

**RECENT RESULTS**  
**ON THE ABRAHAM–LORENTZ–DIRAC EQUATION**

Andrea CARATI, Luigi GALGANI<sup>(\*)</sup>, Jacopo SASSARINI<sup>(\*\*)</sup>

Università di Padova, Dipartimento di Matematica Pura e Applicata

Via Belzoni 7, 35121, Padova, Italy

ABSTRACT

We illustrate two recent results on the Abraham–Lorentz–Dirac equation, which describes the selfinteraction of a classical charged particle with the electromagnetic field in the nonrelativistic approximation. Namely: 1. the series expansions, which are usually given for the solutions in terms of the electric charge, in general are divergent and have asymptotic character; 2. In the scattering by a potential barrier the nonrunaway solutions are not uniquely defined by the initial position and velocity of the particle, and an unlimited number of solution occur; this phenomenon turns out to have some resemblance to tunnel effect.

**1. Introduction.** The researches on classical electrodynamics received a new emphasis in recent years. One of the reasons is the explosion of mathematical results in the theory of classical dynamical systems (ordered and chaotic motions, and so on), which led to the realization that the mathematical possibilities offered by classical physics were not completely exploited up to now. This in turn seems to sustain the hope that some new insights might thus be offered in allowing to understand some difficulties of principle occurring in quantum mechanics and in quantum field theory. An opinion of this type was expressed several times by Dirac, and also by Haag<sup>[1]</sup>; indeed, in the introduction to his famous work on the selfinteraction of the electron the latter author says: “ This often discussed subject will be here reconsidered in light of the difficulties of quantum field theory”. Now, to such an elevated motivation there didn’t follow in fact any really relevant result. This notwithstanding, as other researchers we continued to delve in this

---

<sup>(\*)</sup> Università di Milano, Dipartimento di Matematica, Via Saldini 50, 20133 Milano, Italy

<sup>(\*\*)</sup> Università di Milano, Dipartimento di Fisica, Via Celoria 16, 20133 Milano, Italy

subject, and present here some recent modest results of ours. More precisely, we leave apart 1) some researches we are doing on the wavelike properties of classical charged particles, which are induced by the wave properties of the field accompanying the particles; 2) some results on the rigorous deduction of the Abraham–Lorentz–Dirac equation from the complete Maxwell–Lorentz system describing the interaction of a particle and the electromagnetic field, obtained by our friends Bambusi and Noja <sup>[2]</sup>, a characterization of the electromagnetic field dragged along by a particle in generic motion, which generalizes the well known prescription of Kramers–Pauli–Fierz and was given recently by A. Carati <sup>[3]</sup>, and 3) a discussion of the equipartition problem in classical mechanics in light of the recent progress in classical perturbation theory. <sup>[4]</sup> We instead concentrate our attention on two particular facts related to the Abraham–Lorentz–Dirac equation, namely: a) the divergence of the perturbative series usually considered for its solutions, and b) the description of a related phenomenon having some resemblance to the tunnel effect (or rather the so-called weak reflection effect).

**2. Main facts about the Abraham–Lorentz–Dirac equation.** The equation under discussion is that of Abraham–Lorentz–Dirac for a point electron, which we consider for simplicity’s sake in the nonrelativistic approximation, namely

$$\epsilon \ddot{\mathbf{x}} = \ddot{\mathbf{x}} - \frac{1}{m} \mathbf{F}(\mathbf{x}) ; \quad (1)$$

here  $\mathbf{x}$  is the position vector of the electron and  $m$  its (renormalized) mass,  $\mathbf{F}(\mathbf{x})$  is an external force field, and

$$\epsilon = \frac{2}{3} \frac{e^2}{mc^3} \quad (2)$$

is the “small parameter”,  $e$  and  $c$  being the electron charge and the velocity of light. Equation (1) is an ordinary differential equation in the “extended phase space”  $\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}$ , which for  $\epsilon = 0$  reduces to the “mechanical equation”  $m\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$  in the “ordinary or mechanical phase space”  $\mathbf{x}, \dot{\mathbf{x}}$ , thus losing one order of differentiation. Problems related to equations with such a property, named singular perturbation problems, lead in general to asymptotic expansions about  $\epsilon = 0$ ; this is well known to researchers in fluid dynamics, in many subjects of applied mathematics, and also in quantum electrodynamics, but strangely enough was apparently not stressed by researchers in classical electrodynamics.\*

For eq. (1), conditions of a Cauchy type, which assign position, velocity and acceleration at one time, give as usual existence, uniqueness, and continuity with respect to initial data and to the parameter  $\epsilon$ , for  $\epsilon \neq 0$ . However, as first apparently pointed out by Dirac <sup>[9,10]</sup>, it occurs that for generic initial data the solutions have a runaway character, i.e. have the property that the acceleration  $\mathbf{a}$  diverges. For example, for the free particle the equation reduces to  $\epsilon \dot{\mathbf{a}} = \mathbf{a}$ , which gives  $\mathbf{a}(t) = \mathbf{a}_0 \exp(t/\epsilon)$ . This fact can be seen in

---

\* For the series expansions occurring in quantum electrodynamics, which in general are expected to be divergent and (at most) of asymptotic type, see [5], [6] page 84, and [7] chapter 37. For the series of classical electrodynamics one can instead find statements suggesting a possible analytic character (see [8] page 362).

a particularly perspicuous way, in the general case, if one takes the point of view of the qualitative theory of dynamical systems. Considering for simplicity the case of one degree of freedom, equation (1) can be written in the form

$$\dot{x} = v, \quad \dot{v} = a, \quad \dot{a} = \frac{1}{\epsilon}(a - F(x)/m).$$

So, in the extended phase space  $x, v, a$ , outside a “small” layer about the “slow manifold” defined by  $a - F(x)/m = 0$  the vector field defining the differential equation is “practically infinite” and essentially parallel to the  $a$ -axis, being directed away from the slow manifold. Thus, with a by now standard terminology, the existence of runaways is described by simply saying that for the Abraham–Lorentz–Dirac equation the “fast foliation” is parallel to the acceleration and away from the slow manifold.

Now, the generic appearance of runaways does not exclude the possibility of exceptional solutions with bounded acceleration, and the proposal of Dirac was just to add such a requirement on the solutions. In particular, for scattering problems nonrunaway solutions of (1) are selected by imposing subsidiary conditions of mixed type, namely conditions of ordinary Cauchy type for position and velocity (so defining a unique solution for the corresponding “reduced” or “mechanical” problem  $m\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ ), and an asymptotic outgoing condition for the acceleration, precisely  $\ddot{\mathbf{x}}(t) \rightarrow 0$  for  $t \rightarrow +\infty$ . From the mathematical point of view, the condition of Dirac, together with conditions of a Cauchy type on the initial position and velocity, leads to a problem of Sturm–Liouville type, which in general might admit no solution. However, with ordinary potentials vanishing at infinity the reduced mechanical problem does admit scattering states for suitable initial data, and so the same might be expected to hold also for the Lorentz–Dirac equation (1) with subsidiary conditions of Dirac type, at least for suitable initial data. For example, for the free particle this occurs for  $\mathbf{a}_0 = 0$ .

### 3. The series expansion: its divergence and asymptotic character.

Let us assume the existence of solutions  $\mathbf{x}(t, \epsilon)$  of the Abraham–Lorentz–Dirac equation satisfying the Dirac outgoing condition, and discuss the character of the corresponding series expansions in  $\epsilon$ . If the force field  $\mathbf{F}(\mathbf{x})$  is analytic, the solutions  $\mathbf{x}(t, \epsilon)$  are analytic in  $t$ , and also in  $\epsilon$  for  $\epsilon \neq 0$ ; the problem at hand is then just the analyticity in  $\epsilon$  at  $\epsilon = 0$ . Furthermore, we will consider solutions  $\mathbf{x}(t, \epsilon)$  having at least a singularity in the complex  $t$  plane, which is clearly the generic situation for a nonlinear force field  $\mathbf{F}(\mathbf{x})$ .

The series expansion in  $\epsilon$  for the solution  $\mathbf{x}(t, \epsilon)$  is usually defined through an expansion for the corresponding acceleration  $\ddot{\mathbf{x}}$ . This is given in the form

$$m\ddot{\mathbf{x}}(t, \epsilon) = \sum_{n=0}^N D_t^n \mathbf{F}(\mathbf{x}(t, \epsilon)) \epsilon^n + \mathbf{R}_N(t, \epsilon), \quad (3)$$

where we have denoted  $D_t^n = \frac{d^n}{dt^n}$ , and remainder

$$\mathbf{R}_N(t, \epsilon) = \epsilon^{N+1} \int_0^{+\infty} e^{-u} D_{t+\epsilon u}^{N+1} \mathbf{F}(\mathbf{x}(t + \epsilon u, \epsilon)) du. \quad (4)$$

A simple deduction is the following one: by the variation of constants formula, the Lorentz–Dirac equation (1) with the outgoing irac condition is rewritten in the integro–differential form

$$m\ddot{\mathbf{x}}(t, \epsilon) = \int_t^{+\infty} \frac{e^{(t-s)/\epsilon}}{\epsilon} \mathbf{F}(\mathbf{x}(s, \epsilon)) ds ,$$

and then  $N$  repeated integrations by parts are performed.

The fact that the expansion (3) diverges for any  $\epsilon \neq 0$ , as  $N \rightarrow \infty$ , is an immediate consequence of the assumed existence of a singularity in  $t$  for the solution  $\mathbf{x}(t, \epsilon)$ . Indeed, consider the Taylor expansion of  $\mathbf{F}(\mathbf{x}(t, \epsilon))$  about  $t$ ,

$$\mathbf{F}(\mathbf{x}(t + s, \epsilon)) = \sum_{n=0}^{\infty} \frac{s^n}{n!} D_t^n \mathbf{F}(\mathbf{x}(t, \epsilon)) ,$$

and use the obvious inequality  $|s|^n/n! < |\epsilon|^n$ , which holds for any  $\epsilon \neq 0$  and for any  $s$ , for large enough  $n$ ; thus the convergence of the series expansion (3) for any  $\epsilon$  would imply  $\mathbf{F}(\mathbf{x}(t + s, \epsilon))$  to be an entire function of  $s$ , against the assumption that  $\mathbf{x}(t, \epsilon)$  has a singularity in the complex  $t$  plane.

The discussion about the asymptotic character of the expansion (3)–(??) is a little more delicate. We recall that a series is said to be asymptotic if, for all fixed  $N$ , one has  $|\mathbf{R}_N|/\epsilon^N \rightarrow 0$  as  $\epsilon \rightarrow 0$ . So, preliminarily, since the series is associated to any given solution  $\mathbf{x}(t, \epsilon)$ , one has to define which family of solutions depending on  $\epsilon$  is considered: the natural choice is just to keep fixed the initial data  $\mathbf{x}_0, \dot{\mathbf{x}}_0$  in the mechanical phase space.

From the expression (4) of the remainder it is clear that, in order to prove the asymptotic character of the series, one has to give a bound for all derivatives  $D_t^n \mathbf{x}(t, \epsilon)$ , for  $n \geq 1$  and all  $t$ . An useful control is provided by the assumed singularity of the solution  $\mathbf{x}(t, \epsilon)$  in the variable  $t$ , making use of the Cauchy estimate

$$|D_t^n \mathbf{F}(\mathbf{x}(t, \epsilon))| \leq \mathcal{F}(t, \epsilon) \frac{n!}{d(t, \epsilon)^n} ;$$

here  $d(t, \epsilon)$  is for example half the distance of  $t$  from the nearest singularity of  $\mathbf{x}(t, \epsilon)$ , and  $\mathcal{F}(t, \epsilon) = \sup |\mathbf{F}(\mathbf{x}(z, \epsilon))|$  for  $|z - t| < d(t, \epsilon)$ . This gives immediately the uniform bound

$$\frac{|\mathbf{R}_N(t, \epsilon)|}{\epsilon^N} < \epsilon \tilde{\mathcal{F}}(\epsilon) \frac{(N + 1)!}{\tau(\epsilon)^{N+1}} , \quad (5)$$

where  $\tau(\epsilon)$  is half the amplitude of the analyticity strip of the solution in the complex  $t$  plane, and  $\tilde{\mathcal{F}}(\epsilon) = \sup \mathcal{F}(t, \epsilon)$  for all real  $t$ . So we conclude that the divergent series expansion for the given family of solutions is asymptotic if  $\bar{\tau} > 0$ , where  $\bar{\tau} = \inf_{\epsilon > 0} \tau(\epsilon)$ .

**4. Motions of mechanical type and of nonmechanical type.** It is well known that, for any fixed  $\epsilon$ , the divergence of a series  $\sum_n c_n \epsilon^n$  representing asymptotically a given function does not imply at all that the series be useless. Rather, the expansion can provide a good estimate of the function by just a truncation to an optimal order  $N_{\text{opt}}$ , defined by

the property that the modulus of the remainder has thereby a minimum; moreover, the remainder usually turns out to be exponentially small with  $1/\epsilon$ , if  $N_{\text{opt}} > 0$ .

In our case, with the estimate for the remainder given above, one immediately finds  $N_{\text{opt}}(\epsilon) = [\tau(\epsilon)/\epsilon]$ , where  $[\cdot]$  denotes integer part. Thus the situation strongly depends on the value of  $\tau(\epsilon)/\epsilon$ . Indeed, if such a ratio is larger than 1 one has  $N_{\text{opt}} > 0$ , and correspondingly the remainder turns out to have an exponentially small estimate; precisely one easily finds

$$|\mathbf{R}_{N_{\text{opt}}}(\epsilon)| \leq \tilde{C}[\tau(\epsilon)/\epsilon]^{-1/2} \mathcal{F}(\epsilon) e^{-[\tau(\epsilon)/\epsilon]},$$

with a suitable constant  $\tilde{C}$ . Instead, if the ratio is smaller than 1 one has  $N_{\text{opt}} = 0$ , and the estimate for the remainder just reduces to that given by (5) evaluated for  $N = 0$ , namely

$$|\mathbf{R}_0| \leq \frac{\epsilon}{\tau(\epsilon)} \tilde{\mathcal{F}}(\epsilon),$$

which is not “small”, because  $\frac{\epsilon}{\tau(\epsilon)}$  is then larger than 1.

So we see that the usefulness of the series depends on the value of  $\tau(\epsilon)/\epsilon$ , which is characteristic for any given solution  $\mathbf{x}(t, \epsilon)$  at any given value of  $\epsilon$ . If  $\tau(\epsilon)/\epsilon > 1$ , from the asymptotic expansion of  $\mathbf{x}(t, \epsilon)$  one can extract a partial sum which is a small perturbation of the solution of the corresponding mechanical problem (i.e. that obtained for  $\epsilon = 0$ ). This means that the solution  $\mathbf{x}(t, \epsilon)$  is qualitatively similar to the corresponding mechanical one, or is, as one can say, of “mechanical type”. But a radically different situation occurs if  $\tau(\epsilon)/\epsilon < 1$ , because the expansion becomes then useless, the remainder being no more small. However, this doesn’t mean anything special for the solution  $\mathbf{x}(t, \epsilon)$  of the complete equation, apart from the fact that it is no more a perturbation of the solution of the purely mechanical equation; in such a case we say that the solution is of “nonmechanical type”. In this sense we say that the Abraham–Lorentz–Dirac equation, which takes into account the interaction of a charged particle with “its own” field, should admit solutions of two kinds: those qualitatively similar to purely mechanical ones, and those qualitatively dissimilar. The difference depends on the value of the ratio  $\tau(\epsilon)/\epsilon$  being greater or smaller than 1, where  $\tau(\epsilon)$  is the amplitude of the analyticity strip of the solution in the complex  $t$  plane.

In the paper [11] it was given an a priori estimate for deciding when motions of non mechanical type can be expected for a scattering problem with an external force due to an attractive Coulomb potential. The result is that nonmechanical motions are to be expected if the angular momentum  $l$  is lower than a critical value  $\bar{l}$  given by

$$\bar{l} = 6Z^{2/3} \frac{e^2}{c},$$

where  $Z$  is the atomic number. So, such an estimated threshold of angular momentum is of the the order of  $e^2/c$ , which, as a first approximation in the spirit of perturbation theory, might be considered to be non extremely dissimilar from that of Planck’s constant  $\hbar \simeq 137e^2/c$ . Some speculations were also added about a possible physical interpretation of the threshold as giving a frequency cutoff in the radiated spectrum, comparing it with the Duane–Hunt law and the de Broglie relation.

**5. The problem of the multiple solutions of the Abraham–Lorentz–Dirac equation** The uniqueness problem for the nonrunaway (or physical) solutions of the Abraham–Lorentz–Dirac equation can be formulated in the following way. Given an initial datum  $(\mathbf{x}_0, \mathbf{v}_0)$  in the mechanical phase space, one asks whether there exists a value  $\mathbf{a}_0$  of the acceleration such that the corresponding initial datum  $(\mathbf{x}_0, \mathbf{v}_0, \mathbf{a}_0)$  in the extended phase space gives rise to a motion  $\mathbf{x}(t)$  satisfying the Dirac prescription  $\mathbf{a}(t) \rightarrow 0$  for  $t \rightarrow +\infty$ . Notice that neither existence nor uniqueness is obvious, because this somehow resembles a problem of Sturm–Liouville type. In connection with the discussion made above, one should expect that for small  $\epsilon$  one is confronted with solutions of mechanical type, so that existence and uniqueness should occur, while something new could occur for large  $\epsilon$ , where there might exist solutions of nonmechanical type. We will discuss below how it occurred to us to exhibit a generic mechanism causing nonuniqueness for scattering by barriers at large enough  $\epsilon$ .

From the historical point of view the situation is however rather curious. Indeed, we were mainly referring to a review article by Plass,<sup>[12]</sup> to the book of Rohrlich<sup>[13]</sup> and to the mathematical paper of Hale and Stokes,<sup>[14]</sup> all written in the years 1960–1965. Now, in the paper of Hale and Stokes existence was proven, while the uniqueness problem was left open. On the other hand, through a review article by Erber (see [15], pag. 355) we found that very particular cases of nonuniqueness were already known. Everything goes back to Bopp who, in his beautiful paper<sup>[16]</sup> of the year 1943, in the middle of a discussion of a very general type, solved a very particular example (one–dimensional scattering by a potential step), showing that there exist cases with two solutions corresponding to the same initial mechanical data (position and velocity), one solution corresponding to reflection and the other to transmission. A little variant of that example (a potential step increasing linearly between the two constant values) was studied by R. Haag<sup>[1]</sup> in the year 1955, who showed in particular that nonuniqueness would occur only if the steepness were large enough. By the way, a general mathematical discussion of a very interesting character for the Abraham–Lorentz–Dirac equation is also given there. The same result of Bopp, namely existence of two solutions for a one–dimensional potential step, but for the relativistic Dirac equation, was rediscovered in the year 1976 by Baylis and Huschilt,<sup>[17]</sup> apparently unaware of the previous works of Bopp and Haag. All such examples were studied by the quoted authors in a very elementary way, suited to the particularly simple cases considered. The thesis maintained by Erber is that, among the two “Dirac–type solutions” of Bopp and Haag, by some other physical reason only one should be retained and the other discarded. This has some analogy with an old point of view. Indeed, in discussing the problem of how to eliminate the runaways, it was suggested by Bhabba<sup>[18]</sup> and Rohrlich<sup>[19]</sup> that one should use the “*physical requirement that in the limit  $e \rightarrow 0$  the trajectory of a charged particle should have a limit and that this limit be the trajectory of a corresponding neutral particle (Principle of undetectability of small charges)*” (see [8], page 347; for related problems, see also [20,21]). Now, this just corresponds, in our terms, to eliminating the solutions which we have called of nonmechanical type, by presuming them to be by some reason “unphysical”. On the other hand, we are suggesting that such solutions should just be considered on the same footing as the other ones, so that some new interpretation is required.

The phenomenon of nonuniqueness can be described geometrically in the following way. Consider the subset of the extended phase space corresponding to motions satisfying the Dirac condition, and call it the “physical (or Dirac) manifold”: uniqueness would correspond to the physical manifold being a graph, say  $\mathbf{a} = \mathbf{g}(\mathbf{x}, \mathbf{v})$ , while nonuniqueness means that it is folded; we will indeed show that it has in general infinitely many foldings. Moreover, it will turn out that, in the case of scattering by a barrier, the initial data belonging to different branches of the folded physical manifold and having the same position and velocity give rise alternatively to motions of mechanical and of nonmechanical type, which are transmitted and reflected respectively.

**5. Explanation of the uniqueness property for a one–dimensional barrier by qualitative arguments.** The nonuniqueness property for scattering by a generic one–dimensional barrier was discussed in ref. [22]; some numerical integrations of the equations of motion were first reported, and an explanation was then given in terms of qualitative arguments, typical of the modern theory of dynamical systems. We report here briefly on the qualitative explanation. Everything can be understood by making reference to the particularly simple case of a force vanishing outside a compact domain and linear inside it; consider for example the case with potential energy  $V(x)$  given by

$$V(x) = 1 - x^2 \quad \text{for } |x| < 1, \quad V(x) = 0 \quad \text{for } |x| > 1,$$

with continuity conditions for the acceleration at the points  $x = \pm 1$ . Indeed, by general arguments (i.e. the so called stable manifold theorem), the qualitative results can then be transported to the case of a generic smooth barrier having a single maximum.

Take any initial datum  $z_0$  in the extended phase space with coordinates  $z = (x, v, a)$ , in the region with vanishing force, for definiteness with  $x_0 < 1$ . Thus the particle will proceed by the well known solution of the free particle problem and, depending on  $v_0$  and  $a_0$ , some solutions will be able to reach the point  $x = -1$ . Then one has to solve the linear problem  $\dot{z} = Az$  where  $A$  is the 3 by 3 matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2/\epsilon & 0 & 1/\epsilon \end{pmatrix}.$$

The eigenvalues of  $A$  turn out to be real for small  $\epsilon$ , while an interesting bifurcation occurs at  $\epsilon = \sqrt{2/27}$ , because for larger values of  $\epsilon$  the matrix  $A$  has a real negative eigenvalue and two complex conjugate eigenvalues with positive real part. So, in the case of “large  $\epsilon$ ” to which we now concentrate our attention, for the corresponding linear system the phase space turns out to be the direct sum of a one dimensional “stable linear space”  $E^s$  (corresponding to the real negative eigenvalue) and a two dimensional “unstable linear space”  $E^u$ , the restriction of the system to  $E^u$  being an unstable focus. The dispositions of such linear spaces are easily determined; to fix ideas, for  $\epsilon = 1$  one has the eigenvalue  $-1$  with eigenvector  $(1, -1, 1)$  and the eigenvalues  $1 \pm i$  with eigenspace spanned by  $(1, 1, 0)$  and  $(0, 1, 2)$ . Now the exceptional solution which at  $x = -1$  hits exactly the point of intersection with the straight line  $E^s$  will tend to the origin (which corresponds to the

maximum of the potential) in an infinite time (case of the separatrix). But consider now any other solution coming from left. By linearity, it will have a component tending to the origin and another component spiraling out. Thus there exists a certain time at which it will reach either the plane  $x = -1$  or the plane  $x = 1$ , and after such a time it will become a solution of the free particle problem again. So we have now to impose the limiting Dirac condition, which in this simple case reduces to  $a = 0$ . In other terms, the Dirac or physical solutions will be the exceptional ones which, by spiraling around the separatrix, will reach one of the planes  $x = -1, x = 1$  exactly with  $a = 0$ ; the first ones will be reflected solutions, and the other ones transmitted solutions. Among the solutions of the free particle problem reaching the plane  $x = -1$  from left, those nearer and nearer to the point corresponding to the separatrix will perform a larger and larger number of turns before reaching again one of the planes with  $|x| = 1$ , and so there exist motions making any number of turns. In such a way the surface of the physical solutions (i.e. satisfying the Dirac condition  $a(t) \rightarrow 0$  for  $t \rightarrow +\infty$ ) will be folded, and with infinitely many foldings; moreover, to the “branch” related to a number  $n$  of turns, for a given initial mechanical state there will correspond both one reflected and one transmitted solution.

**6. Analogy with the weak reflection effect, and further comments.** The nonuniqueness property described above appears certainly mathematically interesting, because it has some similarity with the weak reflection effect of quantum mechanics, according to which a particle with energy slightly larger than the maximum of the barrier can be either transmitted or reflected by the barrier. In our case, we find that there exists an energy strip, about a value slightly above the maximum of the barrier, such that for any value of the energy in the strip there exist both transmitted and reflected motions; moreover, for  $\epsilon$  large enough the energy strip turns out to extend up to values relevantly lower than the maximum of the barrier, so that one has even an effect which is similar to the tunnel effect.

This is the similarity. But the difference comes about because the width of the strip is too small for physical values of the parameters. Indeed, the effective “small parameter” in the problem is a pure number that is obtained just by a rescaling, namely the number  $\epsilon'$  defined by

$$\epsilon' = \frac{\epsilon}{\sqrt{mL^2/V_0}},$$

where  $m$  is the mass of the particle,  $L$  a typical length of the potential and  $V_0$  the height of the barrier. By considering typical cases of physical interest, such as that of the alpha-decay, one finds for  $\epsilon'$  values of the order of  $10^{-4}$ , in place of values of the order 1 which are required to exhibit the nonuniqueness phenomenon.

So, from this point of view the phenomenon described above appears to lack of physical interest. We believe however that it has some interest, because it appears to show that classical physics presents a richness of behaviours that were unexpected. In particular, it is now clear that classical electrodynamics for point particles requires intrinsically the introduction of some probabilistic notions, just in order to control the new parameters (here, the initial acceleration), which are needed in addition to the purely mechanical ones, in order to lead to deterministic motions. We hope that the peculiar character of

this new example might prove useful in giving a concrete support to the general discussions of the hidden parameters for quantum mechanics.

## REFERENCES

- [1] R. Haag, *Die Selbstwechselwirkung des Elektrons*, *Z. Naturforsch.* **10 A**, 752–761 (1955).
- [2] D. Bambusi and D. Noja, *Classical electrodynamics of point particles and mass renormalization. Some preliminary results*. *Lett. Math. Phys.* **37**, 449–460 (1996).
- [3] A. Carati, *On the electromagnetic field dragged along by a particle*, preprint
- [4] L. Galgani, *Relaxation Times and the Foundations of Classical Statistical Mechanics in the Light of Modern Perturbation Theory*, lectures given at the Noto School *Non-Linear Evolution and Chaotic Phenomena*, G. Gallavotti and P.F. Zweifel eds. NATO ASI Series B: Vol. 176, Plenum Press, (New York, 1988).
- [5] F.J. Dyson, *Phys. Rev.* **85**, 631 (1952).
- [6] A.B. Migdal, *Qualitative methods in quantum theory*, Addison–Wesley (Reading, 1989).
- [7] J. Zinn–Justin, *Quantum field theory and critical phenomena*, Clarendon Press (Oxford, 1990).
- [8] F. Röhrlich, *The electron: development of the first elementary particle theory*, in J. Mehra ed., *The physicist's conception of nature*, D.Reidel (Dordrecht, 1973).
- [9] P.A.M. Dirac, *Proc. Royal Soc. (London)* **A167**, 148–168 (1938).
- [10] P.A.M. Dirac, *Ann. Inst. Poincaré* **9**, 13 (1938).
- [11] A. Carati and L. Galgani, *Asymptotic character of the series of classical electrodynamics, and an application to bremsstrahlung*, *Nonlinearity* **6**, 905–914 (1993).
- [12] G.N. Plass, *Rev. Mod. Phys.* **33**, 37 (1961).
- [13] F. Röhrlich, *Classical charged particles*, Addison–Wesley (Reading, 1965).
- [14] J.K. Hale and A.P. Stokes, *J. Math. Phys.* **3**, 70 (1962).
- [15] T. Erber, *The classical theory of radiation reaction*, *Fortschr. der Phys.* **9**, 343–392 (1961).
- [16] F. Bopp, *Ann. der Phys.* **42**, 573–608 (1943).
- [17] W.E. Baylis, J.Huschilt, *Phys. Rev.* **D13**, 3237–3230 (1976).
- [18] H.J. Bahbba, *Phys. Rev.* **70**, 759 (1946).
- [19] F. Röhrlich, *Ann. Physics* **13**, 93 (1961).
- [20] P. Caldirola, *Nuovo Cim.* **5**, 99 (1948).
- [21] A. Loinger, *Nuovo Cim.* **6**, 360 (1949).
- [22] A. Carati, P. Delzanno, L. Galgani, J. Sassarini, *Nonuniqueness properties of the physical solutions of the Lorentz–Dirac equation*, *Nonlinearity* **8**, 65–79 (1995).