{Q})$

$$p : h(\mathbb{P}^1_k) \to \mathbb{1} \to h(\mathbb{P}^1_k)$$

is an idempotent morphism and the factorization is a splitting. Since $\mathrm{CHM}^{\mathrm{eff}}(k;\mathbb{Q})$ is pseudo-abelian, then $\mathbb{1} \cong Im(p)$ and we have the decomposition

$$h(\mathbb{P}^1_k) \cong Im(p) \oplus Im(id - p) \cong \mathbb{1} \oplus (id - p)h(\mathbb{P}^1_k).$$

Definition 2.1.20. The object in $\mathrm{CHM}^{\mathrm{eff}}(k;\mathbb{Q})$

$$\mathbb{L} := (id - p)h(\mathbb{P}^1_k)$$

is called the *Lefschetz motive.*

Remark 2.1.21. For any Weil cohomology H^* over k with coefficients in K, since its effective realization functor R^{eff}_{H} is a tensor functor, we have the canonical isomorphisms

$$R^{\mathrm{eff}}_{\mathrm{H}}(h(\mathbb{P}^1_k)) \cong R^{\mathrm{eff}}_{\mathrm{H}}(\mathbb{1} \oplus \mathbb{L}) \cong R^{\mathrm{eff}}_{\mathrm{H}}(\mathbb{1}) \oplus R^{\mathrm{eff}}_{\mathrm{H}}(\mathbb{L}) \cong K \oplus R^{\mathrm{eff}}_{\mathrm{H}}(\mathbb{L}).$$

On the other hand, by proposition 2.1.9, we have the canonical isomorphism

$$R^{\mathrm{eff}}_{\mathrm{H}}(h(\mathbb{P}^1_k)) \cong \mathrm{H}^*(\mathbb{P}^1_k) \cong K \oplus K(-1).$$

We deduce that there exists a canonical isomorphism

$$R^{\mathrm{eff}}_{\mathrm{H}}(\mathbb{L}) \cong K(-1).$$

In other words, for any Weil cohomology, the Lefschetz motive always realizes into the inverse of the Lefschetz module.

Third step: Lefschetz stabilization

The category of effective Chow motives $\mathrm{CHM}^{\mathrm{eff}}(k;\mathbb{Q})$ is a tensor category, but it's not rigid, i.e. not all objects have a dual. In this last step we force the rigidity property on $\mathrm{CHM}^{\mathrm{eff}}(k;\mathbb{Q})$. Given a Weil cohomology H^* over k with coefficients in K, recall from remark 2.1.5 that, for any $X \in SmProj_k$ irreducible of dimension d, the Tate twist allows to rewrite the dual object of $\mathrm{H}^*(X)$ as

$$\mathrm{H}^*(X)^\vee \cong \mathrm{H}^*(X)(d) \cong \mathrm{H}(X)^* \otimes K(1)^{\otimes d}.$$

Since the idea is that the functor h should behave like H^* and since, by remark 2.1.21, the Lefschetz motive $\mathbb{L}$ plays the analogous role of the Lefschetz module $K(-1)$, this suggests that dual objects could be obtained simply by inverting the Lefschetz motive with respect to the tensor product. In other words, we want to make the functor

$$\_ \otimes \mathbb{L} : \mathrm{CHM}^{\mathrm{eff}}(k;\mathbb{Q}) \to \mathrm{CHM}^{\mathrm{eff}}(k;\mathbb{Q}).$$

become an equivalence of categories, so we have to do an $\mathbb{L}$-stabilization construction. This can be done with the formalism of $\mathbb{L}$-spectra, or formally adding the object $\mathbb{L}^{\otimes -1}$, the inverse of $\mathbb{L}$ with respect to tensor product. The latter method leads to the following category.

Definition 2.1.22. We define the *category of Chow motives* with coefficients in $\mathbb{Q}$

$$\mathrm{CHM}(k;\mathbb{Q}),$$

the category with objects

$$ph(X)(r),$$

for each $ph(X) \in \mathrm{CHM}^{\mathrm{eff}}(k;\mathbb{Q})$ and $r \in \mathbb{Z}$, and with morphisms

$$\mathrm{Hom}_{\mathrm{CHM}(k;\mathbb{Q})}(ph(X)(r), qh(Y)(s)) := \varinjlim_{N \gg 0} Hom_{\mathrm{CHM}^{\mathrm{eff}}(k;\mathbb{Q})}(ph(X)\otimes\mathbb{L}^{\otimes N-r}, qh(Y)\otimes\mathbb{L}^{\otimes N-s}).$$

We remark that an alternative notation, that can be found in literature, consists in denoting an object $ph(X)(r)$ by the triple (X, p, r).

The formula for hom-sets in $\mathrm{CHM}(k;\mathbb{Q})$ given in the definition is the formal one of the $\mathbb{L}$-stabilization construction. In this context it holds a more explicit equivalent description, using algebraic correspondences of any degree, given by

$$\mathrm{Hom}_{\mathrm{CHM}(k;\mathbb{Q})}(ph(X)(r), qh(Y)(s)) \cong q \circ \mathrm{CH}^{d+s-r}(X \times_k Y)_{\mathbb{Q}} \circ p,$$

if X is irreducible of dimension d. For general X, we take the direct sum under the irreducible components of X. This also shows that $\mathrm{CHM}(k;\mathbb{Q})$ can be obtained equivalently from the category of correspondences $\mathrm{Cor}_{rat}(k;\mathbb{Q})$, by applying first the stabilization construction and then the pseudo-abelian completion.

The stabilization construction is such that we have a canonical isomorphism

$$\mathbb{1}(-1) \cong \mathbb{L}.$$

Then,

$$\mathbb{1}(1) \cong \mathbb{L}^{\otimes -1}$$

is the inverse with respect to tensor product of the Lefschetz motive, called the *Tate motive.* Moreover, we have canonical isomorphisms, for any $r \in \mathbb{Z}$,

$$eh(X)(r) \cong eh(X) \otimes \mathbb{L}^{\otimes -r}.$$

$\mathrm{CHM}(k;\mathbb{Q})$ is an additive $\mathbb{Q}$-linear category with biproducts

$$ph(X)(r) \oplus qh(Y)(s) \cong \begin{cases} (h(X) \oplus (h(Y) \otimes \mathbb{L}^{r-s}))(r) & \text{if } r \geq s \\ ((h(X) \otimes \mathbb{L}^{s-r}) \oplus h(Y))(s) & \text{else,} \end{cases}$$

where the $\oplus$ on the right hand side is taken in $\mathrm{CHM}^{\mathrm{eff}}(k;\mathbb{Q})$. The zero object is $h(\emptyset)$. The biproduct of morphisms is naturally induced by the one in $\mathrm{CHM}^{\mathrm{eff}}(k;\mathbb{Q})$. Since, as noticed above, the constructions of pseudo-abelian completion and $\mathbb{L}$-stabilization can be reversed, then $\mathrm{CHM}(k;\mathbb{Q})$ is a pseudo-abelian category. Moreover, $\mathrm{CHM}(k;\mathbb{Q})$ is a tensor category with tensor product

$$ph(X)(r) \otimes qh(Y)(s) \cong (p \otimes q)h(X \times_k Y)(r+s).$$

The unit object is $\mathbb{1} = h(Spec(k))$. The tensor product of morphisms, associativity and commutativity constraints are naturally induced by the ones in $\mathrm{CHM}^{\mathrm{eff}}(k;\mathbb{Q})$. This is indeed the right construction to obtain the dual for every object. That is, $\mathrm{CHM}(k;\mathbb{Q})$ is a rigid tensor category. It holds that we have canonical isomorphisms, for any $X \in SmProj_k$ irreducible of dimension d,

$$ph(X)(r)^{\vee} \cong p^t h(X)(d-r),$$

where p^t is the image of $p \in \mathrm{CH}^d(X \times_k X)$ along the pullback map on Chow groups of the the swap morphism of $X \times_k X$. For general X, we take direct sum under the irreducible components of X.

We have a canonical embedding functor

$$\begin{aligned} \mathrm{CHM}^{\mathrm{eff}}(k;\mathbb{Q}) &\hookrightarrow \mathrm{CHM}(k;\mathbb{Q}) \\ ph(X) &\mapsto ph(X)(0) =: ph(X). \end{aligned}$$

By composition with the contravariant monoidal functor $h : SmProj_k^{op} \to \mathrm{CHM}^{\mathrm{eff}}(k;\mathbb{Q})$ described above, we obtain the contravariant monoidal functor, still denoted by h,

$$h : SmProj_k^{op} \xrightarrow{h} \mathrm{CHM}^{\mathrm{eff}}(k;\mathbb{Q}) \hookrightarrow \mathrm{CHM}(k;\mathbb{Q})$$

Since any Weil cohomology H^* uniquely factors through $\mathrm{CHM}^{\mathrm{eff}}(k;\mathbb{Q})$ and since, by remark 2.1.21, the Lefschetz motive $\mathbb{L}$ realizes into $K(-1)$, which is an invertible object of $grVect_K$, then, by universal property of $\mathbb{L}$-stabilization construction, we have that H^* also uniquely factors through $\mathrm{CHM}(k;\mathbb{Q})$

$$\begin{tikzcd}
SmProj_k^{op} \arrow[rrr, "\mathrm{H}^*"] \arrow[dr, "h"] & & & grVect_k \\
& \mathrm{CHM}^{\mathrm{eff}}(k;\mathbb{Q}) \arrow[urr, "R_{\mathrm{H}}^{\mathrm{eff}}"] \arrow[r, hook] & \mathrm{CHM}(k;\mathbb{Q}). \arrow[ur, "R_{\mathrm{H}}"'] &
\end{tikzcd}$$

The functor R_{H}, called the *realization functor*, is a $\mathbb{Q}$-linear tensor functor. Explicitly, for any $X, Y \in SmProj_k$, with X irreducible of dimension d and $r, s \in \mathbb{Z}$, consider a morphism

$$\alpha : h(X)(r) \to h(Y)(s),$$

where $\alpha \in \mathrm{CH}^{d+s-r}(X \times_k Y)_{\mathbb{Q}}$. Notice that, analogously to remark 2.1.8, using Künneth formula and Poincaré duality axioms, we have the following isomorphisms of K-vector spaces

$$\begin{aligned}
\mathrm{H}^{2(d+s-r)}(X \times_k Y)(d+s-r) &\cong \oplus_i \mathrm{H}^{2(d+r)-i}(X)(d+r) \otimes \mathrm{H}^{i+2s}(Y)(s) \cong \\
&\cong \oplus_i \mathrm{H}^{i+2r}(X)(r)^{\vee} \otimes \mathrm{H}^{i+2s}(Y)(s) \cong \\
&\cong \oplus_i \mathrm{Hom}_K(\mathrm{H}^{i+2r}(X)(r), \mathrm{H}^{i+2s}(Y)(s)) \cong \\
&\cong \mathrm{Hom}_{grVect_K}(\mathrm{H}^*(X)(r), \mathrm{H}^*(Y)(s)).
\end{aligned}$$

It holds that the realization of α is given by its image along the composition with the cycle class map

$$\mathrm{CH}^{d+s-r}(X \times_k Y)_{\mathbb{Q}} \xrightarrow{cl^{d+s-r}_{X \times_k Y}} H^{2(d+s-r)}(X \times_k Y)(d+s-r) \cong \mathrm{Hom}_{grVect_K}(\mathrm{H}^*(X)(r), \mathrm{H}^*(Y)(s)). \tag{2.7}$$

For general X we extend by $\mathbb{Q}$-linearity. For the explicit description of the realization of a general morphisms in $\mathrm{CHM}(k;\mathbb{Q})$

$$ph(X)(r) \to qh(Y)(s)$$

we refer to the description given in the second step.

Remark 2.1.23. Given H^* a Weil cohomology over k with coefficients in K, we used Chow groups and cycle class maps to define the category of Chow motives and its realization functor. Viceversa, we can recover Chow groups and cycle class maps from the category of Chow motives as follows. Given $X \in SmProj_k$ irreducible of dimension d, the Chow group of codimension i with rational coefficients can be obtained as hom-sets in the category of Chow motives:

$$\mathrm{CH}^i(X)_{\mathbb{Q}} \cong \mathrm{CH}^i(Spec(k) \times_k X)_{\mathbb{Q}} \cong \mathrm{Hom}_{\mathrm{CHM}(k;\mathbb{Q})}(\mathbb{1}, h(X)(i)),$$

where the first isomorphism is the pullback map pr_X^* on Chow groups. By the above discussion, the map induced on hom-sets by the realization functor

$$\mathrm{Hom}_{\mathrm{CHM}(k;\mathbb{Q})}(\mathbb{1}, h(X)(i)) \to \mathrm{Hom}_{grVect_K}(K, \mathrm{H}^*(X)(i))$$

is given by the cycle class map

$$cl^i_{Spec(k)\times_k X} : \mathrm{CH}^i(Spec(k) \times_k X)_{\mathbb{Q}} \to \mathrm{H}^{2i}(Spec(k) \times_k X)(i).$$

By naturality axiom of the cycle class map, this corresponds, via the isomorphisms given by pullback maps of pr_X, to the cycle class map

$$cl^i_X : \mathrm{CH}^{2i}(X)_{\mathbb{Q}} \to \mathrm{H}^{2i}(X)(i).$$

So, the cycle class maps can be recovered as morphisms induced on hom-sets by the realization functor.

2.2 Mixed motives

2.2.1 A triangulated category of mixed motives

We construct a triangulated category of mixed motives over a field. We follow [Ayo13, §2.1] and the general theory developed in [Ayo07, §4].

Recall that we denote by Sm_k the category of smooth algebraic varieties over k. Notice that, since any object of Sm_k is union of prime spectra of a finitely generated k-algebras, which form a set, then Sm_k is an essentially small category. Consider the category of presheaves of Λ-modules over Sm_k

$$\mathrm{PSh}(Sm_k; \Lambda).$$

This category inherits from the category of Λ-modules an open-wise structure of closed monoidal Λ-linear abelian category. We denote by $\_ \otimes \_$ the tensor product. The unit object is given by Λ, the constant presheaf.

We have a covariant functor

$$\mathbf{\Lambda} : Sm_k \to \mathrm{PSh}(Sm_k; \Lambda),$$

which assigns to each $X \in Sm_k$ the presheaf of Λ-modules over Sm_k represented by X

$$\mathbf{\Lambda}(X) : U \mapsto \Lambda[\mathrm{Hom}_{Sm_k}(U, X)],$$

where $\Lambda[\mathrm{Hom}_{Sm_k}(U, X)]$ denotes the free Λ-module generated by the set $\mathrm{Hom}_{Sm_k}(U, X)$. By Yoneda Lemma and universal property of free Λ-modules, it is such that, for any

$F \in \mathrm{PSh}(Sm_k; \Lambda)$,

$$\mathrm{Hom}_{\mathrm{PSh}(Sm_k;\Lambda)}(\mathbf{\Lambda}(X), F) \cong F(X).$$

If we consider on Sm_k the monoidal structure given by the fiber product over k and unit object $Spec(k)$, then $\mathbf{\Lambda}$ is a monoidal functor. Indeed, for any $U \in Sm_k$,

$$\Lambda[\mathrm{Hom}_{Sm_k}(U, Spec(k))] \cong \Lambda,$$

so $\mathbf{\Lambda}(Spec(k)) \cong \Lambda$, and for any $X, Y \in Sm_k$,

$$\begin{aligned} \Lambda[\mathrm{Hom}_{Sm_k}(U, X \times_k Y)] &\cong \Lambda[\mathrm{Hom}_{Sm_k}(U, X) \times \mathrm{Hom}_{Sm_k}(U, Y)] \cong \\ &\cong \Lambda[\mathrm{Hom}_{Sm_k}(U, X)] \otimes_\Lambda \Lambda[\mathrm{Hom}_{Sm_k}(U, Y)], \end{aligned}$$

so $\mathbf{\Lambda}(X \times_k Y) \cong \mathbf{\Lambda}(X) \otimes \mathbf{\Lambda}(Y)$.

We consider the category of (unbounded) complexes in $\mathrm{PSh}(Sm_k; \Lambda)$

$$\mathrm{Ch}(\mathrm{PSh}(Sm_k; \Lambda)).$$

This category inherits from the category $\mathrm{PSh}(Sm_k; \Lambda)$ a structure of closed monoidal Λ-linear abelian category. We denote by $\_ \otimes \_$ the tensor product. The unit object is given by Λ, the constant presheaf concentrated in degree 0.

We have a covariant monoidal functor

$$\mathbf{\Lambda} : Sm_k \to \mathrm{Ch}(\mathrm{PSh}(Sm_k; \Lambda)),$$

which assigns to any $X \in Sm_k$ the complex given by $\mathbf{\Lambda}(X)$ concentrated in degree 0.

We want to force on objects of $\mathrm{Ch}(\mathrm{PSh}(Sm_k; \Lambda))$ the étale descent and the $\mathbb{A}^1$-homotopy invariance properties. We will do it by localizing the category $\mathrm{Ch}(\mathrm{PSh}(Sm_k; \Lambda))$ with respect to a suitable set of morphisms corresponding to these properties. Naively, this means that we force some morphisms to be isomorphisms. The technical tool we will use to localize is localization of model categories (see [Hir03, §3]).

So, first we need to define a model structure on $\mathrm{Ch}(\mathrm{PSh}(Sm_k; \Lambda))$. We denote by $\mathrm{Ch}(\Lambda)$ the category of (unbounded) complexes of Λ-modules. By [Hov99, §2.3], $\mathrm{Ch}(\Lambda)$ has a model structure such that:

- weak-equivalences are quasi-isomorphisms of complexes of Λ-modules,
- fibrations are epimorphisms of complexes of Λ-modules, i.e. level-wise surjective morphisms of Λ-modules,
- cofibrations are characterized by having the left lifting property with respect to trivial fibrations.

Since the weak-equivalences are the quasi-isomorphisms of chain complexes, the homotopy category of $\mathrm{Ch}(\Lambda)$ with respect to this model structure is equivalent to the (unbounded) derived category of Λ-modules

$$\mathrm{Ho}(\mathrm{Ch}(\Lambda)) \simeq \mathrm{D}(\Lambda).$$

Moreover, this model structure on $\mathrm{Ch}(\Lambda)$ is a stable model structure. The suspension functor is given by the total left derived functor of the shift functor

$$\begin{aligned} [1] : \mathrm{Ch}(\Lambda) &\to \mathrm{Ch}(\Lambda) \\ A_\bullet &\mapsto A_{\bullet-1}, \end{aligned}$$

which is already an equivalence of categories and it is exact, hence it induces an autoequivalence on the homotopy category. The quasi inverse is denoted by $[-1]$ and, for any $p \in \mathbb{Z}$, the p^{th}-iteration of $[1]$ is denoted by $[p]$. Now, notice that

$$\mathrm{Ch}(\mathrm{PSh}(Sm_k; \Lambda)) \cong \mathrm{PSh}(Sm_k; \mathrm{Ch}(\Lambda)),$$

that is, we can think objects of $\mathrm{Ch}(\mathrm{PSh}(Sm_k; \Lambda))$ also as presheaves of complexes of Λ-modules over Sm_k. We have two ways to induce a model structure on $\mathrm{Ch}(\mathrm{PSh}(Sm_k; \Lambda))$ starting from the one on $\mathrm{Ch}(\Lambda)$ described above (see [Ayo07, Def. 4.4.15, Prop. 4.4.16], which can be applied by [Ayo07, Ex. 4.4.24]). We consider the following.

Definition 2.2.1. We define the *projective global model structure*, the model structure on $\mathrm{Ch}(\mathrm{PSh}(Sm_k; \Lambda))$ such that:

- *projective global weak-equivalences* are open-wise weak-equivalences in $\mathrm{Ch}(\Lambda)$,
- *projective global fibrations* are open-wise fibrations in $\mathrm{Ch}(\Lambda)$,
- *projective global cofibrations* are characterized by having the left lifting property with respect to trivial fibrations.

Notice that a morphism in $\mathrm{Ch}(\mathrm{PSh}(Sm_k; \Lambda))$

$$F_\bullet \to G_\bullet$$

is a projective global weak-equivalence (resp. projective global fibration) if and only if, for any $X \in Sm_k$,

$$F_\bullet(X) \to G_\bullet(X)$$

is a weak-equivalence (resp. fibration) in $\mathrm{Ch}(\Lambda)$. In other words, projective global weak-equivalences and projective global fibrations can be checked on global sections. This is why it is called projective global model structure.

Remark 2.2.2. Since exactness in the abelian category $\mathrm{PSh}(Sm_k; \Lambda)$ is open-wise exactness and the notion of a quasi-isomorphism of complexes in an abelian category relies on the notion of exactness, then projective global weak-equivalences are quasi-isomorphisms of chain complexes in the abelian category $\mathrm{PSh}(Sm_k;\Lambda)$. Hence, the homotopy category of $\mathrm{Ch}(\mathrm{PSh}(Sm_k;\Lambda))$ with respect to the projective global model structure is equivalent to the derived category of the abelian category $\mathrm{PSh}(Sm_k;\Lambda)$

$$\mathrm{Ho}(\mathrm{Ch}(\mathrm{PSh}(Sm_k;\Lambda))) \simeq \mathrm{D}(\mathrm{PSh}(Sm_k;\Lambda)).$$

Remark 2.2.3. Since $\mathrm{Ch}(\Lambda)$ is a stable model category, then also the projective global model structure on $\mathrm{Ch}(\mathrm{PSh}(Sm/k;\Lambda))$ is stable (see [Ayo07, Cor. 4.4.21]). The suspension functor is given by the total left derived functor of the shift functor

$$\begin{aligned}[1] : \mathrm{Ch}(\mathrm{PSh}(Sm_k;\Lambda)) &\to \mathrm{Ch}(\mathrm{PSh}(Sm_k;\Lambda)) \\ F_\bullet &\mapsto F_{\bullet-1},\end{aligned}$$

which is already an equivalence of categories and it is exact, hence induces an autoequivalence on the homotopy category. We use the same notations as in $\mathrm{Ch}(\Lambda)$ for the quasi-inverse $[-1]$ and p^{th}-iteration of the shift functor $[p]$.

Now, we define the set of morphisms in $\mathrm{Ch}(\mathrm{PSh}(Sm_k;\Lambda))$ that we want to invert. We start describing the morphisms corresponding to the étale descent property. We need the following definition.

Definition 2.2.4. The *big étale site* on Sm_k is the site with underlying category Sm_k and covering families of an object $X \in Sm_k$ the étale covers of X (definition 1.4.11).

Remark 2.2.5. Recall that we defined an étale cover of a scheme X as a family of étale morphisms of schemes $\{f_i : U_i \to X\}_{i\in I}$, such that $\cup_{i\in I} f_i(U_i) = X$. Taking the disjoint union of the morphisms f_i, we get a surjective étale morphism onto X. Viceversa, a surjective étale morphism onto X forms an étale cover. So, equivalently, an étale cover of X is a surjective étale morphism $f : U \to X$.

We need the notion of hypercovers in the étale toplogy. A reference is [AM69, §8].

Definition 2.2.6. Given X a scheme, an *étale hypercover* of X is a simplicial object in $X_{ét}$, the small étale site over X, augmented in X

$$\begin{tikzcd} \cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & U_2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & U_1 \arrow[r, shift left=1] \arrow[r, shift right=1] & U_0 \arrow[d] \\ & & & X, \end{tikzcd}$$

denoted by $U_* \to X$, such that $U_0 \to X$ is an étale cover of X and, for any $n \geq 0$,

$$U_{n+1} \to (cosk_n tr_n U_*)_{n+1} \tag{2.8}$$

is an étale cover of $(cosk_n tr_n U_*)_{n+1}$.

Recall that the n^{th}-coskeleton functor is right adjoint to the n^{th}-truncation functor

$$tr_n : X_{\acute{e}t}^{\Delta^{op}} \rightleftarrows X_{\acute{e}t}^{\Delta^{op}_{\leq n}} : cosk_n.$$

It exists because $X_{\acute{e}t}$ has all finite products and pullbacks, hence all finite limits. The unit of the adjunction gives the morphism of simplicial objects in $X_{\acute{e}t}$, for any $U_* \in X_{\acute{e}t}^{\Delta^{op}}$,

$$U_* \to cosk_n tr_n U_*,$$

The morphism 2.8 is the morphism between the $(n+1)$-simplexes.

Remark 2.2.7. Étale hypercovers can be thought as a generalization of the Čech nerve $\mathcal{N}U_*$

$$\begin{tikzcd} \cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & U \times_X U \times_X U \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & U \times_X U \arrow[r, shift left=1] \arrow[r, shift right=1] & U \arrow[d] \\ & & & X \end{tikzcd}$$

of an étale cover $U \to X$. Indeed, thinking at U as a simplicial object in $X_{\acute{e}t}$ truncated at level 0, then it holds that (see [AM69, Rmk. 8.5])

$$cosk_0 U \cong \mathcal{N}U_*.$$

Using the universal property of the coskeleton, we deduce that, for any $n \geq 0$,

$$\mathcal{N}U_{n+1} \cong (cosk_n tr_n \mathcal{N}U_*)_{n+1}.$$

Hence, $\mathcal{N}U_* \to X$ is an étale hypercover of X. Viceversa, Čech nerves are exactly those étale hypercovers characterized by having morphisms 2.8, which are isomorphisms. Intuitively, the Čech nerve of an étale cover is the datum of an étale cover and the étale covers given by its successive auto-intersections (i.e. fiber products over X). While, an étale hypercover is the datum of an étale cover and compatible refinements of its successive auto-intersections.

Étale hypercovers give a combinatorial way to compute sheaf cohomology of étale abelian sheaves. Recall that, given $F \in \mathrm{Ab}(X_{\acute{e}t})$, Čech cohomology of X with coefficients in F is a combinatorial notion of cohomology for presheaves. Recall that it is defined as follows. Given $U \to X$ an étale cover of X, consider the simplicial object in $X_{\acute{e}t}$ given by its Čech nerve $\mathcal{N}U_*$. Applying the presheaf F, we obtain the cosimplicial abelian group

$$\begin{tikzcd} F(U) \arrow[r, shift left=1] \arrow[r, shift right=1] & F(U \times U) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & F(U \times U \times U) \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \cdots. \end{tikzcd}$$

Let $F(\mathcal{N}U_*)$ denote the corresponding complex of abelian groups obtained via Dold-Kan correspondence. It is quasi-isomorphic to the Čech complex of F relative to U. The Čech cohomology of X with coefficients in F is defined as

$$\check{\mathrm{H}}^i(X;F) \coloneqq \varinjlim_{U\to X} \mathrm{H}^i(F(\mathcal{N}U_*)),$$

where the direct limit runs over all étale covers of X. This combinatorial notion of cohomology does not always coincide with sheaf cohomology of X with coefficients in F

$$\mathbb{H}^i(X_{et},F) \coloneqq \mathrm{H}^i(\mathbf{R}\Gamma(X,F)).$$

However, if we replace étale covers with étale hypercovers, they do. Notice that, given an étale hypercover $U_* \to X$ of X, analogously to $F(\mathcal{N}U_*)$, we can construct the complex of abelian groups $F(U_*)$.

Theorem 2.2.8 (Verdier's hypercovering Theorem)**.** *Let X be a scheme and $F \in Ab(X_{ét})$. For any $i \geq 0$, there exists a canonical isomorphism*

$$\mathbb{H}^i(X_{ét},F) \cong \varinjlim_{U_*\to X} H^i(F(U_*)),$$

where the direct limit runs over all étale hypercovers of X.

Proof. See [AM69, Thm. 8.16]. □

Remark 2.2.9. By a spectral sequences argument, we deduce that the same isomoprhism holds more generally for $F_\bullet \in \mathrm{Ch}(Ab(X_{ét}))$

$$\mathbb{H}^i(X_{ét},F_\bullet) \cong \varinjlim_{U_*\to X} \mathrm{H}^i(Tot^{\oplus}F_\bullet(U_*)).$$

Now, we come back to our construction. Given an étale hypercover $U_* \to X$ of X, applying the covariant functor $\mathbf{\Lambda}$, we get the augmented simplicial object in $\mathrm{PSh}(Sm_k;\Lambda)$

$$\begin{tikzcd} \cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \mathbf{\Lambda}(U_2) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathbf{\Lambda}(U_1) \arrow[r, shift left=1] \arrow[r, shift right=1] & \mathbf{\Lambda}(U_0) \arrow[d] \\ & & & \mathbf{\Lambda}(X). \end{tikzcd}$$

Via Dold-Kan correspondence, we get the augmented complex in $\mathrm{PSh}(Sm_k;\Lambda)$

$$\begin{tikzcd} \cdots \arrow[r] & \mathbf{\Lambda}(U_2) \arrow[r] & \mathbf{\Lambda}(U_1) \arrow[r] & \mathbf{\Lambda}(U_0) \arrow[d] \\ & & & \mathbf{\Lambda}(X). \end{tikzcd}$$

We denote it by

$$\mathbf{\Lambda}(U_*) \to \mathbf{\Lambda}(X). \tag{2.9}$$

Thought as morphisms in $\mathrm{Ch}(\mathrm{PSh}(Sm_k;\Lambda))$, these are the morphisms corresponding to the étale descent property.

The morphisms corresponding to the $\mathbb{A}^1$-homotopy invariance property are, for any $X \in Sm_k$,

$$\mathbf{\Lambda}(\mathbb{A}^1_X) \to \mathbf{\Lambda}(X), \tag{2.10}$$

obtained by applying the covariant functor $\mathbf{\Lambda}$ to the canonical projection of the fiber product

$$\mathbb{A}^1_X := \mathbb{A}^1_k \times_k X \to X.$$

We consider the following sets of morphisms in $\mathrm{Ch}(\mathrm{PSh}(Sm_k;\Lambda))$ given by all shifts of morphisms of the kind 2.10 and 2.9:

$$\begin{aligned}
&S_{\acute{e}t} := \{\mathbf{\Lambda}(U_*)[n] \to \mathbf{\Lambda}(X)[n] \mid X \in Sm_k,\ U_* \to X \text{ étale hypercover},\ n \in \mathbb{Z}\},\\
&S_{\mathbb{A}^1} := \{\mathbf{\Lambda}(\mathbb{A}^1_X)[n] \to \mathbf{\Lambda}(X)[n] \mid X \in Sm_k,\ n \in \mathbb{Z}\},\\
&S_{(\mathbb{A}^1,\acute{e}t)} := S_{\mathbb{A}^1} \cup S_{\acute{e}t}.
\end{aligned}$$

Notice that, since $\mathrm{Ch}(\Lambda)$ is a left proper cellular model category, then also $\mathrm{Ch}(\mathrm{PSh}(Sm_k;\Lambda))$ is, with the global projective model structure (see [Hir03, Prop. 4.1.5]). Being left proper and cellular are some technical properties that ensure the existence of left Bousfield localizations with respect to a set of morphisms (see [Hir03, Thm. 4.1.1]). Hence, there exists the left Bousfield localization of $\mathrm{Ch}(\mathrm{PSh}(Sm_k;\Lambda))$ with respect to the set of morphisms $S_{(\mathbb{A}^1,\acute{e}t)}$

$$\mathrm{L}_{S_{(\mathbb{A}^1,\acute{e}t)}}\mathrm{Ch}(\mathrm{PSh}(Sm_k;\Lambda)).$$

It is a model category with model structure, called the *projective* $(\mathbb{A}^1, \acute{e}t)$*-local model structure*, such that:

- the underlying category is $\mathrm{Ch}(\mathrm{PSh}(Sm_k;\Lambda))$,

- weak-equivalences are the $S_{(\mathbb{A}^1,\acute{e}t)}$-local equivalences (see [Hir03, Def. 3.1.4]), called the *projective* $(\mathbb{A}^1, \acute{e}t)$*-local weak equivalences*,

- cofibrations are the same of $\mathrm{Ch}(\mathrm{PSh}(Sm_k;\Lambda))$ with projective global model structure,

- fibrations are characterized by having the right lifting property with respect to trivial cofibrations.

Recall that, by definition of localization of a model category, the class of $(\mathbb{A}^1, \acute{e}t)$-local weak-equivalences contains the projective global weak-equivalences and all the morphisms in $S_{(\mathbb{A}^1,\acute{e}t)}$. Recall that a left Bousfield localization is a left localization of model categories. That is, $\mathrm{L}_{S_{(\mathbb{A}^1,\acute{e}t)}}\mathrm{Ch}(\mathrm{PSh}(Sm_k;\Lambda))$ is the universal model category with a Quillen pair

$$\mathrm{Ch}(\mathrm{PSh}(Sm_k;\Lambda)) \rightleftarrows \mathrm{L}_{S_{(\mathbb{A}^1,\acute{e}t)}}\mathrm{Ch}(\mathrm{PSh}(Sm_k;\Lambda)),$$

such that the left Quillen functor maps morphisms in $S_{(\mathbb{A}^1,\acute{e}t)}$ into $(\mathbb{A}^1,\acute{e}t)$-local weak-equivalences. Moreover, the Quillen functors are given by identities. The above Quillen pair induces the total derived adjunction

$$\mathrm{Ho}(\mathrm{Ch}(\mathrm{PSh}(Sm_k;\Lambda))) \rightleftarrows \mathrm{Ho}(\mathrm{L}_{S_{(\mathbb{A}^1,\acute{e}t)}}\mathrm{Ch}(\mathrm{PSh}(Sm_k;\Lambda))).$$

Definition 2.2.10. The homotopy category of a left localization of $\mathrm{Ch}(\mathrm{PSh}(Sm_k;\Lambda))$ with respect to $S_{(\mathbb{A}^1,\acute{e}t)}$

$$\mathbf{DA}^{\mathrm{eff}}_{\acute{e}t}(k;\Lambda) \coloneqq \mathrm{Ho}(\mathrm{L}_{S_{(\mathbb{A}^1,\acute{e}t)}}\mathrm{Ch}(\mathrm{PSh}(Sm_k;\Lambda)))$$

is called the *category of effective étale motivic sheaves.*

Remark 2.2.11. The shift functor

$$\begin{aligned}[1] : \mathrm{L}_{S_{(\mathbb{A}^1,\acute{e}t)}}\mathrm{Ch}(\mathrm{PSh}(Sm_k;\Lambda)) &\to \mathrm{L}_{S_{(\mathbb{A}^1,\acute{e}t)}}\mathrm{Ch}(\mathrm{PSh}(Sm_k;\Lambda)) \\ F &\mapsto F_{\bullet-1},\end{aligned}$$

which is already an equivalence of categories, is exact, hence it induces an autoequivalence on the homotopy category. So, $\mathbf{DA}^{\mathrm{eff}}_{\acute{e}t}(k;\Lambda)$ is a triangulated category. Moreover, the monoidal structure over $\mathrm{Ch}(\mathrm{PSh}(Sm_k;\Lambda))$ is compatible with the projective $(\mathbb{A}^1,\acute{e}t)$-local model structure, hence $\mathbf{DA}^{\mathrm{eff}}_{\acute{e}t}(k;\Lambda)$ is also a monoidal category. We denote the tensor product by $\_ \otimes \_$. The unit object is the constant presheaf $\Lambda_{cst} \cong \mathbf{\Lambda}(Spec(k))$.

The following remark gives an alternative useful description of $\mathbf{DA}^{\mathrm{eff}}_{\acute{e}t}(k;\Lambda)$.

Remark 2.2.12. We defined $\mathbf{DA}^{\mathrm{eff}}_{\acute{e}t}(k;\Lambda)$ as the homotopy category of a left Bousfield localization of a model category. Equivalently, it can be described as the Verdier quotient of a triangulated category (see [Nee01, §2]), which moreover is a left Bousfield localization of triangulated categories (see [Nee01, §9]). First, notice that we can decompose the left Bousfield localization with respect to $S_{(\mathbb{A}^1,et)}$ into two steps, localizing first with respect to $S_{\acute{e}t}$ and then with respect to $S_{\mathbb{A}^1}$. Indeed, consider the left Bousfield localization of $\mathrm{Ch}(\mathrm{PSh}(Sm_k;\Lambda))$ with respect to the set of morphisms $S_{\acute{e}t}$

$$\mathrm{L}_{S_{\acute{e}t}}\mathrm{Ch}(\mathrm{PSh}(Sm_k;\Lambda)).$$

The model structure on this category is called the *projective ét-local model structure.* Since left Bousfield localizations of left proper cellular model categories are again left proper and cellular (see [Hir03, Thm. 4.1.1]), then there exists the left Bousfield localization of $\mathrm{L}_{S_{\acute{e}t}}\mathrm{Ch}(\mathrm{PSh}(Sm_k;\Lambda))$ with respect to the set of morphisms $S_{\mathbb{A}^1}$

$$\mathrm{L}_{S_{\mathbb{A}^1}}\mathrm{L}_{S_{\acute{e}t}}\mathrm{Ch}(\mathrm{PSh}(Sm_k;\Lambda)).$$

By definition of weak-equivalences and cofibrations in left Bousfield localizations, we see that this model category is exactly the left Bousfield localization of $\mathrm{Ch}(\mathrm{PSh}(Sm_k;\Lambda))$ with respect to the set of morphisms $S_{(\mathbb{A}^1,\acute{e}t)}$, so the identity defines a Quillen equivalence

$$\mathrm{L}_{S_{\mathbb{A}^1}}\mathrm{L}_{S_{\acute{e}t}}\mathrm{Ch}(\mathrm{PSh}(Sm_k;\Lambda)) \simeq \mathrm{L}_{S_{\mathbb{A}^1,\acute{e}t}}\mathrm{Ch}(\mathrm{PSh}(Sm_k;\Lambda)).$$

Hence, we have the Quillen pairs

$$\mathrm{Ch}(\mathrm{PSh}(Sm_k;\Lambda)) \rightleftarrows \mathrm{L}_{S_{\acute{e}t}}\mathrm{Ch}(\mathrm{PSh}(Sm_k;\Lambda)) \rightleftarrows \mathrm{L}_{S_{(\mathbb{A}^1,\acute{e}t)}}\mathrm{Ch}(\mathrm{PSh}(Sm_k;\Lambda)).$$

They induce the total derived adjunctions

$$\mathbf{D} \rightleftarrows \mathbf{D}_{\acute{e}t} \rightleftarrows \mathbf{DA}^{\mathrm{eff}}_{\acute{e}t}(k;\Lambda), \tag{2.11}$$

where we denote by

$$\mathbf{D} \coloneqq \mathrm{Ho}(\mathrm{Ch}(\mathrm{PSh}(Sm_k;\Lambda))) \simeq \mathrm{D}(\mathrm{PSh}(Sm_k;\Lambda))$$

and

$$\mathbf{D}_{\acute{e}t} \coloneqq \mathrm{Ho}(\mathrm{L}_{S_{\acute{e}t}}\mathrm{Ch}(\mathrm{PSh}(Sm_k;\Lambda))).$$

Using Verdier's hypercovering Theorem, we see that (see [Vez18, Prop. 3.10]) $\mathbf{D}_{\acute{e}t}$ is equivalent to the derived category of $\mathrm{Sh}_{\acute{e}t}(Sm_k;\Lambda)$, the category of étale sheaves of Λ-modules over Sm_k, i.e. sheaves of Λ-modules over the big étale site on Sm_k,

$$\mathbf{D}_{\acute{e}t} \simeq \mathrm{D}(\mathrm{Sh}_{\acute{e}t}(Sm_k;\Lambda)).$$

With this description, the adjunction

$$\mathbf{D} \underset{}{\overset{a_{\acute{e}t}}{\rightleftarrows}} \mathbf{D}_{\acute{e}t}$$

is the total derived of the adjunction étale sheafification-inclusion. Hence, also $\mathbf{D}_{\acute{e}t}$ is a triangulated category. Now, let $\mathcal{E}_{\mathbb{A}^1}$ be the triangulated subcategory of $\mathbf{D}_{\acute{e}t}$ generated by the objects

$$\mathcal{E}_{\mathbb{A}^1} \coloneqq \langle cone(f) \mid f : \mathbf{\Lambda}_{\acute{e}t}(\mathbb{A}^1_X) \to \mathbf{\Lambda}_{\acute{e}t}(X),\ X \in Sm_k\rangle,$$

where $\mathbf{\Lambda}_{\acute{e}t}(\mathbb{A}^1_X)$ and $\mathbf{\Lambda}_{\acute{e}t}(X)$ denote the étale sheafification of the presheves $\mathbf{\Lambda}(\mathbb{A}^1_X)$ and $\mathbf{\Lambda}(X)$ respectively. It holds that (see [Ayo14a]) $\mathbf{DA}^{\mathrm{eff}}_{\acute{e}t}(k;\Lambda)$ is equivalent to the

Verdier quotient of $\mathbf{D}_{ét}$ over $\mathcal{E}_{\mathbb{A}^1}$

$$\mathbf{DA}^{\mathrm{eff}}_{ét}(k;\Lambda) \simeq \mathbf{D}_{ét}/\mathcal{E}_{\mathbb{A}^1} \simeq \mathbf{D}_{ét}[\mathcal{W}^{-1}_{\mathbb{A}^1}],$$

where

$$\mathcal{W}_{\mathbb{A}^1} \coloneqq \{\alpha \in \mathbf{D}_{ét} \mid cone(\alpha) \in \mathcal{E}_{\mathbb{A}^1}\},$$

is called the class of $\mathbb{A}^1$-*local weak-equivalences*. By the adjunction 2.11, the Verdier quotient functor

$$\mathbf{D}_{ét} \to \mathbf{D}_{ét}/\mathcal{E}_{\mathbb{A}^1} \simeq \mathbf{DA}^{\mathrm{eff}}_{ét}(k;\Lambda)$$

admits a right-adjoint. So, this Verdier quotient is also a left Bousfield localization of triangulated categories. By general theory of Bousfield localization of triangulated categories, the category $\mathbf{DA}^{\mathrm{eff}}_{ét}(k;\Lambda)$ can be identified with the full triangulated subcategory of $\mathbf{D}_{ét}$ of $\mathcal{E}_{\mathbb{A}^1}$-local objects, called $\mathbb{A}^1$-*local objects*,

$$^{\perp}\mathcal{E}_{\mathbb{A}^1} \coloneqq \{G_\bullet \in \mathbf{D}_{ét} \mid \mathrm{Hom}_{\mathbf{D}}(F_\bullet, G_\bullet) = 0,\ \forall F_\bullet \in \mathcal{E}_{ét}\}.$$

Notice that, by definiton of $\mathcal{E}_{\mathbb{A}^1}$ equivalently, $G_\bullet \in \mathbf{D}_{ét}$ is an $\mathbb{A}^1$-local object if, for any $X \in Sm_k$ and $n \in \mathbb{Z}$,

$$\mathrm{Hom}_{\mathbf{D}_{ét}}(\mathbf{\Lambda}_{ét}(X)[n], G_\bullet) \cong \mathrm{Hom}_{\mathbf{D}_{ét}}(\mathbf{\Lambda}_{ét}(\mathbb{A}^1_X)[n], G_\bullet).$$

Notice that

$$\begin{aligned}\mathrm{Hom}_{\mathbf{D}_{ét}}(\mathbf{\Lambda}_{ét}(X)[n],\ \_\,) &\cong \mathrm{H}^0\mathbf{R}\,\mathrm{Hom}_{Sh_{ét}(Sm_k,\Lambda)}(\mathbf{\Lambda}_{ét}(X)[n],\ \_\,) \cong \\ &\cong \mathrm{H}^{-n}\mathbf{R}\Gamma(X,\ \_\,) \cong \mathbb{H}^{-n}(X,\ \_\,) \cong \mathbb{H}^{-n}(X_{ét},\ \_\,),\end{aligned}$$

where the last isomorphism holds because sheaf cohomology computed over the small or big étale site of X is the same (see [Stacks, Tag 03YX]). Hence, being $\mathbb{A}^1$-local for $G_\bullet$ is equivalent to ask that

$$\mathbb{H}^{-n}(X_{ét}, G_\bullet) \cong \mathbb{H}^{-n}((\mathbb{A}^1_X)_{ét}, G_\bullet),$$

where here, by $G_\bullet$, we mean the restriction of $G_\bullet$ to the small étale sites.

By composition of the monoidal functor $\mathbf{\Lambda}$ with the canonical morphisms into the left localization and the homotopy category, we get the covariant functor

$$\mathbf{M} : Sm_k \xrightarrow{\mathbf{\Lambda}} \mathrm{Ch}(\mathrm{PSh}(Sm_k;\Lambda)) \to \mathrm{L}_{S_{(\mathbb{A}^1,ét)}}\mathrm{Ch}(\mathrm{PSh}(Sm_k;\Lambda)), \to \mathbf{DA}^{\mathrm{eff}}_{ét}(k;\Lambda).$$

which is monoidal, since the monoidal structure on $\mathrm{Ch}(\mathrm{PSh}(Sm_k;\Lambda))$ is compatible with the projective $(\mathbb{A}^1, ét)$-local model structure.

Remark 2.2.13. By the equivalent description of $\mathbf{DA}^{\mathrm{eff}}_{ét}(k;\Lambda)$ in remark 2.2.12, equivalently $\mathbf{M}$ is given by the composition of the functor $\mathbf{\Lambda}$, the étale sheafification functor $a_{ét}$, the localization functor into the derived category and the Verdier quotient functor

$$\mathbf{M} : Sm_k \xrightarrow{\mathbf{\Lambda}} \mathrm{Ch}(\mathrm{PSh}(Sm_k;\Lambda)) \xrightarrow{a_{ét}} \mathrm{Ch}(\mathrm{Sh}_{ét}(Sm_k;\Lambda)) \to \mathbf{D}_{ét} \to \mathbf{DA}^{\mathrm{eff}}_{ét}(k;\Lambda).$$

Definition 2.2.14. Given $X \in Sm_k$, the object $\mathbf{M}(X) \in \mathbf{DA}^{\mathrm{eff}}_{ét}(k;\Lambda)$ is called the *effective motive associated* to X.

Definition 2.2.15. Given $X \in Sm_k$ and $Y \subset X$ a smooth closed subvariety, we define the *effective motive of the pair* (X, Y)

$$\mathbf{M}(X, Y) \coloneqq cone(\mathbf{M}(Y) \to \mathbf{M}(X)).$$

Let $x \in \mathbb{G}_m$ be a k-rational point, i.e. a closed point $x : Spec(k) \to \mathbb{G}_m$. We define the *Tate motive*

$$\Lambda(1) \coloneqq \mathbf{M}(\mathbb{G}_m, x)[-1]$$

and, for any $q \geq 0$,

$$\Lambda(q) \coloneqq \Lambda(1)^{\otimes q}.$$

For any $M \in \mathbf{DA}^{\mathrm{eff}}_{ét}(k;\Lambda)$, we denote by

$$M(q) \coloneqq M \otimes \Lambda(q),$$

called the q^{th}-*Tate twist* of M.

Definition 2.2.16. Given an object $M \in \mathbf{DA}^{\mathrm{eff}}_{ét}(k;\Lambda)$, we define, for any $p \in \mathbb{Z}$ and $q \geq 0$,

$$\mathrm{H}^p(M, \Lambda(q)) \coloneqq \mathrm{Hom}_{\mathbf{DA}^{\mathrm{eff}}_{ét}(k;\Lambda)}(M, \Lambda(q)[p])$$

the *étale motivic cohomology groups* of M. In other words, étale motivic cohomology groups are represented by the objects

$$\Lambda(q) \in \mathbf{DA}^{\mathrm{eff}}_{ét}(k;\Lambda)$$

in the category $\mathbf{DA}^{\mathrm{eff}}_{ét}(k;\Lambda)$. For $M = \mathbf{M}(X)$ the effective motive associated to $X \in Sm_k$, we write

$$\mathrm{H}^p(X, \Lambda(q)) \coloneqq \mathrm{H}^p(\mathbf{M}(X), \Lambda(q)),$$

called the *étale motivic cohomology groups* of X.

We only mention that there's a further construction which gains the *category of étale motivic sheaves* (see [Ayo13, §2.1])

$$\mathbf{DA}_{ét}(k, \Lambda).$$

It is a monoidal triangulated category, which contains $\mathbf{DA}^{\mathrm{eff}}_{ét}(k;\Lambda)$ as a triangulated subcategory. Intuitively, it can be thought as the analogous of the construction of the category of Chow motives obtained from the category of effective Chow motives. Indeed it is the triangulated category obtained from $\mathbf{DA}^{\mathrm{eff}}_{ét}(k;\Lambda)$ by inverting the Tate motive $\Lambda(1)$ with respect to the tensor product. So, in $\mathbf{DA}_{ét}(k,\Lambda)$ we also have the object

$$\Lambda(-1) \coloneqq \Lambda(1)^{\otimes -1}.$$

and hence also $\Lambda(q)$ for negative q. It can be proved that the inversion of the Tate motive is sufficient to have duals for any object $M \in \mathbf{DA}_{ét}(k;\Lambda)$. So, $\mathbf{DA}_{ét}(k;\Lambda)$ is a rigid monoidal triangulated category.

One of the reasons why these triangulated categories are considered good candidate triangulated categories of mixed motives is that they are well related to the category of Chow motives. Moreover, motivic cohomology groups are related to Chow groups. Some of these relations are resumed in the following theorem.

Theorem 2.2.17. *If $\mathbb{Q} \subset \Lambda$ and $\mathbb{Q} \subset k$, then there exists a fully faithful tensor contravariant embedding functor*

$$CHM^{\mathit{eff}}(k;\Lambda)^{op} \hookrightarrow \mathbf{DA}^{\mathit{eff}}_{ét}(k;\Lambda),$$

such that

$$\begin{tikzcd} SmProj_k \arrow[r, hook] \arrow[d, "h"] & Sm_k \arrow[d, "\mathbf{M}"] \\ CHM^{\mathit{eff}}(k;\Lambda)^{op} \arrow[r, hook, "R"] & \mathbf{DA}^{\mathit{eff}}_{ét}(k;\Lambda) \end{tikzcd}$$

commutes. The functor R is fully-faithful tensor functor, which maps the Lefschetz motive $\mathbb{1}(-1) \in CHM^{\mathit{eff}}(k;\Lambda)$ into $\Lambda(1)[2] \in \mathbf{DA}^{\mathit{eff}}_{ét}(k;\Lambda)$. An analogous result holds with the non effective version. Moreover, for $p = 2q$, the étale motivic cohomology group of $X \in Sm_k$ computes the Chow group of codimension q with coefficients in Λ

$$CH^q(X)_\Lambda \cong H^{2q}(X, \Lambda(q)).$$

Proof. See [Voe, Prop. 2.1.4] and [Voe02] (or also [And04, Thm. 18.3.1.1] and [MVW06, Cor. 19.2, Prop. 20.1, Rmk. 20.2]), which compare the category of Chow motives to Voevodsky's triangulated category of motives, together with [Ayo13, Thm. B1], which compares Voevodsky's triangulated category of motives to Ayoub's one. $\square$

Notice that, since h is a contravariant functor, while h is covariant, we should think h as a motivic cohomology, while $\mathbf{M}$ as a motivic homology.

2.2.2 The Betti realization functor

One of the good features of the construction of $\mathbf{DA}^{\mathrm{eff}}_{ét}(k;\Lambda)$ is that analogous constructions can be performed also in other similar contexts. Now, instead of the context of smooth algebraic varieties over a field k, we consider the context of smooth complex analytic spaces described in the previous chapter. More precisely, we perform exactly the same constructions of the previous section with the following variations.

- We replace the category Sm_k with $AnSm_{\mathbb{C}}$, the category of smooth analytic spaces described in the previous chapter.
- We replace $\mathbb{A}^1_k \in Sm_k$ with $\mathbb{D}^1 \in AnSm_{\mathbb{C}}$ the open disk in $\mathbb{C}$ of radius 1
$$\mathbb{D}^1 \coloneqq \{z \in \mathbb{C} \mid |z| < 1\}.$$
- We replace the big étale site on Sm_k with the big étale-analytic site on $AnSm_{\mathbb{C}}$, i.e. the site with underlying category $AnSm_{\mathbb{C}}$ and covering families of an object $\mathcal{Y} \in AnSm_{\mathbb{C}}$ the étale-analytic covers of $\mathcal{Y}$ (definition 1.4.18)

Resuming the main steps, we consider the category of complexes of presheaves of Λ-modules over $AnSm_{\mathbb{C}}$
$$\mathrm{Ch}(\mathrm{PSh}(AnSm_{\mathbb{C}}; \Lambda)),$$
which is a stable model tensor category with the *projective global model structure*. Then, we define the sets of morphisms

$$S_{\acute{e}t\text{-}an} \coloneqq \{\mathbf{\Lambda}(\mathcal{W}_*)[n] \to \mathbf{\Lambda}(\mathcal{Y})[n] \mid \mathcal{Y} \in AnSm_{\mathbb{C}},\ \mathcal{W}_* \to \mathcal{Y} \text{ étale-analytic hypercover},\ n \in \mathbb{Z}\},$$
$$S_{\mathbb{D}^1} \coloneqq \{\mathbf{\Lambda}(\mathbb{D}^1 \times \mathcal{Y})[n] \to \mathbf{\Lambda}(\mathcal{Y})[n] \mid \mathcal{Y} \in AnSm_{\mathbb{C}},\ n \in \mathbb{Z}\},$$
$$S_{(\mathbb{D}^1, \acute{e}t\text{-}an)} \coloneqq S_{\mathbb{D}^1} \cup S_{\acute{e}t\text{-}an}.$$

We take the left Bousfield localization
$$\mathrm{L}_{S_{(\mathbb{D}^1,\acute{e}t\text{-}an)}}\mathrm{Ch}(\mathrm{PSh}(AnSm_{\mathbb{C}}; \Lambda)),$$
which is a model category with the *projective* $(\mathbb{D}^1, \acute{e}t\text{-}an)$*-local model structure.* Its homotopy category is a monoidal triangulated category, denoted by
$$\mathbf{AnDA}^{\mathrm{eff}}(\Lambda) \coloneqq \mathrm{Ho}(\mathrm{L}_{S_{(\mathbb{D}^1,\acute{e}t\text{-}an)}}\mathrm{Ch}(\mathrm{PSh}(AnSm_{\mathbb{C}}; \Lambda))).$$

Moreover, it holds the analogous of remark 2.2.12: equivalently, $\mathbf{AnDA}^{\mathrm{eff}}(\Lambda)$ can be described as the Verdier quotient of the derived category of étale-analytic sheaves of Λ-modules $\mathbf{D}_{\acute{e}t\text{-}an} \coloneqq \mathrm{D}(\mathrm{Sh}_{\acute{e}t\text{-}an}(AnSm_{\mathbb{C}}; \Lambda))$
$$\mathbf{AnDA}^{\mathrm{eff}}(\Lambda) \simeq \mathbf{D}_{\acute{e}t\text{-}an}/\mathcal{E}_{\mathbb{D}^1} \simeq \mathrm{D}_{\acute{e}t\text{-}an}[\mathcal{W}_{\mathbb{D}^1}^{-1}],$$
where
$$\mathcal{E}_{\mathbb{D}^1} \coloneqq \langle cone(f) \mid f : \mathbf{\Lambda}_{\acute{e}t\text{-}an}(\mathbb{D}^1 \times \mathcal{Y}) \to \mathbf{\Lambda}_{\acute{e}t\text{-}an}(\mathcal{Y}),\ \mathcal{Y} \in AnSm_{\mathbb{C}}\rangle$$

and
$$\mathcal{W}_{\mathbb{D}^1} := \{\alpha \in \mathbf{D}_{\acute{e}t\text{-}an} \mid cone(\alpha) \in \mathcal{E}_{\mathbb{D}^1}\}.$$

We have a monoidal functor

$$\mathbf{M}_{an} : AnSm_{\mathbb{C}} \xrightarrow{\Lambda} \mathrm{Ch}(\mathrm{PSh}(AnSm_{\mathbb{C}}; \Lambda)) \to \mathbf{AnDA}^{\mathrm{eff}}(\Lambda),$$

constructed analogously to $\mathbf{M}$.

Now, consider the category $\mathbf{DA}^{\mathrm{eff}}_{\acute{e}t}(k; \Lambda)$ for $k = \mathbb{C}$. In this situation we can relate the categories $\mathbf{DA}^{\mathrm{eff}}_{\acute{e}t}(\mathbb{C}; \Lambda)$ and $\mathbf{AnDA}^{\mathrm{eff}}(\Lambda)$ as follows. Consider the analytification functor described in the previous chapter (definition 1.2.14), which restricts to categories of smooth spaces

$$\begin{aligned} an : Sm_{\mathbb{C}} &\to AnSm_{\mathbb{C}} \\ X &\mapsto X(\mathbb{C}). \end{aligned}$$

It induces the adjunction between the categories of presheaves of Λ-modules

$$an_p : \mathrm{PSh}(Sm_{\mathbb{C}}; \Lambda) \rightleftarrows \mathrm{PSh}(AnSm_{\mathbb{C}}; \Lambda) : an^p.$$

Recall that the left adjoint an_p is such that, for any $F \in \mathrm{PSh}(Sm_k; \Lambda)$ and $\mathcal{Y} \in AnSm_{\mathbb{C}}$,

$$an_p F(\mathcal{Y}) = \varinjlim_{\mathcal{Y} \to X(\mathbb{C}) \in AnSm_{\mathbb{C}}} F(X)$$

and the right adjoint an^p is such that, for any $G \in \mathrm{PSh}(AnSm_{\mathbb{C}}; \Lambda)$ and $X \in Sm_{\mathbb{C}}$,

$$an^p G(X) = G(X(\mathbb{C})).$$

This adjunction extends level-wise to an adjunction on categories of complexes

$$an_p : \mathrm{Ch}(\mathrm{PSh}(Sm_{\mathbb{C}}; \Lambda)) \rightleftarrows \mathrm{Ch}(\mathrm{PSh}(AnSm_{\mathbb{C}}; \Lambda)) : an^p. \tag{2.12}$$

Proposition 2.2.18. *The adjuction 2.12 is a Quillen pair with respect to the projective global model structures. It induces the Quillen pair*

$$an_p : L_{S_{(\mathbb{A}^1, \acute{e}t)}} Ch(PSh(Sm_{\mathbb{C}}; \Lambda)) \rightleftarrows L_{S_{(\mathbb{D}^1, \acute{e}t\text{-}an)}} Ch(PSh(AnSm_{\mathbb{C}}; \Lambda)) : an^p. \tag{2.13}$$

with respect to the projective $(\mathbb{A}^1, \acute{e}t)$ and $(\mathbb{D}^1, \acute{e}t\text{-}an)$-local model structures.

Proof. Consider the adjunction 2.12. It is a Quillen pair because the right adjoint an^p is a right Quillen functor, i.e. preserves fibrations and trivial fibrations. This is true by definition of an^p and since projective global fibrations and trivial fibrations can be checked open-wise. To prove that 2.13 is a Quillen pair, consider the composition of Quillen pairs 2.12 and the one given by left localization of $\mathrm{Ch}(\mathrm{PSh}(AnSm_{\mathbb{C}}; \Lambda))$

with respect to $S_{(\mathbb{D}^1,\acute{e}t\text{-}an)}$

$$\mathrm{Ch}(\mathrm{PSh}(Sm_{\mathbb{C}};\Lambda)) \underset{an^p}{\overset{an_p}{\rightleftarrows}} \mathrm{Ch}(\mathrm{PSh}(AnSm_{\mathbb{C}};\Lambda)) \rightleftarrows \mathrm{L}_{S_{(\mathbb{D}^1,\acute{e}t\text{-}an)}}\mathrm{Ch}(\mathrm{PSh}(AnSm_{\mathbb{C}};\Lambda)).$$

By the universal property of localization of model categories, it suffices to prove that an_p maps morphisms in $S_{(\mathbb{A}^1,\acute{e}t)}$ into projective $(\mathbb{D}^1, \acute{e}t\text{-}an)$-local weak-equivalences, i.e. weak-equivalences of $\mathrm{L}_{S_{(\mathbb{D}^1,\acute{e}t\text{-}an)}}\mathrm{Ch}(\mathrm{PSh}(AnSm_{\mathbb{C}};\Lambda))$. Recall that an_p preserves representable functors, that is, for any $X \in Sm_{\mathbb{C}}$,

$$an_p\mathbf{\Lambda}(X) \cong \mathbf{\Lambda}(X(\mathbb{C})).$$

Let $U_* \to X$ be an étale hypercover of X. Then, an_p maps

$$\mathbf{\Lambda}(U_*) \to \mathbf{\Lambda}(X)$$

into

$$\mathbf{\Lambda}(U(\mathbb{C})_*) \to \mathbf{\Lambda}(X(\mathbb{C})).$$

By proposition 1.4.16, $U(\mathbb{C})_* \to X(\mathbb{C})$ is an étale-analytic hypercover of $X(\mathbb{C})$. So, $\mathbf{\Lambda}(U(\mathbb{C})_*) \to \mathbf{\Lambda}(X(\mathbb{C}))$ is a morphism in $S_{(\mathbb{D}^1,\acute{e}t\text{-}an)}$ and then it is a projective $(\mathbb{D}^1, \acute{e}t\text{-}an)$-local weak-equivalence. Moreover, for any $X \in Sm_{\mathbb{C}}$, an_p maps

$$\mathbf{\Lambda}(\mathbb{A}^1_{\mathbb{C}} \times_{\mathbb{C}} X) \cong \mathbf{\Lambda}(\mathbb{A}^1_X) \to \mathbf{\Lambda}(X)$$

into

$$\mathbf{\Lambda}(\mathbb{C} \times X(\mathbb{C})) \to \mathbf{\Lambda}(X(\mathbb{C})).$$

Consider an étale-analytic cover $\mathbb{D}^1 \to \mathbb{C}$. [5] Let $\mathcal{N}\mathbb{D}^1_* \to \mathbb{C}$ be the étale-analytic hypercover given by the Čech nerve. Then,

$$\mathbf{\Lambda}(\mathcal{N}\mathbb{D}^1_*) \to \mathbf{\Lambda}(\mathbb{C})$$

is a morphism in $S_{(\mathbb{D}^1,\acute{e}t\text{-}an)}$, hence it is a projective $(\mathbb{D}^1, \acute{e}t\text{-}an)$-local weak-equivalence. Since $\mathbb{D}^1$ is connected, then the morphism

$$\mathbf{\Lambda}(\mathbb{D}^1) \to \mathbf{\Lambda}(\mathcal{N}\mathbb{D}^1_*),$$

induced by the canonical morphism $\mathbb{D}^1 \to \mathcal{N}\mathbb{D}^1_*$, is a projective global weak-equivalence, hence also a projective $(\mathbb{D}^1, \acute{e}t\text{-}an)$-local weak-equivalence. Hence, the composition

$$\mathbf{\Lambda}(\mathbb{D}^1) \to \mathbf{\Lambda}(\mathcal{N}\mathbb{D}^1_*) \to \mathbf{\Lambda}(\mathbb{C})$$

is a projective $(\mathbb{D}^1, \acute{e}t\text{-}an)$-local weak-equivalence. Tensoring with the identity of

[5] It exists. For example, take the composition of the biholomorphism of $\mathbb{D}^1$ with the Poincaré plane and the holomorphic map $z \mapsto (z-i)^2$.

$\mathbf{\Lambda}(X(\mathbb{C}))$, which is a projective $(\mathbb{D}^1, \acute{e}t\text{-}an)$-local weak-equivalence, we get that

$$\mathbf{\Lambda}(\mathbb{D}^1 \times X(\mathbb{C})) \cong \mathbf{\Lambda}(\mathbb{D}^1) \otimes \mathbf{\Lambda}(X(\mathbb{C})) \to \mathbf{\Lambda}(\mathbb{C}) \otimes \mathbf{\Lambda}(X(\mathbb{C})) \cong \mathbf{\Lambda}(\mathbb{C} \times X(\mathbb{C}))$$

is a projective $(\mathbb{D}^1, \acute{e}t\text{-}an)$-local weak-equivalence. The composition

$$\mathbf{\Lambda}(\mathbb{D}^1 \times X(\mathbb{C})) \to \mathbf{\Lambda}(\mathbb{C} \times X(\mathbb{C})) \to \mathbf{\Lambda}(X(\mathbb{C}))$$

is induced by the canonical projection $\mathbb{D}^1 \times X(\mathbb{C}) \to X(\mathbb{C})$. So, is a morphism in $S_{(\mathbb{D}^1, \acute{e}t\text{-}an)}$ and then it is a projective $(\mathbb{D}^1, \acute{e}t\text{-}an)$-local weak-equivalence. By the two-out-of-three property for weak-equivalences, we deduce that

$$\mathbf{\Lambda}(\mathbb{C} \times X(\mathbb{C})) \to \mathbf{\Lambda}(X(\mathbb{C}))$$

is a projective $(\mathbb{D}^1, \acute{e}t\text{-}an)$-local weak-equivalence. Since the same can be said for all shifted morphisms, this proves the statement. □

The Quillen pair 2.13 induces the total derived adjunction

$$\mathrm{An}^* := \mathbf{L}an_p : \mathbf{DA}^{\mathrm{eff}}_{\acute{e}t}(\mathbb{C}; \Lambda) \rightleftarrows \mathbf{AnDA}^{\mathrm{eff}}(\Lambda) : \mathbf{R}an^p =: \mathrm{An}_*.$$

Notice that, since an_p preserves representable functors, that is

$$\begin{tikzcd} Sm_{\mathbb{C}} \arrow[r, "an"] \arrow[d, "\mathbf{\Lambda}"] & AnSm_{\mathbb{C}} \arrow[d, "\mathbf{\Lambda}"] \\ \mathrm{PSh}(Sm_{\mathbb{C}}; \Lambda) \arrow[r, "an_p"] & \mathrm{PSh}(AnSm_{\mathbb{C}}; \Lambda) \end{tikzcd}$$

commutes, then also

$$\begin{tikzcd} Sm_{\mathbb{C}} \arrow[r, "an"] \arrow[d, "\mathbf{M}"] & AnSm_{\mathbb{C}} \arrow[d, "\mathbf{M}_{an}"] \\ \mathbf{DA}^{\mathrm{eff}}_{\acute{e}t}(\mathbb{C}; \Lambda) \arrow[r, "\mathrm{An}^*"] & \mathbf{AnDA}^{\mathrm{eff}}(\Lambda) \end{tikzcd} \tag{2.14}$$

commutes.

Now, consider the point $\mathbb{C}^0 \in AnSm_{\mathbb{C}}$. We have the inclusion functor of the category of analytic open subsets of $\mathbb{C}^0$ into $SmAn_{\mathbb{C}}$

$$u : \mathrm{Op}(\mathbb{C}^0) \hookrightarrow AnSm_{\mathbb{C}}.$$

As above, it induces an adjunction between categories of presheaves of Λ-modules

$$u_p : \mathrm{PSh}(\mathrm{Op}(\mathbb{C}^0); \Lambda) \rightleftarrows \mathrm{PSh}(AnSm_{\mathbb{C}}; \Lambda) : u^p.$$

Notice that, since $\mathrm{Op}(\mathbb{C}^0)$ is noting but the category with one object $\mathbb{C}^0$ and the identity on it, then we have an equivalence of categories

$$\begin{aligned} \mathrm{PSh}(\mathrm{Op}(\mathbb{C}^0);\Lambda) &\simeq \Lambda\text{-mod} \\ F &\mapsto F(\mathbb{C}^0). \end{aligned}$$

Via this equivalence of categories, the left adjoint u_p is such that, for any $A \in \Lambda$-mod and $\mathcal{Y} \in AnSm_{\mathbb{C}}$,

$$u_pA(\mathcal{Y}) = \varinjlim_{\mathcal{Y}\to\mathbb{C}^0\in AnSm_{\mathbb{C}}} A \cong A.$$

That is, u_p assigns to each Λ-module the associated constant presheaf A_{cst}. For this reason, we denote by $cst := u_p$. The right adjoint u^p is such that, for any $F \in \mathrm{PSh}(AnSm_{\mathbb{C}};\Lambda)$,

$$u^pF(\mathbb{C}^0) = F(u(\mathbb{C}^0)) = F(\mathbb{C}^0).$$

That is, u^p is the global section functor over $\mathbb{C}^0$. For this reason, we denote by $u^p := \Gamma(\mathbb{C}^0, \_\,)$. So, the above adjunction is

$$cst : \Lambda\text{-mod} \rightleftarrows \mathrm{PSh}(AnSm_{\mathbb{C}};\Lambda) : \Gamma(\mathbb{C}^0, \_\,).$$

This adjunction extends level-wise to an adjunction on categories of complexes

$$cst : \mathrm{Ch}(\Lambda) \rightleftarrows \mathrm{Ch}(\mathrm{PSh}(AnSm_{\mathbb{C}};\Lambda)) : \Gamma(\mathbb{C}^0, \_\,). \tag{2.15}$$

Proposition 2.2.19. *The adjunction 2.15 is a Quillen pair with respect to the projective global model structure on Ch(PSh($AnSm_{\mathbb{C}};\Lambda$)). It induces a Quillen equivalence*

$$cst : Ch(\Lambda) \rightleftarrows L_{S_{(\mathbb{D}^1,ét\text{-}an)}} Ch(PSh(SmAn_{\mathbb{C}};\Lambda)) : \Gamma(\mathbb{C}^0, \_\,) \tag{2.16}$$

with respect to the projective ($\mathbb{D}^1$, ét-an)-local model structure on $L_{S_{(\mathbb{D}^1,ét\text{-}an)}}$ Ch(PSh($AnSm_{\mathbb{C}};\Lambda$)).

Proof. Consider the adjunction 2.15. It is a Quillen pair because the right adjoint $\Gamma(\mathbb{C}^0, \_\,)$ is a right Quillen functor, i.e. preserves fibrations and trivial fibrations. This is true because projective global fibrations and trivial fibrations can be checked open-wise. The Quillen pair 2.16 is obtained composing the Quillen pairs

$$cst : \mathrm{Ch}(\Lambda) \underset{\Gamma(\mathbb{C}^0,\_\,)}{\overset{cst}{\rightleftarrows}} \mathrm{Ch}(\mathrm{PSh}(AnSm_{\mathbb{C}};\Lambda)) \rightleftarrows \mathrm{L}_{S_{(\mathbb{D}^1,ét\text{-}an)}}\mathrm{Ch}(\mathrm{PSh}(AnSm_{\mathbb{C}};\Lambda)) : \Gamma(\mathbb{C}^0, \_\,).$$

To prove that it is a Quillen equivalence, we need to prove that the total derived adjunction

$$\mathbf{L}cst : \mathrm{D}(\Lambda) \rightleftarrows \mathbf{AnDA}^{\mathrm{eff}}(\Lambda) : \mathbf{R}\Gamma(\mathbb{C}^0, \_\,),$$

is an equivalence of categories. Notice that, since projective global weak-equivalences can be checked open-wise, then cst takes weak-equivalences in $\mathrm{Ch}(\Lambda)$ into projective global weak-equivalences, which are also projective $(\mathbb{D}^1, \acute{e}t\text{-}an)$-local weak-equivalences. That is, cst is an exact functor, so it doesn't have to be left derived

$$\mathbf{L}cst = cst.$$

We prove that cst is essentially surjective and fully-faithful. We start with the essential surjectivity. The proof consists of the following three steps.

- **Step 1.** We prove that $\mathbf{D} := \mathrm{D}(\mathrm{PSh}(AnSm_{\mathbb{C}}; \Lambda))$ is compactly generated by the objects $\mathbf{\Lambda}(\mathcal{Y})$, for $\mathcal{Y} \in AnSm_{\mathbb{C}}$. It follows that $\mathbf{D}$ coincides with its smallest triangulated subcategory closed by arbitrary direct sums and containing the objects $\mathbf{\Lambda}(\mathcal{Y})$, for $\mathcal{Y} \in AnSm_{\mathbb{C}}$. That is, $\mathbf{D}$ coincides with its triangulated subcategory generated by $\mathbf{\Lambda}(\mathcal{Y})$, for $\mathcal{Y} \in AnSm_{\mathbb{C}}$

$$\mathbf{D} = \langle \mathbf{\Lambda}(\mathcal{Y}) \mid \mathcal{Y} \in AnSm_{\mathbb{C}} \rangle.$$

Recall that, for $\mathbf{D}$, being compactly generated by the objects $\mathbf{\Lambda}(\mathcal{Y})$, for $\mathcal{Y} \in AnSm_{\mathbb{C}}$, means that, for any family $\{F_\bullet^i\}_{i\in I}$ of objects in $\mathbf{D}$

$$\mathrm{Hom}_{\mathbf{D}}(\mathbf{\Lambda}(\mathcal{Y}), \oplus_{i\in I} F_\bullet^i) \cong \oplus_{i\in I} \mathrm{Hom}_{\mathbf{D}}(\mathbf{\Lambda}(\mathcal{Y}), F_\bullet^i)$$

and, if $F_\bullet \in \mathbf{D}$ is such that, for any $\mathcal{Y} \in AnSm_{\mathbb{C}}$ and $n \in \mathbb{Z}$,

$$\mathrm{Hom}_{\mathbf{D}}(\mathbf{\Lambda}(\mathcal{Y})[n], F_\bullet) = 0,$$

then $F_\bullet = 0$. Since

$$\mathrm{Hom}_{\mathbf{D}}(\mathbf{\Lambda}(\mathcal{Y})[n], \_) \cong \mathrm{H}^0 \mathbf{R}\, \mathrm{Hom}_{\mathrm{PSh}(AnSm_{\mathbb{C}};\Lambda)}(\mathbf{\Lambda}(\mathcal{Y})[n], \_) \cong \mathrm{H}^{-n}\mathbf{R}\Gamma(\mathcal{Y}, \_) \cong \mathrm{H}^{-n}\Gamma(\mathcal{Y}, \_),$$

we see that the first condition is satisfied, because cohomology commutes with arbitrary direct sums in the category of Λ-modules. Also the second condition is satisfied because

$$\mathrm{H}^n F_\bullet(\mathcal{Y}) = 0,$$

for any $\mathcal{Y} \in AnSm_{\mathbb{C}}$ and $n \in \mathbb{Z}$, means that $F_\bullet \in \mathrm{Ch}(\mathrm{PSh}(AnSm_{\mathbb{C}}; \Lambda))$ is quasi-isomorphic to 0, that is, $F_\bullet = 0$ in $\mathbf{D}$.

- **Step 2** We prove that

$$\mathbf{AnDA}^{\mathrm{eff}}(\Lambda) = \langle \mathbf{M}_{an}(\mathbb{D}^n) := \mathbf{M}_{an}(\mathbb{D}^1)^{\otimes n} \mid n \in \mathbb{Z} \rangle.$$

Since the triangulated functor

$$\mathbf{D} \to \mathbf{AnDA}^{\mathrm{eff}}(\Lambda)$$

is the identity on objects, then, by step 1, it suffices to prove that, for any $\mathcal{Y} \in AnSm_{\mathbb{C}}$,

$$\mathbf{M}_{an}(\mathcal{Y}) \in \langle \mathbf{M}_{an}(\mathbb{D}^n) \mid n \in \mathbb{Z} \rangle.$$

Given $\mathcal{Y} \in AnSm_{\mathbb{C}}$, notice that we can always construct

$$\mathcal{W}_* \to \mathcal{Y}$$

an étale-analytic hypercover of $\mathcal{Y}$, such that each $\mathcal{W}_i$ is a disjoint union of disks $\mathbb{D}^n := (\mathbb{D}^1)^n$. Indeed, since $\mathcal{Y}$ is smooth, there exists an open cover of $\mathcal{Y}$ given by a disjoint union of open disks of $\mathbb{C}^n$. Hence, we can construct $\mathcal{W}_0 \to \mathcal{Y}$ an étale-analytic cover given by a disjoint union of disks $\mathbb{D}^n$. Then, we construct $\mathcal{W}_*$ taking compatible étale-analytic refinements of successive auto-intersections of $\mathcal{W}_0$, given by disjoint unions of disks $\mathbb{D}^n$. This is possible because open disks of $\mathbb{C}^n$ form a basis of open subsets of $\mathbb{C}^n$ and all open disks of $\mathbb{C}^n$ are biholomorphic to $\mathbb{D}^n$. Since

$$\mathbf{\Lambda}(\coprod_{i \in I} \mathbb{D}^n) \cong \oplus_{i \in I} \mathbf{\Lambda}(\mathbb{D}^n),$$

then,we deduce that

$$\mathbf{\Lambda}(\mathcal{W}_*) \in \langle \mathbf{\Lambda}(\mathbb{D}^n) \mid n \in \mathbb{Z} \rangle.$$

Since in $\mathbf{AnDA}^{\text{eff}}(\Lambda)$

$$\mathbf{M}_{an}(\mathcal{W}_*) \to \mathbf{M}_{an}(\mathcal{Y})$$

is an isomorphism, then

$$\mathbf{M}_{an}(\mathcal{Y}) \in \langle \mathbf{M}_{an}(\mathbb{D}^n) \mid n \in \mathbb{Z} \rangle.$$

- **Step 3** We prove that $\mathbf{AnDA}^{\text{eff}}(\Lambda)$ coincides with its triangulated subcategory generated by Λ_{cst} the constant presheaf

  $$\mathbf{AnDA}^{\text{eff}}(\Lambda) := \langle \Lambda_{cst} \rangle.$$

  Since Λ_{cst} is in the image of the triangulated functor cst, it follows that cst is essentially surjective.

Since in $\mathbf{AnDA}^{\text{eff}}(\Lambda)$

$$\mathbf{M}_{an}(\mathbb{D}^n) \cong \mathbf{M}_{an}(\mathbb{D}^1 \times \mathbb{D}^{n-1}) \to \mathbf{M}_{an}(\mathbb{D}^{n-1})$$

is an isomorphism, then, by induction, we deduce that

$$\mathbf{M}_{an}(\mathbb{D}^n) \cong \mathbf{M}_{an}(\mathbb{C}^0) \cong \Lambda_{cst}.$$

Then, step 3 follows from step 2.

Finally, we prove that *cst* is fully-faithful. Notice that, since $\mathrm{D}(\Lambda)$ is compactly generated by Λ, it suffices to prove that, for any $n \in \mathbb{Z}$,

$$\mathrm{Hom}_{\mathrm{D}(\Lambda)}(\Lambda, \Lambda[n]) \cong \mathrm{Hom}_{\mathbf{AnDA}^{\mathrm{eff}}(\Lambda)}(\Lambda_{cst}, \Lambda_{cst}[n]).$$

We have to compute hom-sets in $\mathbf{AnDA}^{\mathrm{eff}}(\Lambda)$. Recall that, by the analogous of remark 2.2.12, it holds that, for any $F_\bullet, G_\bullet \in \mathbf{D}_{\textit{ét-an}}$ with $G_\bullet$ $\mathbb{D}^1$-local object,

$$\mathrm{Hom}_{\mathbf{AnDA}^{\mathrm{eff}}(\Lambda)}(F_\bullet, G_\bullet) \cong \mathrm{Hom}_{\mathbf{D}_{\textit{ét-an}}}(F_\bullet, G_\bullet).$$

Recall that $G_\bullet \in \mathbf{D}_{\textit{ét-an}}$ is $\mathbb{D}^1$-local if, for any $\mathcal{Y} \in AnSm_{\mathbb{C}}$ and $n \in \mathbb{Z}$,

$$\mathbb{H}^n(\mathcal{Y}_{\textit{ét-an}}, G_\bullet) \cong \mathbb{H}^n((\mathbb{D}^1 \times \mathcal{Y})_{\textit{ét-an}}, G_\bullet).$$

Moreover, recall that by remark 1.4.19, the category of étale-analytic sheaves is equivalent to the one of analytic ones, that is the ones with respect to the classical topology of analytic open subsets. Then, sheaf cohomology of étale-analytic sheaves can be computed equivalently as cohomology of their restriction to the classical analytic site. So, equivalently, $G_\bullet \in \mathbf{D}_{\textit{ét-an}}$ is a $\mathbb{D}^1$-local object if, for any $\mathcal{Y} \in AnSm_{\mathbb{C}}$ and $n \in \mathbb{Z}$,

$$\mathbb{H}^n(\mathcal{Y}_{\textit{ét-an}}, G_\bullet) \cong \mathbb{H}^n((\mathbb{D}^1 \times \mathcal{Y})_{\textit{ét-an}}, G_\bullet),$$

where here, by $G_\bullet$, we mean the restriction of $G_\bullet$ to the small étale-analytic sites. By remark 1.4.19, this is equivalent to ask that

$$\mathbb{H}^n(\mathcal{Y}_{an}, \pi^{an}_{\mathcal{Y}*} G_\bullet) \cong \mathbb{H}^n((\mathbb{D}^1 \times \mathcal{Y})_{an}, \pi^{an}_{\mathcal{Y}*} G_\bullet).$$

Since for any $\mathcal{Y} \in SmAn_{\mathbb{C}}$, by proposition 1.1.4, sheaf cohomology of the constant sheaf $\Lambda_{\mathcal{Y}}$ over $\mathcal{Y}_{an}$ computes the singular cohomology of $\mathcal{Y}$, which is such that

$$\mathbb{H}^n(\mathcal{Y}_{an}, \Lambda_{\mathcal{Y}}) \cong \mathrm{H}^n_{\mathrm{Sing}}(\mathcal{Y}; \Lambda) \cong \mathrm{H}^n_{\mathrm{Sing}}(\mathbb{D}^1 \times \mathcal{Y}; \Lambda) \cong \mathbb{H}^n((\mathbb{D}^1 \times \mathcal{Y})_{an}, \Lambda_{\mathbb{D}^1 \times \mathcal{Y}}),$$

then the constant sheaf $\Lambda^{\textit{ét-an}}_{cst} \in \mathbf{D}_{\textit{ét-an}}$ is a $\mathbb{D}^1$-local object. Notice that $\Lambda^{\textit{ét-an}}_{cst}$ is canonically isomorphic to Λ_{cst} in $\mathbf{D}_{\textit{ét-an}}$, hence also in $\mathbf{AnDA}^{\mathrm{eff}}(\Lambda)$, via the canonical morphism of sheafification. Hence,

$$\begin{aligned}\mathrm{Hom}_{\mathbf{AnDA}^{\mathrm{eff}}(\Lambda)}(\Lambda_{cst}, \Lambda_{cst}[n]) &\cong \mathrm{Hom}_{\mathbf{AnDA}^{\mathrm{eff}}(\Lambda)}(\Lambda^{\textit{ét-an}}_{cst}, \Lambda^{\textit{ét-an}}_{cst}[n]) \cong \\ &\cong \mathrm{Hom}_{\mathbf{D}_{\textit{ét-an}}}(\Lambda^{\textit{ét-an}}_{cst}, \Lambda^{\textit{ét-an}}_{cst}[n]) \cong \\ &\cong \mathrm{Hom}_{\mathbf{D}}(\Lambda_{cst}, \Lambda^{\textit{ét-an}}_{cst}[n]) \cong \mathrm{Hom}_{\mathrm{D}(\Lambda)}(\Lambda, \Lambda[n]),\end{aligned}$$

where the third and the last bijections follow from the total derived adjunctions

$$\mathrm{D}(\Lambda) \underset{\mathbf{R}\Gamma(\mathbb{C}^0,\_)}{\overset{cst}{\rightleftarrows}} \mathbf{D} \overset{a_{\textit{ét-an}}}{\rightleftarrows} \mathbf{D}_{\textit{ét-an}}$$

$\square$

Remark 2.2.20. Given $\mathcal{Y} \in SmAn_{\mathbb{C}}$, let $C_\bullet \in \mathrm{D}(\Lambda)$ be the object corresponding to $\mathbf{M}_{an}(\mathcal{Y}) \in \mathbf{AnDA}^{\mathrm{eff}}(\Lambda)$ into $\mathrm{D}(\Lambda)$, via the equivalence of categories proved in proposition 2.2.19. Then, we have the bijections, for any $n \in \mathbb{Z}$,

$$\mathrm{Hom}_{\mathbf{AnDA}^{\mathrm{eff}}(\Lambda)}(\mathbf{M}_{an}(Y), \Lambda_{cst}[n]) \cong \mathrm{Hom}_{\mathrm{D}(\Lambda)}(C_\bullet, \Lambda[n])$$

Since $\Lambda_{cst} \cong \Lambda_{cst}^{\acute{e}t\text{-}an}$ in $\mathbf{D}_{\acute{e}t\text{-}an}$ is a $\mathbb{D}^1$-local object, as seen in the proof of proposition 2.2.19, then

$$\begin{aligned}\mathrm{Hom}_{\mathbf{AnDA}^{\mathrm{eff}}(\Lambda)}(\mathbf{M}_{an}(Y), \Lambda_{cst}[n]) \cong \mathrm{Hom}_{\mathbf{D}_{\acute{e}t\text{-}an}}(\mathbf{\Lambda}_{\acute{e}t\text{-}an}(\mathcal{Y}), \Lambda_{cst}^{\acute{e}t\text{-}an}[n]) \cong \\ \cong \mathbb{H}^i(\mathcal{Y}_{\acute{e}t\text{-}an}, \Lambda_{cst}^{\acute{e}t\text{-}an}) \cong \mathrm{H}^n_{\mathrm{Sing}}(\mathcal{Y}; \Lambda).\end{aligned}$$

On the other hand

$$\mathrm{Hom}_{\mathrm{D}(\Lambda)}(C_\bullet, \Lambda[n]) \cong \mathrm{H}^n(C_\bullet^\vee),$$

where $C_\bullet^\vee \coloneqq \mathrm{Hom}_\Lambda(C_\bullet, \Lambda)$ denotes the dual of $C_\bullet$. It follows that, in $\mathrm{D}(\Lambda)$, $C_\bullet^\vee$ is isomorphic to the complex of singular cochains of $\mathcal{Y}$ with coefficients in Λ

$$C_\bullet^\vee \cong \mathcal{C}^\bullet_{\mathrm{Sing}}(\mathcal{Y}; \Lambda),$$

hence $C_\bullet$ is the complex of singular chains of $\mathcal{Y}$ with coefficients in Λ

$$C_\bullet \cong \mathcal{C}_\bullet^{\mathrm{Sing}}(\mathcal{Y}; \Lambda).$$

Now, consider $\Lambda = \mathbb{Q}$.

Definition 2.2.21. The composition of triangulated functors

$$\mathrm{R}_{\mathrm{Bet}}^{\mathrm{eff}\ *} : \mathbf{DA}^{\mathrm{eff}}_{\acute{e}t}(\mathbb{C}; \mathbb{Q}) \xrightarrow{\mathrm{An}^*} \mathbf{AnDA}^{\mathrm{eff}}(\mathbb{Q}) \xrightarrow{\mathbf{R}\Gamma(\mathbb{C}^0,\ \_\ )} \mathrm{D}(\mathbb{Q})$$

is called the *(effective) Betti realization functor.*

Remark 2.2.22. The Betti realization functor is monoidal, since it is composition of the total derived of monoidal functors.

Remark 2.2.23. For any $X \in Sm_{\mathbb{C}}$, we have the isomorphisms

$$\mathrm{R}_{\mathrm{Bet}}^{\mathrm{eff}\ *}\mathbf{M}(X) \cong \mathbf{R}\Gamma(\mathbb{C}^0,\ \_\ )\mathrm{An}^*\mathbf{M}(X) \cong \mathbf{R}\Gamma(\mathbb{C}^0,\ \_\ )\mathbf{M}_{an}(X(\mathbb{C})) \cong \mathcal{C}_\bullet^{\mathrm{Sing}}(X(\mathbb{C}); \mathbb{Q}),$$

where the second holds by the commutative diagram 2.14 and the last holds by remark 2.2.20. Hence, the Betti realization functor is such that, for any $n \geq 0$,

$$\mathrm{H}^n((\mathrm{R}_{\mathrm{Bet}}^{\mathrm{eff}\ *}\mathbf{M}(X))^\vee) \cong \mathrm{H}^n(\mathcal{C}_\bullet^{\mathrm{Sing}}(X(\mathbb{C}); \mathbb{Q})^\vee) \cong \mathrm{H}^n(\mathcal{C}^\bullet_{\mathrm{Sing}}(X(\mathbb{C}); \mathbb{Q})) \cong \mathrm{H}^n_{\mathrm{Bet}}(X).$$

We only mention that an analogous construction can be performed with the non-effective variant, obtaining a monoidal triangulated functor

$$\mathrm{R}^*_{\mathrm{Bet}} : \mathbf{DA}_{\acute{e}t}(\mathbb{C};\mathbb{Q}) \to \mathrm{D}(\mathbb{Q}).$$

Since in $\mathbf{DA}_{\acute{e}t}(\mathbb{C};\mathbb{Q})$ we have dual objects, then it holds that, for any $X \in AnSm_{\mathbb{C}}$,

$$\mathrm{H}^n(\mathrm{R}^*_{\mathrm{Bet}}(\mathbf{M}(X)^\vee)) \cong \mathrm{H}^n((\mathrm{R}^*_{\mathrm{Bet}}\mathbf{M}(X))^\vee) \cong \mathrm{H}^n_{\mathrm{Bet}}(X).$$

That is, $\mathrm{R}^*_{\mathrm{Bet}}$ maps $\mathbf{M}(X)^\vee$, to be thought as the cohomological motive associated to X, into a complex which computes the Betti cohomology of X. For this reason it is called the Betti realization functor.

By construction, the Betti realization functor has a right adjoint

$$\mathrm{R}^{\mathrm{eff}\ *}_{\mathrm{Bet}} : \mathbf{DA}^{\mathrm{eff}}_{\acute{e}t}(\mathbb{C};\mathbb{Q}) \underset{\mathrm{An}_*}{\overset{\mathrm{An}^*}{\rightleftarrows}} \mathbf{AnDA}^{\mathrm{eff}}(\mathbb{Q}) \underset{cst}{\overset{\mathbf{R}\Gamma(\mathbb{C}^0,\_)}{\underset{\sim}{\rightleftarrows}}} \mathrm{D}(\mathbb{Q}) : \mathrm{R}^{\mathrm{eff}}_{\mathrm{Bet}*}.$$

Consider $\mathbb{Q}_{cst} \cong \mathbf{M}(Spec(\mathbb{C})) \in \mathbf{DA}^{\mathrm{eff}}_{\acute{e}t}(\mathbb{C};\mathbb{Q})$, the constant presheaf over $Sm_{\mathbb{C}}$, and $An^*\mathbb{Q}_{cst} \cong \mathbf{M}_{an}(\mathbb{C}^0) \in \mathbf{AnDA}^{\mathrm{eff}}(\mathbb{Q})$, the constant presheaf over $AnSm_{\mathbb{C}}$, which are the unit objects of their respective categories. By the adjunction $(\mathrm{R}^{\mathrm{eff}\ *}_{\mathrm{Bet}}, \mathrm{R}^{\mathrm{eff}}_{\mathrm{Bet}*})$, we have the bijections, for any $M \in \mathbf{DA}^{\mathrm{eff}}_{\acute{e}t}(\mathbb{C};\mathbb{Q})$ and $n \in \mathbb{Z}$,

$$\begin{aligned}\mathrm{Hom}_{\mathbf{DA}^{\mathrm{eff}}_{\acute{e}t}(\mathbb{C};\mathbb{Q})}(M, \mathrm{R}^{\mathrm{eff}}_{\mathrm{Bet}*}\mathrm{R}^{\mathrm{eff}\ *}_{\mathrm{Bet}}\mathbb{Q}_{cst}[n]) &\cong \mathrm{Hom}_{\mathbf{DA}^{\mathrm{eff}}_{\acute{e}t}(\mathbb{C};\mathbb{Q})}(M, \mathrm{An}_*\mathrm{An}^*\mathbb{Q}_{cst}[n]) \cong \\ &\cong \mathrm{Hom}_{\mathbf{AnDA}^{\mathrm{eff}}(\mathbb{Q})}(\mathrm{An}^*M, \mathrm{An}^*\mathbb{Q}_{cst}[n]) \cong \\ &\cong \mathrm{Hom}_{\mathbf{AnDA}^{\mathrm{eff}}(\mathbb{Q})}(\mathrm{An}^*M, \mathbb{Q}_{cst}[n]).\end{aligned}$$

In particular, if $M = \mathbf{M}(X)$ for some $X \in Sm_{\mathbb{C}}$, it holds that

$$\begin{aligned}\mathrm{Hom}_{\mathbf{DA}^{\mathrm{eff}}_{\acute{e}t}(\mathbb{C};\mathbb{Q})}(\mathbf{M}(X), \mathrm{R}^{\mathrm{eff}}_{\mathrm{Bet}*}\mathrm{R}^{\mathrm{eff}\ *}_{\mathrm{Bet}}\mathbb{Q}_{cst}[n]) &\cong \mathrm{Hom}_{\mathbf{AnDA}^{\mathrm{eff}}(\mathbb{Q})}(\mathrm{An}^*\mathbf{M}(X), \mathbb{Q}_{cst}[n]) \cong \\ &\cong \mathrm{Hom}_{\mathbf{AnDA}^{\mathrm{eff}}(\mathbb{Q})}(\mathbf{M}_{an}(X(\mathbb{C}))), \mathbb{Q}_{cst}[n]) \cong \\ &\cong \mathrm{H}^n_{\mathrm{Sing}}(X(\mathbb{C});\mathbb{Q}) = \mathrm{H}^n_{\mathrm{Bet}}(X),\end{aligned}$$

where the second isomorphism holds by commutative diagram 2.14 and third was proved in remark 2.2.20. In other words, the object

$$\mathrm{R}^{\mathrm{eff}}_{\mathrm{Bet}*}\mathrm{R}^{\mathrm{eff}\ *}_{\mathrm{Bet}}\mathbb{Q}_{cst} \in \mathbf{DA}^{\mathrm{eff}}_{\acute{e}t}(\mathbb{C};\mathbb{Q})$$

represents the Betti cohomology in the category $\mathbf{DA}^{\mathrm{eff}}_{\acute{e}t}(\mathbb{C};\mathbb{Q})$. Moreover, notice that, since the Betti realization is a monoidal functor, hence preserves the unit, then $\mathrm{R}^{\mathrm{eff}\ *}_{\mathrm{Bet}}\mathbb{Q}_{cst} \cong \mathbb{Q}$. Hence

$$\mathrm{R}^{\mathrm{eff}}_{\mathrm{Bet}*}\mathrm{R}^{\mathrm{eff}\ *}_{\mathrm{Bet}}\mathbb{Q}_{cst} \cong \mathrm{R}^{\mathrm{eff}}_{\mathrm{Bet}*}\mathbb{Q}.$$

2.2.3 The comparison theorem

In analogy with the Betti cohomology, now we define an object in our triangulated category of motives, which represents the algebraic de Rham cohomology.

Definition 2.2.24. We define the *algebraic de Rham complex* $\Omega_k^\bullet$ the object in $\mathbf{DA}^{\mathrm{eff}}_{ét}(k;k)$ given by the complex of presheaves of k-vector spaces over Sm_k

$$\begin{aligned}\Omega_k^\bullet : Sm_k^{op} &\to \mathrm{Ch}(k)\\ X &\mapsto \Gamma(X, \Omega^\bullet_{X/k}),\end{aligned}$$

where $\Omega^\bullet_{X/k}$ is the algebraic de Rham complex of $X \in Sm_k$ defined in the previous chapter (see subsection 1.3.2).

Remark 2.2.25. The complex of presheaves $\Omega_k^\bullet$ is in fact a complex of sheaves over the big étale site on Sm_k . Indeed, for any $X \in Sm_k$, since the covers of X in the big étale site on Sm_k are the same of the small étale site $X_{ét}$, then it suffices to prove that the restriction of $\Omega_k^\bullet$ to $X_{ét}$ is a complex of sheaves. This is true since, for each $p \geq 0$, by remark 1.5.8, the restriction of Ω_k^p to $X_{ét}$ is the étale sheaf $(\Omega^p_{X/k})^{ét}$. So, we can see $\Omega_k^\bullet$ as an object of $\mathbf{D}_{ét}$. Moreover, notice that it is an $\mathbb{A}^1$-local object, i.e., for any $X \in Sm_k$ and $n \in \mathbb{Z}$,

$$\mathbb{H}^n(X_{ét}, (\Omega^\bullet_{X/k})^{ét}) \cong \mathbb{H}^n((\mathbb{A}^n_X)_{ét}, (\Omega^\bullet_{\mathbb{A}^n_X/k})^{ét}).$$

Indeed, by $\mathbb{A}^1$-invariance property (proposition 1.5.6) and étale descent property (proposition 1.5.9) of algebraic de Rham cohomology, it holds that

$$\mathbb{H}^n(X_{ét}, (\Omega^\bullet_{X/k})^{ét}) \cong \mathrm{H}^n_{\mathrm{AdR}}(X/k) \cong \mathrm{H}^n_{\mathrm{AdR}}(\mathbb{A}^1_X/k) \cong \mathbb{H}^n((\mathbb{A}^n_X)_{ét}, (\Omega^\bullet_{\mathbb{A}^n_X/k})^{ét}).$$

Since $\Omega_k^\bullet$ is a $\mathbb{A}^1$-local object, then, by remark 2.2.12, it holds that, for any $X \in Sm_k$ and $n \in \mathbb{Z}$,

$$\begin{aligned}\mathrm{Hom}_{\mathbf{DA}^{\mathrm{eff}}_{ét}(k;k)}(\mathbf{M}(X), \Omega_k^\bullet[n]) &\cong \mathrm{Hom}_{\mathbf{D}_{ét}}(\mathbf{\Lambda}_{ét}(X), \Omega_k^\bullet[n]) \cong\\ &\cong \mathbb{H}^n(X, \Omega_k^\bullet) \cong \mathbb{H}^n(X_{ét}, (\Omega^\bullet_{X/k})^{ét}) \cong \mathrm{H}^n_{\mathrm{AdR}}(X/k).\end{aligned}$$

In other words, the object

$$\Omega_k^\bullet \in \mathbf{DA}^{\mathrm{eff}}_{ét}(k;k)$$

represents the algebraic de Rham cohomology in the category $\mathbf{DA}^{\mathrm{eff}}_{ét}(k;k)$.

Recall that in the previous section we defined the object

$$\mathrm{R}^{\mathrm{eff}}_{\mathrm{Bet}*}\mathbb{Q} \in \mathbf{DA}^{\mathrm{eff}}_{ét}(\mathbb{C};\mathbb{Q}),$$

which represents the Betti cohomology in $\mathbf{DA}^{\mathrm{eff}}_{ét}(\mathbb{C};\mathbb{Q})$. Now, assuming that we have $\sigma : k \hookrightarrow \mathbb{C}$ a field extension, we want to compare these two objects inside the category

$$\mathbf{DA}^{\text{eff}}_{\acute{e}t}(k; \mathbb{C}).$$

First, we need to describe some morphisms between triangulated categories of motives, induced by the change of the base field and extension of scalars.

Since the property of being a smooth algebraic variety is stable under base change, σ induces a functor

$$\begin{aligned} Sm_k &\to Sm_{\mathbb{C}} \\ X &\mapsto X_\sigma := X \times_{Spec(k)} Spec(\mathbb{C}). \end{aligned}$$

It induces the usual adjunction between the categories of presheaves of $\mathbb{Q}$-vector spaces, which extends to the categories of complexes

$$\sigma_p : \mathrm{Ch}(\mathrm{PSh}(Sm_k; \mathbb{Q})) \rightleftarrows \mathrm{Ch}(\mathrm{PSh}(Sm_{\mathbb{C}}; \mathbb{Q})) : \sigma^p. \tag{2.17}$$

Proposition 2.2.26. *The adjunction 2.17 is a Quillen pair with respect to the projective global model structures. It induces the Quillen pair*

$$\sigma_p : L_{(\mathbb{A}^1, \acute{e}t)} Ch(PSh(Sm_k; \mathbb{Q})) \rightleftarrows L_{(\mathbb{A}^1, \acute{e}t)} Ch(PSh(Sm_{\mathbb{C}}; \mathbb{Q})) : \sigma^p. \tag{2.18}$$

with respect to the projective $(\mathbb{A}^1, \acute{e}t)$-local model structures

Proof. The adjunction 2.17 is a Quillen pair because σ^p is a right Quillen functor, i.e. preserves fibrations and trivial fibrations. This is true by definition of σ^p and since projective global fibrations and trivial fibrations can be checked open-wise. To prove that 2.18 is a Quillen pair, using the universal property of localization of model categories, it suffices to prove that σ_p maps morphisms in $S_{(\mathbb{A}^1, \acute{e}t)}$ of $\mathrm{Ch}(\mathrm{PSh}(Sm_k; \mathbb{Q}))$ into morphisms in $S_{(\mathbb{A}^1, \acute{e}t)}$ of $\mathrm{Ch}(\mathrm{PSh}(Sm_{\mathbb{C}}; \mathbb{Q}))$. This is true because σ_p preserves representable functors and morphisms in Sm_k of the kind

$$U_* \to X \qquad \& \qquad \mathbb{A}^1_X \to X$$

are such that their base change along σ

$$U_{\sigma *} \to X_\sigma \qquad \& \qquad \mathbb{A}^1_{X_\sigma} \to X_\sigma$$

are in $Sm_{\mathbb{C}}$, since the properties of being smooth and étale are stable under base change. $\square$

The Quillen pair 2.18 induces the total derived adjunction

$$\sigma^* := \mathbf{L}\sigma_p : \mathbf{DA}^{\text{eff}}_{\acute{e}t}(k; \mathbb{Q}) \rightleftarrows \mathbf{DA}^{\text{eff}}_{\acute{e}t}(\mathbb{C}; \mathbb{Q}) : \mathbf{R}\sigma^p =: \sigma_*.$$

Composing with the adjunction of the Betti realization, we get the adjunction

$$\mathrm{R}^{\mathrm{eff}\ *}_{\mathrm{Bet},\sigma} : \mathbf{DA}^{\mathrm{eff}}_{\acute{e}t}(k;\mathbb{Q}) \underset{\sigma_*}{\overset{\sigma^*}{\rightleftarrows}} \mathbf{DA}^{\mathrm{eff}}_{\acute{e}t}(\mathbb{C};\mathbb{Q}) \underset{\mathrm{R}^{\mathrm{eff}}_{\mathrm{Bet}*}}{\overset{\mathrm{R}^{\mathrm{eff}\ *}_{\mathrm{Bet}}}{\rightleftarrows}} \mathbf{DA}^{\mathrm{eff}}_{\acute{e}t}(\mathbb{C};\mathbb{Q}) : \mathrm{R}^{\mathrm{eff}}_{\mathrm{Bet},\sigma*}.$$

Remark 2.2.27. For any $X \in Sm_k$, it holds that, for any $n \in \mathbb{Z}$

$$\begin{aligned}\mathrm{Hom}_{\mathbf{DA}^{\mathrm{eff}}_{\acute{e}t}(k;\mathbb{Q})}(\mathbf{M}(X), \mathrm{R}^{\mathrm{eff}}_{\mathrm{Bet},\sigma*}\mathbb{Q}[n]) &\cong \mathrm{Hom}_{\mathbf{DA}^{\mathrm{eff}}_{\acute{e}t}(\mathbb{C};\mathbb{Q})}(\sigma^*\mathbf{M}(X), \mathrm{R}^{\mathrm{eff}}_{\mathrm{Bet}*}\mathbb{Q}[n]) \cong \\ &\cong \mathrm{Hom}_{\mathbf{DA}^{\mathrm{eff}}_{\acute{e}t}(\mathbb{C};\mathbb{Q})}(\mathbf{M}(X_\sigma), \mathrm{R}^{\mathrm{eff}}_{\mathrm{Bet}*}\mathbb{Q}[n]) \cong \mathrm{H}^n_{\mathrm{Bet}}(X_\sigma).\end{aligned}$$

That is, the object

$$\mathrm{R}^{\mathrm{eff}}_{\mathrm{Bet},\sigma*}\mathbb{Q} \in \mathbf{DA}^{\mathrm{eff}}_{\acute{e}t}(k;\mathbb{Q})$$

represents the Betti cohomology of the base change along σ in $\mathbf{DA}^{\mathrm{eff}}_{\acute{e}t}(k;\mathbb{Q})$.

Now, let $\Lambda \hookrightarrow \mathbb{C}$ be a field extension. The adjunction extension of scalars-forgetful functor between the categories of Λ and $\mathbb{C}$-vector spaces

$$\_ \otimes_\Lambda \mathbb{C} : Vect_\Lambda \rightleftarrows Vect_{\mathbb{C}} : U.$$

extends to the categories of complexes of presheaves over Sm_k

$$\_ \otimes_\Lambda \mathbb{C} : \mathrm{Ch}(\mathrm{PSh}(Sm_k;\Lambda)) \rightleftarrows \mathrm{Ch}(\mathrm{PSh}(Sm_k;\mathbb{C})) : U. \tag{2.19}$$

Proposition 2.2.28. *The adjunction 2.19 is a Quillen pair with respect to the projective global model structures. It induces the Quillen pair*

$$\_ \otimes_\Lambda \mathbb{C} : L_{(\mathbb{A}^1,\acute{e}t)}Ch(PSh(Sm_k;\Lambda)) \rightleftarrows L_{(\mathbb{A}^1,\acute{e}t)}Ch(PSh(Sm_k;\mathbb{C})) : U. \tag{2.20}$$

with respect to the projective $(\mathbb{A}^1, \acute{e}t)$*-local model structures*

Proof. The adjunction 2.19 is a Quillen pair because the forgetful functor U is a right Quillen functor, i.e. preserves fibrations and trivial fibrations. This is true because projective global fibrations and trivial fibrations can be checked open-wise and projective fibrations and projective trivial fibrations of complexes of $\mathbb{C}$-vector spaces are also of Λ-vectors spaces. To prove that 2.20 is a Quillen pair, using the universal property of localization of model categories, it suffices to prove that $\_ \otimes_\Lambda \mathbb{C}$ maps morphisms in $S_{(\mathbb{A}^1,\acute{e}t)}$ of $\mathrm{Ch}(\mathrm{PSh}(Sm_k;\Lambda))$ into morphisms in $S_{(\mathbb{A}^1,\acute{e}t)}$ of $\mathrm{Ch}(\mathrm{PSh}(Sm_k;\mathbb{C}))$. This is true because $\_ \otimes_\Lambda \mathbb{C}$ maps the presheaf of Λ-vector spaces represented by X in $\mathrm{Ch}(\mathrm{PSh}(Sm_k;\Lambda))$

$$\mathbf{\Lambda}(X) : U \mapsto \Lambda[\mathrm{Hom}_{Sm_k}(U, X)]$$

into the presheaf of $\mathbb{C}$-vector spaces represented by X in $\mathrm{Ch}(\mathrm{PSh}(Sm_k;\mathbb{C}))$

$$\mathbb{C}(X) \cong \mathbf{\Lambda}(X) \otimes_\Lambda \mathbb{C} : U \mapsto \Lambda[\mathrm{Hom}_{Sm_k}(U,X)] \otimes_\Lambda \mathbb{C} \cong \mathbb{C}[\mathrm{Hom}_{Sm_k}(U,X)].$$

□

Notice that $\_ \otimes_\Lambda \mathbb{C}$ in the Quillen pair 2.19 is exact, since it is at the level of vector spaces. Since it also preserves the set of morphisms $S_{(\mathbb{A}^1,ét)}$, then it is also exact in the Quillen pair 2.20. Hence, the Quillen pair 2.20 induces the total derived adjunction

$$\_ \otimes_\Lambda \mathbb{C} : \mathbf{DA}^{\mathrm{eff}}_{ét}(k;\Lambda) \rightleftarrows \mathbf{DA}^{\mathrm{eff}}_{ét}(k;\mathbb{C}) : \mathbf{R}U.$$

Now, we are able to state and prove the comparison result between the representative objects of algebraic de Rham and Betti cohomologies.

Theorem 2.2.29. *Let $\sigma : k \hookrightarrow \mathbb{C}$ be a field extension. Then, there exists a canonical isomorphism in $\mathbf{DA}^{\mathit{eff}}_{ét}(k;\mathbb{C})$*

$$\Omega^\bullet_k \otimes_k \mathbb{C} \cong R^{\mathit{eff}}_{Bet,\sigma*}\mathbb{Q} \otimes_\mathbb{Q} \mathbb{C}.$$

Proof. The canonical isomorphism is obtained by composition of the following canonical isomorphisms.

- We denote by $\Omega^\bullet_{\mathbb{C}^0}$ the object in $\mathbf{AnDA}^{\mathrm{eff}}(\mathbb{C})$ given by the complex of presheaves of $\mathbb{C}$-vector spaces over $AnSm_\mathbb{C}$

  $$\begin{aligned}\Omega^\bullet_{\mathbb{C}^0} : AnSm^{op}_\mathbb{C} &\to \mathrm{Ch}(\mathbb{C})\\ \mathcal{Y} &\mapsto \Gamma(\mathcal{Y},\Omega^\bullet_\mathcal{Y}),\end{aligned}$$

  where $\Omega^\bullet_\mathcal{Y}$ is the analytic de Rham complex of $\mathcal{Y} \in AnSm_\mathbb{C}$. It is in fact a complex of sheaves over the big étale-analytic site on $AnSm_\mathbb{C}$. Indeed, for any $\mathcal{Y} \in AnSm_\mathbb{C}$, since any cover of $\mathcal{Y}$ in the big étale-analytic site is refined by one of the classical small analytic site $\mathcal{Y}_{an}$, then, it suffices to prove that the restriction of $\Omega^\bullet_{\mathbb{C}^0}$ to $\mathcal{Y}_{an}$ is a complex of sheaves. This is true since this restriction is $\Omega^\bullet_\mathcal{Y}$, which is a complex of sheaves over $\mathcal{Y}_{an}$. Consider $\mathbb{C}^{\textit{ét-an}}_{cst} \in \mathbf{AnDA}^{\mathrm{eff}}(\mathbb{C})$, the constant sheaf over the big étale-analytic site on $AnSm_\mathbb{C}$. We have a canonical morphism of complexes of sheaves

  $$\mathbb{C}^{\textit{ét-an}}_{cst} \to \Omega^\bullet_{\mathbb{C}^0},$$

  such that, for any $\mathcal{Y} \in AnSm_\mathbb{C}$, its restriction to the classical small analytic site $\mathcal{Y}_{an}$ is the canonical quasi-isomorphism 1.2 of the holomorphic version of Poincaré Lemma

  $$\mathbb{C}_\mathcal{Y} \to \Omega^\bullet_\mathcal{Y}.$$

Since any cover of $\mathcal{Y}$ in the big étale-analytic site is refined by one of the classical small analytic site $\mathcal{Y}_{an}$, this implies that $\mathbb{C}_{cst}^{\acute{e}t\text{-}an} \to \Omega^{\bullet}_{\mathbb{C}^0}$ is a quasi-isomorphism of sheaves, hence an isomorphism in $\mathbf{D}_{\acute{e}t\text{-}an}$ and also in $\mathbf{AnDA}^{\mathrm{eff}}(\mathbb{C})$. Consider $\mathbb{C}_{cst}$, the constant presheaf over $AnSm_{\mathbb{C}}$, which is canonically isomorphic to $\mathbb{C}_{cst}^{\acute{e}t\text{-}an}$ in $\mathbf{AnDA}^{\mathrm{eff}}(\mathbb{C})$, via the canonical morphism of sheafification. Hence, we have the canonical isomorphisms in $\mathbf{AnDA}^{\mathrm{eff}}(\mathbb{C})$

$$\mathbb{C}_{cst} \cong \mathbb{C}_{cst}^{\acute{e}t\text{-}an} \cong \Omega^{\bullet}_{\mathbb{C}^0}.$$

Applying the functor $\sigma_*\mathrm{An}_*$, we obtain the canonical isomorphism in $\mathbf{DA}^{\mathrm{eff}}_{\acute{e}t}(k;\mathbb{C})$

$$\mathrm{R}^{\mathrm{eff}}_{\mathrm{Bet},\sigma*}\mathbb{C} \cong \sigma_*\mathrm{An}_*\mathbb{C}_{cst} \cong \sigma_*\mathrm{An}_*\Omega^{\bullet}_{\mathbb{C}^0}.$$

Moreover, we have that in $\mathbf{DA}^{\mathrm{eff}}_{\acute{e}t}(k;\mathbb{C})$

$$\mathrm{R}^{\mathrm{eff}}_{\mathrm{Bet},\sigma*}\mathbb{C} \cong \mathrm{R}^{\mathrm{eff}}_{\mathrm{Bet},\sigma*}\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{C}.$$

Indeed, reasoning as in step 1 and 2 of proposition 2.2.19, we see that $\mathbf{DA}^{\mathrm{eff}}_{\acute{e}t}(k;\mathbb{C})$ coincides with its triangulated subcategory generated by motives associated to smooth algebraic varieties over k

$$\mathbf{DA}^{\mathrm{eff}}_{\acute{e}t}(k;\mathbb{C}) = \langle \mathbf{M}(X) \mid X \in Sm_k \rangle.$$

Hence, it suffices to prove that, for any $X \in Sm_k$ and $n \in \mathbb{Z}$,

$$\mathrm{Hom}_{\mathbf{DA}^{\mathrm{eff}}_{\acute{e}t}(k;\mathbb{C})}(\mathbf{M}(X), \mathrm{R}^{\mathrm{eff}}_{\mathrm{Bet},\sigma*}\mathbb{C}[n]) \cong \mathrm{Hom}_{\mathbf{DA}^{\mathrm{eff}}_{\acute{e}t}(k;\mathbb{C})}(\mathbf{M}(X), \mathrm{R}^{\mathrm{eff}}_{\mathrm{Bet},\sigma*}\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{C}[n]).$$

On one hand, by remarks 2.2.20 and 2.2.27, we have that

$$\mathrm{Hom}_{\mathbf{DA}^{\mathrm{eff}}_{\acute{e}t}(k;\mathbb{C})}(\mathbf{M}(X), \mathrm{R}^{\mathrm{eff}}_{\mathrm{Bet},\sigma*}\mathbb{C}[n]) \cong \mathrm{Hom}_{\mathbf{DA}^{\mathrm{eff}}_{\acute{e}t}(\mathbb{C};\mathbb{C})}(\mathbf{M}(X_\sigma), \mathrm{R}^{\mathrm{eff}}_{\mathrm{Bet}*}\mathbb{C}[n]) \cong \mathrm{H}^n_{\mathrm{Sing}}(X_\sigma(\mathbb{C});\mathbb{C}).$$

On the other hand, by remark 2.2.27, we have that

$$\begin{aligned}\mathrm{Hom}_{\mathbf{DA}^{\mathrm{eff}}_{\acute{e}t}(k;\mathbb{C})}(\mathbf{M}(X), \mathrm{R}^{\mathrm{eff}}_{\mathrm{Bet},\sigma*}\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{C}[n]) &\cong \mathrm{Hom}_{\mathbf{DA}^{\mathrm{eff}}_{\acute{e}t}(k;\mathbb{Q})}(\mathbf{M}(X), \mathrm{R}^{\mathrm{eff}}_{\mathrm{Bet},\sigma*}\mathbb{Q}[n]) \otimes_{\mathbb{Q}} \mathbb{C} \cong \\ &\cong \mathrm{H}^n_{\mathrm{Sing}}(X_\sigma(\mathbb{C});\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}.\end{aligned}$$

Then, the wanted isomorphism follows because

$$\mathrm{H}^n_{\mathrm{Sing}}(X_\sigma(\mathbb{C});\mathbb{C}) \cong \mathrm{H}^n_{\mathrm{Sing}}(X_\sigma(\mathbb{C});\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}.$$

So, we obtained the canonical isomorphism in $\mathbf{DA}^{\mathrm{eff}}_{\acute{e}t}(k;\mathbb{C})$

$$\mathrm{R}^{\mathrm{eff}}_{\mathrm{Bet},\sigma*}\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{C} \cong \sigma_*\mathrm{An}_*\Omega^{\bullet}_{\mathbb{C}^0}. \tag{2.21}$$

- We want to prove that there exists a canonical isomorphism in $\mathbf{DA}^{\text{eff}}_{ét}(\mathbb{C};\mathbb{C})$

$$\mathrm{An}_*\Omega^\bullet_{\mathbb{C}^0} \cong \Omega^\bullet_{\mathbb{C}}.$$

Since $\mathbf{DA}^{\text{eff}}_{ét}(\mathbb{C};\mathbb{C})$ coincides with its triangulated subcategory generated by motives associated to smooth algebraic varieties over $\mathbb{C}$, then, it suffices to prove that, for any $X \in Sm_{\mathbb{C}}$ and $n \in \mathbb{Z}$,

$$\mathrm{Hom}_{\mathbf{DA}^{\text{eff}}_{ét}(\mathbb{C};\mathbb{C})}(\mathbf{M}(X), \mathrm{An}_*\Omega^\bullet_{\mathbb{C}^0}[n]) \cong \mathrm{Hom}_{\mathbf{DA}^{\text{eff}}_{ét}(\mathbb{C};\mathbb{C})}(\mathbf{M}(X), \Omega^\bullet_{\mathbb{C}}[n]).$$

On one hand, since $\Omega^\bullet_{\mathbb{C}}$ represents the algebraic de Rham cohomology, then

$$\mathrm{Hom}_{\mathbf{DA}^{\text{eff}}_{ét}(\mathbb{C};\mathbb{C})}(\mathbf{M}(X), \Omega^\bullet_{\mathbb{C}}[n]) \cong \mathrm{H}^n_{\mathrm{AdR}}(X/\mathbb{C}).$$

On the other hand, analogously to remark 2.2.25, we see that $\Omega^\bullet_{\mathbb{C}^0}$ is a $\mathbb{D}^1$-local object. Analogously to the algebraic case, we deduce that $\Omega^\bullet_{\mathbb{C}^0}$ represents the analytic de Rham cohomology. So, we have

$$\begin{aligned}\mathrm{Hom}_{\mathbf{DA}^{\text{eff}}_{ét}(\mathbb{C};\mathbb{C})}(\mathbf{M}(X), \mathrm{An}_*\Omega^\bullet_{\mathbb{C}^0}[n]) &\cong \mathrm{Hom}_{\mathbf{AnDA}^{\text{eff}}(\mathbb{C})}(\mathrm{An}^*\mathbf{M}(X), \Omega^\bullet_{\mathbb{C}^0}[n]) \cong \\ &\cong \mathrm{Hom}_{\mathbf{AnDA}^{\text{eff}}(\mathbb{C})}(\mathbf{M}_{an}(X(\mathbb{C})), \Omega^\bullet_{\mathbb{C}^0}[n]) \cong \\ &\cong \mathrm{Hom}_{\mathbf{D}_{ét\text{-}an}}(\mathbb{C}_{ét\text{-}an}(X(\mathbb{C})), \Omega^\bullet_{\mathbb{C}^0}[n]) \cong \\ &\cong \mathbb{H}^n(X(\mathbb{C})_{an}, \Omega^\bullet_{X(\mathbb{C})}) = \mathrm{H}^n_{\mathrm{dR}}(X(\mathbb{C})).\end{aligned}$$

Then, the wanted isomorphism follows from the comparison theorem between algebraic and analytic de Rham cohomology (theorem 1.5.24). Applying functor σ_* we obtain the canonical isomorphism in $\mathbf{DA}^{\text{eff}}_{ét}(k;\mathbb{C})$

$$\sigma_*\mathrm{An}_*\Omega^\bullet_{\mathbb{C}^0} \cong \sigma_*\Omega^\bullet_{\mathbb{C}}. \tag{2.22}$$

- Finally, we prove that there exists a canonical isomorphism in $\mathbf{DA}^{\text{eff}}_{ét}(k;\mathbb{C})$

$$\Omega^\bullet_k \otimes_k \mathbb{C} \cong \sigma_*\Omega^\bullet_{\mathbb{C}}. \tag{2.23}$$

As in the previous point, it suffices to prove that for any $X \in Sm_k$ and $n \in \mathbb{Z}$,

$$\mathrm{Hom}_{\mathbf{DA}^{\text{eff}}_{ét}(k;\mathbb{C})}(\mathbf{M}(X), \Omega^\bullet_k \otimes_k \mathbb{C}[n]) \cong \mathrm{Hom}_{\mathbf{DA}^{\text{eff}}_{ét}(k;\mathbb{C})}(\mathbf{M}(X), \sigma_*\Omega^\bullet_{\mathbb{C}}[n]).$$

On one hand, we have that

$$\begin{aligned}\mathrm{Hom}_{\mathbf{DA}^{\text{eff}}_{ét}(k;\mathbb{C})}(\mathbf{M}(X), \Omega^\bullet_k \otimes_k \mathbb{C}[n]) &\cong \mathrm{Hom}_{\mathbf{DA}^{\text{eff}}_{ét}(k;\mathbb{Q})}(\mathbf{M}(X), \Omega^\bullet_k[n]) \otimes_k \mathbb{C} \cong \\ &\cong \mathrm{H}^n_{\mathrm{AdR}}(X/k) \otimes_k \mathbb{C}.\end{aligned}$$

On the other hand, we have that

$$\begin{aligned}\mathrm{Hom}_{\mathbf{DA}^{\mathrm{eff}}_{\acute{e}t}(k;\mathbb{C})}(\mathbf{M}(X),\sigma_*\Omega^\bullet_{\mathbb{C}}[n]) &\cong \mathrm{Hom}_{\mathbf{DA}^{\mathrm{eff}}_{\acute{e}t}(\mathbb{C};\mathbb{C})}(\sigma^*\mathbf{M}(X),\Omega^\bullet_{\mathbb{C}}[n]) \cong \\ &\cong \mathrm{Hom}_{\mathbf{DA}^{\mathrm{eff}}_{\acute{e}t}(\mathbb{C};\mathbb{C})}(\mathbf{M}(X_\sigma),\Omega^\bullet_{\mathbb{C}}[n]) \cong \\ &\cong \mathrm{H}^n_{\mathrm{AdR}}(X_\sigma/\mathbb{C}).\end{aligned}$$

Then, the wanted isomorphism follows because we proved that (see the third point in theorem 1.6.2)

$$\mathrm{H}^n_{\mathrm{AdR}}(X/k)\otimes_k\mathbb{C}\cong\mathrm{H}^n_{\mathrm{AdR}}(X_\sigma/\mathbb{C}).$$

Putting together the canonical isomorphisms 2.21, 2.22 and 2.23, we obtain the canonical isomorphism in $\mathbf{DA}^{\mathrm{eff}}_{\acute{e}t}(k;\mathbb{C})$

$$\mathrm{R}^{\mathrm{eff}}_{\mathrm{Bet},\sigma*}\mathbb{Q}\otimes_{\mathbb{Q}}\mathbb{C}\cong\sigma_*\mathrm{An}_*\Omega^\bullet_{\mathbb{C}^0}\cong\sigma_*\Omega^\bullet_{\mathbb{C}}\cong\Omega^\bullet_k\otimes_k\mathbb{C}.$$

□

2.3 The Grothendieck Period Conjecture

In this section we describe a statement of the Grothendieck Period Conjecture. References are [BC14, §1.4] (where is discussed also another formulation, involving the motivic Galois group, and relations between them), [And+20, §1.3] and [And04, §7.5].

2.3.1 The pure case

First, we consider the setting of smooth projective algebraic varieties. Let $\sigma:\overline{\mathbb{Q}}\hookrightarrow\mathbb{C}$ be the inclusion. Given $X\in SmProj_{\overline{\mathbb{Q}}}$, recall that in subsection 2.1.2 we defined the twisted algebraic de Rham cohomology

$$\mathrm{H}^p_{\mathrm{AdR}}(X/\overline{\mathbb{Q}})(q)\coloneqq\mathrm{H}^p_{\mathrm{AdR}}(X/\overline{\mathbb{Q}})$$

and the twisted Betti cohomology

$$\mathrm{H}^p_{\mathrm{Bet}}(X_\sigma)(q)\coloneqq(2\pi i)^q\mathrm{H}^p_{\mathrm{Bet}}(X_\sigma).$$

We also proved the existence of a canonical isomorphism between them, taking coefficients in $\mathbb{C}$,

$$\varpi^{p,q}:\mathrm{H}^p_{\mathrm{AdR}}(X/\overline{\mathbb{Q}})(q)\otimes_{\overline{\mathbb{Q}}}\mathbb{C}\cong\mathbb{H}^p(X_\sigma(\mathbb{C})_{an},\mathbb{C}_{X_\sigma(\mathbb{C})})\cong\mathrm{H}^p_{\mathrm{Bet}}(X_\sigma)(q)\otimes_{\mathbb{Q}}\mathbb{C}.$$

Consider $\{\omega_j\}$ a $\overline{\mathbb{Q}}$-basis of $\mathrm{H}^p_{\mathrm{AdR}}(X/\overline{\mathbb{Q}})$ and $\{\gamma_i\}$ a $\mathbb{Q}$-basis of $\mathrm{H}^{\mathrm{Sing}}_p(X_\sigma(\mathbb{C});\mathbb{Q})$. Hence, $\{(2\pi i)^q\tilde{\gamma}_i\}$ is a $\mathbb{Q}$-basis of $\mathrm{H}^p_{\mathrm{Bet}}(X_\sigma)(q)$, where each $\tilde{\gamma}_i \in \mathrm{H}^p_{\mathrm{Sing}}(X_\sigma(\mathbb{C});\mathbb{Q}) \cong \mathrm{H}^p_{\mathrm{Bet}}(X_\sigma)$ is the dual element of γ_i. Recall from subsection 1.6.2, that the representative matrix of $\varpi^{p,q}$, with respect to the bases $\{\omega_j\}$ and $\{(2\pi i)^q\tilde{\gamma}_i\}$, is given by

$$\left[\frac{1}{(2\pi i)^q}\int_{\gamma_i}\omega_j\right]_{i,j}.$$

That is, we have, for any j,

$$\varpi^{p,q}(\omega_j) = \sum_i \left(\frac{1}{(2\pi i)^q}\int_{\gamma_i}\omega_j\right)(2\pi i)^q\tilde{\gamma}_i.$$

Moreover, we proved the compatibility of $\varpi^{p,q}$ with the cycle class maps (proposition 2.1.15). That is, we have the commutative square, for any $q \geq 0$,

$$\begin{tikzcd} \mathrm{CH}^q(X)_{\mathbb{Q}} \arrow[r, "cl^q_{\mathrm{Bet},X}\circ\tilde{\sigma}^*"] \arrow[d, "cl^q_{\mathrm{AdR},X}"] & \mathrm{H}^{2q}_{\mathrm{Bet}}(X_\sigma)(q) \arrow[d, hook] \\ \mathrm{H}^{2q}_{\mathrm{AdR}}(X/\overline{\mathbb{Q}})(q) \arrow[r, hook] & \mathbb{H}^{2q}(X_\sigma(\mathbb{C})_{an}, \mathbb{C}_{X_\sigma(\mathbb{C})}). \end{tikzcd} \tag{2.24}$$

Notice that this commutative square tells that $\mathbb{Q}$-linear algebraic cycles of X induce $\overline{\mathbb{Q}}[\pi^{-1}]$-linear relations between periods of X as follows. Let $\alpha \in \mathrm{CH}^q(X)_{\mathbb{Q}}$ be the rational class of a $\mathbb{Q}$-linear algebraic cycle of X. We write

$$cl^q_{\mathrm{AdR},X}(\alpha) = \sum_j a_j\omega_j \qquad \& \qquad cl^q_{\mathrm{Bet},X}\circ\tilde{\sigma}^*(\alpha) = \sum_i b_i(2\pi i)^q\tilde{\gamma}_i,$$

with $a_j \in \overline{\mathbb{Q}}$ and $b_i \in \mathbb{Q}$. Then, by commutativity of 2.24, we have that, for any i,

$$\sum_j a_j\frac{1}{(2\pi i)^q}\int_{\gamma_i}\omega_j = b_i,$$

which are $\overline{\mathbb{Q}}[\pi^{-1}]$-linear relations between periods of X. More generally, using Künneth formula, we can see that $\mathbb{Q}$-linear algebraic cycles over products $X^n = X\times_{\overline{\mathbb{Q}}}\cdots\times_{\overline{\mathbb{Q}}} X$ induce some polynomial relations with coefficients in $\overline{\mathbb{Q}}[\pi^{-1}]$ of homogeneous degree n between periods of X.

Grothendieck conjectured that these are in fact all possible polynomial relations between periods. More precisely, the conjecture can be stated as follows. We denote by

$$\mathrm{H}^{p,q}_{\varpi}(X) := \mathrm{H}^p_{\mathrm{AdR}}(X/\overline{\mathbb{Q}})(q)\cap\mathrm{H}^p_{\mathrm{Bet}}(X_\sigma)(q),$$

where the intersection is taken inside $\mathbb{H}^p(X_\sigma(\mathbb{C})_{an}, \mathbb{C}_{X_\sigma(\mathbb{C})})$. As above, an element in $\mathrm{H}^{p,q}_{\varpi}(X)$ induces a $\overline{\mathbb{Q}}[\pi^{-1}]$-linear relation between periods of X. More generally, an element in $\mathrm{H}^{p,q}_{\varpi}(X^n)$ induces a polynomial relation with coefficients in $\overline{\mathbb{Q}}[\pi^{-1}]$ of homogeneous degree n between periods of X. By commutativity of the diagram 2.24, we deduce that the algebraic de Rham and Betti cycle class maps induce a morphism

$$cl^q_{\varpi,X} : \mathrm{CH}^q(X)_{\mathbb{Q}} \to \mathrm{H}^{2q,q}_{\varpi}(X).$$

This map assigns to each algebraic cycle the corresponding polynomial relation between periods induced by it.

Conjecture 2.3.1 (Grothendieck Period Conjecture)**.** *Given $X \in SmProj_{\overline{\mathbb{Q}}}$, we say that the Grothendieck Period Conjecture holds for X, if, for any $q \geq 0$, the morphism*

$$cl^q_{\varpi,X} : CH^q(X)_{\mathbb{Q}} \to H^{2q,q}_{\varpi}(X).$$

is surjective.

The Grothendieck Period Conjecture can be expressed, equivalently, by stating that a certain realization functor from the category of Chow motives is full. Let $\sigma : k \hookrightarrow \mathbb{C}$ be a field extension. Since algebraic de Rham and Betti cohomologies are Weil cohomologies, we have realization functors from the category of Chow motives

$$R_{\mathrm{AdR}} : \mathrm{CHM}(k;\mathbb{Q}) \to Vect_{\overline{\mathbb{Q}}}$$

and

$$R_{\mathrm{Bet},\sigma} : \mathrm{CHM}(k;\mathbb{Q}) \to \mathrm{CHM}(\mathbb{C};\mathbb{Q}) \xrightarrow{R_{\mathrm{Bet}}} Vect_{\mathbb{Q}},$$

where the first arrow maps Chow motives of the kind $h(X)$ into $h(X_\sigma)$.

We consider the $\mathbb{Q}$-linear abelian category

$$Vect_{k,\mathbb{Q}}$$

whose objects are triples $(V_k, V_{\mathbb{Q}}, \varpi)$, where V_k is a finite dimensional k-vector space, $V_{\mathbb{Q}}$ is a finite dimensional $\mathbb{Q}$-vector space and $\varpi : V_k \otimes_k \mathbb{C} \cong V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$ is an isomorphism of $\mathbb{C}$-vector spaces. It is a tensor abelian category with tensor product the usual tensor product of vector spaces and unit $\mathbb{1} := (k, \mathbb{Q}, id_{\mathbb{C}})$. Morphisms of triples are given by morphisms of k-vector spaces between the first components and morphisms of $\mathbb{Q}$-vector spaces between the second components, which are compatible with the isomorphisms given by the third component. The compatibility of algebraic de Rham and Betti cycle class maps with the twisted algebraic de Rham isomorphism allows to define a functor

$$R_{\varpi} : \mathrm{CHM}(k;\mathbb{Q}) \to Vect_{k,\mathbb{Q}},$$

which assigns to Chow motives of the kind $h(X)$, the triple

$$(\oplus_{i\geq 0}\mathrm{H}^i_{\mathrm{AdR}}(X/k), \oplus_{i\geq 0}\mathrm{H}^i_{\mathrm{Bet}}(X_\sigma), \oplus_{i\geq 0}\varpi^i).$$

Notice that composition with forgetful functors, given by the projections to the first and second components, give back the algebraic de Rham realization functor

$$R_{\mathrm{AdR}} : \mathrm{CHM}(k;\mathbb{Q}) \xrightarrow{R_\varpi} Vect_{k,\mathbb{Q}} \to Vect_k$$

and the Betti realization functor

$$R_{\mathrm{Bet},\sigma} : \mathrm{CHM}(k;\mathbb{Q}) \xrightarrow{R_\varpi} Vect_{k,\mathbb{Q}} \to Vect_{\mathbb{Q}}.$$

For this reason, we say that R_ϖ is an *enrichment* of both algebraic de Rham and Betti realization functors.

Definition 2.3.2. The enriched realization functor R_ϖ is called the *de Rham-Betti realization functor*.

Proposition 2.3.3. *The Grothendieck Period Conjecture 2.3.1 is equivalent to ask that the de Rham-Betti realization functor*

$$R_\varpi : CHM(\overline{\mathbb{Q}};\mathbb{Q}) \to Vect_{\overline{\mathbb{Q}},\mathbb{Q}}$$

is full.

Proof. Notice that, given an object $(V_{\overline{\mathbb{Q}}}, V_{\mathbb{Q}}, \varpi) \in Vect_{\overline{\mathbb{Q}},\mathbb{Q}}$, to give a morphism

$$\mathbb{1} = (\overline{\mathbb{Q}}, \mathbb{Q}, id_{\mathbb{C}}) \to (V_{\overline{\mathbb{Q}}}, V_{\mathbb{Q}}, \varpi),$$

is equivalent to given an element in $V_{\overline{\mathbb{Q}}}$ and an element in $V_{\mathbb{Q}}$, which correspond via the isomorphism $\varpi : V_{\overline{\mathbb{Q}}} \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \cong V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$. This is also equivalent to give an element in

$$V_\varpi := V_{\overline{\mathbb{Q}}} \cap V_{\mathbb{Q}},$$

where the intersection is considered inside $V_{\overline{\mathbb{Q}}} \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \cong V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$. Recall from remark 2.1.23, that the algebraic de Rham and Betti cycle class maps can be recoverd as morphisms induced on hom-sets by their respective realization functors:

$$cl^i_{\mathrm{AdR},X} : \mathrm{CH}^i(X)_{\mathbb{Q}} \cong \mathrm{Hom}_{\mathrm{CHM}(\overline{\mathbb{Q}};\mathbb{Q})}(\mathbb{1}, h(X)(i)) \to \mathrm{Hom}_{grVect_{\overline{\mathbb{Q}}}}(\overline{\mathbb{Q}}, \mathrm{H}^*_{\mathrm{AdR}}(X/\overline{\mathbb{Q}})(i)) \cong \mathrm{H}^{2i}_{\mathrm{AdR}}(X/\overline{\mathbb{Q}})(i)$$

and

$$cl^i_{\mathrm{Bet},X} : \mathrm{CH}^i(X_\sigma)_{\mathbb{Q}} \cong \mathrm{Hom}_{\mathrm{CHM}(\mathbb{C};\mathbb{Q})}(\mathbb{1}, h(X_\sigma)(i)) \to \mathrm{Hom}_{grVect_{\mathbb{Q}}}(\mathbb{Q}, \mathrm{H}^*_{\mathrm{Bet}}(X_\sigma)(i)) \cong \mathrm{H}^{2i}_{\mathrm{Bet}}(X_\sigma)(i).$$

Then, we see that the morphism on hom-sets induced by the de Rham-Betti realization is

$$\mathrm{CH}^i(X)_{\mathbb{Q}} \cong \mathrm{Hom}_{\mathrm{CHM}(\overline{\mathbb{Q}};\mathbb{Q})}(\mathbb{1}, h(X)(i)) \to \mathrm{Hom}_{Vect_{\overline{\mathbb{Q}},\mathbb{Q}}}(\mathbb{1}, R_\varpi(h(X)(i))) \cong \mathrm{H}^{2i,i}_\varpi(X),$$

which is exactly the morphism $cl^i_{\varpi,X}$, considered in the statement of the Grothendieck Period Conjecture. So, fullness of the de Rham-Betti realization implies the Grothendieck Period Conjecture. Viceversa, assume that the Grothendieck Period Conjecture holds for any smooth projective algebraic variety over $\overline{\mathbb{Q}}$. For objects of the kind $h(X)(r), h(Y)(s) \in \mathrm{CHM}(\overline{\mathbb{Q}};\mathbb{Q})$, with X irreducible of dimension d, recall from 2.7, that the morphism on hom-sets induced by the algebraic de Rham and Betti realizations are given by suitable components of the corresponding cycle class maps of the product $X \times_k Y$. Then, we see that the morphism induced on hom-sets by the de Rham-Betti realization is given by

$$cl^{d+s-r}_{\varpi} : \mathrm{CH}^{d+s-r}(X \times_k Y) \to \mathrm{H}^{2(d+s-r),d+s-r}_{\varpi}(X \times_k Y),$$

which is surjective, by the Grothendieck Period Conjecture for $X \times_k Y$. For general X, the morphism induced on hom-sets is the direct sum of the ones obtained for each irreducible component of X, so it is surjective. For general objects $ph(X)(r), qh(Y)(s) \in \mathrm{CHM}(\overline{\mathbb{Q}};\mathbb{Q})$, by description of the algebraic de Rham and Betti realizations, we see that the morphism induced on hom-sets by the de Rham-Betti realization is a surjective restriction of the one obtained for $h(X)(r), h(Y)(s) \in \mathrm{CHM}(\overline{\mathbb{Q}};\mathbb{Q})$. So, the de Rham-Betti realization is full. $\square$

2.3.2 The mixed case

Now, we consider the more general setting of smooth algebraic varieties. Given $\sigma : k \hookrightarrow \mathbb{C}$ a field extension, we have a diagram in $\mathbf{DA}^{\mathrm{eff}}_{\acute{e}t}(k;\mathbb{Q})$, for any $q \geq 0$,

$$\begin{tikzcd} \mathbb{Q}(q) \arrow[r, "r^q_{\mathrm{Bet}}"] \arrow[d, "r^q_{\mathrm{AdR}}"] & \mathrm{R}^{\mathrm{eff}}_{\mathrm{Bet},\sigma*}\mathbb{Q}(q) \arrow[d] \\ \Omega^\bullet_k \arrow[r] & \mathrm{R}^{\mathrm{eff}}_{\mathrm{Bet},\sigma*}\mathbb{C}. \end{tikzcd} \tag{2.25}$$

The morphisms are defined as follows. In $\mathrm{R}^{\mathrm{eff}}_{\mathrm{Bet},\sigma*}\mathbb{Q}(q)$, by $\mathbb{Q}(q)$, we mean the object $\mathbb{Q}(q) \coloneqq (2\pi i)^q\mathbb{Q} \in \mathrm{D}(\mathbb{Q})$. Notice that, since $\mathrm{R}^{\mathrm{eff}\ *}_{\mathrm{Bet},\sigma}$ is a triangulated functor, hence preserves exact triangles, then, by remark 2.2.23, $\mathrm{R}^{\mathrm{eff}\ *}_{\mathrm{Bet},\sigma}$ maps $\mathbb{Q}(1)$ into the reduced complex of singular chains of $\mathbb{C}^\times$, shifted by -1. It is canonically quasi-isomorphic to $2\pi i\mathbb{Q}$ concentrated in degree 0. Hence, in $\mathrm{D}(\mathbb{Q})$

$$\mathrm{R}^{\mathrm{eff}\ *}_{\mathrm{Bet},\sigma}\mathbb{Q}(1) \cong 2\pi i\mathbb{Q} = \mathbb{Q}(1).$$

Since $\mathrm{R}^{\mathrm{eff}\ *}_{\mathrm{Bet},\sigma}$ is a monoidal functor (remark 2.2.22), then

$$\mathrm{R}^{\mathrm{eff}\ *}_{\mathrm{Bet},\sigma}\mathbb{Q}(q) \cong \mathrm{R}^{\mathrm{eff}\ *}_{\mathrm{Bet},\sigma}\mathbb{Q}(1)^{\otimes q} \cong (\mathrm{R}^{\mathrm{eff}\ *}_{\mathrm{Bet},\sigma}\mathbb{Q}(1))^{\otimes q} \cong (2\pi i)^q\mathbb{Q} = \mathbb{Q}(q).$$

The upper horizontal morphism is the composition

$$r^q_{\mathrm{Bet}} : \mathbb{Q}(q) \to \mathrm{R}^{\mathrm{eff}}_{\mathrm{Bet},\sigma *}\mathrm{R}^{\mathrm{eff}\ *}_{\mathrm{Bet},\sigma}\mathbb{Q}(q) \cong \mathrm{R}^{\mathrm{eff}}_{\mathrm{Bet},\sigma *}\mathbb{Q}(q),$$

where the first morphism is the unit of the adjunction $(\mathrm{R}^{\mathrm{eff}\ *}_{\mathrm{Bet},\sigma}, \mathrm{R}^{\mathrm{eff}}_{\mathrm{Bet},\sigma *})$. The lower horizontal morphism is the composition

$$\Omega^\bullet_k \to \Omega^\bullet_k \otimes_k \mathbb{C} \cong \mathrm{R}^{\mathrm{eff}}_{\mathrm{Bet},\sigma *}\mathbb{C},$$

where the isomorphism is the comparison theorem 2.2.29. The right vertical morphism is obtained by applying the functor $\mathrm{R}^{\mathrm{eff}}_{\mathrm{Bet},\sigma *}$ to the inclusion

$$\mathbb{Q}(q) = (2\pi i)^q\mathbb{Q} \hookrightarrow \mathbb{C}.$$

We have a canonical isomorphism in $\mathbf{DA}^{\mathrm{eff}}_{\acute{e}t}(k;\mathbb{Q})$ (see [MVW06, Thm. 4.1])

$$\mathbb{Q}(1) \cong \mathcal{O}^\times[-1],$$

where $\mathcal{O}^\times$ is the presheaf of k-vector spaces over Sm_k, such that $X \mapsto \mathcal{O}_X(X)^\times$. We define

$$r^1_{\mathrm{AdR}} : \mathbb{Q}(1) \cong \mathcal{O}^\times[-1] \xrightarrow{d\log} \Omega^\bullet_k.$$

The left vertical morphism is defined as

$$r^q_{\mathrm{AdR}} : \mathbb{Q}(q) \cong \mathbb{Q}(1)^{\otimes q} \xrightarrow{(r^1_{\mathrm{AdR}})^{\otimes q}} (\Omega^\bullet_k)^{\otimes q} \to \Omega^\bullet_k,$$

where the last morphism is given object-wise by product of algebraic forms.

It holds that the square 2.25 is commutative. For any $X \in Sm_k$ and $p \in \mathbb{Z}$, applying the functor $\mathrm{Hom}_{\mathbf{DA}^{\mathrm{eff}}_{\acute{e}t}(k;\mathbb{Q})}(\mathbf{M}(X), \_[p])$ to the commutative square 2.25, we obtain the commutative square

$$\begin{tikzcd}
\mathrm{H}^p(X, \mathbb{Q}(q)) \arrow[r] \arrow[d] & \mathrm{H}^p_{\mathrm{Bet}}(X_\sigma)(q) \arrow[d, hook] \\
\mathrm{H}^p_{\mathrm{AdR}}(X/k) \arrow[r, hook] & \mathbb{H}^p(X_\sigma(\mathbb{C})_{an}, \mathbb{C}_{X_\sigma(\mathbb{C})}).
\end{tikzcd} \tag{2.26}$$

For $p = 2q$ and $X \in SmProj_k$, we get back the commutative square 2.24. This naturally leads to formulate the following generalization of the Grothendieck Period Conjecture 2.3.1 for étale motivic cohomology groups. We denote by

$$\mathrm{H}^{p,q}_{\varpi}(X) := \mathrm{H}^p_{\mathrm{AdR}}(X/k) \cap \mathrm{H}^p_{\mathrm{Bet}}(X_\sigma)(q),$$

where the intersection is taken inside $\mathbb{H}^p(X_\sigma(\mathbb{C})_{an}, \mathbb{C}_{X_\sigma(\mathbb{C})})$. By commutativity of the diagram 2.26, we deduce a morphism

$$\mathrm{H}^p(X, \mathbb{Q}(q)) \to \mathrm{H}^{p,q}_{\varpi}(X).$$

Conjecture 2.3.4 (Generalized Grothendieck Period Conjecture)**.** *Given $X \in Sm_{\overline{\mathbb{Q}}}$, we say that the Grothendieck Period Conjecture holds for X, if, for any $p, q \geq 0$, the morphism*

$$H^p(X, \mathbb{Q}(q)) \to H^{p,q}_{\varpi}(X).$$

is surjective.

Appendix A

Cohomology on sites

Just to recall and fix notations, we collect here some standard definitions and facts in sheaf cohomology theory.

The setting is the following: we consider $\mathcal{C}$ a *site* and *abelian sheaves* over $\mathcal{C}$. We denote their category by $\mathrm{Ab}(\mathcal{C})$. If moreover we have a sheaf of rings $\mathcal{O}$ over $\mathcal{C}$, then we have a *ringed site* $(\mathcal{C}, \mathcal{O})$ and we can consider *sheaves of* $\mathcal{O}$*-modules* over $\mathcal{C}$. We denote their category by $\mathrm{Mod}(\mathcal{O})$, which is a subcategory of $\mathrm{Ab}(\mathcal{C})$. Given R a commutative ring with unit, we denote by R the constant sheaf of rings on $\mathcal{C}$. In this case, $\mathrm{Mod}(R)$ is the category of sheaves of R-modules over $\mathcal{C}$. Notice that taking $R = \mathbb{Z}$, $\mathrm{Mod}(\mathbb{Z})$ is the category $\mathrm{Ab}(\mathcal{C})$. $\mathrm{Mod}(\mathcal{O})$ and $\mathrm{Ab}(\mathcal{C})$ are Grothendieck categories, so we can use Homological Algebra techniques. Recall that, given an additive functor $G : \mathcal{B} \to \mathcal{A}$ between Grothendieck categories, if G is left-exact, we can consider the *total right-derived functor* between bounded below derived categories

$$\mathbf{R}G : \mathrm{D}^+(\mathcal{B}) \to \mathrm{D}^+(\mathcal{A}).$$

It is computed by, for any $X^\bullet \in \mathrm{D}^+(\mathcal{B})$,

$$\mathbf{R}G(X^\bullet) \cong G(I^\bullet),$$

where $I^\bullet \in \mathrm{Ch}^+(\mathcal{B})$ is a complex of injective objects with a quasi-isomorphism $X^\bullet \xrightarrow{\sim} I^\bullet$, called *injective reslution.* Composing with cohomology functors, we obtain the *right-derived functors* of G, for each $i \in \mathbb{Z}$,

$$\mathbf{R}^i G : \mathrm{Ch}^+(\mathcal{B}) \to \mathrm{D}^+(\mathcal{B}) \xrightarrow{\mathbf{R}G} \mathrm{D}^+(\mathcal{A}) \xrightarrow{\mathrm{H}^i} \mathcal{A}.$$

Given an object $X \in \mathcal{B}$, we can think at it as an object of $\mathrm{Ch}^+(\mathcal{B})$, as the complex concentrated in degree 0. This defines a functor

$$\mathcal{B} \to \mathrm{Ch}^+(\mathcal{B}).$$

Composing with this functor, we obtain the family of functors, for each $p \geq 0$,

$$\mathbf{R}^p G : \mathcal{B} \to \mathrm{Ch}^+(\mathcal{B}) \xrightarrow{\mathbf{R}^p G} \mathcal{A},$$

which form a universal δ-functor. More generally, to compute the total right-derived functors of G, instead of $I^\bullet$, we can take $A^\bullet \in \mathrm{Ch}^+(\mathcal{B})$ a complex of G-acyclic objects, i.e. such that, for any $i \in \mathbb{Z}$,

$$\mathbf{R}^p G(A^i) = 0 \qquad \text{for each } p > 0,$$

with a quasi-isomorphism $X^\bullet \xrightarrow{\sim} A^\bullet$, called *G-acyclic resolution.*

Cohomology of abelian sheaves

Let $\mathcal{C}$ be a site. Given an object $U \in \mathcal{C}$, consider the *global sections functor* over U

$$\begin{aligned} \Gamma(U, \_\,) : \mathrm{Ab}(\mathcal{C}) &\to \mathbb{Z}\text{-mod} \\ F &\mapsto F(U). \end{aligned}$$

It is a left-exact functor between Grothendieck categories. So we can consider its total right-derived functor

$$\mathbf{R}\Gamma(U, \_\,) : \mathrm{D}^+(\mathrm{Ab}(\mathcal{C})) \to \mathrm{D}^+(\mathbb{Z}).$$

Composing with cohomology functors, we get the functors, for each $i \in \mathbb{Z}$,

$$\mathbb{H}^i(U, \_\,) : \mathrm{D}^+(\mathrm{Ab}(\mathcal{C})) \xrightarrow{\mathbf{R}\Gamma(U,\_)} \mathrm{D}^+(\mathbb{Z}) \xrightarrow{\mathrm{H}^i} \mathbb{Z}\text{-mod}.$$

Definition A.0.1. Given $F^\bullet \in \mathrm{D}^+(\mathrm{Ab}(\mathcal{C}))$, for each $i \in \mathbb{Z}$, the abelian group

$$\mathbb{H}^i(U, F^\bullet) \coloneqq \mathrm{H}^i(\mathbf{R}\Gamma(U, F^\bullet))$$

is called the *i^{th}-cohomology group of U with coefficients in $F^\bullet$.*

Let $\mathrm{PAb}(\mathcal{C})$ be the category of abelian presheaves over $\mathcal{C}$ and denote by e its terminal object, the constant presheaf of the trivial group. Consider the functor

$$\Gamma(\mathcal{C}, \_\,) \coloneqq \mathrm{Hom}_{\mathrm{PAb}(\mathcal{C})}(e, \_\,) : \mathrm{Ab}(\mathcal{C}) \to \mathbb{Z}\text{-mod},$$

It is a left-exact functor between Grothendieck categories. So we can consider its total right-derived functor

$$\mathbf{R}\Gamma(\mathcal{C}, \_\,) : \mathrm{D}^+(\mathrm{Ab}(\mathcal{C})) \to \mathrm{D}^+(\mathbb{Z}).$$

Composing with cohomology functors, we get the functors, for each $i \in \mathbb{Z}$,

$$\mathbb{H}^i(\mathcal{C}, \_\,) : \mathrm{D}^+(\mathrm{Ab}(\mathcal{C})) \xrightarrow{\mathbf{R}\Gamma(\mathcal{C}, \_\,)} \mathrm{D}^+(\mathbb{Z}) \xrightarrow{\mathrm{H}^i} \mathbb{Z}\text{-mod}.$$

Definition A.0.2. Given $F^\bullet \in \mathrm{D}^+(\mathrm{Ab}(\mathcal{C}))$, for each $i \in \mathbb{Z}$, the abelian group

$$\mathbb{H}^i(\mathcal{C}, F^\bullet) := \mathrm{H}^i(\mathbf{R}\Gamma(\mathcal{C}, F^\bullet))$$

is called the *i^{th}-cohomology group of $\mathcal{C}$ with coefficients in $F^\bullet$.*

Notice that, if $\mathcal{C}$ has terminal object X, then $e = h_X$, the presheaf represented by X. By Yoneda Lemma,

$$\Gamma(\mathcal{C}, \_\,) \cong \Gamma(X, \_\,).$$

Hence, their total right-derived functors are isomorphic and for each, $i \in \mathbb{Z}$,

$$\mathbb{H}^i(\mathcal{C}, \_\,) \cong \mathbb{H}^i(X, \_\,).$$

For example, this is the case of a *localization site* $\mathcal{C}/U$, for some object $U \in \mathcal{C}$, which has terminal object U. The forgetful functor $\mathcal{C}/U \to \mathcal{C}$ is continuous and cocontinuous. Hence, it induces the triple of adjoint functors

$$\begin{tikzcd} \mathrm{Ab}(\mathcal{C}) \arrow[r, "j_U^{-1}"] & \mathrm{Ab}(\mathcal{C}/U). \arrow[l, bend right, "j_{U!}"'] \arrow[l, bend left, "j_{U*}"] \end{tikzcd}$$

Moreover $j_{U!}$ is exact. For any $F \in \mathrm{Ab}(\mathcal{C})$, we denote by

$$F\big|_U := j_U^{-1}F.$$

Notice that, for any $F \in \mathrm{Ab}(\mathcal{C})$,

$$\Gamma(\mathcal{C}/U, F\big|_U) \cong \Gamma(U, F\big|_U) \cong \Gamma(u(U), F) \cong \Gamma(U, F),$$

i.e. we have the commutative diagram of left-exact functors

$$\begin{tikzcd} \mathrm{Ab}(\mathcal{C}) \arrow[rr, "\Gamma(U,\_\,)"] \arrow[dr, "j_U^{-1}"'] & & \mathbb{Z}\text{-mod}. \\ & \mathrm{Ab}(\mathcal{C}/U) \arrow[ur, "\Gamma(\mathcal{C}/U,\_\,)"'] & \end{tikzcd}$$

Recall that, given a composition of left-exact functors between Grothendieck categories

$$\mathcal{C} \xrightarrow{G'} \mathcal{B} \xrightarrow{G} \mathcal{A},$$

if G' takes injective objects into G-acyclic ones (for example, if G is right-adjoint to an exact functor, then it preserves injective objects, which are acyclic with respect to any functor), then we have a natural isomorphism of total right-derived functors

$$\mathbf{R}(GG') \cong \mathbf{R}G \circ \mathbf{R}G'$$

and the associated Grothendieck spectral sequence, for any $X^\bullet \in \mathrm{D}^+(\mathcal{C})$,

$$E_2^{p,q} = (\mathbf{R}^p G \circ \mathbf{R}^q G')(X^\bullet) \Rightarrow \mathbf{R}^{p+q}(GG')(X^\bullet),$$

functorial in $X^\bullet$. Since j_U^{-1} is right-adjoint of $j_{U!}$, which is exact, we can apply this to the above commutative diagram of left-exact functors. We deduce a natural isomorphism of total right-derived functors

$$\mathbf{R}\Gamma(\mathcal{C}/U) \circ \mathbf{R}j_U^{-1} \cong \mathbf{R}\Gamma(U, \_\,).$$

Since j_U^{-1} is exact, then $\mathbf{R}j_U^{-1} = j_U^{-1}$. Hence, for any $F^\bullet \in \mathrm{D}^+(\mathrm{Ab}(\mathcal{C}))$, we have the isomorphisms of abelian groups, for each $i \in \mathbb{Z}$,

$$\mathbb{H}^i(\mathcal{C}/U, F^\bullet\big|_U) \cong \mathbb{H}^i(U, F^\bullet),$$

natural in $F^\bullet$. This tells that cohomology of an object can be seen as cohomology of the associated localization site.

Cohomology of sheaves of modules

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. We have definitions analogous to the ones of in previous paragraph. Given an object $U \in \mathcal{C}$, consider the global sections functor over U

$$\begin{aligned} \Gamma(U, \_\,) : \mathrm{Mod}(\mathcal{O}) &\to \mathcal{O}(U)\text{-mod} \\ F &\mapsto F(U), \end{aligned}$$

It is a left-exact functor between Grothendieck categories. So we can consider its total right-derived functor

$$\mathbf{R}\Gamma(U, \_\,) : \mathrm{D}^+(\mathrm{Mod}(\mathcal{O})) \to \mathrm{D}^+(\mathcal{O}(U)).$$

Composing with cohomology functors, we get the functors, for each $i \in \mathbb{Z}$,

$$\mathbb{H}^i(U, \_\,) : \mathrm{D}^+(\mathrm{Mod}(\mathcal{O})) \xrightarrow{\mathbf{R}\Gamma(U,\_)} \mathrm{D}^+(\mathcal{O}(U)) \xrightarrow{\mathrm{H}^i} \mathcal{O}(U)\text{-mod}.$$

Definition A.0.3. Given $F^\bullet \in \mathrm{D}^+(\mathrm{Mod}(\mathcal{O}))$, for each $i \in \mathbb{Z}$, the $\mathcal{O}(U)$-module

$$\mathbb{H}^i(U, F^\bullet) := \mathrm{H}^i(\mathbf{R}\Gamma(U, F^\bullet)).$$

is called the *i^{th}-cohomology group of U with coefficients in $F^\bullet$*.

Consider the functor

$$\Gamma(\mathcal{C},\ \_\ ) \coloneqq \mathrm{Hom}_{\mathrm{Mod}(\mathcal{O})}(\mathcal{O},\ \_\ ) : \mathrm{Mod}(\mathcal{O}) \to \Gamma(\mathcal{C},\mathcal{O})\text{-mod},$$

where $\Gamma(\mathcal{C},\mathcal{O}) \coloneqq \mathrm{End}_{\mathrm{Mod}(\mathcal{O})}(\mathcal{O})$. It is a left-exact functor of Grothendieck categories. So we can consider its total right-derived functor

$$\mathbf{R}\Gamma(\mathcal{C},\ \_\ ) : \mathrm{D}^+(\mathrm{Mod}(\mathcal{O})) \to \mathrm{D}^+(\Gamma(\mathcal{C},\mathcal{O})).$$

Composing with cohomology functors, we get the functors, for each $i \in \mathbb{Z}$,

$$\mathbb{H}^i(\mathcal{C},\ \_\ ) : \mathrm{D}^+(\mathrm{Mod}(\mathcal{O})) \xrightarrow{\mathbf{R}\Gamma(\mathcal{C},\ \_\ )} \mathrm{D}^+(\mathcal{O}(U)) \xrightarrow{\mathrm{H}^i} \mathcal{O}(U)\text{-mod}.$$

Definition A.0.4. Given $F^\bullet \in \mathrm{D}^+(\mathrm{Mod}(\mathcal{O}))$, for each $i \in \mathbb{Z}$, the $\Gamma(\mathcal{C},\mathcal{O})$-module

$$\mathbb{H}^i(\mathcal{C},F^\bullet) \coloneqq \mathrm{H}^i(\mathbf{R}\Gamma(\mathcal{C},F^\bullet))$$

is called the *i^{th}-cohomology group of $\mathcal{C}$ with coefficients in $F^\bullet$.*

These definitions are not only the analogous of the ones for abelian sheaves. In fact, they coincide with them. Given $F \in \mathrm{Mod}(\mathcal{O})$, denote by F_{ab} the underlying complex of abelian sheaves. This defines a functor

$$(\ \_\ )_{ab} : \mathrm{Mod}(\mathcal{O}) \to \mathrm{Ab}(\mathcal{C}),$$

which is exact and takes injective objects into $\Gamma(U,\ \_\ )$-acyclic objects, for any object $U \in \mathcal{C}$. [1] For any $U \in \mathcal{C}$, consider the functors

$$\Gamma(U,(\ \_\ )_{ab}) : \mathrm{Mod}(\mathcal{O}) \to \mathrm{Ab}(\mathcal{C}) \to \mathbb{Z}\text{-mod}$$

and

$$\Gamma(U,\ \_\ ) : \mathrm{Mod}(\mathcal{O}) \to \mathcal{O}(U)\text{-mod} \to \mathbb{Z}\text{-mod},$$

where the second functor is the forgetful functor, which is exact. These functors are isomorphic, hence also their total right-derived functors are

$$\mathbf{R}\Gamma(U,\ \_\ ) \circ (\ \_\ )_{ab} \cong \mathbf{R}\Gamma(U,\ \_\ ) \circ \mathbf{R}(\ \_\ )_{ab} \cong \mathbf{R}\Gamma(U,(\ \_\ )_{ab}) \cong \mathbf{R}\Gamma(U,\ \_\ )$$

Hence, for any $F^\bullet \in \mathrm{Mod}(\mathcal{O})$ and for each $i \in \mathbb{Z}$,

$$\mathbb{H}^i(U,F^\bullet_{ab}) \cong \mathbb{H}^i(U,F^\bullet).$$

as abelian groups, where the first one is computed as cohomology of a complex of

[1] It can be proved using that $\mathbb{H}^p(U,(\ \_\ )_{ab})$, cohomology of abelian sheaves, and $\mathbb{H}^p(U,\ \_\ )$, cohomology of sheaves of $\mathcal{O}$-modules, are both universal δ-functors $\mathrm{Mod}(\mathcal{O}) \to \mathbb{Z}$-mod, hence they are isomorphic.

abelian sheaves and the second as cohomology of a complex of sheaves of $\mathcal{O}$-modules.

It also holds that
$$\mathbb{H}^i(\mathcal{C}, F^\bullet_{ab}) \cong \mathbb{H}^i(\mathcal{C}, F^\bullet)$$
as abelian groups.

So, dealing with sheaf cohomology groups, we can simply think at complexes of abelian sheaves. If moreover a complex of abelian sheaves is a complex of $\mathcal{O}$-modules, then the cohomology groups of U inherit the additional structure of $\mathcal{O}(U)$-modules.

Functoriality morphisms

Let $f : \mathcal{C} \to \mathcal{D}$ be a morphism of sites, with associated continuous functor $u : \mathcal{D} \to \mathcal{C}$. Consider the inverse image functor
$$f^{-1} : \mathrm{Ab}(\mathcal{D}) \to \mathrm{Ab}(\mathcal{C}).$$
Recall that, for any $G \in \mathrm{Ab}(\mathcal{D})$, the abelian sheaf $f^{-1}G$ is the sheafification of the abelian presheaf over $\mathcal{C}$
$$U \mapsto \varinjlim_{U \to u(V) \in \mathcal{C}} G(V).$$
For any $V \in \mathcal{D}$ and $G \in \mathrm{Ab}(\mathcal{D})$, we have a canonical morphism of abelian groups
$$\Gamma(V, G) \to \varinjlim_{u(V) \to u(V') \in \mathcal{C}} G(V') \to \Gamma(u(V), f^{-1}G),$$
where the second arrow is the canonical morphism of sheafification. This defines a morphism of functors $\mathrm{Ab}(\mathcal{C}) \to \mathbb{Z}$-mod
$$\Gamma(V, \_\,) \to \Gamma(u(V), f^{-1}\,\_\,),$$
which induces a morphisms between their total right-derived functors and hence morphisms of abelian groups, for any $G^\bullet \in \mathrm{D}^+(\mathrm{Ab}(\mathcal{C}))$, for each $i \geq 0$,
$$\mathbb{H}^i(V, G^\bullet) \to \mathbb{H}^i(u(V), f^{-1}G^\bullet), \tag{A.1}$$
natural in $G^\bullet$.

If moreover $f : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ is a morphism of ringed sites, for any $G^\bullet \in \mathrm{D}^+(\mathrm{Mod}(\mathcal{O}_\mathcal{D}))$, the canonical morphism of abelian sheaves
$$f^{-1}G^\bullet \to f^{-1}G^\bullet \otimes_{f^{-1}\mathcal{O}_\mathcal{D}} \mathcal{O}_\mathcal{C} = f^*G^\bullet$$
induces canonical morphisms on cohomology of $u(V)$, for each $i \geq 0$,
$$\mathbb{H}^i(u(V), f^{-1}G^\bullet) \to \mathbb{H}^i(u(V), f^*G^\bullet), \tag{A.2}$$

natural in $G^\bullet$. Composing morphisms A.1 and A.2, we get the canonical morphisms of abelian groups

$$\mathbb{H}^i(V, G^\bullet) \to \mathbb{H}^i(u(V), f^*G^\bullet),$$

natural in $G^\bullet$. It is also a morphism of $\mathcal{O}_{\mathcal{D}}(V)$-modules.

Čech cohomology

In the previous paragraphs we described a notion of cohomology with coefficients in complexes of sheaves over a site. Now, we define a notion of cohomology with coefficients in complexes of abelian presheaves. Let $\mathcal{C}$ be a category (we don't need the notion of site). We denote by $\mathrm{PAb}(\mathcal{C})$ the category of abelian presheaves over $\mathcal{C}$. $\mathrm{PAb}(\mathcal{C})$ is a Grothendieck category, so we can use Homological Algebra techniques. Given an object $U \in \mathcal{C}$, we can still consider the global sections functor over U on the category $\mathrm{PAb}(\mathcal{C})$. However, since it is exact, its total right-derived functor is trivial, so it is not an interesting object. A natural meaningful alternative in the setting of abelian presheaves is given by the following functor. Assume that the category $\mathcal{C}$ has fibered products (we can do without this assumption using Yoneda embedding, but definitions are more involved). Given $\mathcal{U} = \{U_i \to U\}_{i \in I}$ a family of morphisms over U in $\mathcal{C}$, for any finite number of morphisms in $\mathcal{U}$, we denote by

$$U_{i_0 \dots i_n} := U_{i_0} \times_U \cdots \times_U U_{i_n}$$

their fiber product in $\mathcal{C}$. Consider the functor

$$\begin{aligned} \check{\mathrm{H}}^0(\mathcal{U};\ \_\ ) : \mathrm{PAb}(\mathcal{C}) &\to \mathbb{Z}\text{-mod} \\ P &\mapsto ker\left(\prod_{i_0} P(U_{i_0}) \to \prod_{i_0, i_1} P(U_{i_0 i_1})\right). \end{aligned}$$

Since it is a composition of limits, it is left-exact. So we can consider its total right-derived functor

$$\mathbf{R}\check{\mathrm{H}}^0(\mathcal{U};\ \_\ ) : \mathrm{D}^+(\mathrm{PAb}(\mathcal{C})) \to D^+(\mathbb{Z}).$$

Composing with cohomology functors, we get the functors, for each $i \in \mathbb{Z}$,

$$\check{\mathrm{H}}^i(\mathcal{U},\ \_\ ) : \mathrm{D}^+(\mathrm{PAb}(\mathcal{C})) \xrightarrow{\mathbf{R}\check{\mathrm{H}}^0(\mathcal{U};\ \_\ )} \mathrm{D}^+(\mathbb{Z}) \xrightarrow{\mathrm{H}^i} \mathbb{Z}\text{-mod}.$$

Definition A.0.5. Given $P^\bullet \in \mathrm{D}^+(\mathrm{PAb}(\mathcal{C}))$, for each $i \geq 0$, the abelian group

$$\check{\mathrm{H}}^i(\mathcal{U}; P^\bullet) := \mathrm{H}^i(\mathbf{R}\check{\mathrm{H}}^0(\mathcal{U}; P^\bullet))$$

is called the i^{th}-*Čech cohomology group of* U *relative to* $\mathcal{U}$ *with coefficients in* $P^\bullet$.

Besides the usual computation via injective resolutions, Čech cohomology groups can be computed also as the cohomology groups of an explicit complex of abelian groups. Consider the case of a complex of abelian presheaves P concentrated in degree 0. We will see the general case in the following paragraph. We define the complex of abelian groups

$$\check{C}^\bullet(\mathcal{U};P) \ : \ \prod_{i_0} P(U_{i_0}) \to \prod_{i_0,i_1} P(U_{i_0 i_1}) \to \prod_{i_0,i_1,i_2} P(U_{i_0 i_1 i_2}) \to \cdots ,$$

called *Čech complex of P relative to $\mathcal{U}$*. It can be proved that the family of functors

$$\begin{aligned} \mathrm{H}^p\check{C}^\bullet(\mathcal{U};\ \_\ ) : \mathrm{PAb}(\mathcal{C}) &\to \mathbb{Z}\text{-mod} \\ P &\mapsto \mathrm{H}^p(\check{C}^\bullet(\mathcal{U};P)) \end{aligned}$$

is a universal δ-functor. Since

$$\mathrm{H}^0\check{C}^\bullet(\mathcal{U};P) \cong ker\left(\prod_{i_0} P(U_{i_0}) \to \prod_{i_0,i_1} P(U_{i_0 i_1})\right) = \check{\mathrm{H}}^0(\mathcal{U};P),$$

we deduce that $\mathrm{H}^p\check{C}^\bullet(\mathcal{U};\ \_\ )$ is isomorphic to the universal δ-functor given by the right-derived functors of $\check{\mathrm{H}}^0(\mathcal{U};\ \_\ )$

$$\begin{aligned} \mathbf{R}^p\check{\mathrm{H}}^0(\mathcal{U};\ \_\ ) : \mathrm{PAb}(\mathcal{C}) &\to \mathbb{Z}\text{-mod} \\ P &\mapsto \check{\mathrm{H}}^p(\mathcal{U};P). \end{aligned}$$

Hence Čech cohomology in P relative to $\mathcal{U}$ can be computed as cohomology groups of the Čech complex of P relative to $\mathcal{U}$

$$\check{\mathrm{H}}^p(\mathcal{U};P) \cong \mathrm{H}^p(\check{C}^\bullet(\mathcal{U},P)).$$

If moreover $\mathcal{C}$ is endowed with a site structure, we can consider abelian sheaves over $\mathcal{C}$, which are in particular abelian presheaves over $\mathcal{C}$. It is natural to ask if cohomology groups of an object $U \in \mathcal{C}$ with coefficients in an abelian sheaf coincide with Čech cohomology groups of U relative to some cover of U with coefficients in the underlying abelian presheaf. In general they don't, but there always exists a spectral sequence relating them. We will see it in a successive paragraph.

Hyper-cohomology spectral sequences

Recall that to compute the total right-derived functor of a left-exact additive functor $G : \mathcal{B} \to \mathcal{A}$ between Grothendieck abelian categories, we have to take injective resolutions. Given an object $X^\bullet \in D^+(\mathcal{B})$, an injective resolution $X^\bullet \xrightarrow{\sim} I^\bullet$ can be obtained as follows. Assume that $X^\bullet$ is concentrated in non-negative degrees. For each $p \geq 0$, we take an injective resolution of X^p in $\mathcal{B}$

$$X^p \to I^{p,0} \to I^{p,1} \to I^{p,2} \to \cdots .$$

We use the definition of injective object to construct a first quadrant double complex $I^{\bullet,\bullet}$ with these injective resolutions on the p^{th}-column

$$\begin{array}{ccccccc}
\cdots & & \cdots & & \cdots & & \\
\uparrow & & \uparrow & & \uparrow & & \\
I^{0,2} & \to & I^{1,2} & \to & I^{2,2} & \to \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
I^{0,1} & \to & I^{1,1} & \to & I^{2,1} & \to \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
I^{0,0} & \to & I^{1,0} & \to & I^{2,0} & \to \cdots .
\end{array}$$

Consider the total complex $Tot^{\oplus} I^{\bullet,\bullet} \in \mathrm{Ch}^+(\mathcal{B})$. A spectral sequences argument shows that the canonical morphism $X^\bullet \to Tot^{\oplus} I^{\bullet,\bullet}$ is a quasi-isomorphism. Moreover, since finite direct sums of injective objects are injective, $Tot^{\oplus} I^{\bullet,\bullet}$ is a complex of injective objects, hence it is an injective resolution of $X^\bullet$.

Now, consider the problem of computing the cohomology groups

$$\mathrm{H}^i(\mathbf{R}G(X^\bullet)) \cong \mathrm{H}^i(G(I^\bullet)).$$

Choosing the injective resolution $I^\bullet = Tot^{\oplus} I^{\bullet,\bullet}$ constructed above, this is the problem of computing the cohomology groups of the total complex $Tot^{\oplus} G(I^{\bullet,\bullet})$ because

$$\mathrm{H}^i(G(I^\bullet)) = \mathrm{H}^i(G(Tot^{\oplus} I^{\bullet,\bullet})) \cong \mathrm{H}^i(Tot^{\oplus} G(I^{\bullet,\bullet})).$$

Recall that, given a first quadrant double complex, there exist two spectral sequences converging to the cohomology of its total complex. Consider the one whose page 1 is obtained by computing vertical cohomology of the double complex and page 2 the induced horizontal cohomology

$$E_1^{p,q} = \mathrm{H}^q(G(I^{p,\bullet})) \Rightarrow \mathrm{H}^{p+q}(Tot^{\oplus} G(I^{\bullet,\bullet})).$$

Since on columns of $I^{\bullet,\bullet}$ we have injective resolutions of X^p, then

$$\mathrm{H}^q(G(I^{p,\bullet})) \cong \mathbf{R}^q G(X^p).$$

Hence, the above spectral sequence is

$$E_1^{p,q} = \mathbf{R}^q G(X^p) \Rightarrow \mathrm{H}^{p+q}(\mathbf{R}G(X^\bullet)).$$

Applying this spectral sequence to the left-exact functors defining cohomology and Čech cohomology we get

$$E_1^{p,q} = \mathbb{H}^q(U, F^p) \Rightarrow \mathbb{H}^{p+q}(U, \mathcal{F}^\bullet)$$
$$E_1^{p,q} = \check{\mathrm{H}}^q(\mathcal{U}; P^p) \Rightarrow \check{\mathrm{H}}^{p+q}(\mathcal{U}; P^\bullet),$$

where notation is the same of the above corresponding paragraphs. These spectral sequences are useful to reduce from cohomology with coefficients in a complex of abelian sheaves to cohomology with coefficients in a single abelian sheaf, thought as a complex concentrated in degree 0. The former is sometimes called hyper-cohomology, to distinguish it from the latter special case (even if there's no reason to do it). For this reason these spectral sequences are called *hyper-cohomology spectral sequences.* Moreover these spectral sequences are functorial.

Notice that the hyper-cohomology spectral sequence for Čech cohomology also suggests a way to compute Čech cohomology with coefficients in $P^\bullet \in \mathrm{D}^+(\mathrm{PAb}(\mathcal{C}))$ as cohomology of an explicit complex of abelian groups. Assume that $P^\bullet$ is concentrated in non-negative degrees. By functoriality of Čech complex, we can construct a first quadrant double complex of abelian groups $\check{C}^\bullet(\mathcal{U}; \mathcal{P}^\bullet)$ with Čech complexes of P^p on the p^{th}-column

$$\begin{array}{ccccccc}
\cdots & & \cdots & & \cdots & & \\
\uparrow & & \uparrow & & \uparrow & & \\
\prod_{i_0,i_1,i_2} P^0(U_{i_0i_1i_2}) & \to & \prod_{i_0,i_1,i_2} P^1(U_{i_0i_1i_2}) & \to & \prod_{i_0,i_1,i_2} P^2(U_{i_0i_1i_2}) & \to \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
\prod_{i_0,i_1} P^0(U_{i_0i_1}) & \to & \prod_{i_0,i_1} P^1(U_{i_0i_1}) & \to & \prod_{i_0,i_1} P^2(U_{i_0i_1}) & \to \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
\prod_{i_0} P^0(U_{i_0}) & \to & \prod_{i_0} P^1(U_{i_0}) & \to & \prod_{i_0} P^2(U_{i_0}) & \to \cdots .
\end{array}$$

We can consider the spectral sequence converging to the cohomology of its total complex

$$E_1^{p,q} = \mathrm{H}^q(\check{C}^\bullet(\mathcal{U}; P^p)) \Rightarrow \mathrm{H}^{p+q}(Tot^{\oplus}\check{C}^\bullet(\mathcal{U}; P^\bullet)).$$

Since on columns of $\check{C}^\bullet(\mathcal{U}; P^\bullet)$ there are Čech complexes of P^p, then

$$\mathrm{H}^q(\check{C}^\bullet(\mathcal{U}; P^p)) \cong \check{\mathrm{H}}^q(\mathcal{U}; P^p).$$

Hence the above spectral sequence is isomorphic to the hyper-cohomology spectral sequence for Čech cohomology. By uniqueness of the limit, it follows that Čech cohomology can be computed as the cohomology of the total complex of $\check{C}^\bullet(\mathcal{U}; P^\bullet)$

$$\check{\mathrm{H}}^i(\mathcal{U}; P^\bullet) \cong \mathrm{H}^i(Tot^{\oplus}C^\bullet(\mathcal{U}; P^\bullet)).$$

Čech-to-derived spectral sequence and Leray's Theorem.

We describe a spectral sequence relating cohomology groups to Čech cohomology groups. Let $\mathcal{C}$ be a site. Consider the full embedding

$$\iota : \mathrm{Ab}(\mathcal{C}) \hookrightarrow \mathrm{PAb}(\mathcal{C}),$$

which associates to each abelian sheaf over $\mathcal{C}$ its underlying abelian presheaf. It has a left-adjoint given by the sheafification functor, so it is left-exact. Hence we can consider its total right-derived functor

$$\mathcal{H} \coloneqq \mathbf{R}\iota : \mathrm{D}^+(\mathrm{Ab}(\mathcal{C})) \to \mathrm{D}^+(\mathrm{PAb}(\mathcal{C})).$$

Composing with cohomology functors, we get the functors, for each $i \in \mathbb{Z}$,

$$\mathcal{H}^i \coloneqq \mathbf{R}^i\iota : \mathrm{D}^+(\mathrm{Ab}(\mathcal{C})) \xrightarrow{\mathcal{H}} \mathrm{D}^+(\mathrm{PAb}(\mathcal{C})) \xrightarrow{\mathrm{H}^i} \mathrm{PAb}(\mathcal{C}).$$

They are such that, for any $F \in \mathrm{Ab}(\mathcal{C})$, for each $q \geq 0$, $\mathcal{H}^q(F)$ is the abelian presheaf over $\mathcal{C}$

$$U \mapsto \mathbb{H}^q(U, F).$$

Let $U \in \mathcal{C}$ be an object. By definition of sheaves over $\mathcal{C}$, we have that, for any $F \in \mathrm{Ab}(\mathcal{C})$ and for any $\mathcal{U}$ covering over U,

$$\check{\mathrm{H}}^0(\mathcal{U}; F) = F(U) = \Gamma(U, F),$$

i.e. we have the commutative diagram of left-exact functors

$$\begin{tikzcd} \mathrm{Ab}(\mathcal{C}) \arrow[rr, "\Gamma(U,\_)"] \arrow[dr, hook, "\iota"'] & & \mathbb{Z}\text{-mod.} \\ & \mathrm{PAb}(\mathcal{C}) \arrow[ur, "\check{\mathrm{H}}^0(\mathcal{U},\_)"'] & \end{tikzcd}$$

Since ι is right-adjoint to the sheafification functor, which is exact, we have the Grothendieck spectral sequence, for any $F^\bullet \in \mathrm{D}^+(\mathrm{Ab}(\mathcal{C}))$

$$E_2^{p,q} = \check{\mathrm{H}}^p(\mathcal{U}; \mathcal{H}^q(F^\bullet)) \Rightarrow \mathbb{H}^{p+q}(U, F^\bullet),$$

functorial in $F^\bullet$, called *Čech-to-derived spectral sequence*. In case this spectral sequence degenerates, we can obtain isomorphisms between cohomology and Čech cohomology. An example is given by the following theorem.

Theorem A.0.6 (Leray's Theorem)**.** *Let $\mathcal{C}$ be a site and $F^\bullet \in Ab(\mathcal{C})$. Assume that $F^\bullet$ is concentrated in non-negative degrees. Assume that $\mathcal{C}$ has fiber products. Let*

$\mathcal{U} = \{U_i \to U\}_{i\in I}$ be a cover of an object $U \in \mathcal{C}$, such that, for any $p \geq 0$

$$\mathcal{H}^s(F^p)(U_{i_0\ldots i_n}) \cong \mathbb{H}^q(U_{i_0\ldots i_n}, F^p) = 0 \qquad \textit{for each } n \geq 0,\ s > 0.$$

Then, for each $i \in \mathbb{Z}$,

$$\mathbb{H}^i(U; F^\bullet) \cong \check{H}^i(\mathcal{U}; F^\bullet).$$

That is, cohomology of U with coefficients in $F^\bullet$ can be computed as Čech cohomology of U relative to $\mathcal{U}$ with coefficients in $F^\bullet$.

Proof. For any $p \geq 0$, the Čech-to-derived spectral sequence applied to F^p degenerates at page 2, because

$$E_2^{r,s} = \check{\mathrm{H}}^r(\mathcal{U}; \mathcal{H}^s(F^p)) \cong \mathrm{H}^r(\check{C}^\bullet(\mathcal{U}; \mathcal{H}^s(F^p))) \cong \begin{cases} \check{\mathrm{H}}^r(\mathcal{U}; F^p) & \text{if s =0} \\ 0 & \text{if } s > 0. \end{cases}$$

It follows that, for each $q \geq 0$,

$$\check{\mathrm{H}}^q(\mathcal{U}; F^p) \cong \mathbb{H}^q(U, F^p).$$

These isomorphisms define an isomorphism between the hyper-cohomology spectral sequences of Čech cohomolgy and cohomology for $F^\bullet$. By uniqueness of the limit, we deduce that, for each $i \geq 0$,

$$H^i(\mathcal{U}, F^\bullet) \cong H^i(U, F^\bullet).$$

□

Leray spectral sequence.

Let $f : \mathcal{C} \to \mathcal{D}$ be a morphism of sites, with associated continuous functor $u : \mathcal{D} \to \mathcal{C}$. Consider the direct image functor

$$f_* : \mathrm{Ab}(\mathcal{C}) \to \mathrm{Ab}(\mathcal{D}).$$

Recall that, for any $f \in \mathrm{Ab}(\mathcal{C})$, the abelian sheaf f_*F over $\mathcal{D}$ is

$$V \mapsto F(u(V)).$$

It is right-adjoint to the inverse image functor f^{-1}, so is left-exact. Hence, we can consider its total right-derived functor

$$\mathbf{R}f_* : \mathrm{D}^+(\mathrm{Ab}(\mathcal{C})) \to \mathrm{D}^+(\mathrm{Ab}(\mathcal{D})).$$

Composing with cohomology functors, we obtain the functors, for each $i \in \mathbb{Z}$,

$$\mathbf{R}^i f_* : \mathrm{D}^+(\mathrm{Ab}(\mathcal{C})) \xrightarrow{\mathbf{R}f_*} \mathrm{D}^+(\mathrm{Ab}(\mathcal{D})) \xrightarrow{\mathrm{H}^i} \mathrm{Ab}(\mathcal{D}).$$

Definition A.0.7. Given $f : \mathcal{C} \to \mathcal{D}$ a morphism of sites, the functor $\mathbf{R}^i f_*$ is called the *i^{th}-higher direct image functor* of f.

They are such that, for any $F \in \mathrm{Ab}(\mathcal{C})$, for each $p \geq 0$, $\mathbf{R}^p f_* F$ is the sheafification of the abelian presheaf over $\mathcal{D}$

$$V \mapsto \mathbb{H}^p(u(V), F).$$

Notice that, for any $F \in \mathrm{Ab}(\mathcal{C})$ and for any $V \in \mathcal{D}$

$$\Gamma(V, f_* F) = F(u(V)) = \Gamma(u(V), F),$$

i.e. we have the commutative diagram of left-exact functors

$$\begin{tikzcd} \mathrm{Ab}(\mathcal{C}) \arrow[rr, "\Gamma(u(V),\_)"] \arrow[dr, "f_*"'] & & \mathbb{Z}\text{-mod}. \\ & \mathrm{Ab}(\mathcal{D}) \arrow[ur, "\Gamma(V,\_)"'] & \end{tikzcd}$$

Since f_* is right-adjoint to f^{-1}, which is exact, we have the Grothendieck spectral sequence, for any $F^\bullet \in \mathrm{D}^+(\mathrm{Ab}(\mathcal{C}))$,

$$E_2^{p,q} = \mathbb{H}^p(V, \mathbf{R}^q f_* F^\bullet) \Rightarrow \mathbb{H}^{p+q}(u(V), F^\bullet),$$

called the *Leray spectral sequence* for f and $F^\bullet$.

Recall that, given a first quadrant convergent spectral sequence $E_2^{p,q} \Rightarrow \mathrm{H}^{p+q}$, for any $p \geq 0$, we have canonical edge morphisms $E_2^{p,0} \to \mathrm{H}^p$. The edge morphisms of Leray spectral sequence give the canonical morphisms of abelian groups

$$\mathrm{H}^p(V, f_* F^\bullet) \to \mathrm{H}^p(u(V), F^\bullet),$$

natural in $F^\bullet$.

Moreover, if $f : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ is a morphism of ringed sites, then, for any $F^\bullet \in \mathrm{D}^+(\mathrm{Mod}(\mathcal{O}_\mathcal{C}))$, $f_* F^\bullet \in \mathrm{D}^+(\mathrm{Mod}(\mathcal{O}_\mathcal{D}))$ and the above canonical morphisms are also morphisms of $\mathcal{O}_\mathcal{D}(V)$-modules.

Appendix B

Grothendieck's Theory of Chern classes

We follow [Gro58]. Consider a contravariant functor into the category of commutative graded rings

$$A^* : SmProj_k^{op} \to grRing$$
$$X \mapsto A^*(X) = \oplus_{i\geq 0} A^i(X),$$

with a natural transformation of contravariant functors $SmProj_k^{op} \to \mathrm{Ab}$,

$$p_X^1 : \mathrm{Pic} \to A^1,$$

called the *first Chern class*. Recall that, for any $X \in SmProj_k$, $\mathrm{Pic}(X)$, the Picard group of X, is the the abelian group of isomorphism classes of invertible sheaves $\mathcal{O}_X$-modules of X (algebraic vector bundles of rank 1), with multiplication given by tensor product of sheaves of $\mathcal{O}_X$-modules. It defines a contravariant functor $SmProj_k \to \mathrm{Ab}$ with pullback maps given by the pullback of sheaves of modules.

We consider the following axioms.

CC1) For any $i : Z \hookrightarrow X$ closed immersion in $SmProj_k$, where Z has codimension c in X, we an homomorphism of abelian groups, for each $i \geq 0$,

$$i_* : A^i(Z) \to A^{i+c}(X),$$

called *pushforward map.*

CC2) The pushforward map is functorial, that is, for any $Z \overset{i}{\hookrightarrow} Y \overset{j}{\hookrightarrow} X$ closed immersions in $SmProj_k$,

$$(ji)_* = j_* i_*.$$

CC3) (*Projection formula*) For any $i : Z \hookrightarrow X$ closed immersion in $SmProj_k$,

$x \in A^*(X)$ and $y \in A^*(Z)$, it holds

$$y(i_* x) = i_*(i^*(y)x).$$

CC4) (*Projective bundle formula*) Given $X \in SmProj_k$ and $\mathcal{E}$ an algebraic vector bundle over X of rank r (finite locally free sheaf of $\mathcal{O}_X$-modules of rank r), take its projectivization $\pi : \mathbb{P}(\mathcal{E}) \to X$ (see [GW20, §13.7]). Consider the class of the tautological line bundle

$$\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \in \operatorname{Pic}(\mathbb{P}(\mathcal{E}))$$

and denote its first Chern class in $A^1(\mathbb{P}(\mathcal{E}))$ by

$$t \coloneqq p^1_{\mathbb{P}(\mathcal{E})}(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)).$$

Then, the pullback map $p^* : A^*(X) \to A^*(\mathbb{P}(\mathcal{E}))$ induces on $A^*(\mathbb{P}(\mathcal{E}))$ a structure of $A^*(X)$-algebra, such that

$$A^*(\mathbb{P}(\mathcal{E})) \cong A^*(X)[t]/(t^r).$$

CC5) Given $X \in SmProj_k$ and $\mathcal{L} \in Pic(X)$, which admits a regular section $s \in \mathcal{L}(X)$, we denote by $i : Z(s) \hookrightarrow X$ the closed immersion given by the zero scheme associated to s, which has codimension 1 in X. Then, in $A^1(X)$

$$i_*(1_{Z(s)}) = p^1_X(\mathcal{L}),$$

where $1_{Z(s)} \in A^0(Z(s))$ is the unit of $A^*(Z(s))$.

With these data and axioms, given $X \in SmProj_k$ and $\mathcal{E}$ an algebraic vector bundle over X, let

$$t \coloneqq p^1_{\mathbb{P}(\mathcal{E})}(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)).$$

By axiom CC4), we have that $t^r \in A^r(\mathbb{P}(\mathcal{E}))$ can be written uniquely in $A^*(\mathbb{P}(\mathcal{E}))$ as

$$t^r = (-1)^{r+1}a_r + (-1)^r a_{r-1}t + \cdots + a_1 t^{r-1} = \sum_{i=1}^{r}(-1)^{i+1}a_i t^{r-i},$$

with $a_i \in A^i(X)$, for $i = 1, \ldots, r$. We define the i^{th}-*Chern class* of $\mathcal{E}$

$$c^i_X(\mathcal{E}) \coloneqq \begin{cases} 1 & \text{for } i = 0, \\ a_i & \text{for } i = 1, \ldots, r, \\ 0 & \text{for } i > r. \end{cases}$$

Let

$$\mathrm{K}^0(X) \coloneqq \mathrm{K}_0(\mathrm{Vect}(X)),$$

be the Grothendieck group of the category of algebraic vector bundles over X (finite locally free sheaves of $\mathcal{O}_X$-modules), called the 0^{th}-algebraic K-group of X. It is a ring with sum and multiplication given by direct sum and tensor product of sheaves of $\mathcal{O}_X$-modules. There exists a map, for any $X \in SmProj_k$,

$$\begin{aligned} c_X : \mathrm{K}^0(X) &\to A^*(X) \\ [\mathcal{E}] &\mapsto \sum_{i\geq 0} c_X^i(\mathcal{E}), \end{aligned}$$

called the *total Chern class* of X, which is characterized by the following properties:

- It is a group homomorphism, where we consider on $\mathrm{K}^0(X)$ the additive structure and on $A^*(X)$ the multiplicative structure.
- It's natural in X, i.e. commutes with pullback maps.
- For any $\mathcal{L} \in \mathrm{Pic}(X)$, $c_X([\mathcal{L}]) = 1 + p_X^1(\mathcal{L})$.

This implies that each Chern class can be extended to a map on $K^0(X)$ (not a group homomorphism!)

$$c_X^i : K^0(X) \to A^i(X).$$

Moreover, if $A^*(X)$ is a graded $\mathbb{Q}$-algebra, there exists a map, for any $X \in SmProj_k$

$$ch_X : \mathrm{K}^0(X) \to \prod_{i\geq 0} A^i(X),$$

called the *Chern character* of X, which is characterized by the following properties:

- It is a ring homomorphism.
- It's natural in X, i.e. commutes with pullback maps.
- For any $\mathcal{L} \in \mathrm{Pic}(X)$, $ch_X([\mathcal{L}]) = \sum_{i\geq 0} \frac{1}{i!} p_X^1(\mathcal{L})^i$.

An important example is given by the Chow group (see [Gro58, §4.3]). Gothendieck's theory of Chern classes, applied to this case, shades light on the connection between algebraic cycles and algebraic vector bundles of an algebraic variety. Consider the functor

$$\begin{aligned} \mathrm{CH}^* : SmProj_k^{op} &\to grRing \\ X &\mapsto \mathrm{CH}^*(X) = \oplus_{i\geq 0}\mathrm{CH}^i(X), \end{aligned}$$

with first Chern class

$$p_X^1 : \mathrm{Pic}(X) \to \mathrm{CH}^1(X)$$

the isomorphism of abelian groups between Cartier and Weil divisors (see [GW20, Thm. 11.40]), which holds because X is locally factorial, by corollary 1.4.23. Axioms CC1)-CC5) are satisfied (notice that axiom CC5) tells that the first Chern class is exactly the isomorphism between Cartier and Weil divisors), so we can apply the above constructions. Taking rational coefficients, i.e. considering the functor

$$\begin{aligned} \mathrm{CH}^*_{\mathbb{Q}} : SmProj_k^{op} &\to grRing \\ X &\mapsto \mathrm{CH}^*(X)_{\mathbb{Q}} = \oplus_{i\geq 0}\mathrm{CH}^i(X) \otimes_{\mathbb{Z}} \mathbb{Q}, \end{aligned}$$

which still satisfies axioms CC1)-CC5), we can also construct the Chern character, for any $X \in SmProj_k$,

$$ch_X : \mathrm{K}^0(X) \to \mathrm{CH}^*(X)_{\mathbb{Q}}.$$

Further, we see that we can refine the Chern character to an isomorphism of graded rings. Let

$$\mathrm{K}_0(X) \coloneqq \mathrm{K}_0(\mathrm{Coh}(\mathcal{O}_X))$$

be the Grothendieck group of the category of coherent sheaves of $\mathcal{O}_X$-modules. It is an abelian group with sum given by direct sum of sheaves of $\mathcal{O}_X$-modules. Since $\mathcal{O}_X$ is a coherent sheaf of $\mathcal{O}_X$-modules (proposition 1.2.20), then any algebraic vector bundle over X is a coherent sheaf of $\mathcal{O}_X$-modules. Hence we have the inclusion $\mathrm{Vect}(X) \hookrightarrow \mathrm{Coh}(X)$, which induces the group homomorphism

$$\begin{aligned} \mathrm{K}^0(X) &\to \mathrm{K}_0(X) \\ [\mathcal{E}] &\mapsto [\mathcal{E}], \end{aligned}$$

called the *Cartan homomorphism*. For X smooth, a theorem of Serre (see [BS58, Cor. 10]) asserts that any $F \in \mathrm{Coh}(X)$ admits a finite resolution of algebraic vector bundles, i.e. there exists an exact sequence of sheaves of $\mathcal{O}_X$-modules

$$0 \to \mathcal{E}_n \to \cdots \to \mathcal{E}_0 \to F \to 0,$$

where each $\mathcal{E}_i$ is an algebraic vector bundle. This allows to define a ring homomorphism

$$\begin{aligned} \mathrm{K}^0(X) &\to \mathrm{K}_0(X) \\ [F] &\mapsto \sum_{k=0}^{n}(-1)^k[\mathcal{E}_k], \end{aligned}$$

which is the inverse of the Cartan homomorphism. So, in this case, $K_0(X)$ is also a ring, with multiplication induced by the one on $K^0(X)$. The ring $\mathrm{K}_0(X)$ is useful because we can define the *coniveau filtration* on it:

$$\mathrm{K}_0^{(\nu)}(X) \coloneqq \langle [F] \in \mathrm{K}_0(X) \mid codim_X(supp(F)) \geq \nu \rangle,$$

which is a descending filtration. The filtration respects the product on $K_0(X)$, i.e. the graded group associated

$$gr^*\mathrm{K}_0(X) \coloneqq \oplus_{\nu\geq 0} gr^\nu K_0(X) = \oplus_{\nu\geq 0}\mathrm{K}_0^{(\nu)}(X)/\mathrm{K}_0^{(\nu+1)}(X)$$

is a graded ring. The Chern character for the rational Chow group factors through the composition of the Cartan isomomorphism with the canonical morphism into the graded ring

$$K^0(X) \cong K_0(X) \to gr^*K_0(X).$$

That is, we have a commutative diagram of rings

$$\begin{tikzcd} \mathrm{K}^0(X) \arrow[r, "ch_X"] \arrow[d] & \mathrm{CH}^*(X)_{\mathbb{Q}}, \\ gr^*\mathrm{K}_0(X) \arrow[ur, "\psi_X"] & \end{tikzcd}$$

where ψ_X is also a morphism of graded rings. Moreover, we can define a group homomorphism, for each $i \geq 0$,

$$\begin{aligned} \mathrm{CH}^i(X) &\to \mathrm{K}_0^{(i)}(X) \\ [Z] &\to [\mathcal{O}_Z], \end{aligned}$$

where by $[\mathcal{O}_Z]$ we mean the class of the coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{O}_X/\mathcal{I}_Z$, where $\mathcal{I}_Z \subset \mathcal{O}_X$ is the sheaf of ideals corresponding to $Z \subset X$ closed subscheme. By definition of product in $\mathrm{CH}^*(X)$, it linearly extends to a morphism of rings

$$\mathrm{CH}^*(X) \to \mathrm{K}_0(X),$$

which respects the filtrations (on $\mathrm{CH}^*(X)$, we mean the filtration induced by the decomposition, which is such that the graded ring associated is $\mathrm{CH}^*(X)$ itself). Hence, we have the induced morphism on graded rings

$$\varphi_X : \mathrm{CH}^*(X) \to gr^*\mathrm{K}_0(X).$$

Taking rational coefficients, the graduation by coniveau on $\mathrm{K}_0(X)$ splits, that is, the canonical ring homomorphism

$$\mathrm{K}_0(X)_{\mathbb{Q}} \to gr^*\mathrm{K}_0(X)_{\mathbb{Q}}$$

is an isomorphism. Using Grothendieck-Riemann-Roch's Theorem (see [BS58]) applied to the closed immersions $Z \hookrightarrow X$ in $SmProj_k$, we have that $(\psi_X)_{\mathbb{Q}}$ and

$(\varphi_X)_{\mathbb{Q}}$ are such that, in each degree $i \geq 0$,

$$(\varphi_X)_{\mathbb{Q}} \circ (\psi_X)_{\mathbb{Q}} = (i-1)!id \qquad \& \qquad (\psi_X)_{\mathbb{Q}} \circ (\varphi_X)_{\mathbb{Q}} = (i-1)!id.$$

Hence, we have the commutative diagram of ring isomorphisms

$$\begin{tikzcd} \mathrm{K}^0(X)_{\mathbb{Q}} \arrow[r, "(ch_X)_{\mathbb{Q}}"] \arrow[d] & \mathrm{CH}^*(X)_{\mathbb{Q}} \arrow[dl] \\ gr^*\mathrm{K}_0(X)_{\mathbb{Q}} & \end{tikzcd}$$

where

$$\begin{aligned} \mathrm{CH}^*(X)_{\mathbb{Q}} &\to gr^*\mathrm{K}_0(X)_{\mathbb{Q}} \\ [Z] &\mapsto \frac{1}{(i-1)!}[\mathcal{O}_Z] \end{aligned}$$

in each degree $i \geq 0$, is also an isomorphism of graded rings. So, with rational coefficients, the Chern character of the Chow group gives an isomorphism of rings between the Chow group and the 0^{th} K-group.

Given any other contravariant functor

$$\begin{aligned} A^* : SmProj_k^{op} &\to grRing \\ X &\mapsto A^*(X) = \oplus_{i\geq 0} A^i(X), \end{aligned}$$

with a natural transformation of contravariant functors $SmProj_k^{op} \to \mathrm{Ab}$

$$p_A^1 : \mathrm{Pic} \to A^1,$$

which satisfies the axioms CC1)-CC5) and such that, for any $X \in SmProj_k$, $A^*(X)$ is a graded $\mathbb{Q}$-algebra, consider the Chern character

$$ch_{A,X} : \mathrm{K}^0(X) \to A^*(X).$$

Precomposing with the inverse of the rational Chern character for the Chow group, we obtain a morphism of graded rings

$$cl_{A,X} : \mathrm{CH}^*(X)_{\mathbb{Q}} \xrightarrow{(ch_X)_{\mathbb{Q}}^{-1}} \mathrm{K}^0(X)_{\mathbb{Q}} \xrightarrow{(ch_{A,X})_{\mathbb{Q}}} A^*(X).$$

Explicitly, it is such that, for any $[Z] \in \mathrm{CH}^i(X)$,

$$cl^i_{A,X}([Z]) = \frac{1}{(i-1)!} c^i_{A,X}([\mathcal{O}_Z]),$$

where we see the i^{th}-Chern class $c^i_{A,X}$ as a map from $\mathrm{K}_0(X)$ via the Cartan isomorphism

$$c^i_{A,X} : \mathrm{K}_0(X) \cong \mathrm{K}^0(X) \to A^i(X).$$

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