Stable Non-standard Imprecise Probabilities

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Abstract. Stability arises as the consistency criterion in a betting interpretation for hyperreal imprecise previsions, that is imprecise previsions (and probabilities) which may take infinitesimal values. The purpose of this work is to extend the notion of *stable coherence* introduced in [8] to conditional hyperreal imprecise probabilities. Our investigation extends the de Finetti-Walley operational characterisation of (imprecise) prevision to conditioning on events which are considered "practically impossible" but not "logically impossible".

1 Introduction and Motivation

This paper combines, within a logico-algebraic setting, several extensions of the imprecise probability framework which we aim to generalise so as to represent infinitesimal imprecise probabilities on fuzzy events.

Imprecise conditional probabilities, as well as imprecise conditional previsions, have been investigated in details by Walley [12] in the case where the conditioning event ψ is boolean and has non-zero lower probability. There, a de Finetti-style interpretation of upper and lower probability and of upper and lower prevision in terms of bets is proposed. In Walley's approach, the conditional upper prevision, $U(x|\psi)$ of the gamble x given the event ψ is defined to be a number α such that $U(x_\psi \cdot (x - \alpha)) = 0$, where $x_\psi = 1$ if ψ is true and $x_\psi = 0$ if ψ is false. When the lower probability of ψ is non-zero, there is exactly one α satisfying the above condition, and hence, the upper conditional prevision is well-defined. Likewise, the lower conditional prevision of x given ψ is the unique β such that $L(x_\psi \cdot (x - \beta)) = 0$. However, the uniqueness of α and β is only guaranteed if the lower probability of ψ is not zero, otherwise, there might be infinitely many solutions of the above equations.

In terms of bets, the rationality of an assessment of upper probabilities (or of upper previsions) corresponds to the absence of inadmissible bets, that is, of bets for which there is an alternative strategy for the gambler which ensures to him a strictly better payoff whatever the outcome of the experiment will be. In the case of conditional upper and lower previsions, however, the absence of inadmissible bets might be due to the fact that the conditioning event has lower

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probability zero (remind that when the conditioning event is false the bet is invalidated, and hence the payoff of *any* bet on the conditional event is zero). So, the presence of events with lower probability zero might force the absence of inadmissible bets even in non-rational assessments. For instance, if we chose a point at random on the surface of Earth, and ϕ denotes the event: *the point belongs to the western hemisphere* and ψ denotes the event *the point belongs to the equator*, the assessment $\phi|\psi \mapsto 0$, $\neg\phi|\psi \mapsto 0$ avoids inadmissible bets (in case the point does not belong to the equator), but is not rational.

The goals of this paper are the following:

(1) To provide for a treatment of conditional upper and lower previsions when the conditioning event is many-valued and when the conditioning event has probability zero. For probabilities in the usual sense, this goal has been pursued in [8].

(2) To model conditional or unconditional bets in which truth values and betting odds may be non-standard. In particular, to every non-zero event with probability zero we assign an infinitesimal non-zero probability, so that we may avoid conditioning events with probability zero. Then taking standard parts we obtain a probability (or, in the case of many-valued events, a state) in the usual sense.

(3) The basic idea is the following: we replace the concept of coherence (absence of inadmissible bets) by a stronger concept, namely, *stable coherence*. Not only inadmissible bets are ruled out, but, in addition, the absence of inadmissible bets is preserved if we modify the assessment by an infinitesimal in such a way that no lower probability assessments equal to zero are allowed for events which are not impossible. The main result (Theorem 5) will be that stable coherence for an assessment of conditional upper probability corresponds to the existence of a non-standard upper probability which extends the assessment modulo an infinitesimal and assigns a non-zero lower probability to all non-zero events.

Our main result has important foundational consequences. For stable coherence allows us to distinguish between events which are regarded as practically impossible and events which are indeed logically impossible. It is well-known that this subtle but crucial difference can only be captured by so-called *regular* probability functions which are characterised by Shimony's notion of *strict coherence*. A companion paper will address this point in full detail.

For reasons of space all proofs are omitted from this version of the paper.

2 Algebraic Structures for Non-standard Probability

We build on [8], which can be consulted for further background on MV-algebra and related structures.[1] Specifically, we work in the framework of unital lattice ordered abelian groups, which can be represented as algebras[2] of bounded

[1] [2] provides a basic introduction and [11] a more advanced treatment, including states and their relationship with coherence. The basic notions of universal algebras we use are provided by [1].

[2] Algebras will be usually denoted by boldface capital letters (with the exception of the standard MV-algebra on $[0,1]$ and the standard PMV^+-algebra on $[0,1]$ which are denoted by $[0,1]_{MV}$ and by $[0,1]_{PMV}$, respectively) and their domains will be denoted by the corresponding lightface capital letters.

functions from a set X into a non-standard extension, $\mathbf{R}^*$, of $\mathbf{R}$. Hence, their elements may be interpreted as bounded random variables (also called *gambles* in [12]). The set of bounded random variables is closed under sum, subtraction and under the lattice operations. Since we are interested in extending the framework of [8] to *imprecise conditional* prevision and probability, we also need a product operation. It turns out that appropriate structure is constituted by the c-s-u-f integral domains.

Definition 1. *A commutative strongly unital function integral domain (c-s-u-f integral domain for short) is an algebra* $\mathbf{R} = (R, +, -, \vee, \wedge, \cdot, 0, 1)$ *such that:*

(i) $(R, +, -, \vee, \wedge, 0)$ *is a lattice ordered abelian group and* 1 *is a strong unit of this lattice ordered group.*

(ii) $(R, +, -, \cdot, 0, 1)$ *is a commutative ring with neutral element* 1.

(iii) The identity $x^+ \cdot (y \vee z) = (x^+ \cdot y) \vee (x^+ \cdot z)$ *holds, where* $x^+ = x \vee 0$.

(iv) The quasi identity: $x^2 = 0$ implies *that* $x = 0$ *holds.*

Remark 1. In [7] it is shown that every c-s-u-f integral domain embeds into an algebra of the form $(\mathbf{R}^*_{fin})^H$, where $\mathbf{R}^*$ is an ultrapower of the real field, $\mathbf{R}_{fin}$ is the c-s-u-f domain consisting of all finite elements of $\mathbf{R}^*$, and H is an index set[3]. In particular, every c-s-u-f integral domain embeds into the product of totally ordered integral domains, and this fact justifies the name of these structures.

Let $(\mathbf{G}, u)$ be a unital lattice ordered group. We define $\Gamma(\mathbf{G}, u)$ to be the algebra whose domain is the interval $[0, u]$, with the constant 0 and with the operations $\sim x = u - x$ and $x \oplus y = (x + y) \wedge u$. Moreover if h is a homomorphism of unital lattice ordered abelian groups $(\mathbf{G}, u)$ and $(\mathbf{G}', u')$ (i.e., a homomorphism of lattice ordered groups such that $h(u) = u'$), we define $\Gamma(h)$ to be the restriction of h to $\Gamma(\mathbf{G}, u)$. Likewise, given a c-s-u-f integral domain $(\mathbf{F}, u)$ and denoting by $\mathbf{F}^-$ the underlying lattice ordered abelian group, we define $\Gamma_R(\mathbf{F}, u)$ to be $\Gamma(\mathbf{F}^-, u)$ equipped with the restriction of $\cdot$ to $\Gamma(\mathbf{F}^-, u)$. Moreover given a homomorphism h of c-s-u-f integral domains from $(\mathbf{F}, u)$ into $(\mathbf{F}', u')$ we denote by $\Gamma_R(h)$ its restriction to $\Gamma_R(\mathbf{F}, u)$.

Theorem 1. *(1) (see [10]).* Γ *is a functor from the category of unital lattice ordered abelian groups into the category of MV-algebras. Moreover* Γ *has an adjoint* Γ^{-1} *such that the pair* (Γ, Γ^{-1}) *is an equivalence of categories.*

(2) (see [7]). Γ_R *is a functor from the category of c-s-u-f integral domains into the category of PMV*$^+$*-algebras. Moreover* Γ_R *has an adjoint* Γ_R^{-1} *such that the pair* $(\Gamma_R, \Gamma_R^{-1})$ *is an equivalence of categories.*

Remark 2. Theorem 1 tells us that the algebra of gambles, represented by a unital lattice ordered abelian group or of a c-s-u-f integral domain, is completely determined by the algebra of $[0, u]$-valued gambles, whose elements may be regarded as many-valued events. MV and PMV$^+$-algebras provide rich semantics for the logic of many-valued events.

[3] Of course, the embedding sends u, the neutral element for product, in to 1.

Let $\mathbf{A}$ be an MV-algebra, and let $(\mathbf{G}, u)$ be a lattice ordered unital abelian group such that $\Gamma(\mathbf{G}, u) = \mathbf{A}$. Let $\mathbf{R}^*$ be an ultrapower of the real field, and let $\mathbf{R}^*_{fin}$ be the set of all finite elements of $\mathbf{R}^*$ and let $[0,1]^* = \Gamma(\mathbf{R}^*_{fin}, 1)$.

We say that $\mathbf{R}^*$ is $(\mathbf{G}, u)$-*amenable* if for all $g \in G$, if $g \neq 0$, then there is a homomorphism h from $\mathbf{G}$ into $\mathbf{R}^*_{fin}$, considered as a lattice ordered abelian group, such that $h(u) = 1$ and $h(g) \neq 0$. We say that $[0,1]^*$ is $\mathbf{A}$-*amenable* if for all $a \in A$, if $a \neq 0$, then there is a homomorphism h from $\mathbf{A}$ into $[0,1]^*$, such that $h(u) = 1$ and $h(g) \neq 0$.

Lemma 1. *Let $(\mathbf{G}, u)$ be a unital lattice ordered abelian group, let $\mathbf{R}^*$ be an ultrapower of $\mathbf{R}$ and $[0,1]^* = \Gamma(\mathbf{R}^*, 1) = \Gamma(\mathbf{R}^*_{fin}, 1)$. Then $[0,1]^*$ is $\Gamma(\mathbf{G}, u)$-amenable iff $\mathbf{R}^*$ is $(\mathbf{G}, u)$-amenable.*

In [8], the following result is shown:

Proposition 1. *For every MV-algebra $\mathbf{A}$, an $\mathbf{A}$-amenable ultrapower $[0,1]^*$, of $[0,1]$ exists.*

It follows:

Corollary 1. *For each unital lattice ordered abelian group $(\mathbf{G}, u)$, a $(\mathbf{G}, u)$-amenable ultrapower $\mathbf{R}^*$ of $\mathbf{R}$ exists.*

In the sequel, we will need MV-algebras with product or c-s-u-f domains in order to treat conditional probability in an algebraic setting. Moreover we need to treat probabilities in terms of bets in such a way that zero probabilities will be replaced by infinitesimal probabilities. We would like to have a richer structure in which not only MV-operations or lattice ordered group operations, but also product and hyperreal numbers are present. The construction presented in the next lines provides for such structures.

Definition 2. *Let $(\mathbf{R}^*_{fin}, 1)$ be $(\mathbf{G}, u)$-amenable, let H be the set of all homomorphisms from $(\mathbf{G}, u)$ into $(\mathbf{R}^*_{fin}, 1)$, and let Φ be defined, for all $g \in \mathbf{G}$, by $\Phi(g) = (h(g) : h \in H)$. By $\Pi(\mathbf{G}, u, \mathbf{R}^*_{fin})$, we denote the subalgebra of $(\mathbf{R}^*_{fin})^H$ (with respect to the lattice ordered group operations and to product in $\mathbf{R}^*_{fin}$) generated by $\Phi(\mathbf{G})$ and by the elements of $\mathbf{R}^*_{fin}$, thought of as constant maps from H into $\mathbf{R}^*$ (in the sequel, by abuse of language, we denote by α the function from H into $\mathbf{R}^*_{fin}$ which is constantly equal to α).*

Likewise, if $\mathbf{A}$ is an MV-algebra and $[0,1]^*$ is an ultrapower of $[0,1]_{MV}$ which is $\mathbf{A}$-amenable, if H is the set of all homomorphisms from $\mathbf{A}$ into $[0,1]^*$ and Φ is defined, for all $a \in A$ by $\Phi(a) = (h(a) : h \in H)$, then $\Pi(\mathbf{A}, [0,1]^*)$ denotes the subalgebra of $([0,1]^*)^H$ generated by $[0,1]^*$ and by $\Phi(\mathbf{A})$.

It can be proved that both $(\mathbf{G}, u)$ and $\mathbf{R}^*_{fin}$ are embeddable into $\Pi(\mathbf{G}, u, \mathbf{R}^*_{fin})$ and both $\mathbf{A}$ and $[0,1]^*$ are embeddable in $\Pi(\mathbf{A}, [0,1]^*)$, see [8].

Lemma 2. $\Pi(\mathbf{G}, u, \mathbf{R}^*_{fin}) = \Gamma^{-1}(\Pi(\Gamma(\mathbf{G}, u), [0,1]^*))$.

3 Fuzzy Imprecise Probabilities over the Hyperreals

We are now in a position to introduce the notion of fuzzy imprecise hyperprevisions, which, as usual, lead in a special case, to probabilities. We will focus on *upper* hyper previsions and probabilities (*lower* notions will be obtained as usual). Naturally enough, our first step requires us to extend propositional valuations to the hyperreals.

Definition 3. *Let* $(\mathbf{G}, u)$ *be a unital lattice ordered abelian group and suppose that* $\mathbf{R}^*$ *is* $(\mathbf{G}, u)$ *amenable. Let* $\mathbf{A} = \Gamma(\mathbf{G}, u)$ *and* $[0,1]^* = \Gamma_R(\mathbf{R}^*_{fin}, 1)$*,* $\mathbf{A}^* = \Pi(\mathbf{A}, [0,1]^*)$*,* $\mathbf{G}^* = \Pi(\mathbf{G}, u, \mathbf{R}^*_{fin})$*. A* hypervaluation *on* $\mathbf{G}^*$ *(resp., on* $\mathbf{A}^*$*) is a homomorphism* v^* *from* $\mathbf{G}^*$ *into* $\mathbf{R}^*_{fin}$ *(resp., from* $\mathbf{A}^*$ *into* $[0,1]^*$*) such that for every* $\alpha \in \mathbf{R}^*_{fin}$ *(resp., in* $[0,1]^*$*),* $v^*(\alpha^*) = \alpha$*. A* hyperprevision *on* $(\mathbf{G}, u)$ *is a function* P^* *from* $\mathbf{G}^*$ *into* $\mathbf{R}^*_{fin}$ *such that for all* $\alpha, \in \mathbf{R}^*_{fin}$ *and* $x, y \in \mathbf{G}^*$*, the following conditions hold:*
(1) $P^*(\alpha^* x) = \alpha P^*(x)$.
(2) if $x \geq y$*, then* $P^*(x) \geq P^*(y)$.
(3) $P^*(x + y) = P^*(x) + P^*(y)$.
(4) $P^*(u) = 1$.
(5) There are hypervaluations v, w *such that* $v(x) \leq P^*(x) \leq w(x)$.

A hyperstate *on* $\mathbf{A}^*$ *is a map* S^* *from* $\mathbf{A}^*$*) into* $[0,1]^*$ *such that, for all* $\alpha \in [0,1]^*$ *and for all* $x, y \in \mathbf{A}^*$*, the following conditions hold:*
(a) $S^*(u) = 1$

(b) $S^*(\alpha^* \cdot x) = \alpha \cdot S^*(x)$

(c) if $x \odot y = 0$*, then* $S^*(x \oplus y) = S^*(x) + S^*(y)$

(d) there are hypervaluations v, w *such that* $v(x) \leq S^*(x) \leq w(x)$.

Definition 4. *An* upper hyperprevision *is a function* U^* *on* $\mathbf{G}^*$ *which satisfies (2), (4), (5), with* P^* *replaced by* U^**, and*
(1)' $U^*(\alpha^* x) = \alpha U^*(x)$*, provided that* $\alpha \geq 0$.
(3)' $U^*(x + y) \leq U^*(x) + U^*(y)$.
(6) $U^*(x + \alpha) = U^*(x) + \alpha$.

An upper hyperstate *on* $\mathbf{A}^*$ *is a function* U_0^* *from* $\mathbf{A}^*$ *into* $[0,1]^*$ *such that, for all* $x, y \in \mathbf{A}^*$ *and for all* $\alpha \in [0,1]^*$*, the following conditions hold:*

(i) $U_0^*(u) = 1$.

(ii) If $x \leq y$*, then* $U_0^*(x) \leq U_0^*(y)$.

(iii) $U_0^*(\alpha \cdot x) = \alpha \cdot U_0^*(x)$ *and if* $\alpha \odot x = 0$*, then* $U_0^*(x \oplus \alpha) = U_0^*(x) + \alpha$.

(iv) $U_0^*(x \oplus y) \leq U_0^*(x) \oplus U_0^*(y)$.

(v) $U_0^*(x \oplus \alpha) = U_0^*(a) + \alpha$ *whenever* $x \odot \alpha = 0$.
(vi) There are hypervaluations v, w *such that* $v(x) \leq U_0^*(x) \leq w(x)$.

Remark 3. (1) A valuation on a lattice ordered abelian group $\mathbf{G}^*$ (resp., on $\mathbf{A}^*$) is a homomorphism v from $\mathbf{G}^*$ into $\mathbf{R}$ (resp., from $\mathbf{A}$ into $[0,1]_{PMV}$) such that $v(\alpha) = \alpha$ for every standard real α. Moreover, a prevision on $\mathbf{G}^*$ is a map into $\mathbf{R}$ which satisfies (2), (3) and (4), as well as (1) for all standard α and (5) with hypervaluations replaced by valuations. Likewise, a state on $\mathbf{A}^*$ is a map into $[0,1]$ which satisfies (a) and (c), as well as (b) for all standard α and (d) with hypervaluations replaced by valuations.

Moreover, an upper prevision on $\mathbf{G}^*$ is a map into $\mathbf{R}$ which satisfies (2), (4) and (3)', as well as (1)' and (6) for all standard α and (5) with hypervaluations replaced by valuations. Finally, an upper hyperstate on $\mathbf{A}^*$ is a map into $[0,1]$ which satisfies (i), (ii) and (iv), as well as (iii) and (v) for all standard α and (vi) with hypervaluations replaced by valuations.

Hence, hypervaluations, hyperprevisions, hyperstates, upper hypeprevisions and upper hyperstates are natural non-standard generalizations of valuations, previsions, states, upper previsions and upper states, respectively.

(2) The restriction to $\mathbf{A}^*$ of a hyperprevision P^* (resp., an upper hyperprevision U^*) on $\mathbf{G}^*$, is a hyperstate (resp., an upper hyperstate). Moreover, a hyperstate S^* (resp, a hyper upper state U_0^*) on $\mathbf{A}^*$ has a unique extension P^* (resp., U^*) to a hyper prevision (resp., to a hyper upper prevision) on $\mathbf{G}^*$. Indeed, given $a \in \mathbf{G}^*$, there are positive integers M, N such that $0 \leq \frac{a+N}{M} \leq u$. So, $\frac{a+N}{M} \in \mathbf{A}^*$, and it suffices to define $P^*(a) = M \cdot S^*(\frac{a+N}{M}) - N$, and $U^*(a) = M \cdot U_0^*(\frac{a+N}{M}) - N$. Note that in [5] it is shown that the definition does not depend on the choice of the integers M and N such that $\frac{a+N}{M} \in \mathbf{A}$.

(3) Let U_0^* be an upper hyperstate on $\mathbf{A}^*$ and U^* be the unique upper hyperprevision on $\mathbf{G}^*$ extending U_0^*. Then for all $x \in \mathbf{A}^*$, the upper hyperprobability, $U_0^*(x)$, of x, is a number α such that the upper hyperprevision, $U^*(x-\alpha)$ of the gamble $x - \alpha$, is 0. Indeed, $U^*(x-\alpha) = U^*(x) - \alpha$ (because α is a constant), and hence, $U_0^*(x) = U^*(x) = \alpha$ iff $U^*(x-\alpha) = 0$. This means that the upper hyperprevision of a gamble x is a number α such that the upper hyperprevision of the payoff of the gambler when he bets 1 with betting odd α, namely $x - \alpha$, is 0.

(4) Given an upper hyperprevision U^*, its corresponding lower hyperprevision is $L^*(x) = -U^*(-x)$. Likewise, if U_0^* is an upper hyperstate, its corresponding lower hyperstate is given by $L_0^*(x) = 1 - U_0^*(\neg x)$.

(5) If U^* is an upper hypeprevision, then $U^*(x) = U^*(x+y-y) \leq U^*(x+y) + U^*(-y)$, and hence, $U^*(x) + L^*(y) \leq U^*(x+y) \leq U^*(x) + U^*(y)$. Likewise, if U_0^* is an upper hyperstate and L_0^* is its corresponding lower hyperstate and if $x \odot y = 0$, then $U_0^*(x) + L_0^*(y) \leq U_0^*(x \oplus y) \leq U_0^*(x) + U_0^*(y)$.

We now present a betting interpretation of hyper upper previsions, which will lead to the appropriate notion of coherence. We begin by recalling a characterisation of coherence as *avoiding inadmissible bets* given in [6] (the terminology "bad bet" was used there.)

Definition 5. *Let* $(\mathbf{G}, u)$ *be a unital lattice ordered abelian group, and let* $\Lambda = x_1 \mapsto \alpha_1, \ldots, x_n \mapsto \alpha_n$ *be an assessment of upper previsions on the bounded* *random variables* $x_1, \ldots, x_n \in \mathbf{G}$. *The associated betting game is as follows: the gambler can bet only non-negative numbers* $\lambda_1, \ldots, \lambda_n$ *on* $x_1, \ldots, x_n$*, and the payoff for the bookmaker corresponding to the valuation* v *on* $(\mathbf{G}, u)$ *will be* $\sum_{i=1}^{n} \lambda_i \cdot (\alpha_i - v(x_i))$.

Let W *be a set of valuations on* $(\mathbf{G}, u)$. *An* inadmissible W bet *is a bet* $\mu_i \geq 0$ *on* x_i *(for some* $i \leq n$*) such that there is a system of non-negative bets* $\lambda_1, \ldots, \lambda_n$ *which guarantees a better payoff to the gambler, independently of the valuation* $v \in W$*, that is, for every valuation* $v \in W$*,* $\sum_{j=1}^{n} \lambda_j \cdot (v(x_j) - \alpha_j) > \mu_i \cdot (v(x_i) - \alpha_i)$. *An* inadmissible bet *is an inadmissible* W *bet, where* W *is the set of all valuations on* $(\mathbf{G}, u)$.

The assessment Λ *is said to be* W coherent *if it excludes inadmissible* W*-bets, and* coherent *if it excludes inadmissible bets.*

In [6] it is shown that an assessment of upper probability avoids inadmissible bets iff it can be extended to an upper prevision. The result was shown first, although in a different setting, by Walley in [12].

In [6] it is also shown that given gambles $x_1, \ldots, x_m$ and given an upper prevision U, for $i = 1, \ldots, m$ there is a prevision P_i such that $P_i(x_i) = U(x_i)$ and $P_i(x) \leq U(x)$ for every gamble x. Moreover, as shown in [11], there are valuations $v_{i,j}$ and non-negative reals $\lambda_{j,i}$, $i = 1, \ldots, m+1$, $j = 1, \ldots, m$ such that for $j = 1, \ldots, m$, $\sum_{i=1}^{m+1} \lambda_{i,j} = 1$ and for $h, j = 1, \ldots, m$, $s_j(x_h) = \sum_{i=1}^{m+1} \lambda_{i,j} v_{i,j}(x_h)$. In other words, we can assume that each P_i is a convex combination of valuations.

Hence, coherence for upper previsions is equivalent to the following condition:

Theorem 2. *Let* $\Lambda = x_1 \mapsto \alpha_1, \ldots, x_m \mapsto \alpha_m$ *be an assessment as in Definition 5. Then* Λ *is coherent (i.e., avoids inadmissible bets) iff there are valuations* $v_{i,j}$: $j = 1, \ldots, m$, $i = 1, \ldots, m+1$ *and non-negative real numbers* $\lambda_{i,j}$: $j = 1, \ldots, m$, $i = 1, \ldots, m+1$*, such that, letting for* $j = 1, \ldots, m$*,* $P_j(x) = \sum_{i=1}^{m+1} \lambda_{i,j} v_{i,j}(x)$*, the following conditions hold:*

(i) For $j = 1, \ldots, m$*,* $\sum_{i=1}^{m+1} \lambda_{i,j} = 1$.
(ii) For $j = 1, \ldots, m$*,* $P_i(x_j) \leq \alpha_j$.
(iii) $P_i(x_i) = \alpha_i$.

In words, Λ *avoids inadmissible bets iff there are* m *convex combinations,* $P_1, \ldots, P_m$*, of valuations, such that for* $j = 1, \ldots, m$*,* $\alpha_j = \max\{P_h(x_j) : h = 1, \ldots, m\}$.

The result above[4] may be extended to non-standard assessments, to hypervaluations and to upper hyper previsions. First of all, we consider a $(\mathbf{G}, u)$-amenable ultrapower, $\mathbf{R}^*$, of $\mathbf{R}$, and we set $\mathbf{G}^* = \Pi(\mathbf{G}, u, \mathbf{R}^*_{fin})$. Then we consider a hyperassessment $\Lambda := x_1 \mapsto \alpha_1, \ldots, x_n \mapsto \alpha_n$ with $x_1, \ldots, x_n \in \mathbf{G}^*$. Let W be a set of hypervaluations. We say that Λ is W*-coherent* if it rules out inadmissible W-bets, that is, for $i = 1, \ldots, n$ and for every $\lambda, \lambda_1, \ldots, \lambda_n \geq 0$, there is a hypervaluation $v^* \in W$ such that $\lambda \cdot (v^*(x_i) - \alpha_i) \geq \sum_{j=1}^{n+1} \lambda_j \cdot (v^*(x_j) - \alpha_j)$. We

[4] Our coherence criterion resembles very closely a number of similarly-minded generalisations of de Finetti's own notion of coherence, among others that of [3].

say that Λ is $\mathbf{R}^*$-*coherent* if it is W-coherent, where W is the set of all hypervaluations on $\mathbf{G}^*$, and that Λ is *coherent* if it is $\mathbf{R}^\circ$-coherent for some ultrapower, $\mathbf{R}^\circ$, of $\mathbf{R}^*$. Similar definitions can be given for assessments of upper hyperprobability on algebras of the form $\mathbf{A}^* = \Pi(\mathbf{A}, [0,1]^*)$ (in this case, hypervaluations are homomorphisms from $\mathbf{A}^*$ into $[0,1]^*$ which preserve the elements of $[0,1]^*$, and (upper) hyperprevisions must be replaced by (upper) hyperstates).

Recall that there is a bijection between upper hyperprevisions on $\mathbf{G}^* = \Pi(\mathbf{G}, u, \mathbf{R}^*_{fin})$ and upper hyperstates on $\mathbf{A}^* \Gamma_R(\Pi(\mathbf{G}, u, \mathbf{R}^*_{fin}))$: the restriction to $\mathbf{A}^*$ of an upper hyperprevision is an upper hyperstate, and every upper hyperstate on $\mathbf{A}^*$ has a unique extension to an upper hyperprevision U^* on $\mathbf{G}^*$.

By a similar argument, there is a bijection between coherent assessments on $\mathbf{G}^*$ and coherent assessments on $\mathbf{A}^*$. Indeed, clearly, a coherent assessment on $\mathbf{A}^*$ is also a coherent assessment on $\mathbf{G}^*$. Conversely, given any assessment $\Lambda =: x_1 \mapsto \alpha_1, \ldots, x_k \mapsto \alpha_k$ on $\mathbf{G}^*$, there are integers M_i, N_i, with $M_i > 0$, such that $0 \le \frac{x_i+N_i}{M_i} \le u$. Now let $a_i = \frac{x_i+N_i}{M_i}$, and let Λ_0 be the assessment: $\Lambda_0 =: a_1 \mapsto \frac{\alpha_1+N_1}{M_1}, \ldots, a_k \mapsto \frac{\alpha_k+N_k}{M_k}$ on $\mathbf{A}^*$. Then Λ avoids inadmissible bets iff Λ_0 avoids inadmissible bets, and Λ can be extended to an upper hyperprevision U^* iff Λ_0 extends to its restriction U^*_0 to $\mathbf{A}^*$, which is an upper hyperstate on $\mathbf{A}^*$. Hence, *in the sequel we will often identify upper hyperprevisions on* $\mathbf{G}^*$ *with their restriction to* $\mathbf{A}^*$*, and the assessment* Λ *on* $\mathbf{G}^*$ *with its corresponding assessment* Λ_0 *on* $\mathbf{A}^*$.

Theorem 3. *Let* $\Lambda = x_1 \mapsto \alpha_1, \ldots, x_m \mapsto \alpha_m$ *be a hyperassessment on an algebra of the form* $\mathbf{G}^*$ *(hence,* $\alpha_i \in \mathbf{R}^*_{fin}$*). Then the following are equivalent:*

(1) Λ *is coherent.*

(2) There is an upper hyperprevision U^* *s.t. for* $i = 1, \ldots, m$, $U^*(x_i) = \alpha_i$.

(3) There are hypervaluations $v^*_{i,j} : j = 1, \ldots, m$, $i = 1, \ldots, m+1$ *and non-negative hyperreal numbers (possibly, in an ultrapower of* $\mathbf{R}^*$*)* $\lambda_{i,j} : j = 1, \ldots, m$, $i = 1, \ldots, m+1$, *such that, letting for* $j = 1, \ldots, m$, $P^*_j(x) = \sum_{i=1}^{m+1} \lambda_{i,j} v^*_{i,j}(x)$, *the following conditions hold:*
(i) For $j = 1, \ldots, m$, $\sum_{i=1}^{m+1} \lambda_{i,j} = 1$.
(ii) For $j = 1, \ldots, m$, $P^*_i(x_j) \le \alpha_j$.
(iii) $P^*_i(x_i) = \alpha_i$.

We express this fact saying that Λ *is the supremum of* m *convex combinations of* $m+1$ *hypervaluations*

We conclude this section with a result that, up to infinitesimals, a coherent assessment can be extended by a faithful upper hyperprevision (an upper hyperprevision U^* is faithful if $U^*(x) = 1$ implies $x = 1$, or equivalently if $L^*(x) = 0$ implies $x = 0$, where $L^*(x) = 1 - U^*(\neg x)$ is the lower prevision associated to U^*).

Theorem 4. *For every coherent assessment* $a_1 \mapsto \alpha_1, \ldots, a_n \mapsto \alpha_n$ *of upper previsions on* $(\mathbf{G}, u)$ *there is a faithful upper hyperprevision* U^* *on* $\mathbf{G}^*$ *such that for* $i = 1, \ldots, n$, $U^*(a_i) - \alpha_i$ *is infinitesimal.*

4 Conditional Imprecise Non-standard Probabilities

[9] gives the following betting interpretation of conditional probability: when the conditioning event ψ is many-valued, we assume that betting on $\phi|\psi$ is like betting on ϕ with the proviso that only a part of the bet proportional to the truth value $v(\psi)$ of ψ will be valid. Hence, the gambler's payoff corresponding to the bet λ in a conditional bet on $\phi|\psi$ is $\lambda v(\psi)(v(\phi) - \alpha)$, where α is the betting odd. If $\lambda = 1$, and if we identify any formula with its truth value, the payoff is expressed by $\psi(\phi - \alpha)$. Hence, the upper conditional probability of ϕ given ψ is obtained by imposing the upper prevision of the payoff to be zero. That is, the upper conditional probability $U_0^*(\phi|\psi)$ of ϕ given ψ must be a number α such that $U^*(\psi(\phi-\alpha)) = 0$, where U^* is the unique upper hyperprevision which extends U_0^*.

A desirable condition is that for a given hyper upper probability U_0^* there is a unique α such that $U_0^*(\psi(\phi - \alpha)) = 0$. Clearly one cannot expect this condition to hold when for instance $U_0^*(\psi) = 0$. Indeed, the random variable $\phi - \alpha$ is bounded (it takes values in $[-1, 1]$), and hence for any choice of $\alpha \in [0, 1]$,

$-\psi \leq \psi(\phi - \alpha) \leq \psi$, and

$0 = -U^*(\psi) \leq U^*(-\psi) \leq U^*(\psi(\phi - \alpha)) \leq U^*(\psi) = 0$.

This equality holds independently of α, and hence we are in the bad situation where any α might serve as an upper probability of $\phi|\psi$. We will see that such an inadmissible situation is avoided when $U_0^*(\neg\psi) < 1$, or equivalently, when the lower prevision $L_0^*(\psi)$ is strictly positive.

Lemma 3. *Suppose $L_0^*(\psi) > 0$. Then there is at most one α such that $U^*(\psi(\phi-\alpha)) = 0$.*

The argument used to prove the lemma shows that if $L_0^*(\psi) = L^*(\psi) > 0$, then the upper conditional probability $U_0^*(\phi|\psi)$, if it exists, can be uniquely recovered from the (unconditional) upper hyperstate U_0^*. In the standard case, such a conditional upper probability is shown to exist by a continuity argument, while it is not clear whether it exists in the non-standard case (we have shown uniqueness, not existence). However, for a given finite assessment we will prove that such a conditional upper probability exists, in a sense which will be made precise in Theorem 5 below. Before discussing it, we introduce completeness.

Definition 6. *An assessment of conditional upper probability is said to be* complete *if for any betting odd $\phi|\psi \mapsto \alpha$ on a conditional event $\phi|\psi$, it also contains a betting odd $\neg\psi_i \mapsto \beta_i$ on the negation of the conditioning event ψ.*

A complete assessment $\Lambda : \phi_i|\psi_i \mapsto \alpha_i$, $\neg\psi_i \mapsto \beta_i, i = 1, \ldots, n$ of conditional upper probability is said to be stably coherent *if there is a hyperassessment*

$$\Lambda' : \phi_i|\psi_i \mapsto \alpha_i',\ \neg\psi_i \mapsto \beta_i',\ i = 1, \ldots, n$$

which avoids inadmissible bets, differs from Λ by an infinitesimal and such that, for every i, $\beta_i' < 1$.

The requirement of a betting odd β on the negation of the conditioning event and not on the conditioning event itself may look strange. However, imposing a betting odd for the *upper* hyperprevision of the negation of ψ is the same as imposing the betting odd $1-\beta$ for the *lower* hyperprevision of ψ. So, in a complete assessment we really impose conditions on the lower prevision of the conditioning event.

Stable coherence is the consistency criterion for conditional hyper upper probabilities: when the probability assigned to the conditioning events is 0, it is quite possible that any assignment to conditional events avoids inadmissible bets. Now stable coherence says that inadmissible bets are also avoided if the bookmaker changes the lower probabilities of the conditioning events by an infinitesimal so that a positive number is assigned to them. Our main theorem shows that stable coherence corresponds to faithful upper hyperprevisions.

Theorem 5. *Let $\Lambda : \phi_i|\psi_i \mapsto \alpha_i, \neg\psi_i \mapsto \beta_i, i = 1, \ldots, n$ be an assessment of conditional upper probability. Then Λ is stably coherent if there is a* faithful *hyper upper prevision U^* s.t. for $i = 1, \ldots, n$, $U^*(\neg\psi_i)) - \beta_i$ is an infinitesimal, and $U^*(\psi_i(\phi_i - \alpha'_i)) = 0$ for some α'_i such that $\alpha_i - \alpha'_i$ is infinitesimal.*

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