

Aspherical Complex Surfaces, the Singer Conjecture, and Gromov's Inequality $\chi \geq |\sigma|$

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Abstract

We discuss the Singer conjecture and Gromov's inequality $\chi \geq |\sigma|$ for aspherical complex surfaces. We give a proof of the Singer conjecture for aspherical complex surface with residually finite fundamental group that does not rely on Gromov's Kähler groups theory. Without the residually finiteness assumption, we observe that this conjecture can be proven for all aspherical complex surfaces except possibly those in Class VII_0^+ (a positive answer to the global spherical shell conjecture would rule out the existence of aspherical surfaces in this class). We also sharpen Gromov's inequality for aspherical complex surfaces that are not in Class VII_0^+ . This is achieved by connecting the circle of ideas of the Singer conjecture with the study of Reid's conjecture.

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1 Introduction and Main Results

A significant part of modern Riemannian geometry deals with the interaction between curvature and topology of smooth manifolds. As beautifully recounted in Marcel Berger's panoramic book on Riemannian geometry (see in particular Chapter 12 in [Ber03]), Heinz Hopf was the first to investigate the connections between topology and curvature in a general and systematic way. Surprisingly, some of the questions that Hopf posed in the 1930's remain unanswered. A well-known example is the following problem on the sign of the Euler characteristic of aspherical manifolds.

Conjecture 1.1 (Hopf Conjecture). *If X is a closed aspherical manifold of dimension $2n$, then:*

$$(-1)^n \chi_{top}(X) \geq 0.$$

Thanks to the uniformization theorem for Riemann surfaces, Conjecture 1.1 is true when $n = 1$. On the other hand, this problem is still completely open when $n = 2$, but if X^4 is a closed, non-positively curved 4-manifold, then John Milnor proved that Conjecture 1.1 is indeed true in this case, see [Ber03, Chapter 12, Note 12.3.1.1]. There are several families of closed aspherical 4-manifolds which do not admit non-positively curved metrics. For example, non-flat nilmanifolds cannot admit such a metric by [Yau71, Corollary A] – of

course, these have Euler characteristic zero, so the Hopf conjecture holds for them.

During the 1970's, Isadore Singer suggested an approach to Conjecture 1.1 via the study of L^2 -harmonic forms on the topological universal cover of X . More precisely, he proposed the following.

Conjecture 1.2 (Singer Conjecture). *If X is a closed aspherical manifold of real dimension $2n$, then the L^2 -Betti numbers are:*

$$b_k^{(2)}(X; \tilde{X}) = \begin{cases} (-1)^n \chi_{top}(X) & \text{if } k = n \\ 0 & \text{if } k \neq n \end{cases}$$

where $\pi: \tilde{X} \rightarrow X$ is the topological universal cover of X .

Initially, Singer conjectured that $b_k^{(2)}(X; \tilde{X})$ is the only potentially non-zero L^2 -Betti number, but thanks to Atiyah's L^2 -index theorem [Ati76], its value is determined. Since $b_k^{(2)}(X; \tilde{X}) \geq 0$, an affirmative solution to Conjecture 1.2 would also settle Conjecture 1.1. For more details on this circle of ideas, we refer to Shing-Tung Yau's influential list of main open problems in geometry [SY94, Section VII, Problem 10]. We also refer to Wolfgang Lück's book [Lüc02] for the definition of L^2 -Betti numbers and for a comprehensive account on the history of the Singer conjecture. Interestingly, Conjecture 1.2 is not known to be true for $n = 2$ even under the assumption that X^4 is non-positively curved.

As observed by Mikhael Gromov in [Gro93, Section 8], Conjecture 1.2 implies an *effective* version of Conjecture 1.1 in dimension four. More precisely, one can state the following intriguing conjecture regarding the geography of aspherical 4-manifolds.

Conjecture 1.3 (Gromov's Inequality). *If X is a closed, oriented, aspherical 4-manifold, then:*

$$\chi_{top}(X) \geq |\sigma(X)|,$$

where $\sigma(X)$ is the signature of X .

In this paper, we study Conjectures 1.1, 1.2, and 1.3 on closed, aspherical 4-manifolds that admit a complex structure. Our knowledge of compact complex surfaces via the Kodaira-Enriques classification is a powerful tool in this case. For example in [JK93, Theorem 2], Johnson-Kotschick show that any complex surface X satisfies the inequality $\chi_{top} \geq |\sigma|$ unless X is a ruled surface over a curve of genus $g \geq 2$. Since ruled surfaces are *not* aspherical, we therefore have that Conjecture 1.3 (and then also Conjecture 1.1) is true for aspherical complex surfaces. With that said, we do believe that a deeper study of Conjecture 1.3 is warranted even for complex surfaces. First, Conjecture 1.3 is quite crude when compared with other geometric inequalities constraining the geography of vast classes of 4-manifolds. For example, the equality cases in the Hitchin-Thorpe

inequality for Einstein 4-manifolds and the Bogomolov-Miyaoka-Yau inequality for minimal surfaces of general type are neatly characterized. On the other hand, there is no (conjectural) characterization of the equality case in Conjecture 1.3. Moreover, one may wonder if it is useful to sharpen Gromov's inequality to a tighter constraint on the geography of aspherical 4-manifolds that do *not* satisfy $\chi_{top} = \sigma = 0$. We do have a *quite* satisfactory answer to all such questions when X is an aspherical complex surface.

Theorem 1.4. *Let X be a closed, aspherical, complex surface. We have the following possibilities for its Euler characteristic $\chi_{top}(X)$ and signature $\sigma(X)$:*

- $\chi_{top}(X) = -\sigma(X) > 0$ in which case X is a minimal Class VII_0^+ surface violating the global spherical shell conjecture;
- $\chi_{top}(X) \geq \frac{9}{5}|\sigma(X)|$ and $\chi_{top}(X) > 0$ in which case X is of general type;
- $\chi_{top}(X) = \sigma(X) = 0$ in all other cases.

In particular, we see that if the global spherical shell conjecture is true, then Gromov's inequality is always strict for closed, aspherical, complex surfaces unless the Euler characteristic and signature are both zero. Furthermore, we obtain a factor $9/5 > 1$ in front of the absolute value of the signature in all of the remaining cases.

Next, we address the big elephant in the room: is Conjecture 1.2 true for closed, aspherical, complex surfaces? We observe that Gromov's characterization of closed Kähler manifolds with non-vanishing first L^2 -Betti number, when combined with the Kodaira-Enriques classification, suffices to show this conjecture holds true for all closed, aspherical, complex surfaces that are not in Class VII_0^+ , see Theorem 4.2. Frustratingly enough, the validity of Conjecture 1.2 also stumbles upon the existence of aspherical surfaces in Class VII_0^+ . We conclude by providing a proof of the following.

Theorem 1.5. *Let X be a closed, aspherical complex surface with residually finite fundamental group, and let $\tilde{X}$ be the topological universal cover. The L^2 -Betti numbers are:*

$$b_k^{(2)}(X; \tilde{X}) = \begin{cases} \chi_{top}(X) & \text{if } k = 2 \\ 0 & \text{if } k \neq 2 \end{cases}$$

Our proof of Theorem 1.5 does not rely upon Gromov's theory of Kähler groups. It combines the study of the Albanese map, Lück's approximation theorem and the Kodaira-Enriques classification.

In real dimension greater than or equal to four, there is a plethora of examples of closed aspherical manifolds whose fundamental group is not residually finite. Such examples can be constructed with the so-called Davis reflection trick

[Dav83], see for example [Mes90]. It seems to be currently unknown whether or not there are examples of aspherical smooth projective varieties with non-residually finite π_1 . Indeed, the examples of Toledo [Tol93] and Catanese-Kollár [CK90] of smooth projective varieties with non-residually finite π_1 appear not to be aspherical. It is currently unknown whether the non-positively curved smooth minimal toroidal compactifications of ball quotients identified in [DiC12, Theorem A], or the negatively curved branched covers constructed in [ST22, Theorem 1.5] have residually finite π_1 .

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2 Aspherical Complex Surfaces

In this section, we give a brief overview of those compact complex surfaces which are aspherical. These surfaces have contractible universal cover or equivalently, π_k vanishes for $k > 1$. By [ADC23, Lemma 2], such surfaces are minimal. We will work through the Kodaira-Enriques classification by Kodaira dimension.

Kodaira dimension $-\infty$: In the Kähler case, such surfaces are rational or ruled. The former consists of $\mathbb{CP}^2$ and Hirzebruch surfaces $\Sigma_n = \mathbb{P}_{\mathbb{CP}^1}(\mathcal{O} \oplus \mathcal{O}(n))$. These are all simply connected, so they are their own universal covers. As they are not contractible, rational surfaces are not aspherical. Ruled surfaces are holomorphic fiber bundles with fiber $\mathbb{CP}^1$ and structure group $PGL(2, \mathbb{C})$ over a smooth connected curve C of positive genus. Every such surface is the projectivisation of a rank two holomorphic vector bundle over C . From the long exact sequence in homotopy, it follows that ruled surfaces have non-zero π_2 . In fact, if $\tilde{C} \rightarrow C$ denotes the universal covering of C , pulling back the $\mathbb{CP}^1$ -bundle by this map exhibits ruled surfaces have universal cover $\mathbb{CP}^1 \times \tilde{C}$ – since $\tilde{C}$ is Stein, we have $H^1(\tilde{C}, \mathcal{PGL}(2, \mathbb{C})) = 0$ and hence the $\mathbb{CP}^1$ bundle over $\tilde{C}$ is trivial.

A non-Kähler surface with Kodaira dimension $-\infty$ is called a *Class VII surface*. A minimal such surface is called a *Class VII_0 surface*, and if furthermore

the second Betti number is positive, then it is called a *Class VII₀⁺ surface*. A Class VII₀ surface with second Betti number zero is biholomorphic to a Hopf surface or an Inoue-Bombieri surface, see [Bog76], [Bog82], [LYZ94], and [Tel94]. Hopf surfaces have universal cover $\mathbb{C}^2 \setminus \{0\}$ which is not contractible, while Inoue-Bombieri surfaces have universal cover $\mathbb{C} \times \mathbb{H}$ which is contractible, so they are aspherical.

A *spherical shell* in a complex surface is an open subset biholomorphic to a neighbourhood of S^3 in $\mathbb{C}^2 \setminus \{0\}$. If the complement is connected, then it is called a *global spherical shell*. A surface which admits a global spherical shell is a deformation of a primary Hopf surface¹ blownup at finitely many points [Kat78, Theorem 1] – note that such surfaces are not aspherical. The global spherical shell conjecture asserts that all Class VII₀⁺ surfaces contains a global spherical shell. The conjecture remains open with some progress for small values of b_2 , see [Tel05], [Tel10], [Tel18]. It is not yet known if there exists an aspherical Class VII₀⁺ surface (it would necessarily violate the global spherical shell conjecture).

Since class VII surfaces have first Betti number 1, such surfaces have $\chi_{top}(X) = b_2(X)$. Furthermore, as they are non-Kähler, we see that $b^+(X) = 2h^{2,0}(X) = 0$ and hence $\sigma(X) = -b^-(X) = -b_2(X)$. So Inoue-Bombieri surfaces have $\chi_{top}(X) = \sigma(X) = 0$, while aspherical class VII₀⁺ surfaces have $\chi_{top}(X) = -\sigma(X) = b_2(X) > 0$.

Kodaira dimension 0: In the Kähler case, there are two families: tori and their quotients (bi-elliptic surfaces), and K3 surfaces and their quotients (Enriques surfaces). The former have universal cover $\mathbb{C}^2$ and are therefore aspherical, while the latter have K3 surfaces as their universal cover and hence are not aspherical.

In the non-Kähler realm, such surfaces are primary Kodaira surfaces and their quotients (secondary Kodaira surfaces). Primary Kodaira surfaces are holomorphic principal elliptic curve bundles over a smooth connected genus one curve. It follows from the long exact sequence in homotopy that $\pi_k = 0$ for $k > 1$. Just as in the case of ruled surfaces, we can also identify the universal cover of Kodaira surfaces as $\mathbb{C}^2$ by pulling back such a bundle by the universal covering of the base. A description of primary Kodaira surfaces as quotients of $\mathbb{C}^2$ by a group of affine transformations was given by Suwa [Suw75, Theorem 2].

Kodaira dimension 1: A compact surface X is called an *elliptic surface* if there is a smooth connected curve C and a holomorphic map $\pi : X \rightarrow C$ such that the generic fiber is a smooth genus one curve; the map π is called an *elliptic fibration*. We call an elliptic surface X *relatively minimal* if there are no -1 curves in the fibers of π – every elliptic surface is an iterated blowup of a relatively minimal elliptic surface. Every surface of Kodaira dimension 1 is elliptic (see [Wal86, Lemma 7.2(a)]), but there are also elliptic surfaces

¹A Hopf surface X is called *primary* if $\pi_1(X) \cong \mathbb{Z}$. Such surfaces are diffeomorphic to $S^1 \times S^3$.

of Kodaira dimension $-\infty$ and Kodaira dimension 0. An elliptic surface with Kodaira dimension 1 is called a *properly elliptic surface*.

The non-generic fibers of a relatively minimal elliptic fibration $\pi : X \rightarrow C$, called exceptional fibers, were classified by Kodaira, see [Kod63, Theorem 6.2]. Aside from multiples of a smooth genus one curve (known as a multiple fibers with smooth reduction), every other possibility is a configuration of (possibly singular) rational curves. The elliptic fibration induces an orbifold structure on C by declaring images of multiple fibers as cone points whose order is the multiplicity of the fiber. We denote the orbifold Euler characteristic and orbifold fundamental group of C by $\chi^{\text{orb}}(C)$ and $\pi_1^{\text{orb}}(C)$ respectively.

Proposition 2.1. *An elliptic surface $X \rightarrow C$ is aspherical if and only if it is relatively minimal with no exceptional fibers other than multiple fibers with smooth reduction, and X has Kodaira dimension 0 or 1.*

Proof. If X is aspherical, then it is minimal (and hence relatively minimal) by [ADC23, Lemma 2]. Furthermore, if X is Kähler, then X contains no rational curves, so the only exceptional fibers must be multiple fibers with smooth reduction. In the non-Kähler case, the same is true by [Wal86, Lemma 7.2(b)]. By [FM94, Lemma I.3.18 (ii)], we have $\chi(\mathcal{O}_X) = 0$. If $\chi^{\text{orb}}(C) > 0 = \chi(\mathcal{O}_X)$, then $\kappa(X) = -\infty$ by [Wal86, Lemma 7.1]. An aspherical surface with Kodaira dimension $-\infty$ is either Inoue-Bombieri or a Class VII₀⁺ surface. The former can't be elliptic as they contain no complex curves, and the latter can't be elliptic as they satisfy $c_1^2 < 0$. Therefore $\chi^{\text{orb}}(C) \leq 0$ and hence X has Kodaira dimension 0 or 1 by [Wal86, Lemma 7.1].

Conversely, if $X \rightarrow C$ is relatively minimal with no exceptional fibers other than multiple fibers with smooth reduction, and X has Kodaira dimension 0 or 1, then $\chi^{\text{orb}}(C) \leq \chi(\mathcal{O}_X) = 0$. Therefore C is a good orbifold, i.e. there is a finite orbifold covering $C' \rightarrow C$ where C' is a manifold. Pulling back $X \rightarrow C$ by this map induces an elliptic fibration $X' \rightarrow C'$ with no multiple fibers such that X' is a finite unramified cover of X , see [BHPV04, Proposition III.9.1]. Since $\chi(\mathcal{O}_{X'}) = 0$, all the fibres of $X' \rightarrow C'$ are isomorphic by [BHPV04, Proposition V.12.2] and the remark which precedes it, and hence $X' \rightarrow C'$ is locally trivial by [FG65]. As the orbifold Euler characteristic is multiplicative under orbifold coverings, we have $\chi^{\text{orb}}(C') \leq 0$ and hence C' has positive genus. Applying the long exact sequence of homotopy groups, we see that X' is aspherical. $\square$

Remark 2.2. In the course of the proof, we showed that an aspherical elliptic surface X has a finite cover X' which is a holomorphic fiber bundle over a Riemann surface of positive genus, with an elliptic curve fiber. The structure group of this bundle is the automorphism group of the elliptic curve. Since translations form a finite index subgroup, there is a finite cover $C'' \rightarrow C'$ such that the pullback of $X' \rightarrow C'$ gives a bundle $X'' \rightarrow C''$ whose structure group reduces to the group of translations. That is, the bundle $X'' \rightarrow C''$ is a holomorphic principal elliptic bundle. Just as in the discussion of ruled surfaces, it follows

that the universal cover of X'' , and hence X , is biholomorphic to $\mathbb{C} \times \mathbb{C}$ if X has Kodaira dimension 0, or $\mathbb{C} \times \mathbb{H}$ if X has Kodaira dimension 1.

Corollary 2.3. *Aspherical elliptic surfaces contain no rational curves.*

Proof. If X is an aspherical elliptic curve, then any map $\mathbb{CP}^1 \rightarrow X$ lifts to the universal cover since $\mathbb{CP}^1$ is simply connected. As described above, the universal cover is an open subset of $\mathbb{C}^2$, so the lift must be constant. $\square$

There are examples of elliptic surfaces which contain finitely many rational curves, and examples with infinitely many, see [BFO23, section 5].

Note, there are elliptic surfaces with Kodaira dimension $-\infty$, but none of them are aspherical (they are either rational, ruled, or Hopf). As for Kodaira dimension 2, none of them are elliptic.

There are non-aspherical elliptic surfaces in Kodaira dimensions 0 and 1. By combining Proposition 2.1 with [Wal86, Lemma 7.2(b)], such surfaces must be Kähler. In Kodaira dimension 0, such surfaces are the elliptic K3 surfaces and all Enriques surfaces, while for Kodaira dimension 1, homotopy K3 surfaces and Dolgachev surfaces provide examples. One can construct many more Kodaira dimension 1 examples as follows (the stated examples arise this way). Choose an elliptic surface with an exceptional fiber which is not a multiple fiber with smooth reduction (equivalently, has positive Euler characteristic). Applying logarithmic transformations decreases the value of $\chi^{\text{orb}}(C)$, so by [Wal86, Lemma 7.1], the result will eventually have Kodaira dimension 1.

Kodaira dimension 2: Aspherical surfaces with Kodaira dimension 2 exist, but as with most problems regarding general type surfaces, we have nothing even close to a classification. Indeed, the list of known aspherical surfaces of general type is not particularly rich even if there are reasons to expect such surfaces exist in great profusion. The list includes ball quotients (e.g., fake projective planes), surfaces isogeneous to product of curves, Kodaira fibrations, Mostow-Siu surfaces, and certain branched covers of ball quotients due to Domingo-Stover [ST22, Theorem 1.5]. We refer to the paper of Bauer-Catanese [BC18] for more details. The list of aspherical surfaces of general type also includes the vast majority of smooth minimal toroidal compactifications of ball quotients, see [DiC12, Theorem A]. In all of these examples, when the signature is explicitly computed one has that $\sigma \geq 0$. It seems currently unknown whether or not an aspherical complex surface of general type can have negative signature. In conclusion, we can summarize this discussion into a table.

$\kappa(X)$	$b_1(X)$	List	$\chi_{top}(X)$	$\sigma(X)$
$-\infty$	odd	Inoue-Bombieri Potential Class VII ₀ ⁺ examples	0 Positive	0 Negative
0	even odd	Tori and quotients All	0 0	0 0
1	even odd	Some All	0 0	0 0
2	even	Some	Positive	?

3 Singer Conjecture for Surfaces with Residually Finite Fundamental Group

In this section, we show that the Singer conjecture holds true for closed aspherical complex surfaces with residually finite fundamental group. The proof we present here does not rely on Gromov's characterization of Kähler manifolds with non-vanishing first L^2 -Betti number [Gro89]. We rely upon the study of the Albanese map given in [DL23a] and [DL23b], and on Lück's approximation theorem which we now briefly recall.

Let X be a manifold with $\Gamma \stackrel{\text{def}}{=} \pi_1(X)$ residually finite. We consider a sequence of nested, normal, finite index subgroups $\{\Gamma_i\}$ of Γ such that $\cap_i \Gamma_i$ is the identity element. This sequence is usually called a *cofinal filtration* of Γ . Let $\pi_i: X_i \rightarrow X$ be the finite regular cover of X associated to Γ_i . Lück's approximation theorem [Lüc94] ensures that

$$\lim_{i \rightarrow \infty} \frac{b_k(X_i)}{\deg \pi_i} = b_k^{(2)}(X; \tilde{X}), \quad (1)$$

where $b_k(X_i)$ denotes the k^{th} Betti number of X_i , and $b_k^{(2)}(X; \tilde{X})$ is the L^2 -Betti number of X computed with respect to the universal cover $\tilde{X}$. Thus, the limit in (1) always exists and it is *independent* of the cofinal filtration. We refer to the ratio $b_k(X_i)/\deg \pi_i$ as the *normalized k^{th} -Betti number* of the cover $\pi_i: X_i \rightarrow X$. Conjecture 1.2 is then equivalent to the sub-degree growth of Betti numbers along a tower of covers associated to a cofinal filtration.

We start with the following proposition that is not limited to complex dimension two.

Proposition 3.1. *Let X be an aspherical smooth projective variety. Assume that $\pi_1(X)$ is residually finite and there exists a cofinal tower of coverings $\pi_i: X_i \rightarrow X$ such that the images $a_{X_i}(X_i)$ of the Albanese maps are either points or curves in $\text{Alb}(X_i)$. We then have*

$$\lim_{i \rightarrow \infty} \frac{b_1(X_i)}{\deg \pi_i} = 0.$$

Proof. Clearly, we just need to study the case where $b_1(X_i) \neq 0$ from some point on in the cofinal tower. Recall that if $a_{X_i}(X_i)$ is a curve, it must be smooth, connected, and its genus equals $\frac{1}{2}b_1(X_i)$. For simplicity sake, from now on we assume that $a_{X_i}(X_i)$ is a curve for any $i \geq 0$ in the cofinal tower. Moreover, we set $S := a_X(X) = a_{X_0}(X_0)$ and $S_i := a_{X_i}(X_i)$. Due to the universal property of the Albanese variety, there is a map $a_{\pi_i} : S_i \rightarrow S$ such that the following diagram commutes:

$$\begin{array}{ccc} X_i & \xrightarrow{a_{X_i}} & S_i \\ \pi_i \downarrow & & \downarrow a_{\pi_i} \\ X & \xrightarrow{a_X} & S. \end{array}$$

Since π_i is unramified, for any $i \geq 1$, the branching locus B_i of a_{π_i} is contained in the (finite) set of critical values of a_X . In particular, there exists a positive constant $C > 0$ such that for all i we have $\#(B_i) \leq C$. Thus the degree of the ramification divisor R_i of a_{π_i} is bounded by $\deg R_i \leq C \cdot \deg(a_{\pi_i})$ for any i . By using the Riemann–Hurwitz formula, we have

$$b_1(S_i) = 2 \deg(a_{\pi_i}) \cdot \chi(\omega_S) + \deg R_i + 2 \leq \deg(a_{\pi_i}) \cdot (2\chi(\omega_S) + C) + 2.$$

Since $b_1(X_i) = b_1(S_i)$, dividing by $\deg(a_{\pi_i}) > 0$ yields

$$\frac{b_1(X_i)}{\deg(a_{\pi_i})} \leq 2\chi(\omega_S) + C + \frac{2}{\deg a_{\pi_i}} \leq 2\chi(\omega_S) + C + 2. \quad (2)$$

Next, let k_i be the minimal degree of the restriction of π_i to a general fiber of a_{X_i} . Note that $\{k_i\}_{i \in \mathbb{N}}$ is a sequence of non-decreasing positive integers and

$$\deg \pi_i \geq k_i \cdot \deg(a_{\pi_i}) \quad \forall i. \quad (3)$$

We claim that

$$\lim_{i \rightarrow \infty} k_i = \infty. \quad (4)$$

By contradiction, as the tower of coverings $\pi_i : X_i \rightarrow X$ is cofinal, the X_i 's converge to the topological universal cover $\tilde{X}$ (cf. [DD19, §3]). Now equip the covers X_i with the metrics induced by a fixed Kähler metric on the base, given by an ample line bundle L on X , via pullback. Moreover, let G_i be a general fiber of a_{X_i} such that $k_i = \deg(\pi_i|_{G_i})$. In this way the volume of G_i is computed as

$$(\pi_i^* L \cdot G_i) = (L \cdot \pi_{i*} G_i) = k_i \cdot (L \cdot F) \quad \forall i.$$

If the k_i 's were bounded, there would exist an integer $N > 0$ such that $(\pi_i^* L \cdot G_i) < N$ for all i ; but this contradicts [DD19, Proposition 3.3] (note that if X

is aspherical, then $\pi_1(X)$ is large, see [LMW21, Proposition 6.7] and [Kol93, Proposition 2.12.1]). In conclusion, by (2), (3) and (4) it follows that

$$0 \leq \frac{b_1(X_i)}{\deg \pi_i} \leq \frac{b_1(X_i)}{k_i \deg(a_{\pi_i})} \leq \frac{2\chi(\omega_S) + C + 2}{k_i}.$$

Taking the limit as $i \rightarrow \infty$ concludes the proof. $\square$

We can now give a proof of Theorem 1.5 stated in the introduction.

Proof of Theorem 1.5. Since $b_0^{(2)}(X; \tilde{X}) = b_4^{(2)}(X; \tilde{X}) = 0$ and the alternating sum of L^2 -Betti numbers is equal to $\chi_{top}(X)$, it is enough to show that $b_1^{(2)}(X; \tilde{X}) = 0$ by Poincaré duality. Moreover, we can assume that X is minimal by the asphericity assumption (see [ADC23, Lemma 2]). We divide the proof into several steps according to the Kodaira dimension (using the results of Section 2).

To begin with, suppose $\text{Kod}(X) = -\infty$. In the non-Kähler case, X is of class VII. All such surfaces have $b_1(X) = 1$. Moreover any finite covering of a surface of class VII is again of class VII (see for example [FM94, Proposition II.1.21] or [Dür05, Lemma 5.1]). Thus, the vanishing of $b_1^{(2)}(X; \tilde{X})$ follows immediately from Lück's approximation [Lüc94]. In the Kähler case, no surface with $\text{Kod}(X) = -\infty$ is aspherical.

As discussed in Section 2, aspherical complex surfaces of Kodaira dimension 0 are finitely covered by either a torus or a primary Kodaira surface. It follows that the fundamental groups of such surfaces contain a normal subgroup isomorphic to $\mathbb{Z}^2$. Hence, the L^2 -Betti numbers vanish by a classical result of Cheeger-Gromov [CG86, Corollary 0.6], see also [Lüc02, Theorem 1.44].

The aspherical complex surfaces of Kodaira dimension 1 are the properly elliptic surfaces with no exceptional fibers other than multiple fibers with smooth reduction. By Remark 2.2, such surfaces are finitely covered by holomorphic elliptic curve bundle, and hence their fundamental groups also contain a normal subgroup isomorphic to $\mathbb{Z}^2$. Again, this implies that the L^2 -Betti numbers vanish.

For surfaces of general type, we first recall that they have to be projective (see [BHPV04, p.189]), and we can use the Albanese map if the surface is irregular (i.e. $b_1(X) \neq 0$). Then, we proceed by using Lück's approximation [Lüc94] on a cofinal tower. If none of the covers in the tower is irregular, then the vanishing of $b_1^{(2)}(X; \tilde{X})$ is immediate and the result follows. In the other cases, we use either Proposition 3.1, or Theorem [DL23b, Theorem 1.3] specialized to complex dimension two. Recall that in complex dimension two, a_X is semismall if and only if it is generically finite onto its image. $\square$

4 On the Singer Conjecture for Complex Surfaces

In [Gro89], Gromov shows that if (X, ω) is a closed, Kähler manifold with $b_1^{(2)}(X; \mathbb{C}) \neq 0$, then $\pi_1(X)$ is *commensurable* to the fundamental group of a compact surface of genus $g \geq 2$. For more details about this important result, we refer to [ABR92] and the nice book [ABCKT96, Chapter 4] on Kähler groups.

Gromov's theorem implies that no aspherical Kähler surface (X^2, ω) can have non-vanishing $b_1^{(2)}$. Indeed, if this was the case then a finite cover of X , say X' , would have the same fundamental group as a hyperbolic Riemann surface, say C . Since both X' and C are aspherical with isomorphic fundamental groups, they are homotopy equivalent [Lüc12, Theorem 2.1], which is clearly not possible as $H_4(X'; \mathbb{Z}) \neq H_4(C; \mathbb{Z}) = 0$. Let's summarize this discussion into a theorem.

Theorem 4.1 (Gromov). *The Singer conjecture is true for closed, aspherical, Kähler surfaces.*

We can now combine some parts of the proof of Theorem 1.5 with Theorem 4.1 to prove the following.

Theorem 4.2. *The Singer conjecture is true for closed, aspherical, complex surfaces that are not in Class VII_0^+ .*

Proof. By Theorem 4.1, we only need to check the non-Kähler case. First note that all Inoue-Bombieri surfaces have solvable fundamental group – in fact, they are all solvmanifolds, see [Has05, Theorem 1]. Since solvable groups are amenable, it follows that the L^2 -Betti numbers vanish [Lüc02, Corollary 6.75]. This leaves only the minimal non-Kähler surfaces of Kodaira dimension 0 and 1, all of which are elliptic. As in the proof of Theorem 1.5, it follows from Remark 2.2 that the fundamental group of such a surface contains a normal subgroup isomorphic to $\mathbb{Z}^2$. In this case, the L^2 -Betti numbers vanish by a classical result of Cheeger-Gromov [CG86, Corollary 0.6], see also [Lüc02, Theorem 1.44]. $\square$

Note, if one could show that the fundamental group of a Class VII_0^+ surface was residually finite, then we could apply the argument in the proof of Theorem 1.5 to extend Theorem 4.2 to all complex surfaces.

The big elephant in the room. Aspherical Class VII_0^+ surfaces conjecturally do not exist. That said, their cohomological structure seems somewhat simple. This motivates the following.

Question 4.3. Assume there are aspherical Class VII_0^+ surfaces. Can we prove the Singer conjecture holds for them?

5 Reid's Conjecture and Gromov's Inequality

In this section, we prove Theorem 1.4.

By the discussion in section 2, the only aspherical surfaces with Kodaira dimension $-\infty$ are Inoue-Bombieri surfaces and potential aspherical class VII_0^+ surfaces. As we have seen, the former satisfy $\chi_{\text{top}}(X) = \sigma(X) = 0$, while the latter satisfy $\chi_{\text{top}}(X) = -\sigma(X) = b_2(X) > 0$. On the other hand, the aspherical surfaces of Kodaira dimension 0 or 1 all have $\chi_{\text{top}}(X) = \sigma(X) = 0$. This only leaves surfaces of general type.

Note, the Bogomolov-Miyaoka-Yau inequality states that for a general type surface X we have $\chi_{\text{top}}(X) \geq 3\sigma(X)$. However, it is not true that $\chi_{\text{top}}(X) \geq 3|\sigma(X)|$ for every such X . For example, let X_d be a smooth degree d hypersurface of $\mathbb{CP}^3$. Note that X_d is a surface of general type for $d \geq 5$, and a simple characteristic class argument shows that $\chi_{\text{top}}(X_d) = d^3 - 4d^2 + 6d$ and $\sigma(X_d) = -\frac{1}{3}(d-2)d(d+2)$. So, for example, we have $\chi_{\text{top}}(X_5) = 55$ and $\sigma(X_5) = -35$ so $3|\sigma(X_5)| = 105 > 55 = \chi_{\text{top}}(X_5)$. In fact, the proposed inequality is violated by X_d for all $d \geq 5$ (also $d = 3, 4$, but these are not surfaces of general type). Of course, none of these examples are aspherical since they are simply connected by the Lefschetz Hyperplane Theorem.

If the signature is non-negative, then of course $\chi_{\text{top}}(X) \geq 3\sigma(X)$ is equivalent to $\chi_{\text{top}}(X) \geq 3|\sigma(X)|$. The discrepancy occurs, as in the examples above, when the signature is negative.

Question 5.1. Does there exist an aspherical complex surface with negative signature? (Either has to be a counterexample to the global spherical shell conjecture or a surface of general type).

This question is yet to be answered, so we continue on our quest to find an inequality relating $\chi_{\text{top}}(X)$ and $|\sigma(X)|$. To do so, we need to recall the circle of ideas related to Reid's conjecture, see for example [BHPV04, Chapter VII]. We also refer to the beautiful survey [MLP12] of Mendes Lopes-Pardini on the geography of irregular surfaces for much more on this fascinating topic.

Conjecture 5.2 (Reid). *Let X be a minimal surface of general type such that $K_X^2 < 4\chi_{\text{hol}}$, where χ_{hol} is the holomorphic Euler characteristic. Then $\pi_1(X)$ is either finite, or it is commensurable with the fundamental group of a curve.*

As shown by Horikawa [Hor76] and Reid [Rei79], Conjecture 5.2 holds true under the stronger assumption $K_X^2 < 3\chi_{\text{hol}}$. We therefore can observe the following.

Proposition 5.3. *Let X be an aspherical surface of general type. We then have $K_X^2 \geq 3\chi_{\text{hol}}$.*

Proof. By [ADC23, Lemma 2], X must be minimal. Now an aspherical surface must have infinite π_1 , see for example [Lüc12, Lemma 4.1]. As was observed at the beginning of section 4, $\pi_1(X)$ cannot be commensurable with the fundamental group of a curve. Since Conjecture 5.2 holds true for minimal surfaces of general type satisfying $K_X^2 < 3\chi_{hol}$, we conclude that

$$K_X^2 \geq 3\chi_{hol},$$

for any aspherical surface of general type. $\square$

We can now prove the desired inequality relating $\chi_{top}(X)$ and $|\sigma(X)|$ for aspherical general type surfaces.

Lemma 5.4. *Let X be an aspherical surface of general type. We then have:*

$$\chi_{top}(X) \geq \frac{9}{5}|\sigma|.$$

Proof. Recall that

$$K_X^2 = 2\chi_{top}(X) + 3\sigma(X), \quad \chi_{hol}(X) = \frac{\chi_{top}(X) + \sigma(X)}{4}.$$

By using Proposition 5.3, we obtain

$$\chi_{top}(X) \geq \frac{9}{5}(-\sigma(X)),$$

which, combined with the Bogomolov-Miyaoka-Yau inequality, gives $\chi_{top}(X) \geq \frac{9}{5}|\sigma(X)|$. $\square$

This completes the proof of Theorem 1.4.

Note that for a minimal surface of general type X , we have $c_1^2(X) > 0$ from which it follows that $\chi_{top}(X) > \frac{3}{2}(-\sigma(X))$. If the signature is negative, the inequality $\chi_{top}(X) \geq \frac{9}{5}(-\sigma(X))$ is stronger. For example, if $\sigma(X) = -3$, the former inequality implies $\chi_{top}(X) \geq 5$ while the latter implies $\chi_{top}(X) \geq 6$.

Remark 5.5. Reid's conjecture implies the slightly better bound $\chi_{top}(X) \geq 2|\sigma(X)|$.

Note that we actually have $\chi_{top}(X) \geq \frac{9}{5}|\sigma(X)|$ for all aspherical complex surfaces, except any potential Class VII₀⁺ examples (in all other cases, $\chi_{top}(X) = \sigma(X) = 0$).

It is tantalizing to ask what is the optimal constant $a > 0$, such that $\chi_{top}(X) \geq a|\sigma(X)|$ for all aspherical surfaces of general type. As remarked above, we currently seem not to know any example of aspherical surfaces of

general type with negative signature. If this is not an accident simply due to our lack of good examples, but a true fact of nature, by using the Bogomolov-Miyaoka-Yau inequality we would have

$$\chi_{top}(X) \geq 3\sigma(X) \geq 0$$

where the first inequality is saturated if and only if X is a ball quotient. Notice that given a minimal surface of general type X with $\sigma(X) > 0$, the reversed oriented 4-manifold $\overline{X}$ can never admit a complex structure compatible with the orientation. This follows from Seiberg-Witten theory, see Theorem 2 in [Kot97]. Thus, in order to give a positive answer to Question 5.1, a genuinely new example of a surface of general type would need to be constructed, or alternatively one would need to provide an aspherical counterexample to the global spherical shell conjecture!

References

- [ABCKT96] J. Amorós, M. Burger, K. Corlette, D. Kotschick, D. Toledo. Fundamental groups of compact Kähler manifolds. Math. Surveys Monogr., 44 *American Mathematical Society, Providence, RI*, 1996, xii+140 pp.
- [ABR92] Arapura, D. and Bressler, P. and Ramachandran, M. *On the fundamental group of a compact Kähler manifold* **68** (1992), no. 3, 477-488
- [ADC23] M. Albanese, L. F. Di Cerbo. Aspherical 4-manifolds, complex structures, and Einstein metrics. *arXiv:2303.13219v3*.
- [Ati76] M. F. Atiyah. Elliptic operators, discrete groups and von Neumann algebras. *Colloque “Analyse et Topologie” en l’Honneur de Henri Cartan (Orsay, 1974)*, pp. 43-72, Astérisque, No. 32-32, Soc. Math. France, Paris, 1976.
- [BFO23] G. Barbaro, F. Fagiolo, A. Ortiz. A Survey on Rational Curves on Complex Surfaces. *arXiv:2209.04229v2*.
- [BHPV04] W. P. Barth, K. Hulek, C. A. M. Peters, A. Van de Ven. Compact complex surfaces, 2nd ed., *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*, vol. 4, Springer-Verlag, Berlin, 2004.
- [BC18] I. Bauer, F. Catanese. On rigid complex surfaces and manifolds *Adv. Math.* **333**, (2018), 620-669.

- [Ber03] M. Berger. A panoramic view of Riemannian geometry. *Springer-Verlag, Berlin*, 2003. xxiv+824 pp.
- [Bog76] F. A. Bogomolov. Classification of surfaces of class VII_0 with $b_2 = 0$. *Izv. Akad. Nauk SSSR Ser. Mat.* **40** (1976), no.2, 273–288, 469.
- [Bog82] F. A. Bogomolov. Surfaces of class VII_0 and affine geometry. *Izv. Akad. Nauk SSSR Ser. Mat.* **46** (1982), no. 4, 710–761, 896.
- [CK90] F. Catanese, J. Kollár. Trento Examples. Proceedings, Lecture Notes in Mathematics 1515. *Springer* 1990, pp. 134-139.
- [CG86] J. Cheeger, M. Gromov. L^2 -cohomology and group cohomology. *Topology* (**25**) (1986), 198-215.
- [Dav83] M. W. Davis. Groups generated by reflections and aspherical manifolds not covered by Euclidean space. *Ann. of Math.* **117** (1983), 293-324.
- [DiC12] L. F. Di Cerbo. Finite-volume complex-hyperbolic surfaces, their toroidal compactifications, and geometric applications. *Pacific J. Math.* **255** (2012), no. 2, 305-315.
- [DD19] G. Di Cerbo, L. F. Di Cerbo. On Seshadri constants of varieties with large fundamental group. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (**19**) (2019), no. 1, 335-344.
- [DL23a] L. F. Di Cerbo, L. Lombardi. L^2 -Betti Numbers and Convergence of Normalized Hodge Numbers via the Weak Generic Nakano Vanishing Theorem. Online first in *Ann. Inst. Fourier (Grenoble)* (2023).
- [DL23b] L. F. Di Cerbo, L. Lombardi. Singer Conjecture for Varieties with Semismall Albanese Map and Residually Finite Fundamental Group. *arXiv:2303.06709v1*.
- [Dür05] M. Dürr. Fundamental groups of elliptic fibrations and the invariance of the plurigenera for surfaces with odd first Betti numbers. *Expo. Math.* **23** (2005), 1-22.
- [FG65] W. Fischer, F. Grauert. Lokal-triviale Familien kompakter komplexer Mannigfaltigkeiten. *Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II* **1965** (1965), 89–94.
- [FM94] R. Friedman and J. Morgan. Smooth four-manifolds and complex surfaces. *Ergebnisse der Mathematik und ihrer Grenzgebiete* **27**, Springer-Verlag (1994)
- [Gro89] M. Gromov. Sur le groupe fondamental d’une variété Kählérienne. *C. R. Acad. Sci. Paris Sér. I Math.* **308** (1989), no. 3, 67-70

- [Gro91] M. Gromov. Kähler hyperbolicity and L_2 -Hodge theory. *J. Differential Geom.* **33** (1991), no. 1, 263-292.
- [Gro93] M. Gromov. *Geometric group theory. Volume 2: Asymptotic invariants of infinite groups. Proceedings of the symposium held at the Sussex University, Brighton, July 14-19, 1991*, Lond. Math. Soc. Lect. Note Ser., vol. 182, Cambridge: Cambridge University Press, 1993 (English).
- [Has05] K. Hasegawa. Complex and Kähler Structures on Compact Solvmanifolds. *J. Symplectic Geom.* **3** (2005), no. 4, 749–767.
- [Hor76] E. Horikawa. Algebraic surfaces of general type with small c_1^2 . I *Ann. Math.* **104** (1976), 357-387.
- [JK93] F. E. A. Johnson, D. Kotschick. On the signature and Euler characteristic of certain four-manifolds. *Math. Proc. Camb. Phil. Soc.* **114** (1993), no. 3, 431-437.
- [Kat78] M. Kato. Compact complex manifolds containing “global” spherical shells. I. *Proceedings of the International Symposium on Algebraic Geometry (Kyoto Univ., Kyoto, 1977)*, pp. 45–84, Kinokuniya Book Store, Tokyo, 1978.
- [Kod63] K. Kodaira. On Compact Analytic Surfaces: II. *Ann. of Math. (2)* **77** (1963), 563–626.
- [Kol93] J. Kollár. Shafarevich maps and plurigenera of algebraic varieties. *Invent. Math.* **113** (1993), no. 1, 177-215.
- [Kot97] D. Kotschick. Orientations and geometrisations of compact complex surfaces. *Bull. London Math. Soc.* **29** (1997), no. 2, 145-149.
- [LYZ94] J. Li, S.-T. Yau, F. Zheng. On projectively flat Hermitian manifolds. *Comm. Anal. Geom.* **2** (1994), no.1, 103–109.
- [LMW21] Y. Liu, L. Maxim, B. Wang. Aspherical manifolds, Mellin transformation and a question of Bobadilla-Kollár. *J. Reine Angew. Math.* **781** (2021), 1-18.
- [LP12] M. M. Lopes, R. Pardini. The geography of irregular surfaces. *Current developments in algebraic geometry* 349-378. Math. Sci. Res. Inst. Publ., 59 *Cambridge University Press, Cambridge*, 2012.
- [Lüc94] W. Lück. Approximating L^2 -invariants by their finite-dimensional analogues. *Geom. Funct. Anal.* **4** (1994), no.4, 455-481.
- [Lüc02] W. Lück. L^2 -invariants: theory and applications to geometry and K -theory. *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*, 44. *Springer-Verlag, Berlin*, 2002.

- [Lüc12] W. Lück. Aspherical manifolds. *Bull. of the Manifolds Atlas* (2012), 1-17
- [MLP12] M. Mendes Lopes, R. Pardini. The geography of irregular surfaces. *Current developments in algebraic geometry* 349-378. Math. Sci. Res. Inst. Publ., 59 *Cambridge University Press, Cambridge*, 2012.
- [Mes90] G. Mess, Examples of Poincaré Duality Groups. *Proc. Amer. Math. Soc.* **110** (1990), no.4, 1145–1146.
- [Rei79] M. Reid, π_1 for surfaces with small K^2 , in *Algebraic geometry, Copenhagen, 1978*, Springer Lect. Notes in Math. **732** (1979), 534-544.
- [SY94] R. Schoen, S.-T. Yau, Lectures on differential geometry. Conference Proceedings and Lecture Notes in Geometry and Topology, I. *International Press, Cambridge, MA*, 1994.
- [ST22] M. Stover, D. Toledo. Residual finiteness for central extensions of lattices in $PU(n, 1)$ and negatively curved projective varieties. *Pure Appl. Math. Q.* **18** (2022), no. 4, 1771-1797.
- [Suw75] T. Suwa. Compact Quotient Spaces of $\mathbb{C}^2$ by Affine Transformation Groups. *J. Differential Geometry* **10** (1975), 239–252.
- [Tel94] A. Teleman. Projectively flat surfaces and Bogomolov’s theorem on class VII_0 surfaces. *Internat. J. Math.* **5** (1994), no.2, 253–264.
- [Tel05] A. Teleman. Donaldson theory on non-Kählerian surfaces and class VII surfaces with $b_2 = 1$. *Invent. Math.* **162** (2005), no.3, 493–521.
- [Tel10] A. Teleman. Instantons and curves on class VII surfaces. *Ann. of Math. (2)* **172** (2010), no.3, 1749–1804.
- [Tel18] A. Teleman. Donaldson theory in non-Kählerian geometry. *Modern geometry: a celebration of the work of Simon Donaldson*, 363–392. Proc. Sympos. Pure Math., 99. *American Mathematical Society, Providence, RI*, 2018
- [Tol93] D. Toledo. Projective varieties with non-residually finite fundamental group. *Inst. Hautes Études Sci. Publ. Math.* (1993), no. 77, 103-119.
- [Wal86] C. T. C. Wall. Geometric structures on compact complex analytic surfaces. *Topology* **25** (1986), no. 2, 119-153.
- [Yau71] S.-T. Yau. On the fundamental group of compact manifolds of non-positive curvature. *Ann. of Math. (2)* **93** (1971), 579-585.