

A logical characterization of coherence for imprecise probabilities

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ABSTRACT

Whilst supported by compelling arguments, the representation of uncertainty by means of (subjective) probability does not enjoy a unanimous consensus. A substantial part of the relevant criticisms point to its alleged inadequacy for representing *ignorance* as opposed to uncertainty. The purpose of this paper is to show how a strong justification for taking belief as probability, namely the Dutch Book argument, can be extended naturally so as to provide a logical characterization of coherence for *imprecise probability*, a framework which is widely believed to accommodate some fundamental features of reasoning under ignorance. The appropriate logic for our purposes is an algebraizable logic whose equivalent algebraic semantics is a variety of MV-algebras with an additional internal unary operation representing upper probability (these algebras will be called UMV-algebras).

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1. Introduction

According to Frank Knight, the author of the classic 1921 volume *Risk, Uncertainty and Profit*, a decision under risk is one for which a “known” or otherwise “correct” probability distribution can be identified by the decision maker. Uncertain, on the other hand, is a situation in which probability cannot be defined with sufficient precision. Whenever this happened, back at the time, insurance companies were very likely to declare the corresponding amounts as uninsurable. Over the past century, the general attitude towards risk and uncertainty has changed dramatically, especially owing to the introduction of sophisticated mathematical techniques of risk management. Yet, as recent evidence coming from the financial markets painfully shows, the view according to which a “known probability distribution” contains no uncertainty is not quite right. Whether precise or not, probability is not an intrinsic property of specific events.

Suppose a die is being rolled. One thing is to be uncertain about the face that will eventually show up. One quite different thing, is to be uncertain about whether the die is *fair* or unbiased. We can rather naturally refer to *first order* and *second order* uncertainty, respectively. In the former case we are uncertain about some (presently unknown) state of affairs. In the latter we are uncertain about our uncertainty. We can illustrate informally the key idea as follows.

First order uncertainty can be very directly referred to our lack of information, knowledge, etc. about the correct answer to some well-posed question, e.g. which country will have (relative to its GDP) the largest public debt by the end of 2015. Presumably, no one is able to give a definite answer to this question before 2016, and perhaps not even then. Meanwhile, however, say a financial rating agency, should be in a position to make an estimate which can be used to make relevant decisions. This forecast amounts to provide a tentative answer to the well-posed question. The way in which the answer is tentative reflects first order uncertainty.

Suppose now that the same question is being asked to a professional financial analyst and to the occasional *Financial Times* reader (independently). It would not be surprising if the former assessed her uncertainty with much higher precision,

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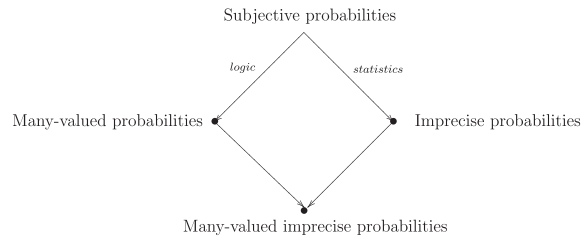


Fig. 1. A stylized landscape of probabilistic uncertain reasoning. The statistical and logical traditions highlight two rather distinct aspects in the generalization of probabilistic reasoning to higher level forms of uncertainty. The characterization proposed in this paper in many ways can be seen as a synthesis of such traditions.

confidence or indeed reliability, than the latter. Put another way, the occasional FT reader might just feel that his information (knowledge about the relevant facts, competence, etc.) is just *not sufficient* for him to give a reasoned or even informed answer. In this picture then, second order uncertainty refers to the assessment that an agent makes about her own uncertainty.

How can we *measure* first and second order uncertainty? Reasoning under first order uncertainty, or simply *uncertainty*, has been modelled for over three centuries by the calculus of probability. However it was not until the late 1920's the key foundational questions about the meaning and the interpretation of probability were put directly under scrutiny. The work of Bruno de Finetti and Frank P. Ramsey is particularly important, in this respect. They put forward a view according to which probability can be *justified* as a mathematical measure of one individual's uncertainty by referring to the concept of coherence. This view, which is often referred to as *bayesianism*, takes probability to be:

- (1) Subjective: it is a measure of an individual's degrees of belief (and hence uncertainty) with respect to some relevant facts.
- (2) Single-valued: for all events of interest an agent is in a position to indicate a point $\alpha \in [0, 1]$.
- (3) Logically boolean: events can only occur or fail to occur, and hence the corresponding propositions can only be either true or false.
- (4) Constrained only by coherence: in case many probability distributions are consistent with one's information, no formal principle can be justifiably applied to break the tie.

This paper endorses fully (1), challenges (2) and (3) while remaining neutral on (4). Indeed we agree with the bayesian claim to the effect that probability is, from the foundational point of view, the strongest candidate for defining rational belief under uncertainty. Unlike radical subjectivists, however, we restrict the validity of this claim *only* to first order uncertainty. In other words, we argue that in many interesting situations, the practical need arises to put forward coherent probability assignments which fall short of being single valued, or which pertain to nonbinary events. This, as we will see, calls for an adequate model of second order uncertainty and in particular, a model of probabilistic *imprecision*.

The issue of extending subjective probability so as to encompass imprecision has been pursued at least by two distinct traditions, which can be conveniently labelled as statistical and logical, respectively. As illustrated in Fig. 1, they have a common root in the subjective interpretation of probability, but they diverge in the formal development and, as a consequence, in their applications to reasoning under uncertainty. To the best of our knowledge, no systematic comparison between those two traditions has ever been put forward. This paper fills this gap by introducing framework in which the key concepts pertaining to both traditions can be adequately represented.

Specifically, we introduce a logico-algebraic setting in which *coherence* for upper and lower probabilities (and previsions) is given a *purely logical* characterization. As we shall put forward, the appropriate logic for our purposes is an algebraizable logic whose equivalent algebraic semantics is a variety of MV-algebras with an additional unary operation representing upper probabilities (these algebras will be called UMV-algebras).

The paper is organized as follows. In the remainder of this section we provide a sketchy outline of the bayesian characterization of reasoning under first order uncertainty and its logical and statistical extensions to second order uncertainty.¹ In Section 2 we firstly recall the basic definitions and concepts of MV-algebras and algebraizable logics and then we introduce an algebraic framework which formalizes the key concepts pertaining to the logical and statistical traditions. In Section 3 we translate the main concepts introduced in Section 2 in the framework of universal algebra and algebraic logic. To this purpose, we introduce UMV-algebras, and prove a completeness result with respect to satisfiability. This result enables us to prove that the coherence of an assessment is equivalent to the logical coherence of an equational theory over the equational logic of UMV-algebras. Then in Section 4 we introduce an algebraic framework for gambles and upper previsions. This is the structure of UG-algebras that is, divisible abelian ℓ -groups with strong unit with an internal upper prevision. We show that a categorical equivalence exists between UMV-algebras and UG-algebras. This equivalence allows us to interpret the equational logic UG for UG-algebras into the equational logic UMV for UMV-algebras. As a consequence we can find a logical

¹ For further background and historic references on the topics covered in this section, the interested reader may wish to consult the volumes [15, 18, 5].

counterpart of coherent assessments on divisible and unital ℓ groups inside the equational logic of UMV-algebras. Finally, Section 5 is devoted to the complexity of the logics UMV and UG.

1.1. Rationality as coherence via the betting scheme

In his two-volumes monograph [8] (published in English in 1974, but collecting results which span over almost five decades) Bruno de Finetti couples the mathematical presentation of the calculus of probability, with an unprecedented analysis of its foundations, culminating in a cohesive presentation of his *subjectivist interpretation*. While formally unquestionable, Kolmogorov's axiomatic characterization of probability functions fails to give us particular insights as to the *meaning* of probability, which is just defined as anything satisfying the axioms. This failure is particularly relevant when it comes to discussing the practical importance of probabilistic judgments from statistics, to decision theory, to everyday life, since they all involve the assessment of one's own uncertainty. Bridging the gap between the 'everyday use' of the concept of probability and its mathematical properties is one of the explicit aims of de Finetti's work. This is why, among the many frameworks which have been proposed to characterize subjective probability, which include among others [35, 12], we choose to focus here on de Finetti's.

The key intuition in de Finetti's operational definition of probability consists in viewing probability as a *price*. Specifically, the probability that an agent assigns to an event ϕ corresponds to the price $\alpha = P(\phi)$ that she would be willing to pay in order to secure the right to receive 1 if ϕ occurs and nothing otherwise (hence suffering a net loss of α). Probability-as-price then, can be claimed to be a natural *measure* of the agent's first order uncertainty about ϕ . The formalization of the betting scheme, which we shall describe precisely in this section, constitutes a crucial step in proving this claim (see Theorem 1.1).

de Finetti's betting scheme provides both an operational definition of probability as price and a direct way of defining the *logical constraint* that an agent's beliefs should satisfy in order to avoid manifestly irrational behaviour. The idea is as follows. Suppose that you are willing to place bets in such a way that you are led to loose *no matter what the outcome* of the relevant events will turn out to be. Under the assumption that your betting dispositions reflect your subjective beliefs,² your willingness to face sure loss is a clear indication of the *inconsistency* of your beliefs and hence, arguably, of your *irrationality*. The essence of de Finetti's proof (a result which had already been stated by Ramsey [33]) is that the necessary and sufficient conditions for your degrees of belief equally not to be inconsistent are normality, finite additivity and non-negativity (see Theorem 1.1 for a rigorous statement).

The betting scheme and the corresponding criterion of coherence (or consistency) therefore constitute the core conceptual machinery of subjective probability and, as such, lie at the heart of our probabilistic model of uncertain (first and second order) reasoning. We interpret the formal results of this paper as showing that the conceptual machinery devised by de Finetti for modelling first order uncertainty can be extended to construct mathematical models of more sophisticated notions of uncertainty, which nonetheless rest on the very same epistemological grounds.

In order to introduce a formalization of the betting scheme which we shall use, modulo suitable adjustments, throughout the paper, we need a little notation. Let X be a random quantity about whose actual value an agent is (first order) uncertain. The *prevision* $E(X)$ is defined as the price that an individual considers fair in exchange for the random quantity X . Whenever the range of the random quantity X is restricted to the binary set $\{0, 1\}$, the fair price for X amounts to its *probability* $P(X)$. For ease of exposition we shall characterize the notion of fair price for probability, rather than prevision, recalling however, that both de Finetti and Walley take previsions as primitive. Indeed, previsions on bounded (continuous) random variables will be the object of our model, see Section 2.1.

A *classical event* is a two-valued random quantity, whose values range over $\{0, 1\}$. From the logical point of view then, an event is a classical (boolean-valued) proposition which can either take the value "true" (1) or "false" (0).

We can now introduce the *classical betting game*. Let B and G be two players, let us say the *bookmaker* and the *gambler*, respectively. Let ϕ be an event in some suitable language and let $v(\phi)$ be the truth value of ϕ (so $v(\phi) = 1$, if ϕ occurs, 0 otherwise and no other possibility is given). We are interested in defining the probability that B assigns to the event ϕ . In the light of the above discussion a natural choice consists in equating $P(\phi)$ to $\alpha \in [0, 1]$ so that α is the betting odd for which the bookmaker is willing to accept any bets on (or against) ϕ placed by the gambler.³ That is, *after choosing* α the bookmaker is *forced* to accept any bets placed by the gambler. Placing a bet, for G , means choosing a real-valued λ giving rise to the following possible payoffs for the gambler:

- If $\lambda > 0$, G is actually betting *on* ϕ , so he pays $\lambda\alpha$ to the bookmaker and receives λ , if $v(\phi) = 1$, and nothing otherwise.
- If $\lambda < 0$ then G is betting *against* ϕ , so he receives $\lambda\alpha$ from the bookmaker and pays back λ , if $v(\phi) = 1$, and nothing otherwise.

The rules of the classical betting game take the form of a complete contract which, it is worth stressing, force the bookmaker to accept any bets placed by the gambler. Since she can choose not only the magnitude, but also the sign of

² This is sometimes referred to as "behavioural interpretation" of subjective degrees of belief.

³ In the real betting games, sometimes the betting quotient $\alpha : (1 - \alpha)$ is used in place of the betting odd α . The betting quotient represents the amount of money that the gambler must pay to have a net gain of 1 in case of win, while the betting odd α is the amount of money the gambler has to pay to get back 1 (with a net gain of $1 - \alpha$) in case of win.

the stakes λ , the gambler, by choosing whether to bet on a certain event or against it, effectively chooses which role will eventually be played by the bookmaker (i.e. her payoff matrix). Since, of course, the bookmaker, knows this, she cannot rationally choose betting odds α that could give rise to a non-zero expected payoff for either sides of the game, for she is not the one choosing which will be eventually her own side! The unique price α which is so determined, is indeed a *fair betting odd*.

This line of reasoning justifies the *condition of coherence* which de Finetti couples with the classical betting scheme (see [8, Chapter 3]). Intuitively, this condition requires B to choose the betting odds in such a way as to guard herself from sure loss. More precisely, let S be a *finite set* of events, and for each $\phi \in S$, let α_ϕ be the betting odds for which the bookmaker is willing to accept any bets on (or against) ϕ placed by the gambler. An *assessment* is a function assigning to each $\phi \in S$ a real number α_ϕ . A *system of bets* is defined to be a map $\phi \mapsto \lambda_\phi$ from S into \mathbf{R} . Then the condition of coherence can be defined as follows:

Definition 1.1 (*Coherence*). An assessment is said to be *coherent* if there is no system of bets $\phi \mapsto \lambda_\phi$ which causes to B a *sure loss*, that is, such that $\sum_{\phi \in S} \lambda_\phi (\alpha_\phi - v(\phi)) < 0$ for every valuation v .

We are now in a position to state the Dutch Book Theorem, whose first proof appeared in [9], a result to the effect that finite additivity is a necessary condition for coherence.

Theorem 1.1 (*Dutch Book Theorem*). An assessment $\phi \mapsto \alpha_\phi : \phi \in S$, where S is a *finite set of events*, is *coherent* iff there is a *probability distribution on the algebra of events generated by S* which extends it.

Although de Finetti's betting scheme is well-known, a few comments are in order here.

1. Individuals perceive risk in their own subjective way and there are many circumstances in which they exhibit *risk aversion*. de Finetti was well aware of this (back in 1931!) when he introduced his betting scheme. To counter the potential distortions arising from the aversion (or its contrary) to risk, the betting scheme is devised for a *series of bets* and it contains the explicit clause to the effect that the bookmaker, once she has announced the price p , is forced to accept indefinitely many bets, either on or against the events. This, in de Finetti's words marks the 'essential distinction' between a one-off bet and the situation in which an individual is willing to bet 'systematically and unlimitedly'.
2. Note that in order to represent such a disposition, the definition of a fair betting quotient can really only arise by taking *the bookmaker's point of view*. However used they may be to betting, gamblers cannot reasonably be thought of being willing (much less forced!) to accept any bets at a given price. It is, on the other hand, the bookmaker's job to do exactly that. This asymmetry between gamblers and bookmakers is therefore fundamental in order to justify the betting scheme as an appropriate framework to define *subjective probability as price*. We shall generalize the betting game to its *imprecise* counterpart by keeping the bookmaker's side, unlike Walley who takes the gambler's point of view. As we shall see, however, the two approaches are trivially dual to one another. Shafer and Vovk also take the gambler's point of view in their game-theoretic characterization of probability [36] but the focus of their attention is on the characterization of *objective probability*. Indeed its central use of Cournot's principle, which loosely stated implies that a small-probability event can be discarded as "practically impossible", puts their approach to probability into a completely different framework than the one put forward in the present paper.
3. When referring to the betting game in de Finetti's framework, we should be very cautious in referring, as it is sometimes done in the literature, to the classical betting scheme as being *reversible*. What should be stressed is that the classical betting game is sequential. As the gambler has the possibility of choosing the direction of the bet *after* reading the odds, she can unilaterally impose a payoff-matrix swap to the bookmaker. In the presence of second order uncertainty (see Theorem 3.5), gamblers will cease to have this privilege.
4. Owing to its cogency, we follow de Finetti in assuming that the stakes involved in the betting game correspond to actual money (in some currency). So, in order to avoid the potential complications arising from the diminishing marginal utility of money, we need to make further restriction to the effect that stakes should be *small*, what de Finetti refers to as the *rigidity hypothesis* (see [8, pp. 77–78]). This of course applies to the interpretation of probability and not to the calculus. As a consequence, the current framework is not committed to representing any specific attitude towards risk.
5. Since this paper aims at extending de Finetti's framework, we naturally focussed on his betting scheme and the resulting criterion of coherence. This is certainly not the only framework supporting a subjectivist foundation of probability. For one thing, de Finetti himself supported (see [8, Chapter 3]) an alternative (but mathematically equivalent) criterion based on *proper scoring rules*, which however is not directly relevant to our present discussion. As already mentioned in passing, Ramsey put forward, independently of de Finetti, the decision-theoretic foundation for subjective probability. Building on both those pioneering contributions, Savage axiomatised [35] the core of what is now referred to as *bayesian decision theory*.

We finally mention two alternative justifications for subjective probability, which only partially overlap with the issues discussed in the present paper. The former is originally due to Cox (see [31] for a detailed account and an improved proof). By imposing suitable constraints on the notion of conditional degrees of belief, the Cox–Paris theorem

establishes that the resulting measure must be isomorphic to some (scaled) probability function. The latter is due to Fine [12], whose framework grounds the justification of subjective probability on the comparison of qualitative probabilities. His framework provides an alternative foundation to upper and lower probabilities, which differs from the present one by focussing on the issue of *measurement* rather than the *interpretation* of probability. See e.g. [40,23] for Fine’s approach.

Remark 1.1 (*Coherence and completeness*). In classical logic, as well as in many nonclassical logics, a *Model Existence Lemma* can be proved to the effect that a (maximally) consistent set of formulas has a model. This Lemma usually provides a key step to proving completeness results. It is natural to ask whether the notion of coherence⁴ just introduced yields analogous consequences. Theorem 1.1, by establishing that if an unconditional assignment is coherent, then there exists an unconditional probability distribution which extends it, appears to be in strict analogy with the Model Existence Lemma. An immediate way to spell this analogy out would be to take the probability distribution as the *model* in the Model Existence Lemma and coherence as its *syntactic* counterpart. Howson has gone some way towards this interpretation in a couple of recent papers [19,20].

Our logico-algebraic framework presents a different picture. As it will become apparent with the completeness result in Section 3, in our context probability, being represented by a modality, is distinctively syntactic while coherence, defined in terms of (good and bad) bets play the role of the semantic element.

1.2. Probability over many-valued events: the logical tradition

The operational definition of probability via fair betting odds rests on two distinctly logical notions. First of all, the payoff matrices determined by the betting game depend essentially on the *truth value* of the (sentence representing the) event for which the odds are given. This means that a logical semantics is at work in the betting scheme. For de Finetti, as we recalled above, the logic of events is unquestionably *two-valued*: it is a defining feature of events that they can only be either true or false, and that the conditions under which this happens are well-specified in advance. Secondly, the Dutch Book argument is based essentially on the notion of coherence (or consistency). This, as noted in Remark 1.1, leads to an interesting and, to some extent, surprising connection with logic and the notion of logical completeness. Although the importance of logical consistency in probabilistic reasoning has not entirely escaped the logicians’ attention (see, e.g. [19,20,31,16]), we believe that much still needs to be understood about this connection.

From the logical point of view, probabilistic reasoning presupposes *classical* logic. Hence it is completely natural to ask whether probability continues to be justified as a representation of rational degrees of belief when the underlying logic is *non-classical*. Of course there are many “non-classical” logics, so the answer must begin by fixing a specific variation of classical logic.

An important step forward in this direction came with the proof that Theorem 1.1 can be, rather naturally indeed, generalized to a number of two-valued *possible-world* semantics [32]. This result⁵ triggered a number of papers ([29,22,1] among others⁶), which established that the Dutch Book Theorem, and hence the consistency criterion defined by de Finetti for rational subjective beliefs, could be generalized to various *many-valued* logics. The reason this turned out to be very useful lies in the fact that many-valued logics are natural candidates to provide an appropriate semantics to represent probabilistic reasoning about events which fall short of being binary, a requirement about which de Finetti was not ready to make concessions of any sort. If, as we recalled, the statistical tradition objects to the bayesian view that probability need not come under the form of a single real number, the (many-valued) logical tradition suggests that we, as uncertain reasoners, confront more often than not events whose verification comes in degrees.⁷

Example 1.1 (*Probability over many valued events*). Consider an experiment of tossing a die. Then, it makes sense (e.g., in a bet where the gambler’s payoff is proportional to the outcome) to consider the event *the outcome will be high* (for a die, of course). While in classical logic we can only distinguish between high and not high, many-valued logic allows us to distinguish between 5 and 6, giving different truth values to the sentences *6 is high* and *5 is high*.

The natural starting point for introducing our framework is Mundici’s extension of de Finetti’s criterion to many-valued events. In [29] probability over many-valued events is given an operational interpretation which is entirely analogous to that given by de Finetti for binary (or *crisp*) events, recalled in Section 1.1. The crucial difference is that, in the many-valued extension, the truth value $v(\phi)$ of an event need not necessarily be either 0 or 1, but it may be an intermediate value. As

⁴ A small terminological remark is in order. de Finetti uses the Italian *coerenza* which has been translated sometimes as *coherence*, sometimes as *consistency*. We do not attempt at a philological interpretation here, and take consistency to be fully synonymous with coherence, at least for the purposes of the present discussion. It appears that de Finetti was happy with both translations.

⁵ Interestingly enough Paris’s generalization of the Dutch Book Theorem [32] was presented at the 2001 Symposium on Imprecise Probabilities.

⁶ See also [14] for a groundbreaking study of the many-valued case.

⁷ It should be emphasised, however, that the problem of defining probability over non-classical events raises a number of difficult foundational questions, as illustrated e.g. in [25].

a consequence, events are now represented as elements of an MV-algebra, or equivalently, they are equivalence classes of formulas of Łukasiewicz logic \mathbb{L} modulo provable equivalence in a theory over \mathbb{L} .⁸

Thus the operational interpretation of probability over MV-events is essentially the same as in the classical case, and so is the condition of coherence which still reads as avoiding sure loss for the bookmaker (see below for a precise formulation). To close the circle then, we need an analogue of a probability distribution for many-valued events. It turns out that the appropriate notion is the concept of a *state* on an MV-algebra (see the next section for all the relevant definitions).

Definition 1.2 (State [29]). A *state* on an MV-algebra \mathbf{A} is a function s from \mathbf{A} into $[0, 1]$ such that for all $x, y \in \mathbf{A}$, $s(1) = 1$ and whenever $x \odot y = 0$, then $s(x \oplus y) = s(x) + s(y)$.

As a consequence of an important theorem proved by Panti and Kroupa, states may be represented as integrals. More precisely, let \mathbf{A} be a semisimple MV-algebra, let $X_{\mathbf{A}}$ be the set of all homomorphisms from \mathbf{A} into the standard MV-algebra $[0, 1]_{MV}$ on $[0, 1]$, equipped with the hull-kernel topology (to be defined below). Then, $X_{\mathbf{A}}$ is a compact Hausdorff space. Moreover any element a of \mathbf{A} can be identified with the function a^* from $X_{\mathbf{A}}$ into $[0, 1]$ defined for all $v \in X_{\mathbf{A}}$, by $a^*(v) = v(a)$. Notice that every a^* is continuous. Now Panti [30] and Kroupa [21] independently proved the following:

Theorem 1.2 (Panti–Kroupa). To every state s on \mathbf{A} we can associate a unique Borel regular probability measure μ on $X_{\mathbf{A}}$ such that for all $a \in \mathbf{A}$, $s(a) = \int_{X_{\mathbf{A}}} a^* d\mu$.

This result justifies the claim that states constitute a very natural generalization of probability measures when classical (boolean) events are extended to their many-valued counterpart.

The formal definition of coherence for assessments over MV-events reads as follows:

Definition 1.3. Let \mathbf{A} be an MV-algebra, let $S = \{\phi_1, \dots, \phi_n\} \subseteq \mathbf{A}$, and let $\alpha_1, \dots, \alpha_n \in [0, 1]$. Then the assessment $\Lambda : \phi_i \mapsto \alpha_i : i = 1, \dots, n$ is said to be *coherent* if there are no $\lambda_1, \dots, \lambda_n \in \mathbf{R}$ such that, for every valuation v from \mathbf{A} into $[0, 1]_{MV}$, $\sum_{i=1}^n \lambda_i (\alpha_i - v(\phi_i)) < 0$.

Equipped with the appropriate notion of probability, a generalization of Theorem 1.1 is proved in [29]:

Theorem 1.3 (Mundici). An assessment on an arbitrary set S of many valued events (regarded as elements of an MV-algebra \mathbf{A}) is coherent if and only if there is a state on \mathbf{A} which extends it.

1.3. Relaxing the bayesian dogma: the statistical tradition

de Finetti stressed at various points that no higher orders of uncertainty can be meaningfully described. For instance he insisted that

Among the answers that do not make sense, and cannot be admitted are the following: “I do not know”, “I am ignorant of what the probability is”, “in my opinion the probability does not exist”. Probability (or prevision) is not something which in itself can be known or not known: it exists in that it serves to express, in a precise fashion, for each individual, his choice in his given state of ignorance. To imagine a greater degree of ignorance which would justify the refusal to answer would be rather like thinking that in a statistical survey it makes sense to indicate, in addition to those whose sex is unknown, those for whom one does not even know “whether the sex is unknown or not”. ([8, p. 82])

Many, including the present authors, disagree with de Finetti on this. While Theorem 1.1 provides a solid justification for taking *beliefs* as subjective probability, the same argument does not immediately support the view that subjective probability can equally well serve as a justified measure of subjective *ignorance*. Indeed, it is precisely because probability is a subjective quantification of one’s own uncertainty that second order uncertainty makes sense. If an agent feels, for instance, that she is evaluating the probability of an event using information which is only loosely related to the event in question, it is certainly wise of her to hedge, literally as we shall see, her bets.

One natural way of modelling this kind of ignorance, which conceptually fits with the subjective probabilistic setting,⁹ consists in relaxing what Peter Walley appropriately describes as the *bayesian dogma of precision*, that is the assumption that probability should be single-valued. The idea is therefore that of extending coherent assessment to *intervals* of real numbers which will lead us to define *upper* and *lower probabilities* (and previsions).¹⁰

⁸ Note that the choice of \mathbb{L} is mainly due to the fact that it is the only many-valued logic whose connectives are continuous.

⁹ We do not insist here on those models (e.g. Dempster–Shafer’s) which begin by rejecting the bayesian point of view as being conceptually flawed. We acknowledge that critics of radical (i.e. *à la* de Finetti) bayesianism have a point, but fail to acknowledge that by separating first and second order uncertainty, the representation of the latter can be grounded safely on the bayesian foundations.

¹⁰ The idea of probability intervals and sets of probabilities goes back at least to [38] and was given an early formalization by Williams [41]. See [24] and [39] for further historic references.

Example 1.2. Suppose an agent is wondering whether to accept bets on (or against) an event ϕ , when she realizes that her relevant knowledge about ϕ is very scarce. Recalling our opening example, ϕ could stand for “In 2015 Japan will have (relative to its GDP) the largest public debt in the world”. Our agent might reasonably feel reluctant to make a definite probability assignment to ϕ and choose to consult three experts instead. Suppose that their assessments are 60%, 50% and 40% respectively. According to these assessments, it would seem entirely reasonable of her to bet on ϕ if the betting odd is at least 60% and to bet against ϕ if the betting odd is at most 40%. For intermediate betting odds, it may be preferable not to bet at all.

The idea of a probability interval bounded by lower and upper probabilities is arrived at by Walley via *desirable gambles*, which we now briefly recall with the goal of emphasizing the decision-theoretic flavour of the approach.

Definition 1.4. Given a non-empty set of possible outcomes Ω , a *gamble* is a bounded real-valued function on Ω . Let $\mathcal{L}(\Omega)$ be the set of all gambles on Ω . Then for all $\lambda \in \mathbb{R}$ and $X \in \mathcal{L}(\Omega)$, λX is the gamble defined by $(\lambda X)(\omega) = \lambda X(\omega)$. For all $X, Y \in \mathcal{L}(\Omega)$, $X + Y$ is the gamble defined by $(X + Y)(\omega) = X(\omega) + Y(\omega)$.

$X(\omega)$ is intuitively interpreted as the amount of (monetary) utility that an agent will receive depending on the uncertain outcome ω as a consequence of buying the gamble X . Thus gambles, just as de Finetti’s previsions, inherit their linearity directly by the linearity of prices. Walley suggests (see [39, Section 2.2.3]) that a gamble X is *desirable* for the gambler if she is inclined to accept X whenever it is offered (for free) to her. On the other hand, X is *not desirable*, if she does *not accept* it, meaning either rejecting or being undecided about it. Walley then introduces the following conditions as rationality constraints on the set of gambles (again, from the point of view of the gambler):

- (D0) If $\sup X = \sup\{X(\omega) | \omega \in \Omega\} < 0$ then X is not desirable.
- (D1) If $\inf X = \inf\{X(\omega) | \omega \in \Omega\} > 0$ then X is desirable.
- (D2) If X is desirable and λ is a positive real number then λX is desirable.
- (D3) If X and Y are each desirable then $X + Y$ is desirable.

The decision-theoretic criterion of desirability for gambles serves a twofold purpose. On the one hand it allows us to interpret lower and upper previsions in terms of the gambler’s disposition to bet. On the other hand, it justifies coherence as a rationality requirement for a prevision.

We end this section by recalling the key definitions and results of Walley’s theory.

Definition 1.5. A *lower prevision* \underline{P} is a real-valued function defined on an arbitrary subset $S \subseteq \mathcal{L}(\Omega)$ of all gambles.

Although, by definition, lower previsions are arbitrary real valued functions on a finite set of gambles, we will be following Walley in focussing on lower previsions representing, for every gamble X , the supremum price μ such that $X - \mu$ is desirable to the gambler.

Note that the obvious duality allows us to define lower and upper previsions as follows.

Definition 1.6. Suppose that \underline{P} is a lower prevision defined on a linear subspace $S \subseteq \mathcal{L}(\Omega)$. Then its *conjugate upper prevision* \bar{P} is defined on the same domain by $\bar{P}(X) = -\underline{P}(-X)$.

The interpretation of $\underline{P}(X)$ as the supremum price μ for which $X - \mu$ is desirable immediately gives us the following:

$$\begin{aligned} \bar{P}(X) &= -\sup\{\mu : -X - \mu \in \mathcal{D}\} \\ &= \inf\{-\mu : -X - \mu \in \mathcal{D}\} \\ &= \inf\{\alpha : \alpha - X \in \mathcal{D}\}, \end{aligned}$$

where \mathcal{D} is the set of desirable gambles. It follows that $\bar{P}(X)$ is the infimum price α for which $\alpha - X$ is desirable, i.e. such that the gambler is willing to sell X in return for α . Just as the lower prevision $\underline{P}(X)$ is a supremum buying price for X , the upper prevision $\bar{P}(X)$ is an infimum selling price for X . Observe that if $\underline{P}(X) = \bar{P}(X)$ then we are back to de Finetti’s notion of fair price.

Definition 1.7 (Coherence condition for lower previsions (Definition 2.5.1 of [39])). A lower prevision \underline{P} defined on an arbitrary subset $S \subseteq \mathcal{L}(\Omega)$ is *coherent* if

$$\sup \left\{ \sum_{j=1}^n (X_j - \underline{P}(X_j)) - m(X_0 - \underline{P}(X_0)) \right\} \geq 0,$$

for every non-negative integers m, n and for every $X_0, X_1, \dots, X_n \in S$.

Walley uses the desirability criterion for gambles to justify coherence as a rationality requirement. In fact if $m = 0$, then coherence reduces to avoiding sure loss. If $m > 0$ and $n = 0$, it simply means that all the gambles $X_0 - \mu$ with $\mu < \inf X_0$ are desirable to the gambler. Finally, if $m, n > 0$, coherence is violated just when there are gambles X_0, X_1, \dots, X_n such that, if $X_i - \underline{P}(X_i)$ is desirable for $i = 1, \dots, n$, then, on the grounds of the axioms of desirability, the gambler should be willing to buy X_0 for a price greater than $\underline{P}(X_0)$.

Since the concept of a *desirable gamble* is the fundamental idea underpinning Walley’s characterization of coherent lower (and upper) previsions, we naturally refer to as Walley’s generalization of subjective probability as *decision-theoretic*.

If lower previsions are defined on a linear subspace of $\mathcal{L}(\Omega)$, then coherence can be characterized as follows:

Theorem 1.4 (Theorem 2.5.5 of [39]). *A lower prevision \underline{P} defined on an linear subspace of $\mathcal{L}(\Omega)$ is coherent if and only if it satisfies the following axioms:*

- (P1) $\underline{P}(X) \geq \inf X$ when $X \in \mathcal{L}(\Omega)$.
- (P2) $\underline{P}(\lambda X) = \lambda \underline{P}(X)$ when $X \in \mathcal{L}(\Omega)$ and $\lambda > 0$.
- (P3) $\underline{P}(X + Y) \geq \underline{P}(X) + \underline{P}(Y)$ when $X, Y \in \mathcal{L}(\Omega)$.

Walley’s central result about lower (upper) previsions ensures that coherent lower previsions defined on an arbitrary set $S \subseteq \mathcal{L}(\Omega)$ always have a coherent extension to the class of all gambles.

Definition 1.8 (Definition 3.1.1 of [39]). Suppose \underline{P} is a lower prevision defined on S . Then \underline{E} , the *natural extension* of \underline{P} , is defined on the set of all gambles by:

$$\underline{E}(X) = \sup \left\{ \alpha : X - \alpha \geq \sum_{j=1}^n \lambda_j (X_j - \underline{P}(X_j)) \quad n \geq 0, X_j \in S, \lambda_j \geq 0, \alpha \in \mathbb{R} \right\}$$

The natural extension of \underline{P} is trivial unless \underline{P} avoids sure loss. If \underline{P} incurs sure loss then $\sum_{j=1}^n \lambda_j (X_j - \underline{P}(X_j))$ can be made arbitrarily large, hence α can be made arbitrarily large and $\underline{E} = \infty$. Thus \underline{P} avoids sure loss iff its natural extension \underline{E} is finite.

Theorem 1.5 (Theorem 3.1.2 of [39]). *Suppose \underline{P} is a lower prevision defined on S that avoids sure loss. Then its natural extension \underline{E} on $\mathcal{L}(\Omega)$ has the following properties:*

- (1) \underline{E} is a coherent lower prevision on $\mathcal{L}(\Omega)$.
- (2) \underline{E} agrees with \underline{P} on S iff \underline{P} is coherent.
- (3) \underline{E} is the minimal coherent lower prevision on \mathcal{L} that dominates \underline{P} on S .
- (4) If \underline{P} is coherent then \underline{E} is the minimal coherent extension of \underline{P} to \mathcal{L} .

2. Algebraic tools

In order to keep this work self-contained we provide here a list of notions which will be used in the remainder of the paper. For further details on Universal Algebra, the reader is referred to [3].

A class of algebras of the same type is said to be a *variety* if it has an equational axiomatization, and a *quasivariety* if it is axiomatized by a set of quasiequations, i.e., of formulas of the form:

$$\text{if } \eta_1 \text{ and } \dots \text{ and } \eta_n, \text{ then } \eta,$$

where η_1, \dots, η_n and η are equations.

A *congruence* of a universal algebra \mathbf{A} is an equivalence θ on \mathbf{A} which is compatible with the operations of \mathbf{A} , that is, whenever f is an operation of \mathbf{A} and $a_1, b_1, \dots, a_n, b_n$ are elements of \mathbf{A} , if $(a_i, b_i) \in \theta$ for $i = 1, \dots, n$, then $(f(a_1, \dots, a_n), f(b_1, \dots, b_n)) \in \theta$. The congruence $\{(a, a) : a \in \mathbf{A}\}$ is called the *trivial congruence*.

Given an algebra \mathbf{A} and a direct product $\prod_{i \in I} \mathbf{A}_i$ of algebras of the same type, an injective homomorphism h from \mathbf{A} into $\prod_{i \in I} \mathbf{A}_i$ is said to be a *subdirect embedding* if for all $i \in I$ the composition $\pi_i \circ h$ is an epimorphism, where π_i denotes the i th projection. An algebra \mathbf{A} is said to be *subdirectly irreducible* iff for every subdirect embedding h of \mathbf{A} into $\prod_{i \in I} \mathbf{A}_i$, there is an $i \in I$ such that $h \circ \pi_i$ is an isomorphism between \mathbf{A} and \mathbf{A}_i . It is a well-known fact of Universal Algebra that an algebra \mathbf{A} is subdirectly irreducible if it has a minimum non-trivial congruence. Birkhoff’s subdirect representation theorem [3, Theorem 8.6] says that every algebra has a subdirect embedding into a direct product of subdirectly irreducible algebras.

We will work in the abstract algebraic logic framework of [2]. Thus n -ary connectives are identified with n -ary operations and formulas are identified with terms. Unless specified otherwise, a *valuation* of formulas of a propositional logic L into an algebra \mathbf{A} of the same type is defined to be a homomorphism from the algebra of formulas of L into \mathbf{A} .

The logics used in this paper are *algebraizable* in the sense of [2]. Indeed, any such logic L has a binary connective \leftrightarrow and a constant 1 such that the relation \equiv defined by $\phi \equiv \psi$ iff $\vdash_L \phi \leftrightarrow \psi$ is a congruence of the algebra of formulas, and the

theorems of L are precisely those formulas ϕ such that $\phi \equiv 1$. Thus we can construct the *Lindenbaum algebra* \mathbf{L} of L , that is, the quotient of the algebra of formulas modulo \equiv .

In this way, the variety \mathcal{L} generated by \mathbf{L} is the equivalent algebraic semantics of L . We will use the same notation for connectives of L and for the operations of \mathcal{L} , and hence formulas of L will be identified with terms of \mathcal{L} . For every formula ϕ and for every equation ε of the form $\psi = \zeta$, ϕ^e denotes the equation $\phi = 1$ and ε^f denotes the formula $\psi \leftrightarrow \zeta$. Moreover, given a set Γ of formulas and a set Δ of equations, Γ^e denotes the set $\{\phi^e : \phi \in \Gamma\}$ and Δ^f denotes the set $\{\varepsilon^f : \varepsilon \in \Delta\}$. Then we have: (1) $\vdash_L (\phi^e)^f \leftrightarrow \phi$; (2) in \mathcal{L} , the equation $(\varepsilon^f)^e$ is equivalent to ε , and (3) $\Gamma \vdash_L \phi$ iff $\Gamma^e \models_{\mathcal{L}} \phi^e$ and $\Delta \models_{\mathcal{L}} \varepsilon$ iff $\Delta^f \vdash_L \varepsilon^f$.

Note that the above properties imply the strong completeness of \mathbf{L} with respect to \mathcal{L} : given any set Γ of sentences and a sentence ϕ , we have that $\Gamma \vdash_L \phi$ iff for every $\mathbf{A} \in \mathcal{L}$ and for every valuation v in \mathbf{A} , if $v(\psi) = 1$ for all $\psi \in \Gamma$, then $v(\phi) = 1$.

For basic notions about Łukasiewicz logic, the reader is referred to [17]. We now review some basic facts about the class of MV-algebras that is the equivalent algebraic semantics of \mathbf{L} in the sense of [2].

An *MV-algebra* is an algebra $\mathbf{A} = (A, \oplus, \neg, 0, 1)$ such that:

- (MV1) $(A, \oplus, 0)$ is a commutative monoid.
- (MV2) $1 \oplus x = 1$.
- (MV3) $\neg\neg x = x$.
- (MV4) $x \oplus \neg(\neg y \oplus x) = y \oplus \neg(\neg x \oplus y)$.

Thus MV-algebras have the binary operation \oplus , the unary operation \neg and the constants 0 and 1 as primitives. The operations $\rightarrow, \odot, \ominus, \leftrightarrow, \wedge$ and \vee are defined as follows: $x \rightarrow y = \neg x \oplus y$, $x \odot y = \neg(\neg x \oplus \neg y)$, $x \ominus y = \neg(x \rightarrow y)$, $x \leftrightarrow y = (x \rightarrow y) \odot (y \rightarrow x)$, $x \wedge y = x \odot (x \rightarrow y)$ and $x \vee y = x \oplus (y \ominus x)$.

Let $[0, 1]_{MV}$ denote the algebra $([0, 1], \oplus, \neg, 0, 1)$, where $x \oplus y = \min\{x + y, 1\}$ and $\neg x = 1 - x$. Then, due to Chang's completeness theorem [4] the class of MV-algebras coincides with the variety generated by $[0, 1]_{MV}$, and the set of theorems of Łukasiewicz logic coincides with the set of all formulas ϕ such that $v(\phi) = 1$ for every valuation v into $[0, 1]_{MV}$. Thus, in order to verify that an equation is valid in all MV-algebras, it suffices to verify it in $[0, 1]_{MV}$.

In any MV-algebra, we define, for every natural number n , the operation $(n)x$ by induction on n as follows: $(0)x = 0$ and $(n + 1)x = (n)x \oplus x$. In a similar fashion we define, for every formula ϕ , the formula $(n)\phi$.

A *filter* of an MV-algebra \mathbf{A} is a subset F of A such that $1 \in F$ and for all $a, b, c, d \in A$, if $a, b \in F$ then $a \odot b \in F$ and if $c \in F$ and $c \leq d$, then $d \in F$. A filter is said to be *proper* if $0 \notin F$, and *maximal* iff it is maximal among all proper filters. The set of all filters of \mathbf{A} is denoted by $F(\mathbf{A})$, and the set of all maximal filters of \mathbf{A} is denoted by $M(\mathbf{A})$. For every $F \in F(\mathbf{A})$, C_F denotes the set $\{M \in M(\mathbf{A}) : F \subseteq M\}$. The *hull kernel topology* is the topology on $M(\mathbf{A})$ whose closed sets are precisely the sets of the form C_F for some $F \in F(\mathbf{A})$. Note that $M(\mathbf{A})$ with the hull kernel topology is a compact Hausdorff space.

In MV-algebras, filters correspond to congruences: given a congruence θ , the set $F_\theta = \{x : (x, 1) \in \theta\}$ is a filter, and given a filter F , the set $\{(x, y) : x \leftrightarrow y \in F\}$ is a congruence. Moreover the maps $\theta \mapsto F_\theta$ and $F \mapsto \theta_F$ are mutually inverse isomorphisms between the congruence lattice and the filter lattice of an MV-algebra. The quotient of \mathbf{A} modulo the congruence θ_F determined by a filter F is indifferently denoted by \mathbf{A}/θ_F or by \mathbf{A}/F , and the equivalence class of an element a modulo θ_F is indifferently denoted by a/θ_F or by a/F .

A filter F is maximal iff \mathbf{A}/F is a *simple* algebra, that is, its only congruences are $\{(a, a) : a \in A\}$ and $A \times A$. We recall [5] that an MV-algebra \mathbf{A} is simple iff it can be embedded into $[0, 1]_{MV}$; in this case, the embedding is unique. Thus for any MV-algebra \mathbf{A} , $M(\mathbf{A})$ is bijective to the set X_A of homomorphisms v from \mathbf{A} into $[0, 1]_{MV}$: to each $M \in M(\mathbf{A})$ we associate the homomorphism $a \mapsto a/M$ thought of as an element of $[0, 1]_{MV}$. Such a bijection becomes a homeomorphism if we equip X_A with the topology induced by the product topology on $[0, 1]^A$. Hence, the spaces $M(\mathbf{A})$ and X_A will be often identified in the sequel.

An MV-algebra \mathbf{A} is *semisimple* if and only if the co-radical of \mathbf{A} , that is $\bigcap M(\mathbf{A})$, is equal to $\{1\}$. The elements of a semisimple MV-algebra \mathbf{A} may be represented as continuous functions from $M(\mathbf{A})$ into $[0, 1]$.

An MV-algebra \mathbf{A} is said to be *divisible* if for all $x \in A$ there is a $y \in A$ such that $(n)y = x$ and $(n - 1)y \odot y = 0$. Such a y , which is uniquely determined by x and n , is denoted by $\frac{x}{n}$. In a divisible MV-algebra, if $0 \leq n \leq m$ and $m > 0$, then $\frac{n}{m}x$ denotes $(n)\frac{x}{m}$.

A *lattice ordered abelian group* (ℓ -group for short) is an algebra $\mathbf{G} = (G, +, -, 0, \vee, \wedge)$ where $(G, +, -, 0)$ is an abelian group, (G, \vee, \wedge) is a lattice and the equations $x + (y \vee z) = (x + y) \vee (x + z)$ and $x + (y \wedge z) = (x + y) \wedge (x + z)$ hold.

In an ℓ -group we define nx by induction on n as follows: $0x = 0$ and $(n + 1)x = nx + x$. We also define $(-n)x = -(nx)$.

A *strong unit* of an ℓ -group \mathbf{G} is an element 1_G of G such that for all $x \in G$ there is a natural number n such that $x \leq n1_G$. An abelian ℓ -group with a strong unit is called *unital*. In the sequel we will sometimes omit the subscript G when there is no danger of confusion. An ℓ -group is said to be *divisible* if for every $x \in G$ and for every positive natural number n there is a y , denoted by $\frac{x}{n}$, such that $ny = x$, and *2-divisible* if the above property holds when $n = 2$. In a divisible unital ℓ -group $(\mathbf{G}, 1)$, we also define $\pm \frac{n}{m}x = n(\pm \frac{x}{m})$ and $\pm \frac{n}{m} = \pm n \frac{1}{m}$.

A *convex subgroup* of a unital ℓ -group $(\mathbf{G}, 1)$ is a lattice ordered subgroup \mathbf{H} of \mathbf{G} such that for every $y \in \mathbf{G}$ and every $x \in \mathbf{H}$, if $|y| \leq |x|$ then $y \in \mathbf{H}$ (recall that for every $x \in \mathbf{G}$, $|x| = x \vee -x$). A convex subgroup is said to be *proper* if it is not equal to \mathbf{G} , and *maximal* iff it is maximal among all proper convex subgroups. The set of all maximal convex subgroups of \mathbf{G} is denoted by $M(\mathbf{G})$. For every $a \in \mathbf{G}$, C_a denotes the set $\{\mathbf{H} \in M(\mathbf{G}) : a \in \mathbf{H}\}$. The *hull kernel topology* is the smallest topology on $M(\mathbf{G})$ for which the sets of the form C_a for some $a \in \mathbf{G}$ are closed. Note that $M(\mathbf{G})$ with the hull kernel topology is a compact Hausdorff space.

We recall that a unital ℓ -group $(\mathbf{G}, 1)$ is *semisimple* if and only if the radical of \mathbf{G} , that is $\bigcap M(\mathbf{G})$, is equal to $\{0\}$.

Given a unital ℓ -group $(\mathbf{G}, 1)$, $\Gamma(\mathbf{G}, 1)$ denotes the algebra with domain $\{x \in G : 0 \leq x \leq 1\}$ with operations $x \oplus y = (x + y) \wedge 1$ and $\neg x = 1 - x$. Then $\Gamma(\mathbf{G}, 1)$ is an MV-algebra. Moreover given a ℓ -group homomorphism h from $(\mathbf{G}, 1_G)$ into $(\mathbf{H}, 1_H)$ such that $h(1_G) = 1_H$, we denote by $\Gamma(h)$ the restriction of h to $\Gamma(\mathbf{G}, 1_G)$. In this way, Γ becomes a functor from the category of unital ℓ -groups into the category of MV-algebras. Moreover Γ has an inverse Γ^{-1} , and the pair (Γ, Γ^{-1}) constitutes an equivalence between the category of unital ℓ -groups and the category of MV-algebras [27]. Note that $(\mathbf{G}, 1)$ is divisible iff $\Gamma(\mathbf{G}, 1)$ is a divisible MV-algebra.

In the sequel, given a unital ℓ -group $(\mathbf{G}, 1)$, X_G will denote the set of all homomorphisms from \mathbf{G} into $(\mathbf{R}, 1)$, equipped with the topology induced by the product topology on \mathbf{R}^G . Note that X_G is homeomorphic to $M(\mathbf{G})$, and hence it is a compact Hausdorff space.

2.1. Putting things together: a logico-algebraic framework

Our goal is now to connect the logical and algebraic approach, based on the notion of state on an MV-algebra, with the statistical approach, based on the notion of coherent upper and lower previsions. As a final product of our attempt, we will give a logical characterization of coherence of assessments of upper and lower probability and of upper and lower previsions.

As proved in [11] gambles can be framed in an algebraic setting; that is, gambles will be represented as elements of a divisible and unital ℓ -group. Although very simple, such structures are very rich: indeed, a divisible unital ℓ -group $(\mathbf{G}, 1)$ is a vector lattice over the rational field, and X_G is a compact Hausdorff space. Moreover, if in addition \mathbf{G} is semisimple, then every element g of it can be identified with its Gelfand transform \hat{g} , that is, with the function from X_G into \mathbf{R} defined for all $v \in X_G$, by $\hat{g}(v) = v(g)$. Moreover, each Gelfand transform \hat{g} is a continuous function from X_G into \mathbf{R} , and hence \mathbf{G} , identified with $\hat{\mathbf{G}} = \{\hat{g} : g \in G\}$, becomes a subspace of $C(X_G, \mathbf{R})$, the space of all continuous functions from X_G into \mathbf{R} . Finally, $\hat{\mathbf{G}}$ is dense in $C(X_G, \mathbf{R})$ with respect to the topology of uniform convergence.

As already noted by Walley (see [39, Appendix F]) (linear) previsions¹¹ may then be defined as linear, positive and normalized functionals over the set of all gambles. Since, in our framework, the set of all gambles is a divisible and unital ℓ -group $(\mathbf{G}, 1_G)$, a prevision on \mathbf{G} is an operator E from \mathbf{G} into \mathbf{R} such that for all $g, h \in G$ and for all $\lambda, \mu \in \mathbf{R}$, satisfies the following conditions:

- (i) $E(\lambda g + \mu h) = \lambda E(g) + \mu E(h)$ (linearity).
- (ii) $E(1_G) = 1$ (normality).
- (iii) $g \geq 0$ implies $E(g) \geq 0$ (positivity).

Since infinitesimal elements (i.e., elements a such that for every natural number n , $na < 1_G$) are mapped into 0 by any prevision, we can assume without loss of generality that \mathbf{G} has no infinitesimals, and hence that it is semisimple. It follows that every prevision on \mathbf{G} has a unique extension to a prevision over $C(X_G, \mathbf{R})$, and hence previsions on a divisible and unital ℓ -group \mathbf{G} constitute an algebraic framework which also offers the tools from functional analysis necessary to deal with the basic principles of probability.

Many-valued events may be considered as special cases of random variables, and the corresponding probabilities may be regarded as their expected values. But unlike boolean algebras and probability measures, MV-algebras and states on them allow us to reconstruct the whole algebra of gambles and the corresponding prevision. In other words, not only the restriction to $\Gamma(\mathbf{G}, 1_G)$ of a prevision on $(\mathbf{G}, 1)$ is a state on $\Gamma(\mathbf{G}, 1_G)$, but conversely any state on $\Gamma(\mathbf{G}, 1_G)$ has a unique extension to a prevision on $(\mathbf{G}, 1)$. This means that every property of states on MV-algebras has a natural translation to a property of previsions over the corresponding ℓ -group of gambles.

In particular de Finetti's betting scheme and Mundici's characterization of coherence for assessments of MV-events have a natural extension to a characterization of coherence for assessments on gambles. More precisely:

Definition 2.1. An assessment over a finite set $S = \{\phi_1, \dots, \phi_n\}$ of gambles (represented as elements of a divisible unital ℓ -group $(\mathbf{G}, 1_G)$) is a function $\Pi : \phi_i \mapsto \alpha_i, i = 1, \dots, n$ from S into \mathbf{R} . The assessment Π is said to be *coherent* iff there is no system of bets leading B to a sure loss whatever the values of the events ϕ_i are. More formally, if for any choice of real numbers $\lambda_1, \dots, \lambda_n$, there is a homomorphism $v \in X_G$ such that $\sum_{i=1}^n \lambda_i(\alpha_i - v(\phi_i)) \geq 0$.

¹¹ Recall that Walley, partly following de Finetti, uses "gamble" and "prevision" in place of "random variable" and "expected value", respectively, a terminology to which we shall conform for the rest of this paper.

Then, Mundici’s theorem naturally translates to:

Theorem 2.1. *Let $(\mathbf{G}, 1_G)$ be a divisible and unital ℓ -group, and let $\Pi : \phi \mapsto \alpha_\phi$ be an assessment from a finite set $S \subseteq G$ into \mathbf{R} . Then, Π is coherent iff it can be extended to a prevision on $(\mathbf{G}, 1_G)$.*

The next step consists of an algebraic treatment of imprecise probabilities and previsions. The first part of this plan has been done in [11] and will be summarized below.

As a first approach an imprecise prevision on a divisible and unital ℓ -group $(\mathbf{G}, 1)$ is represented by a set of previsions. Every set Σ of previsions gives rise to two functionals $\Sigma^* : G \mapsto \mathbb{R}$ and $\Sigma_* : G \mapsto \mathbb{R}$ defined by $\Sigma^*(f) = \sup_{E \in \Sigma} E(f)$ and $\Sigma_*(f) = \inf_{E \in \Sigma} E(f)$ for all $f \in \mathbf{G}$. Previsions over a divisible and unital ℓ -group $(\mathbf{G}, 1)$ can be regarded as elements of $C(X_G, \mathbf{R})^*$ (the dual space of $C(X_G, \mathbf{R})$), which can be equipped with the weak* topology, that is, the smallest topology such that for all $f \in C(X_G, \mathbf{R})$, the function $f^*(F) = F(f)$ ($F \in C(X_G, \mathbf{R})^*$), is continuous (see [34]).

If Σ is a convex and weak*-closed set of previsions, the supremum and the infimum are attained, that is, for all $f \in \mathbf{G}$ $\Sigma^*(f) = \max_{E \in \Sigma} E(f)$ and $\Sigma_*(f) = \min_{E \in \Sigma} E(f)$.

As a second approach imprecise previsions on a divisible and unital ℓ -group $(\mathbf{G}, 1)$ are axiomatically defined as follows:

Definition 2.2 [11]. An upper prevision on a divisible and unital ℓ -group $(\mathbf{G}, 1)$ is a map U from \mathbf{G} into \mathbb{R} such that for all $f, g \in \mathbf{G}$, the following properties are satisfied:

- (1) $f \leq g$ implies $U(f) \leq U(g)$.
- (2) $U(f + q) = U(f) + q$ for every constant $q \in \mathbb{Q}$.
- (3) $U(\lambda f) = \lambda U(f)$, for all positive λ .
- (4) $U(f + g) \leq U(f) + U(g)$.

Given an upper prevision U , its conjugate lower prevision L on $(\mathbf{G}, 1)$ is defined dually by $L(f) = -U(-f)$ for all $f \in \mathbf{G}$.

Remark 2.1. Note that axioms (1) and (2) can be replaced by $U(f) \leq \sup(f)$, thus imprecise previsions on divisible and unital ℓ -group introduced in this way, satisfy Walley’s axioms for coherent imprecise previsions defined on the space of all gambles. Hence, it is important to notice that our terminology now needs departing slightly from Walley’s, according to which upper and lower previsions are both functions from gambles to the reals. While mathematically sound, this fails to convey the intuition, which we are keen to preserve, that upper and lower previsions are, in general, distinct.

The equivalence between the two formulations of upper and lower probabilities (as operators satisfying axioms (1a), (1b), (2), (3) and (4) and as suprema and infima of weak*-closed and convex sets of previsions) is showed by:

Theorem 2.2 [11]. (1) *Let U be an upper prevision and L its conjugate lower prevision over a divisible unital ℓ -group $(\mathbf{G}, 1)$. Then, for every prevision E on $(\mathbf{G}, 1)$, we have $E(f) \leq U(f)$ for all $f \in G$ iff $L(f) \leq E(f)$ for all $f \in G$.*
 (2) *The set Σ of all previsions E such that $E(f) \leq U(f)$ for all f is a weak*-closed and convex subset of $C(X, \mathbf{R})^*$. Moreover $\Sigma^* = U$ and $\Sigma_* = L$.*

Remark 2.2. For the sake of precision, Theorem 2.2 was proved for 2-divisible unital ℓ groups, but it holds a fortiori when full divisibility is assumed. Moreover, Walley proves a similar result (Theorem 3.3.3. of [39]), but with reference to vector spaces and not to ℓ -groups.

Lower and upper previsions (and probabilities) can alternatively be introduced by suitably extending the betting scheme introduced in Section 1.1. It can be immediately observed that in the special case in which upper and lower previsions coincide, the appropriate betting game is again the classical one. Otherwise, we must give up the fairness condition. Specifically, in the imprecise betting game, the gambler can only choose the magnitude of stakes λ , not its sign. As an immediate consequence, the gambler is not in a position unilaterally decide to swap payoff matrices with the bookmaker. This can certainly be seen as an element of realism of this extended betting scheme. So, as noted in [11], the imprecise betting game is just as its classical counterpart except that the upper prevision of a gamble ϕ is defined as the infimum β such that a rational bookmaker would accept to pay $v(\phi)$ in return for the gambler’s payment of β . As we remarked when presenting the Dutch Book Theorem, Walley takes the point of view of the gambler¹², hence the characterization of lower prevision of a gamble ϕ is given in [39] as the supremum of all betting odds α such that a rational gambler would accept to pay α in exchange for $v(\phi)$. Of course, the two criteria are equivalent modulo exchanging the roles of the players and replacing lower previsions by upper previsions. As already noted in [39], this loss of symmetry between the gambler’s and the bookmaker’s payoff matrices, results in the inadequacy, in general, of the coherence criterion as described above (see Definition 1.1). An appropriate refinement of the criterion can be given by referring to the notion of bad and good bets.

¹² A perspective which has more recently been also followed by [36].

Definition 2.3 (*Bad (good) bet*). Let Δ be an assessment on a finite set S of gambles such that for all $\phi \in S$, $\Delta(\phi) = \alpha_\phi$. A *bad bet* for the gambler G is a bet $\lambda_\phi > 0$ on $\phi \in S$ such that there is a system of bets $\lambda_{\phi_i} \geq 0$ on $\phi_i \in S$, $i = 1, \dots, n$ which gives G a better payoff, that is, such that $\sum_{i=1}^n \lambda_{\phi_i} (\alpha_{\phi_i} - v(\phi_i)) > \lambda_\phi (\alpha_\phi - v(\phi))$ for every valuation v . A *good bet* for G is a bet $\lambda_\phi > 0$ on $\phi \in S$ such that there is a system of bets $\lambda_{\phi_i} \geq 0$ on $\phi_i \in S$, $i = 1, \dots, n$ which gives G a worse payoff, that is, such that $\sum_{i=1}^n \lambda_{\phi_i} (\alpha_{\phi_i} - v(\phi_i)) < \lambda_\phi (\alpha_\phi - v(\phi))$ for every valuation v .

The existence of a bad bet does not imply G 's irrationality (indeed, G need not choose that bad bet). It rather means that the assessment made by the bookmaker B was not rational, because he could have made his assessment more attractive for G by reducing the betting odd α_ϕ without any loss of money when G plays a rational strategy. On the other hand, the nonexistence of a bad bet implies that B , cannot reduce his betting odds without an expected loss of money, even if G plays a rational strategy. This justifies our choice of taking the coherence criterion for upper previsions as the nonexistence of bad bets.

Theorem 2.3 [11]. Let $\Pi : \phi \mapsto \alpha_\phi$ be an assessment on a divisible unital ℓ -group. Then:

- (1) Π can be extended to an upper prevision iff there is no bad bet based on Π .
- (2) Π can be extended to a lower prevision if there is no good bet based on Π .
- (3) Π can be extended to a prevision if there is neither a good bet nor a bad bet based on Π .

The assumption on divisibility may be replaced by 2-divisibility. Moreover, the divisibility assumption can be removed if we replace upper previsions (lower previsions respectively) by maxima (minima respectively) of weak*-closed and convex sets of previsions.

The nonexistence of a good bet for an assessment on a finite set of gambles is equivalent to Walley's criterion if the supremum in Definition 1.7 is attained. Note also that Theorem 2.3 extends Theorem 1.5 to the framework of unital ℓ -groups.

Upper and lower probabilities over a divisible MV-algebra \mathbf{A} can be equivalently presented in at least three different ways: as operators satisfying a suitable set of axioms (which will be listed below), as suprema (infima respectively) of weak*-closed and convex sets of states, and as restrictions of upper (lower respectively) previsions on $\Gamma^{-1}(\mathbf{A})$. We start from the axiomatic definition.

Definition 2.4 (cf. [11]). An *upper probability* on an MV-algebra \mathbf{A} is a map u from \mathbf{A} into $[0, 1]$ such that for all $x, y \in \mathbf{A}$, the following properties are satisfied:

- (1) $u(1) = 1$.
- (2) $u(x \oplus y) \leq u(x) + u(y)$.
- (3) $u(qx) = qu(x)$ for each rational q in $[0, 1]$.
- (4) $x \leq y$ implies $u(x) \leq u(y)$.
- (5) $u(x \oplus q) = u(x) + q$ for each rational q in $[0, 1]$ such that $x \odot q = 0$.

A *lower probability* on \mathbf{A} is an operator l of the form $l(x) = 1 - u(\neg x)$ for some upper probability u .

Upper probabilities u over an MV-algebra \mathbf{A} can also be thought of as restrictions to \mathbf{A} of upper previsions on $\Gamma^{-1}(\mathbf{A})$. Indeed, for every upper prevision U on a divisible and unital ℓ group $(\mathbf{G}, 1_G)$, the restriction of U to $\Gamma(\mathbf{G}, 1_G)$ is an upper probability. Conversely, every upper probability u over $\Gamma(\mathbf{G}, 1_G)$ has exactly one extension to an upper prevision on $(\mathbf{G}, 1_G)$. A similar statement holds for lower previsions and lower probabilities.

Hence, Theorem 2.3 naturally extends to upper and lower probabilities on MV-algebras.

Theorem 2.4 (cf. [11]). Let Π be an assessment on a subset S of a divisible (or even 2-divisible) MV-algebra. Then:

- (1) There is no bad bet based on Π iff Π can be extended to an upper probability.
- (2) There is no good bet on Π iff Π can be extended to a lower probability.
- (3) There is neither a good bet nor a bad bet for Π iff Π can be extended to a state.

Moreover divisibility may be removed if upper and lower probabilities are replaced by maxima and minima of weak*-closed and convex sets of states.

While it is possible to give an axiomatic definition of upper and lower previsions on an ℓ -group without the assumption of divisibility or of 2-divisibility, [namely, restricting axiom (2) of upper probabilities to natural numbers λ and axiom (1b) to integer q] in such a way that Theorem 2.2 still holds, this is not the case for MV-algebras. Indeed:

1. If we drop divisibility, then axioms (3) and (5) in Definition 2.3 cannot be expressed in our language (e.g., they do not make sense in a boolean algebra), and if we drop them we obtain operators u which need not be suprema of any set of states. For instance, let X be a three element set and define u on the powerset of X as follows: $u(\emptyset) = 0$, $u(X) = 1$ and $u(Y) = \frac{1}{2}$ for any non-empty proper subset Y of X . Then u is normalized, monotonic and subadditive, but it cannot be the supremum of any set of states.
2. If we drop divisibility, then we can construct [39] a boolean algebra with two sets of states on it which have the same supremum, but such that the sets of their extensions to the enveloping divisible group have distinct suprema. Hence, an upper probability (regarded as supremum of a set of states) on an MV-algebra need not have a unique extension to its enveloping ℓ -group. To the contrary, uniqueness is ensured when the MV-algebra is divisible.

3. Algebra and logic of upper probabilities

The results of [11] provide a very convenient algebraic setting for the treatment of upper and lower probabilities and previsions, and hence they connect Walley’s statistical approach with the algebraic approach by Mundici and others. However, we do not have yet a logic of imprecise probabilities. At the same time, we have an algebraic description of upper and lower probabilities and of upper and lower previsions, but the structures we need are not yet algebras in the sense of Universal Algebra. The point is that states, previsions, upper and lower probabilities and upper and lower previsions are not operations of the algebra, but rather external operators, mapping the algebra into the reals or into $[0, 1]$.

In order to overcome this difficulty, following a strategy developed in [13] for states, we will replace the external operators of upper and lower probability by internal operations of the algebra, thus getting a variety of universal algebras, whose members are called UMV-algebras, and hence an algebraizable logic corresponding to such quasivariety. In this section we will see that the usual upper and lower probability operators can be conveniently represented in UMV-algebras, and hence we will obtain an algebraizable logic in which the relevant properties of upper and lower probabilities are preserved.

Definition 3.1. A divisible MV-algebra with an internal upper probability, UMV-algebra for short, is a pair (\mathbf{A}, u) where \mathbf{A} a divisible MV-algebra and u is a unary operation satisfying the following conditions:

- (1) $u(1) = 1$.
- (2) $u(t) = t$ for every term t such that every variable in t occurs under the scope of u .¹³
- (3) $u(qx) = qu(x)$ for every rational $q \in [0, 1]$.
- (4) $u(x) \leq u(x \vee y)$.¹⁴
- (5) $u(x \oplus y) \leq u(x) \oplus u(y)$.
- (6) If q is a rational in $[0, 1]$ and $x \odot q = 0$, then $u(x \oplus q) = u(x) \oplus q = u(x) + q$, where $+$ is interpreted in $\Gamma^{-1}(\mathbf{A})$.

The class of UMV-algebras will be denoted by \mathcal{UMV} .

We now introduce a propositional logic $\text{\textcircled{L}U}$ whose equivalent algebraic semantics is \mathcal{UMV} . We define a valuation on a \mathcal{UMV} -formula into (\mathbf{A}, u) to be a homomorphism from the algebra of \mathcal{UMV} -formulas into (\mathbf{A}, u) . Then $\text{\textcircled{L}U}$ is semantically characterized as the set of all formulas ϕ which are evaluated to 1 by any valuation in any UMV-algebra. Consequence relation in $\text{\textcircled{L}U}$ can be characterized semantically in a similar way, that is, ϕ is a consequence of Γ iff for every valuation v in any UMV-algebra, if $v(\psi) = 1$ for every $\psi \in \Gamma$, then $v(\phi) = 1$.

An axiomatization of $\text{\textcircled{L}U}$ is given by:

- (1) Axioms of Łukasiewicz logic.
- (2) Divisibility axioms: $(n) \frac{\phi}{n} \leftrightarrow \phi$ and $\neg((n-1) \frac{\phi}{n} \odot \frac{\phi}{n})$ for every $n > 1$.
- (3) The axiom $u(\psi \oplus \phi) \rightarrow (u(\psi) \oplus u(\phi))$.
- (4) The axiom $u(\psi) \rightarrow u(\phi \vee \psi)$.
- (5) The axiom $u(q\phi) \leftrightarrow qu(\phi)$ for every rational $q \in [0, 1]$.
- (6) The axiom $u(\phi) \leftrightarrow \phi$ for all formulas ϕ whose variables occur only under the scope of u .
- (7) The rules Modus Ponens $\frac{\phi \quad \phi \rightarrow \psi}{\psi}$, $\frac{\phi \rightarrow \psi}{u(\phi) \rightarrow u(\psi)}$ and $\frac{q \rightarrow \neg \psi}{u(\phi \oplus q) \leftrightarrow (u(\phi) \oplus q)}$, q any rational in $[0, 1]$.

However, we find it more convenient to work in an equational logic, called UMV, which is defined below.

Definition 3.2. The axioms of UMV are axioms (1), . . . , (6) of UMV-algebras, plus $x = x$. Its rules are usual Birkhoff’s rules for equational logic, that is: $\frac{x=y, \quad x=z}{y=z}$ and $\frac{x_1=y_1 \quad \dots \quad x_n=y_n}{f(x_1, \dots, x_n)=f(y_1, \dots, y_n)}$ for every n -ary function symbol of UMV.

We will also consider the fragment UMV^- of UMV whose language is restricted to the terms which are combinations, in the language of divisible MV-algebras, of terms of the form $u(t)$, t a term of divisible MV-algebras.

¹³ It is easily seen that axiom (2) can be replaced by the schemata $u(u(x) \oplus u(y)) = u(x) \oplus u(y)$ and $u(\neg u(x)) = \neg u(x)$.

¹⁴ Here \leq is the order of the MV-algebra.

We want to relate divisible MV-algebras with an upper probability and UMV-algebras. We first show how to obtain a UMV-algebra from a divisible MV-algebra with an upper probability u . To this purpose, we need some auxiliary notions.

Definition 3.3. Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be MV-algebras. A *bimorphism* from $\mathbf{A} \times \mathbf{B}$ into \mathbf{C} is a map β from $\mathbf{A} \times \mathbf{B}$ into \mathbf{C} satisfying the following conditions, for all $a, a_1, a_2 \in \mathbf{A}$ and for all $b, b_1, b_2 \in \mathbf{B}$:

- (1) $\beta(1, 1) = 1$.
- (2) $\beta(0, b) = \beta(a, 0) = 0$.
- (3) The maps $f(x) = \beta(a, x)$ and $g(y) = \beta(y, b)$ are lattice homomorphisms from \mathbf{A} into \mathbf{C} and from \mathbf{B} into \mathbf{C} respectively.
- (4) If $a_1 \odot a_2 = 0$, then $\beta(a_1 \oplus a_2, b) = \beta(a_1, b) \oplus \beta(a_2, b)$, and if $b_1 \odot b_2 = 0$, then $\beta(a, b_1 \oplus b_2) = \beta(a, b_1) \oplus \beta(a, b_2)$.

The *tensor product* (cf. [28]) of two MV-algebras \mathbf{A} and \mathbf{B} is an MV-algebra (which is shown to exist and to be unique up to isomorphism) $\mathbf{A} \otimes \mathbf{B}$ with the following property: there is a bimorphism $(x, y) \mapsto x \otimes y$ from $\mathbf{A} \times \mathbf{B}$ into $\mathbf{A} \otimes \mathbf{B}$ such that for every bimorphism β from $\mathbf{A} \times \mathbf{B}$ into an MV-algebra \mathbf{C} , there is a unique homomorphism h from $\mathbf{A} \otimes \mathbf{B}$ into \mathbf{C} such that, for all $a \in \mathbf{A}$ and for all $b \in \mathbf{B}$, $h(a \otimes b) = \beta(a, b)$.

It is well known [28] that the maps Φ_1 and Φ_2 defined, for all $a \in \mathbf{A}$ and for all $b \in \mathbf{B}$, by $\Phi_1(a) = a \otimes 1$ and $\Phi_2(b) = 1 \otimes b$ are embeddings of \mathbf{A} (\mathbf{B} respectively) into $\mathbf{A} \otimes \mathbf{B}$. In the sequel, we will call Φ_1 and Φ_2 the *canonical embeddings* of \mathbf{A} (\mathbf{B} respectively) into $\mathbf{A} \otimes \mathbf{B}$.

It follows from [28] that for any MV-algebra \mathbf{A} , the algebra $[0, 1]_{MV} \otimes \mathbf{A}$ is a divisible MV-algebra.

Theorem 3.1. Let \mathbf{A} be a divisible MV-algebra and u an upper probability on \mathbf{A} . Let Φ_1 and Φ_2 be the canonical embeddings of $[0, 1]_{MV}$ (\mathbf{A} respectively) into $\mathbf{B} = [0, 1]_{MV} \otimes \mathbf{A}$. Then there is an upper probability u° on \mathbf{B} , which makes \mathbf{B} an UMV-algebra. Moreover, for all $a \in \mathbf{A}$, $\Phi_1(u(a)) = u^\circ(\Phi_2(a))$.

Proof. Since $\Phi_2(\mathbf{A})$ is a subalgebra of \mathbf{B} , by Mundici [27], every state s on $\Phi_2(\mathbf{A})$ can be extended to a state s^* of \mathbf{B} . Now let S be the set of all states σ of $\Phi_2(\mathbf{A})$ of the form $\sigma(\Phi_2(a)) = s(a)$, where s is a state on \mathbf{A} such that $s(a) \leq u(a)$ for all $a \in \mathbf{A}$. Let S^* be the set of all states s^* on \mathbf{B} such that s^* extends some $s \in S$. An easy computation shows that S^* is a weak- $*$ closed and convex subset of the dual space of $C(X_{\mathbf{B}}, [0, 1])$. Thus letting for every $b \in \mathbf{B}$, $u^*(b) = \max\{s^*(b) : s^* \in S^*\}$, we have that u^* is an upper probability on \mathbf{B} . It follows from the construction that for all $a \in \mathbf{A}$, $u^*(\Phi_2(a)) = u(a)$. Finally, let for all $b \in \mathbf{B}$ $u^\circ(b) = u^*(b) \otimes 1$. Using the fact that u^* is an upper probability, we obtain that u° is an internal upper probability which makes \mathbf{B} a UMV-algebra. Moreover, for all $a \in \mathbf{A}$, $u^\circ(\Phi_2(a)) = u^*(\Phi_2(a)) \otimes 1 = \Phi_1(u(a))$, as desired. \square

Remark 3.1. In [13], the authors present a proof that for every state s on an MV-algebra \mathbf{A} there is an internal state s° on $[0, 1]_{MV} \otimes \mathbf{A}$ such that for all $a \in \mathbf{A}$, $\Phi_1(s(a)) = s^\circ(\Phi_2(a))$. The proof has a gap since it is based on the wrong assumption that every element of $[0, 1]_{MV} \otimes \mathbf{A}$ has the form $\alpha \otimes a$ with $\alpha \in [0, 1]$ and $a \in \mathbf{A}$. The proof of Theorem 3.1 works a fortiori if u is a state, and hence it fixes the gap.

We now present a method for obtaining a divisible MV-algebra with an upper probability (into $[0, 1]$) from a UMV-algebra. To begin with, we prove that \mathcal{UMV} is in fact a variety.

Theorem 3.2. \mathcal{UMV} is the class of divisible MV-algebras with an operator u satisfying Eqs. (1)–(5) in Definition 3.1 plus the identity

$$(6') \quad u\left(\frac{n}{m} \oplus \left(y \ominus \left(\frac{n}{m} \odot y\right)\right)\right) = \frac{n}{m} \oplus u\left(y \ominus \left(\frac{n}{m} \odot y\right)\right) \quad (m > 0, 0 \leq n \leq m).$$

Proof. We prove that (6') is equivalent to (6). Indeed, $\frac{n}{m} \odot \left(y \ominus \left(\frac{n}{m} \odot y\right)\right) = 0$ and from (6) we get (6'). Conversely if $\frac{n}{m} \odot y = 0$, then from (6') we get $u\left(\frac{n}{m} \oplus y\right) = u\left(\frac{n}{m} \oplus \left(y \ominus \left(\frac{n}{m} \odot y\right)\right)\right) = \frac{n}{m} \oplus u\left(y \ominus \left(\frac{n}{m} \odot y\right)\right) = \frac{n}{m} \oplus u(y)$. \square

Lemma 3.1. If (\mathbf{A}, u) is a UMV-algebra, then the image of \mathbf{A} under u (denoted by $u(\mathbf{A})$) is the domain of a subalgebra of (\mathbf{A}, u) .

Proof. This follows directly from axiom (2) of UMV-algebras. \square

Given a UMV-algebra (\mathbf{A}, u) , a congruence of its MV-reduct \mathbf{A} need not be a congruence of (\mathbf{A}, u) , and hence not all MV-filters determine a congruence of (\mathbf{A}, u) . The filters corresponding to congruences of (\mathbf{A}, u) are precisely the MV-filters F such that if $x \rightarrow y \in F$, then $u(x) \rightarrow u(y) \in F$. Those filters will be called *UMV-filters* in the sequel.

Lemma 3.2. (1) For every congruence θ of a UMV-algebra (\mathbf{A}, u) , the set $F_\theta = \{x : (x, 1) \in \theta\}$ is a UMV-filter.
(2) For every UMV-filter F of (\mathbf{A}, u) , the set $\theta_F = \{(x, y) : x \leftrightarrow y \in F\}$ is a congruence of (\mathbf{A}, u) .

Hence, the maps $\theta \mapsto F_\theta$ and $F \mapsto \theta_F$ are mutually inverse isomorphisms between the lattice of congruences and the lattice of UMV-filters of (\mathbf{A}, u) .

Proof. (1) We already know that F_θ is an MV-filter, and it remains to prove that if $(x \rightarrow y, 1) \in \theta$, then $(u(x) \rightarrow u(y), 1) \in \theta$. Now $(x \rightarrow y, 1) \in \theta$ implies $(x \odot (x \rightarrow y), x) \in \theta$, and hence $(x \wedge y, x) \in \theta$. It follows $(u(x \wedge y), u(x)) \in \theta$, and finally $(u(x \wedge y) \rightarrow u(y), u(x) \rightarrow u(y)) \in \theta$. Since $u(x \wedge y) \rightarrow u(y) = 1$, we get the claim.

(2) We already know that θ_F is an MV-congruence, and it is left to prove that if $(x, y) \in \theta_F$, then $(u(x), u(y)) \in \theta_F$. Now $(x, y) \in \theta_F$ implies $(x \rightarrow y, 1) \in \theta_F$. Hence, $x \rightarrow y \in F$ and $u(x) \rightarrow u(y) \in F$ as F is an UMV-filter. By the same argument, $u(y) \rightarrow u(x) \in F$, and $(u(x), u(y)) \in \theta_F$. \square

Lemma 3.3. *Let (\mathbf{A}, u) be a UMV-algebra. Then, the MV-filter of \mathbf{A} generated by any MV-filter of $u(\mathbf{A})$ is a UMV-filter.*

Proof. Let F be an MV-filter of $u(\mathbf{A})$ and let G be the MV-filter of \mathbf{A} generated by F . Suppose $x \rightarrow y \in G$. Then, there are $u(a_1), \dots, u(a_n) \in F$ (not necessarily distinct) such that $u(a_1) \odot \dots \odot u(a_n) \leq x \rightarrow y$. Let $c = u(a_1) \odot \dots \odot u(a_n)$. Then $c \in F$, $u(c) = c$ and $c \leq x \rightarrow y$. By the residuation property, $x \leq c \rightarrow y$, and by (4) and (5) in Definition 3.1, $u(x) \leq u(c \rightarrow y) \leq u(\neg c) \oplus u(y)$. Now $\neg c \in u(\mathbf{A})$, and hence $u(\neg c) = \neg c$. Hence, $u(x) \leq c \rightarrow u(y)$, and by the residuation property again, $c \leq u(x) \rightarrow u(y)$. Hence, $u(x) \rightarrow u(y) \in G$. \square

Corollary 3.1. *If (\mathbf{A}, u) is a subdirectly irreducible UMV-algebra, then $u(\mathbf{A})$ is totally ordered.*

Proof. Let $a, b \in u(\mathbf{A})$, where (\mathbf{A}, u) is a subdirectly irreducible UMV-algebra, and suppose, by way of contradiction, that a and b are incomparable with respect to order. Let F be the minimal non-trivial UMV-filter of (\mathbf{A}, u) and let $c \in F \setminus \{1\}$. Then, $a \rightarrow b \in u(\mathbf{A})$, $b \rightarrow a \in u(\mathbf{A})$ and the MV-filter generated by any of them is a non-trivial UMV-filter, and hence it contains c . It follows that for some n , $(a \rightarrow b)^n \leq c$, $(b \rightarrow a)^n \leq c$ and hence $1 = (a \rightarrow b)^n \vee (b \rightarrow a)^n \leq c$, a contradiction. \square

Now let (\mathbf{A}, u) be a UMV-algebra and let M be a maximal filter of $u(\mathbf{A})$. Then, $u(\mathbf{A})/M$ is a subalgebra of $[0, 1]_{MV}$. Let N be the filter of \mathbf{A} generated by M . By Lemma 3.3, N is a UMV-filter of (\mathbf{A}, u) . Let u_N be the internal upper probability of $(\mathbf{A}, u)/N$, that is, $u_N(a/N) = u(a)/N$.

Lemma 3.4. *Up to isomorphism, u_N is an upper probability (into $[0, 1]$) on \mathbf{A}/N .*

Proof. Since $N \cap u(\mathbf{A}) = M$, the map $u(a)/M \mapsto u(a)/N$ is an isomorphism from $u(\mathbf{A})/M$ onto $u_N(\mathbf{A}/N)$, and hence $u_N(\mathbf{A}/N)$ is isomorphic to a subalgebra of $[0, 1]_{MV}$, and modulo this isomorphism, u_N can be regarded as a map of \mathbf{A}/N into $[0, 1]$. Moreover the axioms of UMV-algebras imply that u_N satisfies the axioms of upper probabilities. \square

By the above construction, starting from a UMV-algebra (\mathbf{A}, u) we have obtained a divisible MV-algebra \mathbf{A}/N and an upper probability u_N (taking values in $[0, 1]$) on \mathbf{A}/N , as desired.

Definition 3.4. Let \mathcal{K} be a class of UMV-algebras, S be a set of equations and ε be an equation (in the language of UMV-algebras). Then, S is said to be *satisfiable* in \mathcal{K} iff there is a UMV-algebra (\mathbf{A}, u) in \mathcal{K} and a valuation v into (\mathbf{A}, u) such that $(\mathbf{A}, u), v \models \varepsilon$ for every equation ε in S . We say that an equation δ is a *semantic consequence* of S in \mathcal{K} if for every UMV-algebra (\mathbf{A}, u) in \mathcal{K} and for every valuation v in (\mathbf{A}, u) , if $(\mathbf{A}, u), v \models \varepsilon$ for every equation ε in S , then $(\mathbf{A}, u), v \models \delta$.

Definition 3.5. A (nontrivial) UMV-algebra (\mathbf{A}, u) is said to be a *standard UMV-algebra* if $u(\mathbf{A})$ is simple (hence, $u(\mathbf{A})$ is isomorphic to a subalgebra of $[0, 1]_{MV}$).

In a standard UMV-algebra (\mathbf{A}, u) , $u(\mathbf{A})$ is isomorphic to a subalgebra of $[0, 1]_{MV}$, and hence u can be regarded as an upper probability (taking values in $[0, 1]$). This motivates the choice of the name *standard*.

There are at least two ways of expressing strong completeness of an equational logic L with respect to a class \mathcal{K} of algebras: the most common way is to define L to be complete with respect to \mathcal{K} if for every set S of equations and for every equation ε , ε is a semantic consequence of S in \mathcal{K} iff ε is derivable from S in L . But there is an alternative definition, that is, L is complete with respect to \mathcal{K} iff every consistent set of equations (with respect to L) is satisfiable in \mathcal{K} . In the case of an equational logic, consistency is expressed as the underviability of $x = y$ (an equation meaning that all terms are equal).

As regards to UMV, we have both kinds of completeness if \mathcal{K} is the class of all UMV-algebras. This follows from the strong completeness theorem for classical logic and from the fact that the models of UMV are precisely the UMV-algebras. We will prove that if \mathcal{K} denotes the class of all standard UMV-algebras, the second kind of completeness still holds, but the first one does not.

Theorem 3.3. *A set S of UMV-equations is consistent iff it is satisfiable in the class of all standard UMV-algebras.*

Proof. For the non-trivial direction, suppose S is consistent. Hence, S is satisfiable in a non-trivial UMV-algebra (\mathbf{A}_0, u_0) by some valuation v . Without loss of generality, we may suppose that every equation in S has the form $\phi = 1$ for some formula ϕ . Hence, we have $v(\phi) = 1$ for all ϕ such that $\phi = 1$ is in S . We prove that S is satisfiable in a standard UMV-algebra. Let M be a maximal filter of $u_0(\mathbf{A}_0)$ and N be the MV-filter of \mathbf{A}_0 generated by M . Let u_N be defined for all a/N in \mathbf{A}_0/N by

$u_N(a/N) = (u_0(a))/N$. Then $u_N(\mathbf{A}_0/N)$ is simple and $(\mathbf{A}_0/N, u_N)$ is a standard UMV-algebra. Finally, let for every variable p , $v_N(p) = (v(p))/N$. Then, v_N extends to a valuation on $(\mathbf{A}_0/N, u_N)$ such that $v_N(\phi) = 1$ for all ϕ such that $\phi = 1$ is in S . It follows that S is satisfiable in a standard UMV-algebra. \square

The other form of standard completeness (i.e., if an equation ε is a semantic consequence of a set of equations S with respect to the class of standard UMV-algebras, then ε is derivable from S), does not hold.

Theorem 3.4. *There is a set S of UMV-equations and an equation ε such that ε is a semantic consequence of S with respect to the class of all standard UMV-algebras but ε is not derivable from S in UMV.*

Proof. Let $S = \{u(p) \rightarrow u(q)^n = 1 : n \in \omega\}$, let $\phi = \neg u(p) \vee u(q)$ and let ε be the equation $\phi = 1$. Let v be a valuation in a standard UMV-algebra such that $v(\phi) < 1$. Then $0 < v(p)$ and $u(q) < 1$. It follows $v(u(q)^n) = 0$ for n sufficiently large and $v(u(p) \rightarrow u(q)^n) < 1$. Hence, S is not satisfied by v . It follows that every valuation v in a standard UMV-algebra which satisfies S also satisfies ε .

On the other hand, there is a UMV-algebra (\mathbf{A}, u) which is non-standard, and a valuation v such that $v(\phi) < 1$ and $v(\psi) = 1$ for all ψ such that $\psi = 1$ is in S . Indeed, let \mathbf{A} be a nontrivial ultrapower of $[0, 1]_{MV}$ and u be the identity on \mathbf{A} . Moreover, let v be such that $v(p) = v(u(p)) = \frac{1}{2}$ and $v(q) = v(u(q))$ is a co-infinitesimal, i.e., $v(q) < 1$ and for all n , $v(q)^n > \neg(v(q))$. It is readily seen that (\mathbf{A}, u) and v meet our requirements, i.e., they satisfy S and they do not satisfy ε . \square

Problem: Does the previous incompleteness result hold when S is finite?

We now come to a logical characterization of coherence of an assessment of upper probability (thus, coherence means absence of a bad bet). Now events are identified with classes of formulas of divisible Łukasiewicz logic modulo provable equivalence or equivalently with terms of divisible MV-algebras modulo provable equality, or even as elements of the countably generated free divisible MV-algebra. When there is no danger of confusion, we identify a formula (term respectively) with its equivalence class modulo provable equivalence (modulo provable equality respectively). For every formula ϕ and real number $\alpha \in [0, 1]$, $U_{(\phi, \alpha)}$ denotes the set of UMV-equations $\{q \leq u(\phi), u(\phi) \leq r : q, r \in \mathbf{Q}, 0 \leq q \leq \alpha \leq r \leq 1\}$. Moreover, $L_{(\phi, \alpha)}$ denotes the set $\{q \leq \neg u(\neg\phi), \neg u(\neg\phi) \leq r : q, r \in \mathbf{Q}, 0 \leq q \leq \alpha \leq r \leq 1\}$ and $S_{(\phi, \alpha)}$ denotes $U_{(\phi, \alpha)} \cup L_{(\phi, \alpha)}$. Note that if α is rational, then $U_{(\phi, \alpha)}$, $L_{(\phi, \alpha)}$ and $S_{(\phi, \alpha)}$ may be replaced by $\{\alpha = u(\phi)\}$ (by $\{\alpha = \neg u(\neg\phi)\}$, and by $\{\alpha = u(\phi), \alpha = \neg u(\neg\phi)\}$ respectively). Given an assessment $\Delta: \phi \mapsto \alpha_\phi : \phi \in K$, let

$$\begin{aligned} U_\Delta &= \bigcup \{U_{(\phi, \alpha_\phi)} : (\phi, \alpha_\phi) \in \Delta\} \\ L_\Delta &= \bigcup \{L_{(\phi, \alpha_\phi)} : (\phi, \alpha_\phi) \in \Delta\} \\ S_\Delta &= U_\Delta \cup L_\Delta. \end{aligned}$$

Note that if K is finite and for all $\phi \in K$, α_ϕ is rational, then the sets U_Δ , L_Δ and S_Δ may be replaced by finite sets.

Theorem 3.5. *Let Δ be an assessment on a set K of (equivalence classes of) formulas of Łukasiewicz logic with $\Delta(\phi) = \alpha_\phi$. Then:*

- (1) *There is no bad bet based on Δ iff U_Δ is logically coherent over UMV.*
- (2) *There is no good bet based on Δ iff L_Δ is logically coherent over UMV.*
- (3) *There is neither a good bet nor a bad bet based on Δ iff S_Δ is logically coherent over UMV.*

Proof. (1) If there is no bad bet based on Δ , then there is an upper probability u^* on the Lindenbaum algebra \mathbf{L}_{div} of divisible Łukasiewicz logic such that for all $\phi \in K$, $u^*(\phi) = \alpha_\phi$. But then letting $\mathbf{L}^\circ = [0, 1]_{MV} \otimes \mathbf{L}_{div}$ and $u^\circ(\alpha \otimes \psi) = (\alpha u^*(\psi) \otimes 1)$ we obtain a UMV-algebra such that for all $\phi \in K$, $u^\circ(1 \otimes \phi) = u^*(\phi) \otimes 1 = \alpha_\phi \otimes 1$. Moreover letting for every propositional variable p , $v^\circ(p) = 1 \otimes p$, we obtain a valuation on \mathbf{L}° such that for all $\phi \in K$, $v^\circ(u(\phi)) = u^*(\phi) \otimes 1 = \alpha_\phi \otimes 1$. It follows that every equation in U_Δ is satisfied by v° . Hence, U_Δ is logically coherent.

Conversely, if U_Δ is logically coherent, then by Theorem 3.3 it is satisfiable in a standard UMV-algebra (\mathbf{A}, u) by a suitable valuation, v say. Since $u(\mathbf{A})$ is simple, without loss of generality, we can assume that it is a sublgebra of $[0, 1]_{MV}$. Hence, u maps \mathbf{A} into $[0, 1]$ and it is an upper probability. Since every equation in U_Δ is satisfied, it follows that for all $\phi \in K$, $u(\phi) = \alpha_\phi$. Hence, Δ can be extended to an upper probability and there is no bad bet based on Δ .

(2) Similar to the proof of (1), with upper probabilities and bad bets replaced by lower probabilities and good bets.

(3) Suppose that there is neither a good bet nor a bad bet based on Δ . Then Δ may be extended to a state s^* on \mathbf{L}_{div} , and we can construct a UMV-algebra $(\mathbf{L}^\circ, u^\circ)$, with $\mathbf{L}^\circ = [0, 1]_{MV} \otimes \mathbf{L}_{div}$, and u° defined by $u^\circ(\alpha \otimes \psi) = (\alpha s^*(\psi) \otimes 1)$. Moreover letting for every propositional variable p , $v^\circ(p) = 1 \otimes p$, we obtain a valuation on $(\mathbf{L}^\circ, u^\circ)$ such that for all $\phi \in K$, $v^\circ(u(\phi)) = s^*(\phi) \otimes 1 = \alpha_\phi \otimes 1$. Since s^* is actually a state, and hence $s^*(\dots) = \neg s^*(\dots)$, it follows that every equation in S_Δ is satisfied. Hence, S_Δ is logically coherent.

Conversely, if S_Δ is logically coherent, then it may be satisfied in a standard UMV-algebra, and imitating the proof of (1) we get an upper probability u^* such that letting $l^*(\phi) = 1 - u^*(\neg(\phi))$, for every $\phi \in S$, $u^*(\phi) = l^*(\phi) = \alpha_\phi$. It follows that there is a closed and convex set Σ of states such that for all $\phi \in S$, $\alpha_\phi = \max \{s(\phi) : s \in \Sigma\} = \min \{s(\phi) : s \in \Sigma\}$. Hence, for all $s \in \Sigma$ and for all $\phi \in S$, $\alpha_\phi = s(\phi)$. Hence, Δ can be extended to a state (choose any $s \in \Sigma$), and there is neither a good bet nor a bad bet based on Δ . \square

4. Algebra and logic of upper and lower previsions

We now want to introduce an equational logic for reasoning about gambles and upper and lower previsions. As said before, we will represent gambles as elements of divisible and unital ℓ -groups. Moreover as in the case of UMV-algebras, we will replace real-valued upper previsions by *internal* operations satisfying a suitable set of identities which are suggested by the algebraic properties of upper previsions. In this way, we are led to the notion of UG-algebra defined below.

Definition 4.1. A UG-algebra is an algebra $(\mathbf{G}, 1, U)$ where $(\mathbf{G}, 1)$ is a divisible and unital ℓ -group and U is a unary operation on \mathbf{G} satisfying the following equations:

- (1) $U(1) = 1$.
- (2) $U(qx) = qU(x)$ for every rational $q \geq 0$.
- (3) $U(x + y) \leq U(x) + U(y)$.
- (4) $U(x) \leq U(x \vee y)$.
- (5) $U(x + q) = U(x) + q$ for every rational q .
- (6) $U(t) = t$ for every term t whose variables only occur under the scope of U .

Conditions (1) ... (6) are all equational, but we cannot express by equations (and not even by first order formulas) the fact that 1 is a strong unit. However, we will see that modulo a different presentation of elements of a UG-algebra, it is possible to represent the basic concepts about upper and lower previsions inside an equational logic.

To begin with, we will establish a categorical equivalence between the category of UMV-algebras (with morphisms the homomorphisms) and the category of UG-algebras, again with morphisms the homomorphisms. We start with a lemma connecting divisible unital ℓ -groups with the corresponding divisible MV-algebras.

Lemma 4.1. (1) For every divisible MV-algebra \mathbf{A} and for every $g \in \Gamma^{-1}(\mathbf{A})$ there are a natural number n , an integer z and an element $x \in A$ such that $g = nx + z$.

(2) Given natural numbers n, m and p , integers a, b and variables x, y , we can compute natural numbers h, k, l, q , integers c, d, e, f and terms $r(x, y), s(x), t(x, y), w(x)$ of divisible MV-algebras such that for every divisible MV-algebra \mathbf{A} and for every $x_0, y_0 \in A$, the equations $(nx_0 + a) + (my_0 + b) = hr(x_0, y_0) + c, -(nx_0 + a) = ks(x_0) + d, (nx_0 + a) \vee (my_0 + b) = lt(x_0, y_0) + e$ and $p(qw(x_0) + f) = nx_0 + a$ hold in $\Gamma^{-1}(\mathbf{A})$.

(3) Given two natural numbers n, m and two integers a, b , we can compute two equations $E(x, y)$ and $L(x, y)$ in the variables x, y and in the language of divisible MV-algebras such that for every divisible MV-algebra \mathbf{A} and for every $x_0, y_0 \in A, E(x_0, y_0)$ is satisfied in \mathbf{A} iff $nx_0 + a = my_0 + b$ holds in $\Gamma^{-1}(\mathbf{A})$, and $L(x_0, y_0)$ holds in \mathbf{A} iff $nx_0 + a \leq my_0 + b$ holds in $\Gamma^{-1}(\mathbf{A})$.

(4) For every equation $\varepsilon(x_1, \dots, x_n)$ in the variables x_1, \dots, x_n in the language of divisible ℓ -groups we can compute an equation $S_\varepsilon(x_1, \dots, x_n)$ in the language of divisible MV-algebras such that for every divisible MV-algebra \mathbf{A} and for every $x_1^0, \dots, x_n^0 \in A, \varepsilon(x_1^0, \dots, x_n^0)$ is satisfied in $\Gamma^{-1}(\mathbf{A})$ iff $S_\varepsilon(x_1^0, \dots, x_n^0)$ is satisfied in \mathbf{A} .

Proof. (1) Since 1 is a strong unit of $\Gamma^{-1}(\mathbf{A})$, there is a natural number k such that $g \geq -k$. Hence, $g + k \geq 0$. Moreover, by the same reason there is a natural number h such that $g + k \leq h$. Hence, $0 \leq \frac{g+k}{h} \leq 1$, and $\frac{g+k}{h} \in A$. Moreover, $g = h(\frac{g+k}{h}) - k$, and the claim is proved.

(2) Let $h = n + m + 1$ and $c = a + b$. Then, $(nx + a) + (my + b) = h(\frac{nx}{h} \oplus \frac{my}{h}) + c$. Thus letting $r(x, y) = (n)\frac{x}{h} \oplus (m)\frac{y}{h}$ we have $(nx + a) + (my + b) = hr(x, y) + c$.

Moreover $-(nx + a) = n(1 - x) - a - n$. Therefore, letting $s(x) = \neg x, k = n$ and $d = -a - n$ we have $-(nx + a) = n(s(x)) + d$.

Next, let $e = \min \{a, b, 0\}$. Then, $(nx + a) \vee (mx + b) = ((nx + a - e) \vee (mx + b - e)) + e$. Now let $i = a - e$ and $j = b - e$. Then, $i, j \geq 0$ and $(nx + a) \vee (mx + b) = ((nx + i) \vee (mx + j)) + e$. Finally, let $l = m + n + i + j + 1$ and $t(x, y) = ((n)\frac{x}{l} \oplus (i)\frac{1}{l}) \vee ((m)\frac{y}{l} \oplus (j)\frac{1}{l})$. Then, $(nx + a) \vee (mx + b) = lt(x, y) + e$.

Of course, we can do a similar computation for \wedge , using the fact that $(nx + a) \wedge (mx + b) = -(- (nx + a) \vee - (mx + b))$.

For the last claim of (2) let f and p' be the Euclidean quotient and the remainder of the division of a by p (hence, $0 \leq p' < p$ and $a = pf + p'$). If $n = 0$, then let $q = 1$ and $w(x) = \frac{p'}{p}$. It follows $p(qw(x) + f) = p\frac{p'}{p} + pf = a = nx + a$. If $n > 0$, then let $w(x) = \frac{x}{p} \oplus \frac{p'}{np}$ and $q = n$. Then $p(nw(x) + f) = nx + p' + pf = nx + a$.

(3) Suppose first $a \leq b$. Then $nx + a = my + b$ iff $nx \geq my$ and $nx = my + b - a$. Let $M = n + m + |a| + |b| + 1$. Then, the last equation is equivalent to $(n)\frac{x}{M} = (m)\frac{y}{M} \oplus \frac{b-a}{M}$. If $b < a$, by a similar argument $nx + a = my + b$ reduces

to $(m) \frac{y}{M} = (n) \frac{x}{M} \oplus \frac{a-b}{M}$. Moreover, the condition $nx + a \leq mx + b$ reduces to $(n) \frac{x}{M} \leq (m) \frac{y}{M} \oplus \frac{b-a}{M}$ if $b \geq a$ and to $(m) \frac{y}{M} \geq (n) \frac{x}{M} \oplus \frac{a-b}{M}$ if $a > b$.

(4) Applying iteratively the reductions in (2) we can write $\varepsilon(x_1, \dots, x_n)$ as an equation of the form $Kt + Z = K't' + Z'$ where t, t' are MV-terms in the variables x_1, \dots, x_n , K and K' are natural numbers and Z, Z' are integers. The desired set $S_\varepsilon(x_1, \dots, x_n)$ is obtained applying the procedure described in (3) to the equation $Kt + Z = K't' + Z'$. \square

Definition 4.2. Given a UG-algebra $(\mathbf{G}, 1, U)$, we denote by U_* the restriction of U to $\Gamma(\mathbf{G}, 1)$. Note that U_* maps $\Gamma(\mathbf{G}, 1)$ into itself, and hence it is an operation on $\Gamma(\mathbf{G}, 1)$.

Conversely, given a UMV-algebra (\mathbf{A}, u) we define, for every $x \in A$, for every natural number n and for every integer a , $u^*(nx + a) = nu(x) + a$.

Lemma 4.2. (1) Let (\mathbf{A}, u) be a UMV-algebra and let $(\mathbf{G}, 1) = \Gamma^{-1}(\mathbf{A})$. Then, u^* is a well defined operation on $(\mathbf{G}, 1)$ and makes $(\mathbf{G}, 1)$ a UG-algebra. Moreover, u^* is the unique operator on $(\mathbf{G}, 1)$ which extends u and makes $(\mathbf{G}, 1)$ a UG-algebra. Hence, $(u^*)_* = u$.

(2) Let $(\mathbf{G}, 1, U)$ be a UG-algebra. Then, $(\Gamma(\mathbf{G}, 1), U_*)$ is a UMV-algebra, and $(U_*)^* = U$.

Proof. (1) We first prove that u^* is well defined. Suppose $nx + a = my + b$. Without loss of generality, we may assume $a \leq b$. Hence, $nx = my + b - a$, and letting $M = n + m + |b| + |a| + 1$, we have $\frac{n}{M}x = \frac{m}{M}y \oplus \frac{b-a}{M}$. Since $\frac{m}{M}y \odot \frac{b-a}{M} = 0$, we deduce $\frac{n}{M}u(x) = \frac{m}{M}u(y) + \frac{b-a}{M}$, and hence $nu(x) = mu(y) + b - a$ and $u^*(nx + a) = u^*(my + b)$. Hence, u^* is well defined.

We verify that u^* satisfies the defining properties of UG-algebras. That $u^*(1) = 1$ is trivial. We verify homogeneity. Let $q = \frac{h}{p}$ be any non-negative rational number, and let $nx + a \in G$ where $x \in A$, n is a natural number and a is an integer. We only treat the case $n > 0$, the case $n = 0$ being similar and easier. Hence, $\frac{1}{p}(nx + a) = n \left(\frac{x}{p} \oplus \frac{p'}{np} \right) + f$, where $0 \leq p' < k$ and $a = fp + p'$. Then,

$$\begin{aligned} u^*(qx) &= u^* \left(hn \left(\frac{x}{p} \oplus \frac{p'}{np} \right) + hf \right) = hn \left(u \left(\frac{x}{p} \oplus \frac{p'}{np} \right) \right) + hf \\ &= \frac{h}{p}(nu(x) + p' + pf) = \frac{h}{p}(nu(x) + a) = qu^*(nx + a). \end{aligned}$$

We verify subadditivity. Let $X = nx + a$ and $Y = my + b$. Then, $X + Y = M \left(\frac{nx}{M} \oplus \frac{my}{M} \right) + (a + b)$, where $M = n + m + 1$. Hence,

$$\begin{aligned} u^*(X + Y) &= M \left(u \left(\frac{nx}{M} \oplus \frac{my}{M} \right) \right) + a + b \leq M \left(u \left(\frac{nx}{M} \right) + u \left(\frac{my}{M} \right) \right) + a + b \\ &= nu(x) + mu(y) + a + b = u^*(X) + u^*(Y). \end{aligned}$$

We verify monotonicity. Let $X = nx + a$ and $Y = my + b$, and let $M = n + m + |a| + |b| + 1$. Then $X \leq Y$ iff either $a \leq b$ and $(n) \frac{x}{M} \leq (m) \frac{y}{M} \oplus \frac{b-a}{M}$ or $a > b$ and $(m) \frac{y}{M} \geq (n) \frac{x}{M} \oplus \frac{a-b}{M}$. In the first case, $\frac{n}{M}u(x) \leq \frac{m}{M}u(y) \oplus \frac{b-a}{M} = \frac{m}{M}u(y) + \frac{b-a}{M}$, and $u^*(X) = nu(x) + a \leq mu(y) + b = u^*(Y)$.

In the latter case, $\frac{m}{M}u(y) \geq \frac{n}{M}u(x) \oplus \frac{a-b}{M} = \frac{n}{M}u(x) + \frac{a-b}{M}$, and $u^*(X) = nu(x) + a \leq mu(y) + b = u^*(Y)$.

Finally, we verify that for every rational number q one has $u^*(X + q) = u^*(X) + q$, where X has the form $nx + a$, n a non-negative integer, a an integer and $x \in A$. Suppose $q = \frac{c}{k}$ with $k > 0$. Let us write $ka + c$ as $ka + c = pk + r$, where $0 \leq r < k$. If $n = 0$, then $X + q = a + \frac{c}{k} = 1 \frac{r}{k} + p$, and $u^*(X) = \frac{r}{k} + p = a + \frac{c}{k} = u^*(X) + q$. If $n > 0$, then $X + q = nk \left(\frac{x}{k} \oplus \frac{r}{k^2n} \right) + p$. Since $\frac{x}{k} \odot \frac{r}{k^2n} = 0$,

$$u^*(X + q) = nk \left(u \left(\frac{x}{k} \right) + \frac{r}{k^2n} \right) + p = nu(x) + \frac{r}{k} + p = nu(x) + a + \frac{c}{k} = u^*(X) + q.$$

Finally, it is clear that the restriction of u^* to \mathbf{A} is u , that is, $(u^*)_* = u$.

(2) It follows directly from the axioms of UG-algebras that if $(\mathbf{G}, 1, U)$ is a UG-algebra, then U_* makes $\Gamma(\mathbf{G}, 1)$ a UMV-algebra. Moreover $(U_*)^*$ and U are both extensions of U_* to $(\mathbf{G}, 1)$ which make $(\mathbf{G}, 1)$ a UG-algebra. Hence, for any element $nx + a$ of \mathbf{G} , where n is a natural number, a is an integer and $x \in \mathbf{A}$, it must be $U(nx + a) = nU(x) + a = nU_*(x) + a = (U_*)^*(nx + a)$. \square

By essentially repeating the same proof, we obtain that:

Lemma 4.3. (1) Given an upper probability u (taking values in $[0, 1]$) on a divisible MV-algebra \mathbf{A} , there is a unique upper prevision u^* on $\Gamma^{-1}(\mathbf{A})$ which extends u . This upper prevision is defined, for every $x \in \mathbf{A}$, for every natural number n and for every integer a , by $u^*(nx + a) = nu(x) + a$.

(2) For every upper prevision U on a divisible ℓ -group with strong unit $(\mathbf{G}, 1)$, its restriction U_* to $\Gamma(\mathbf{G}, 1)$ is an upper probability over $\Gamma(\mathbf{G}, 1)$.

Lemma 4.4. Let (Γ, Γ^{-1}) be Mundici's equivalence between the category of MV-algebras and the category of unital ℓ -groups. Then:

- (1) If h is a homomorphism from a UMV-algebra (\mathbf{A}, u_A) into a UMV-algebra (\mathbf{B}, u_B) , then $\Gamma^{-1}(h)$ is a homomorphism from $(\Gamma^{-1}(\mathbf{A}), u_A^*)$ into $(\Gamma^{-1}(\mathbf{B}), u_B^*)$.
- (2) If k is a homomorphism from a UG-algebra $(\mathbf{G}, 1_G, U_G)$ into a UG-algebra $(\mathbf{H}, 1_H, U_H)$, then $\Gamma(k)$ is a homomorphism from $(\Gamma(\mathbf{G}, 1_G), U_{G*})$ into $(\Gamma(\mathbf{H}, 1_H), U_{H*})$.

Proof. (1) We already know that $\Gamma^{-1}(h)$ is the unique morphism of unital ℓ -groups from $\Gamma^{-1}(\mathbf{A})$ into $\Gamma^{-1}(\mathbf{B})$ which extends h . This implies that for every natural number n , for every integer a and for every $x \in \mathbf{A}$ we must have $\Gamma^{-1}(h)(nx + a) = nh(x) + a$. Hence,

$$\begin{aligned} \Gamma^{-1}(h)(u_A^*(nx + a)) &= \Gamma^{-1}(h)(nu_A(x) + a) \\ &= nh(u_A(x)) + a = nu_B(h(x)) + a = u_B^*(\Gamma^{-1}(h)(nx + a)). \end{aligned}$$

Hence, $\Gamma^{-1}(h)$ is a homomorphism of UG-algebras.

(2) We already know that $\Gamma(k)$ is a MV-homomorphism from $\Gamma(\mathbf{G}, 1_G)$ into $\Gamma(\mathbf{H}, 1_H)$, and it is left to prove that $\Gamma(k)$ is compatible with U_{G*} . Now if $x \in \Gamma(\mathbf{G}, 1_G)$, then $\Gamma(k)(x) = k(x)$ and $U_{G*}(x) = U_G(x)$. Hence,

$$\Gamma(k)(U_{G*}(x)) = k(U_{G*}(x)) = k(U_G(x)) = U_G(k(x)) = U_{G*}(\Gamma(k)(x)),$$

and the claim is proved. \square

Definition 4.3. We define a pair of functors $(\Gamma_U, \Gamma_U^{-1})$ from the category of UG-algebras into the category of UMV-algebras and viceversa as follows:

- (1) For every UG-algebra $(\mathbf{G}, 1, U)$ and for every UMV-algebra (\mathbf{A}, u) we set $\Gamma_U(\mathbf{G}, 1, U) = (\Gamma(\mathbf{G}, 1), U_*)$ and $\Gamma_U^{-1}(\mathbf{A}, u) = (\Gamma^{-1}(\mathbf{A}), u^*)$.
- (2) For every morphism h from a UG-algebra $(\mathbf{G}, 1_G, U_G)$ into a UG-algebra $(\mathbf{H}, 1_H, U_H)$, we define $\Gamma_U(h) = \Gamma(h)$, and for every morphism k from a UMV-algebra (\mathbf{A}, u_A) into a UMV-algebra $(\mathbf{B}, 1_B)$, we define $\Gamma_U^{-1}(k) = \Gamma^{-1}(k)$.

Theorem 4.1. The pair $(\Gamma_U, \Gamma_U^{-1})$ is an equivalence of categories.

Proof. It suffices to prove that $\Gamma_U(\Gamma_U^{-1}(\mathbf{A}, u))$ is isomorphic to (\mathbf{A}, u) , that $\Gamma_U^{-1}(\Gamma_U(\mathbf{G}, 1, U))$ is isomorphic to $(\mathbf{G}, 1, U)$, and that Γ_U is full and faithful. The first two claims follow from the equalities $(u^*)_* = u$ and $(U_*)^* = U$ (and from the fact that (Γ, Γ^{-1}) is an equivalence between the categories of MV-algebras and of unital ℓ -groups).

It remains to prove that Γ_U is full and faithful. Now if h is a morphism in the category of UG-algebras from $(\mathbf{G}, 1_G, U_G)$ into $(\mathbf{H}, 1_H, U_H)$, then it is the unique homomorphism from $(\mathbf{G}, 1_G, U_G)$ into $(\mathbf{H}, 1_H, U_H)$ which extends $\Gamma_U(h)$, and hence Γ_U is faithful. Moreover every homomorphism k from $\Gamma_U(\mathbf{G}, 1_G, U_G)$ into $\Gamma_U(\mathbf{H}, 1_H, U_H)$ is the restriction to $\Gamma_U(\mathbf{G}, 1_G, U_G)$ of a morphism from $(\mathbf{G}, 1_G, U_G)$ into $(\mathbf{H}, 1_H, U_H)$, and hence Γ_U is full. This settles the claim. \square

Our goal is now to introduce a logic for the treatment of UG-algebras and hence of gambles and of upper and lower previsions. As we said at the beginning of this section, the main problem is that UG-algebras do not constitute an equational class. However, the categorical equivalence between UMV-algebras and UG-algebras allows us, modulo a different presentation of terms of UG-algebras, to find an equational logic for them and to interpret it in UMV. The main idea is to present UG-terms in the canonical form $nt + a$ where t is a UMV-term, n is a natural number and a is an integer. This is allowed by Lemmas 4.1 and 4.2. Those lemmas also allow us to define UG-operations in such a way that every UG-term reduces to a term $nt + a$ in canonical form.

Definition 4.4. The equational logic UG is defined as follows (in the sequel, t and s denote arbitrary UMV-terms, m and n denote arbitrary natural numbers, a and b denote arbitrary integers, k denotes any strictly positive integer, and v, y, z and w denote arbitrary-UG terms):

- (1) The set of terms of UG is the smallest set T such that:
- (1a) For every n, t and a as shown above, $nt + a$ is in T .
 - (1b) If v, w are in T , then $v + w, -v, v \vee w, v \wedge w, \frac{v}{k}$ and $U(v)$ are in T .
- (2) The axioms of UG are as follows:
- (2a) Suppose that $a \leq b$, and that $(n) \frac{t}{M} x = (m) \frac{s}{M} \oplus \frac{b-a}{M}$ is identically satisfied in all UMG-algebras, where $M = n + m + |a| + |b| + 1$. Then, $nt + a = ms + b$ is an axiom of UG.
 - (2b) $(nt + a) + (ms + b) = M((n) \frac{t}{M} \oplus (m) \frac{s}{M}) + (a + b)$, where $M = m + n + 1$.
 - (2c) $-(nt + a) = n(\neg t) - (n + a)$.
 - (2d) $\frac{1}{k}(0t + a) = r \frac{1}{k} + q$, where q and r are integers such that $0 \leq r < k$ and $a = qk + r$.
 - (2e) $\frac{1}{k}(nt + a) = n(\frac{t}{k} \oplus (r) \frac{1}{nk}) + q$, where $n > 0$ and q and r are integers such that $0 \leq r < k$ and $a = qk + r$.
 - (2f) $(nt + a) \vee (ms + b) = lt' + e$, where $e = \min\{a, b, 0\}$, $l = m + n + (a - e) + (b - e) + 1$, and $t' = ((n) \frac{t}{l} \oplus \frac{a-e}{l}) \vee ((m) \frac{s}{l} \oplus \frac{b-e}{l})$.
 - (2g) $(nt + a) \wedge (ms + b) = ls' + e$, where $e = \min\{a, b, 0\}$, $l = m + n + (a - e) + (b - e) + 1$, and $s' = ((n) \frac{t}{l} \vee \frac{a-e}{l}) \wedge ((m) \frac{s}{l} \oplus \frac{b-e}{l})$.
 - (2h) $U(nt + a) = nu(t) + a$.
- (3) The inference rules of UG are:
- (3a) $\frac{v=w}{w=v}$ and $\frac{y=v \quad v=w}{y=w}$.
 - (3b) $\frac{y=v}{-y=-v}, \frac{y=v}{\frac{y}{n}=\frac{v}{n}}$, n any positive integer, $\frac{v=w}{U(v)=U(w)}$ and $\frac{y=v \quad w=z}{y \circ w = v \circ z}$, where \circ denotes any of $+$, \vee or \wedge .

The axioms and rules of UG allow us to transform any UG-term into one of the form $nt + a$, where t is an UMG-term, n is a natural number and a is an integer. Moreover, axiom (2a) allows us to reduce an equality of the form $nt + a = ns + b$ into an equality of UMG-algebras. Hence, any equality $\varepsilon(x_1, \dots, x_n)$ of UG is transformed into an equality $\varepsilon^*(x_1, \dots, x_n)$ such that for every UG-algebra $(\mathbf{G}, 1, U)$ and for every $x_1, \dots, x_n \in \Gamma_U(\mathbf{G}, 1, U)$ one has that $\varepsilon(x_1, \dots, x_n)$ holds in $(\mathbf{G}, 1, U)$ iff $\varepsilon^*(x_1, \dots, x_n)$ holds in $\Gamma(\mathbf{G}, 1, U)$.

Note that the assumption that all variables have to be interpreted in $\Gamma_U(\mathbf{G}, 1, U)$ is crucial. Henceforth, by the term *UG-valuation* we mean an evaluation v of UG-terms such that $0 \leq v(x) \leq 1$ for every variable x . Validity, satisfiability and semantic consequence in UG are defined accordingly.

Now let Δ be a set of equations of UG and ε be an equation of UG. Let Δ^* be the set of all equations of the form ξ^* with $\xi \in \Delta$. Then ε is a semantic consequence of Δ in UG iff ε^* is a semantic consequence of Δ^* in UMG. Indeed, if there is a UG-valuation v in $\Gamma(\mathbf{G}, 1, U)$ which validates Δ and invalidates ε in $(\mathbf{G}, 1, U)$, then the same valuation (thought of as a UMG-valuation) validates Δ^* and invalidates ε^* in $\Gamma(\mathbf{G}, 1, U)$, and viceversa. Moreover, a UG-valuation v validates Δ in $(\mathbf{G}, 1, U)$ iff v (thought of as a UMG-valuation) validates Δ^* in $\Gamma(\mathbf{G}, 1, U)$. It follows that Δ is satisfiable in the class of UG-algebras iff Δ^* is satisfiable in the class of UMG-algebras.

Conversely, let for every UMG equation $\varepsilon : t = s$, ε° be the equation $1t + 0 = 1s + 0$, and let for every set Σ of equations, $\Sigma^\circ = \{\gamma^\circ : \gamma \in \Sigma\}$. Then, ε is a semantic consequence of Σ in UMG iff ε° is a semantic consequence of Σ° in UG. Moreover Σ is satisfiable in the class of UMG-algebras iff Σ° is satisfiable in the class of UG-algebras.

It follows:

Theorem 4.2.

- (i) There is a polynomial time computable map $*$ from the set of UG-equations into the set of UMG-equations which preserves both semantic consequence and satisfiability.
- (ii) There is a polynomial time computable map $^\circ$ from the set of UMG-equations into the set of UG-equations which preserves both semantic consequence and satisfiability.

Moreover:

Theorem 4.3. To any assessment $\Pi : t_i \mapsto \alpha_{t_i} : i = 1, \dots, n$ on a finite set $S = \{t_1, \dots, t_n\}$ of (equivalence classes of) UG-terms we can associate three sets of UG-equations U_Π, L_Π and S_Π such that:

- (1) U_Π is logically coherent in UG iff U_Π^* is logically coherent in UMG iff there is no bad bet based on Π .
- (2) L_Π is logically coherent in UG iff L_Π^* is logically coherent in UMG iff there is no good bet based on Π .
- (3) S_Π is logically coherent in UG iff S_Π^* is logically coherent in UMG iff there is no bad bet based on Π .

Proof. We only prove (1), the proofs of (2) and of (3) being similar. First of all, every term t_i occurring in S can be written in the form $n_i s_i + a_i$, where the n_i is a natural number, a_i is an integer and s_i is a UMG-term. Let U_Π be the set of all equations of the form $r \leq U(t_i) \leq q$ with r, q rational numbers, $r \leq \alpha_{t_i} \leq q$, and with $i = 1, \dots, n$. Hence U_Π^* is equivalent to the set of all equations of the form $r' \leq u(s_i) \leq q'$ with r', q' rational numbers in $[0, 1]$, $r' \leq \frac{\alpha_i - a_i}{n_i} \leq q'$. Let Π^* be the map

$s_i \mapsto \frac{\alpha_i - a_i}{n_i}$. Then, Π^* is an assessment on UMV-terms and an easy computation shows that there is no bad bet based on Π iff there is no bad bet based on Π^* . The latter condition is in turn equivalent to the satisfiability of the set U_{Π^*} . Hence, there is no bad bet based on Π iff there is no bad bet based on Π^* iff U_{Π^*} is logically coherent over UMV iff U_{Π} is coherent over UG.

The proofs of (2) and (3) are similar. \square

5. UMV-algebras and tSMV-algebras

In this section we define an interpretation of UMV-algebras into the class of tSMV-algebras, t being a suitable natural number depending on the terms to be interpreted. This interpretation will allow us to extend to UMV-algebras some complexity results from [10].

Definition 5.1. For every integer $t \geq 1$ we define a *tSMV-algebra* as a system $(\mathbf{A}, \sigma_1, \dots, \sigma_t)$ where \mathbf{A} is an MV-algebra and $\sigma_i, i = 1, \dots, t$ are unary operations on \mathbf{A} such that, for every $i, k, h = 1, \dots, t$, the following equations are satisfied:

- (1) $\sigma_i(0) = 0$.
- (2) $\sigma_i(\neg x) = \neg \sigma_i(x)$.
- (3) $\sigma_i(\sigma_k(x) \oplus \sigma_h(y)) = \sigma_k(x) \oplus \sigma_h(y)$.
- (4) $\sigma_i(x \oplus y) = \sigma_i(x) \oplus \sigma_i(y \odot (x \odot y))$.

Remark. The SMV-algebras introduced in [13] occur as a particular case of tSMV-algebras when $t = 1$. For all $1 \leq i, j \leq t$, $\sigma_i(A)$, the image of A under σ_i , is the domain of an MV-subalgebra of \mathbf{A} , denoted by $\sigma_i(\mathbf{A})$. Moreover $\sigma_i(\mathbf{A}) = \sigma_j(\mathbf{A})$, cf. [10]. Therefore, we feel free to speak about $\sigma_i(\mathbf{A})$ without specifying the index i .

As shown in [10], there is a close connection between MV-algebras with t states and tSMV-algebras. On one hand, given a tSMV-algebra $(\mathbf{A}, \sigma_1, \dots, \sigma_t)$ and an MV-ultrafilter, M , of $\sigma_i(\mathbf{A})$, the quotient $\sigma_i(\mathbf{A})/M$ may be identified with a subalgebra of $[0, 1]_{MV}$, and hence, for $i = 1, \dots, t$, the map $a \mapsto \sigma_i(a)/M$ is a state on \mathbf{A} .

Conversely, given an MV-algebra \mathbf{A} , and t states s_1, \dots, s_t on it, we can obtain a tSMV-algebra with MV-reduct $\mathbf{B} = [0, 1]_{MV} \otimes \mathbf{A}$ defining, for $i = 1, \dots, t$ an internal state s_i° on \mathbf{B} according to Theorem 3.1 and Remark 3.1, thus obtaining an MV-algebras with t internal states, and hence a tSMV-algebra.

In tSMV-algebras, congruences are associated to t-filters.

Definition 5.2. Let $(\mathbf{A}, \sigma_1, \dots, \sigma_t)$ be a tSMV-algebra. A subset F of A is said to be a *t-filter*, if F is a filter of its MV-reduct \mathbf{A} , and, for every $x \in F$, and for $i = 1, \dots, t$, $\sigma_i(x) \in F$.

For every t-filter F of a tSMV-algebra $(\mathbf{A}, \sigma_1, \dots, \sigma_t)$, the relation

$$\theta_F = \{(x, y) \in A \times A : (x \rightarrow y) \odot (y \rightarrow x) \in F\}$$

is a congruence of $(\mathbf{A}, \sigma_1, \dots, \sigma_t)$. Conversely, for every congruence θ of $(\mathbf{A}, \sigma_1, \dots, \sigma_t)$, the set

$$F_\theta = \{x \in A : (x, 1) \in \theta\}$$

is a t-filter of $(\mathbf{A}, \sigma_1, \dots, \sigma_t)$. Moreover:

Theorem 5.1 [10]. *The above defined maps $\tau : F \mapsto \theta_F$ and $\lambda : \theta \mapsto F_\theta$ are mutually inverse isomorphisms between the lattice of t-filters, and that of congruences of $(\mathbf{A}, \sigma_1, \dots, \sigma_t)$.*

Proposition 5.1 [10]. *Let $(\mathbf{A}, \sigma_1, \dots, \sigma_t)$ be a subdirectly irreducible tSMV-algebra. Then the MV-algebra $\sigma_i(\mathbf{A})$ is linearly ordered.*

The next result about the computational complexity of the satisfiability relation of tSMV-equations will be used in this section in order to determine the computational complexity of coherence of rational-valued assessments over UMV-algebras.

Theorem 5.2 [10]. *The problem of checking the satisfiability of a tSMV-equation is NP-complete.*

Although it is possible to introduce an algebraizable propositional logic whose equivalent algebraic semantics is the variety of tSMV-algebras, we find it more convenient to work in the equational logic of tSMV-algebras, called tSMV, which is defined below.

Definition 5.3. The axioms of tSMV are axioms (1), . . . , (4) of tSMV-algebras, plus $x = x$. Its rules are the usual Birkhoff's rules of equational logic (cf. Definition 3.2).

Now we concentrate on the relation between satisfiability and validity of UMV^- -equations and tSMV-equations. Recall that a UMV^- -term is a term of UMV -algebras of the form $\gamma(u(\psi_1), \dots, u(\psi_n))$, where $\gamma(x_1, \dots, x_n)$ and ψ_1, \dots, ψ_n are MV-terms. If ϕ is a UMV^- -term, then ϕ^* denotes the tSMV-term obtained by replacing in ϕ each occurrence of $u(\psi_i)$ by $\bigvee_{j=1}^t \sigma_j(\psi_i)$.

Theorem 5.3. Given an UMV^- -term ϕ with at most t subterms beginning with the symbol u , the equation $\phi = 1$ is satisfiable in UMV iff $\phi^* = 1$ is satisfiable in tSMV.

Proof. Let $\gamma(x_1, \dots, x_n)$ be an MV-term and let $\gamma^\circ = \gamma(u(x_1), \dots, u(x_n))$. We claim that $\gamma = 1$ is MV satisfiable iff $\gamma^\circ = 1$ is UMV -satisfiable. Since MV satisfiability is NP-hard, and $\gamma \mapsto \gamma^\circ$ is a polynomial time computable map, this will settle the claim. Now suppose that $\gamma^\circ = 1$ is satisfied in an UMV -algebra (\mathbf{A}, u) under the valuation v . Define a valuation v^* in \mathbf{A} by $v^*(x_i) = v(u(x_i))$. Then we have $v^*(\gamma) = v(\gamma^\circ) = 1$.

Conversely, let v be a valuation into an MV-algebra \mathbf{A} such that $v(\gamma) = 1$. Let u be the identity function on \mathbf{A} . Then, (\mathbf{A}, u) is a UMV -algebra, and the same valuation v , thought of as a UMV valuation, satisfies $v(\gamma^\circ) = 1$. \square

Since the map * defined as above is a polynomial time computable map which preserves satisfiability, using Theorem 5.2 we obtain the following result:

Lemma 5.1. The problem of checking the satisfiability of a UMV^- equation $\phi = 1$ in UMV is in NP.

Lemma 5.2. The problem of checking the satisfiability of a UMV^- equation $\phi = 1$ in UMV is NP-hard.

Proof. Let $\gamma(x_1, \dots, x_k)$ be an MV-term and consider $\gamma^\circ = \gamma(u(x_1), \dots, u(x_k))$. If $\gamma^\circ = 1$ is satisfiable in a UMV -algebra (\mathbf{A}, u) , then $\gamma = 1$ is satisfiable in the MV-algebra $[0, 1]_{MV}$. In fact, thanks to Theorem 3.3 u can be regarded as an upper probability taking values in $[0, 1]$, thus it is sufficient to consider a valuation v such that $v(x_i) = u(x_i)$.

Conversely, let v be an evaluation in $[0, 1]_{MV}$ such that $v(\gamma) = 1$. Let id be the identity function on $[0, 1]_{MV}$. Then id makes $[0, 1]_{MV}$ a UMV -algebra, and the same valuation v (thought as UMV -valuation) satisfies $\gamma^\circ = 1$ in the UMV -algebra $([0, 1]_{MV}, id)$.

Since the map $^\circ$ is polynomial time computable, NP-hardness of the satisfiability problem for UMV^- -equations follows from the NP-hardness of satisfiability of $[0, 1]_{MV}$ -equations. \square

Combining these two lemmas we obtain that:

Theorem 5.4. Satisfiability in UMV of UMV^- -equations is NP-complete.

Now let us define a UG^- term to be one of the form $\gamma(U(\psi_1), \dots, U(\psi_n))$, where $\gamma(x_1, \dots, x_n)$, ψ_1, \dots, ψ_n are MV terms, and a UG^- equation to be an equation of the form $s = t$ where s and t are UG^- terms. Thanks to Theorem 4.2 we easily obtain the following result:

Corollary 5.1. Satisfiability of UG^- -equations in the equational logic of UG -algebras is NP-complete.

The previous theorem ensures that in the equational logic of UMV -algebras and of tSMV-algebras the satisfiability-problem has the same computational complexity. But if we fix a $t \geq 1$, satisfiability of an equation $\phi = 1$ in UMV and satisfiability of $\phi^* = 1$ in tSMV are two distinct concepts. In fact, the following result holds:

Proposition 5.2. Given $t \geq 1$

- (i) There is an UMV -equation $\phi = 1$ such that $\phi = 1$ is not valid in UMV but $\phi^* = 1$ is valid in tSMV.
- (ii) There is an UMV -equation $\xi = 1$ such that $\xi = 1$ is satisfiable in UMV but $\xi^* = 1$ is not satisfiable in tSMV.

Proof. Given a $t \geq 1$, let us consider $t + 1$ distinct propositional variables p_1, \dots, p_{t+1} .

(i) Denote by ϕ the UMV -term

$$\bigvee_{i,j \leq t+1, i \neq j} (u(p_i \oplus p_j) \leftrightarrow u(p_i) \oplus u(p_j)).$$

Then $\phi = 1$ is not valid in UMV. In fact, let (\mathbf{A}, u) be the UMV-algebra where \mathbf{A} is the MV-algebra of the functions from $[0, 1]$ into $[0, 1]$ and u is defined as $u(f) = \sup(f)$. Then consider a valuation v on \mathbf{A} such that $v(p_i) : [0, 1] \rightarrow [0, 1]$ is the following function:

$$v(p_i)(x) = \begin{cases} \frac{1}{2}, & \frac{1}{i+2} < x < \frac{1}{i+1}; \\ 0, & \text{elsewhere.} \end{cases}$$

Thus, being for every $i, j \leq (t + 1)$ $v(u(p_i \oplus p_j)) = \frac{1}{2}$ and $v(u(p_i)) \oplus v(u(p_j)) = 1$, the equation $\phi = 1$ is not valid in UMV.

Now let us consider an arbitrary non-trivial tSMV-algebra $(\mathbf{A}, \sigma_1, \dots, \sigma_t)$ and let us denote by ϕ^* the tSMV-term obtained by replacing each subterm $u(\chi_i)$ of ϕ by $\bigvee_{h=1}^t \sigma_h(\chi_i)$. Then $\phi^* = 1$ is valid in tSMV.

In fact, $\exists i, j \leq (t + 1)$ and $\exists k \leq t$ such that

$$\sigma_k(p_i) = \bigvee_{h \leq t} \sigma_h(p_i) \quad \text{and} \quad \sigma_k(p_j) = \bigvee_{h \leq t} \sigma_h(p_j). \tag{1}$$

Being $p_i \odot p_j = 0$ if $i \neq j$, equalities (1) imply the claim.

(ii) By claim (i), the equation $(\phi^2)^* = 0$ is not satisfiable in tSMV, because of the validity of $\phi^* = 1$. On the other hand, it follows from the proof of (i) that there is a valuation v into a UMV-algebra such that $v(\phi) = \frac{1}{2}$, and hence $v(\phi^2) = 0$. Thus, $\phi = 0$ is satisfiable in UMV. \square

6. Conclusions and further work

We have introduced a logico-algebraic framework to provide a logical characterization of coherence for imprecise probability. This framework, which brings together ideas and results from the general theory of coherent lower previsions and the theory of probability on MV-algebras, is a very natural candidate for the characterization of rational belief, in the general case of reasoning under second order uncertainty. Indeed, our framework has important consequences on the foundations of uncertain reasoning. By providing a purely logical notion of coherence for imprecise probabilities, we have effectively extended the expressive power of the standard Bayesian paradigm, and in particular de Finetti's betting scheme, so as to meet the challenge posed by the representation of (some aspects of) probabilistic ignorance. We did so by suggesting how the role of the Dutch Book Theorem in supporting the Bayesian approach, as spelt out here, is unquestionable *only* for first order uncertainty. We have then illustrated how a more sophisticated betting scheme and hence a refined notion of coherence are needed in order to model imprecision and vagueness. By providing such notions, have shown that the characterization of rational belief under second order uncertainty, including ignorance, need not imply a rejection of the Bayesian approach.

We conclude this paper by pointing to the future lines of research which we envisage would follow naturally from this initial investigation. The focus of this paper was restricted to *unconditional* prevision and probability. Whilst the conditional case has already received a substantial attention in the field of coherent lower previsions [41,39,26],¹⁵ this has not been the case for the logical tradition. We plan to fill-in this gap by extending, in a future paper, our logico-algebraic framework to the conditional case where the conditioning events take values in $[0, 1]$.

Another extremely promising line of research emerges from the detailed formal comparison between the results presented here and two mathematically related, yet foundationally rather distinct, frameworks. In particular, the recent work on imprecise probability trees [7] opens up to a very interesting connection between the theory of imprecise probability and the game-theoretic foundations put forward in [37]. As noted above, our interpretation of probability diverges rather substantially from that of Shafer and Vovk's, but the unifying framework put forward in [7] could shed some very interesting light the characterization of coherence for second order uncertainty, as outlined in this paper.

Finally, as noted in Remark 1.1, we would like to mention that we can interpret the connection between logic and probability in terms of the latter providing model-theoretic counterparts of the syntactic, logical notion of coherence. Our completeness result however, gives probability a distinctively syntactic role which is quite akin to the one arising within the order-theoretic framework of [6]. By arguing that the belief structures based on classical logics are embedded in those of lower previsions, de Cooman effectively shows that probability models can play the syntactic role of maximally consistent sets in (analogues of) the Model Existence Lemma. Although he considers only two-valued propositional logics, we believe an accurate mathematical comparison between his framework and the present one could lead to bringing the logical and statistical traditions even closer to providing a unified model of rational reasoning under second order uncertainty.

¹⁵ Walley shows that given an unconditional lower prevision, when coherent conditional extensions exist, they can be characterized by a generalised Bayes rule.

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