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Nonlocality of two- and three-mode continuous variable systems

Alessandro Ferraro and Matteo G A Paris

Dipartimento di Fisica and Unità INFM, Università di Milano, Italy

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Abstract
We address the nonlocality of fully inseparable three-mode Gaussian states generated either by bilinear three-mode Hamiltonians or by a sequence of bilinear two-mode Hamiltonians. Two different tests revealing nonlocality are considered, in which the dichotomic Bell operator is represented by the displaced parity and by the pseudospin operator respectively. Three-mode states are also considered as a conditional source of two-mode non-Gaussian states, whose nonlocality properties are analysed. We found that the non-Gaussian character of the conditional states allows violation of Bell's inequalities (by parity and pseudospin tests) stronger than with a conventional twin-beam state. However, the non-Gaussian character is not sufficient to reveal nonlocality through a dichotomized quadrature measurement strategy.

Keywords: nonlocality, Bell's inequalities, continuous variables

1. Introduction

Einstein, Podolsky and Rosen (EPR) formulated their famous argument about the completeness of quantum mechanics in the framework of continuous variable systems [1]. However, after Bohm gave a dichotomized version of it [2], the debate concerning nonlocality moved to systems described by discrete variables, leading Bell to formulate his celebrated inequalities in a dichotomized fashion [3]. Recently the increasing importance of continuous variable systems led many authors to explore the nonlocality issue in its original setting, where dichotomic observables to test Bell's inequalities are not uniquely determined. The attempts to translate Bell's inequalities to continuous variable systems clarified the fact that crucial in a nonlocality test is the existence of a set of dichotomized bounded observables used to perform the test itself, from which the so called 'Bell operator' is derived. The more debated question has dealt with the nonlocality of the normalized version of the original EPR state, i.e. the twin-beam (TWB) state of radiation produced by spontaneous downconversion in a parametric amplifier [4]. The nonlocality of the TWB state was not clear for a long time. Using the Wigner function approach, Bell argued that the original EPR state, and as a consequence the TWB too, does not exhibit nonlocality because its Wigner function is positive, and therefore represents a local hidden variable description [5]. More recently, Banaszek and Wodkiewicz [6] showed instead how to reveal nonlocality of the EPR state through the measurement of the displaced parity operator. Furthermore, a subsequent work of Chen *et al* [7] showed that TWB's violation of Bell's inequalities may achieve the maximum value admitted by quantum mechanics upon a suitable choice of the measured observables. Indeed, the amount of violation crucially depends on the kind of Bell operator adopted in the analysis, ranging from no violation to maximal violation for the same (entangled) quantum state.

Systems which involve only two parties are the simplest settings in which to study violation of local realism in quantum mechanics. A more complex scenario arises if multipartite systems are considered. Studying the peculiar quantum features of these systems is worthwhile in view of their relevance in the development of quantum communication technology, e.g. to manipulate and distribute information in a quantum communication network [8, 9]. Although the study of multipartite nonlocality has originated without the use of inequalities [10], an approach to derive Bell inequalities has also been developed [11] for these systems and applied to characterize their entanglement properties [12]. Being originally developed in the framework of discrete variables, these multiparty Bell inequalities have also found application in the characterization of continuous variable systems [13, 14].

The aim of this paper is to apply the various approaches hitherto developed to test the nonlocality of two- and three-mode continuous variable systems. We will consider tripartite

Gaussian states as well as non-Gaussian bipartite states. In the first case strong violation of Bell inequalities is found, allowing the Bell factor to reach values of $\mathcal{B} \simeq 3$, while in the second case enhancement of nonlocality is obtained in comparison with the TWB case.

The paper is organized as follows. In section 2 we review the different approaches to test nonlocality in the framework of continuous variables and introduce notation that will be used throughout the paper. The three-mode states we are interested in are introduced in section 3, and their violation of local realism is analysed in sections 3.1 and 3.2. In section 4, the tripartite states are considered as sources for conditional generation of non-Gaussian bipartite states, whose nonlocal proprties are then studied in sections 4.1, 4.2 and 4.3. Finally, the main results obtained are summarized in section 5, which closes the paper with some concluding remarks.

2. Nonlocality tests for continuous variables

In this section we will briefly recall the inequalities imposed by local realism in the cases of our interest. Let us start by focusing our attention on a bipartite system. Let $m(\alpha_1) = \pm 1$ and $m(\alpha_1') = \pm 1$ denote two possible outcomes of two possible measurements on the first subsystem and similarly $m(\alpha_2) = \pm 1$ and $m(\alpha_2') = \pm 1$ for the second subsystem. The essential feature of these measurements is that they are local, dichotomic and bounded. The Bell combination

$$F_2 \equiv m(\alpha_1)m(\alpha_2) + m(\alpha_1)m(\alpha_2') + m(\alpha_1')m(\alpha_2) - m(\alpha_1')m(\alpha_2') \quad (1)$$

under the assumption of local realism gives rise to the well known Bell–CHSH inequality [15]:

$$\mathcal{B}_2 \equiv |E(\alpha_1, \alpha_2) + E(\alpha_1, \alpha_2') + E(\alpha_1', \alpha_2) - E(\alpha_1', \alpha_2')| \leqslant 2, \quad (2)$$

where $E(\alpha_1, \alpha_2)$ is the correlation function between the measurement results, i.e., the expectation value of the products of the results of the experiments $m(\alpha_1)$ and $m(\alpha_2)$.

In the case of an n-partite system, a nonlocality test is possible using the Bell–Klyshko inequalities [11, 12] which provides a generalization of inequality (2). These inequalities are based on the following recursively defined linear combination:

$$F_n \equiv \tfrac{1}{2}[m(\alpha_n) + m(\alpha_n')]F_{n-1} + \tfrac{1}{2}[m(\alpha_n) - m(\alpha_n')]F_{n-1}', \quad (3)$$

where $m(\alpha_n) = \pm 1$ and $m(\alpha_n') = \pm 1$ refer to measurements on the nth party of the system, and F_n' denotes the same expression as F_n but with all the α_j and α_j' exchanged. In the case of a three-partite system, local realism assumption imposes the following inequality from combination (3):

$$\mathcal{B}_3 \equiv |E(\alpha_1, \alpha_2, \alpha_3') + E(\alpha_1, \alpha_2', \alpha_3) + E(\alpha_1', \alpha_2, \alpha_3) - E(\alpha_1', \alpha_2', \alpha_3')| \leqslant 2, \quad (4)$$

where again $E(\alpha_1, \alpha_2, \alpha_3)$ is the correlation function between the measurement results. Quantum mechanical systems can violate inequalities (2) and (4) by a maximal amount given by, respectively, $\mathcal{B}_2 \leqslant 2\sqrt{2}$ and $\mathcal{B}_3 \leqslant 4$ (see, e.g., [12]).

We now briefly review three different strategies to reveal quantum nonlocality in the framework of continuous variable systems. Recall that in the case of a discrete bipartite system, for example a spin-$\frac{1}{2}$ two-particle system, the local dichotomic bounded observable usually taken into account is the spin of the particle in a fixed direction, say $\mathbf{d}$. Hence the correlation between two measurements performed over the two particles is $E(\mathbf{d_1}, \mathbf{d_2}) = \langle \mathbf{d_1}\boldsymbol{\sigma} \otimes \mathbf{d_2}\boldsymbol{\sigma}\rangle$, where the operator $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is decomposed on the Pauli matrix base and $\mathbf{d_1}$, $\mathbf{d_2}$ are two unit vectors. The quantum Bell operator analogue to F_2 in equation (1) is then given by the expression

$$\mathcal{B}_{2,\mathrm{sp}} = \mathbf{d_1}\boldsymbol{\sigma} \otimes \mathbf{d_2}\boldsymbol{\sigma} + \mathbf{d_1'}\boldsymbol{\sigma} \otimes \mathbf{d_2}\boldsymbol{\sigma} + \mathbf{d_1}\boldsymbol{\sigma} \otimes \mathbf{d_2'}\boldsymbol{\sigma} - \mathbf{d_1'}\boldsymbol{\sigma} \otimes \mathbf{d_2'}\boldsymbol{\sigma}. \quad (5)$$

Consider now a n-partite continuous variable system identified by the creation operator $a_j^\dagger$ and the annihilation operator a_j ($j = 1, \ldots, n$) with associated boson commutation relations. Following the original argument by EPR it is quite natural to attempt to reveal the nonlocality of this system by trying to infer quadratures of one subsystem from those of the others. From now on, we will refer to this procedure as a 'homodyne nonlocality test', as quadrature measurements of radiation field are performed through homodyne detection. Here we identify the quadrature $x_j(\theta)$ according to the definition $x_j^\theta = \frac{1}{\sqrt{2}}(a_j e^{-i\theta} + a_j^\dagger e^{i\theta})$. As they are local but neither bounded nor dichotomic, quadrature observables are not immediately suitable to perform a nonlocality test based on Bell's inequalities. The procedure to make them bounded and dichotomic is quite arbitrary and consists in the assignment of two domains D_+ and D_- to each observable [16]. When the result of a quadrature measurement falls in the domain $D_\pm$ the value ± 1 is associated with it. Usually the choice $D_\pm = \mathbb{R}^\pm$ is considered, though a choice suitable to the system under investigation may be preferable. Considering a bipartite system we can introduce the following quantities:

$$\begin{aligned} P_{++}(x_1^\theta, x_2^\varphi) &= \int_{D_+} dx_1^\theta \int_{D_+} dx_2^\varphi \, P(x_1^\theta, x_2^\varphi) \\ P_{+-}(x_1^\theta, x_2^\varphi) &= \int_{D_+} dx_1^\theta \int_{D_-} dx_2^\varphi \, P(x_1^\theta, x_2^\varphi) \\ P_{-+}(x_1^\theta, x_2^\varphi) &= \int_{D_-} dx_1^\theta \int_{D_+} dx_2^\varphi \, P(x_1^\theta, x_2^\varphi) \\ P_{--}(x_1^\theta, x_2^\varphi) &= \int_{D_-} dx_1^\theta \int_{D_-} dx_2^\varphi \, P(x_1^\theta, x_2^\varphi), \end{aligned} \quad (6)$$

where $P(x_1^\theta, x_2^\varphi)$ is the joint probability distribution of the quadratures x_1^θ and x_2^φ. We can now identify the homodyne correlation function $E_H(\theta, \varphi)$ as

$$E_H(\theta, \varphi) = P_{++}(x_1^\theta, x_2^\varphi) + P_{--}(x_1^\theta, x_2^\varphi) - P_{+-}(x_1^\theta, x_2^\varphi) - P_{-+}(x_1^\theta, x_2^\varphi), \quad (7)$$

which can be straightforwardly used to construct the Bell combination $\mathcal{B}_{2,H}$ of equation (2) and to perform the nonlocality test. The main problem of pursuing such a nonlocality test is that it is not suitable in the case of systems described by a positive Wigner function, as the TWB state of radiation defined as $|X\rangle = \sqrt{1 - X^2} \sum_n X^n |n\, n\rangle$, where $X = \tanh r$ and r is the squeezing parameter. Indeed, a positive Wigner function can be interpreted as a hidden phase-space probability distribution, preventing violation of the Bell–CHSH inequality unless the measured observables have an unbounded Wigner representation, which is not the case of the dichotomized quadrature measurement described above. Considering in fact that $P(x_1^\theta, x_2^\varphi)$ can be determined as a marginal distribution from the Wigner function, one can write from equations (6) and (7)

$$E_H(\theta,\varphi) = \int dx_1^\theta\, dx_2^\varphi\, dx_1^{\theta+\frac{\pi}{2}}\, dx_2^{\varphi+\frac{\pi}{2}}\, \mathrm{sgn}(x_1^\theta, x_2^\varphi) \times W(x_1^\theta, x_1^{\theta+\frac{\pi}{2}}, x_2^\varphi, x_2^{\varphi+\frac{\pi}{2}}), \tag{8}$$

where the integration is performed over the whole phase-space and without loss of generality we have considered $D_\pm = \mathbb{R}^\pm$. Equation (8) itself is indeed a local hidden variable description of the correlation function, hence obeying inequality (2).

In order to overcome this obstacle different strategies have been considered by many authors, based essentially on parity measurements. Banaszek and Wodkiewicz [6] demonstrated the nonlocality of the TWB, considering as a local observable on subsystem j the parity operator on the state displaced by α_j (hence we will refer to this procedure as a 'displaced parity (DP) nonlocality test'), which is dichotomic and bounded:

$$\Pi(\boldsymbol{\alpha}) = \bigotimes_{j=1}^{n} D_j(\alpha_j)(-1)^{n_j} D_j^\dagger(\alpha_j). \tag{9}$$

In the above formula, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$, while $n_j = a^\dagger a$ and $D_j(\alpha_j) = \exp[\alpha_j a_j^\dagger - \alpha_j^* a_j]$ denote the number operator and the phase space displacement operator for the subsystem j. Hence the correlation function reads

$$E_{\mathrm{DP}}(\boldsymbol{\alpha}) = \langle \Pi(\boldsymbol{\alpha})\rangle, \tag{10}$$

from which Bell's combinations $\mathcal{B}_{2,\mathrm{DP}}$ in equation (2) and $\mathcal{B}_{3,\mathrm{DP}}$ in equation (4) can be easily reconstructed in the cases $n = 2, 3$. The reason why this procedure would also be able to reveal nonlocality in the case of quantum states characterized by a positive Wigner function is clear using the following relation:

$$W(\boldsymbol{\alpha}) = \left(\frac{2}{\pi}\right)^n \langle \Pi(\boldsymbol{\alpha})\rangle. \tag{11}$$

Indeed, the analogue of equation (8) is

$$E_{\mathrm{DP}}(\boldsymbol{\alpha}) = \int d^{2n}\boldsymbol{\lambda} \left(\frac{2}{\pi}\right)^n W(\boldsymbol{\alpha})\delta^{(2n)}(\boldsymbol{\alpha} - \boldsymbol{\lambda}). \tag{12}$$

The Dirac-δ distribution being unbounded, inequations (2) and (4) are no longer necessarily valid for $\mathcal{B}_{2,\mathrm{DP}}$ and $\mathcal{B}_{3,\mathrm{DP}}$. The maximal violation found with this procedure for a EPR state is $\mathcal{B}_{2,\mathrm{DP}} \simeq 2.32$ [17], still far from the maximum violation admitted by quantum mechanics.

Another strategy, developed by Chen *et al* [7], shares a similar behaviour as the one described above, allowing us to reveal the nonlocality for quantum states with positive Wigner function. Interestingly, this type of nonlocality test, which we will refer to as a 'pseudospin (PS) nonlocality test', admits a maximum violation for the EPR state. It can be seen as a generalization to continuous variable systems of the one introduced by Gisin and Peres for the case of discrete variable systems [18], hence, for the case of a pure bipartite system, it is equivalent to an entanglement test [17]. Let us consider the following set of operators, known as *pseudospins* in view of their commutation relations, $\mathbf{s^j} = (s_x^j, s_y^j, s_z^j)$ acting on the jth subsystem

$$\begin{aligned} s_z^j &= \sum_{n=0}^{\infty} \big(|2n+1\rangle_j\langle 2n+1| - |2n\rangle_j\langle 2n|\big), \\ s_x^j \pm s_y^j &= 2s_\pm^j, \\ \mathbf{d}^j\mathbf{s}^j &= s_z^j \cos\theta^j + \sin\theta^j (e^{i\varphi^j} s_-^j + e^{-i\varphi^j} s_+^j), \end{aligned} \tag{13}$$

where $s_-^j = \sum_{n=0}^{\infty} |2n\rangle_j\langle 2n+1| = (s_+^j)^\dagger$ and $\mathbf{d}^j$ is a unit vector associated with the angles θ^j and φ^j. In analogy to the spin-$\frac{1}{2}$ system and defining $\mathbf{d} = (\mathbf{d}^1, \ldots, \mathbf{d}^n)$ the correlation function is simply given by

$$E_{\mathrm{PS}}(\mathbf{d}) = \left\langle \bigotimes_{j=1}^{n} \mathbf{d}^j\mathbf{s}^j \right\rangle, \tag{14}$$

from which the Bell combinations $\mathcal{B}_{2,\mathrm{PS}}$ and $\mathcal{B}_{3,\mathrm{PS}}$ are evaluated. Different representations of the spin-$\frac{1}{2}$ algebra have also been discussed in the recent literature [19, 20]. In particular in [20] it has been pointed out that different representations lead to different expectation values of the Bell operators. Hence, the violation of the Bell inequality for continuous variable systems turns out to depend on, besides orientational parameters, also configurational ones. In the following sections we will also consider the pseudospin operators $\boldsymbol{\Pi}^j = (\Pi_x^j, \Pi_y^j, \Pi_z^j)$ taken into account in [20], which have the following Wigner representation:

$$\begin{aligned} W_{\Pi_x^j} &= \mathrm{sgn}\, x_j \qquad W_{\Pi_y^j} = -\delta(x_j)\mathcal{P}\frac{1}{y_j} \\ W_{\Pi_z^j} &= -\pi\delta(x_j)\delta(y_j), \end{aligned} \tag{15}$$

where $x_j = x_j^0$, $y_j = x_j^{\frac{\pi}{2}}$ and $\mathcal{P}$ stands for the 'principal value'. The correlation function obtained using operators $\boldsymbol{\Pi}^j$ will be indicated as $E'_{\mathrm{PS}}(\mathbf{d}) = \langle \otimes_{j=1}^n \mathbf{d}^j \boldsymbol{\Pi}^j\rangle$.

3. Three-mode nonlocality

In this section we will analyse the nonlocal properties of tripartite Gaussian states. In particular, we will consider two classes of states, the first one proposed by Van Loock and Braunstein [13], the second one proposed in [21]. The reason why we consider these two classes is that the first is a very natural and scalable way to produce multimode entanglement using only passive optical elements and single squeezers, while the second one is the simplest way to produce three-mode entanglement using a single nonlinear optical device. Indeed, both states can be achieved experimentally [22, 23]. As concerns the first class of states, it is generated with the aid of three single-mode squeezed states combined in a 'tritter' (a three-mode generalization of a beam-splitter). The evolution is then ruled by a sequence of single- and two-mode quadratic Hamiltonians. As a consequence, being generated from vacuum, the three-mode entangled state is Gaussian and its Wigner function is given by

$$W_S(\mathbf{x},\mathbf{y}) = \frac{1}{\pi^3} \exp\left[-(\mathbf{x},\mathbf{y})\mathbf{C}^{-1}\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix}\right], \tag{16}$$

where $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2, y_3)$ are the positions and momenta of the three modes and $\mathbf{C}^{-1}$ is the inverse of the covariance matrix, whose explicit expression reads

$$\mathbf{C} = \begin{pmatrix} \mathcal{R} & \mathcal{S} & \mathcal{S} & 0 & 0 & 0 \\ \mathcal{S} & \mathcal{R} & \mathcal{S} & 0 & 0 & 0 \\ \mathcal{S} & \mathcal{S} & \mathcal{R} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{T} & -\mathcal{S} & -\mathcal{S} \\ 0 & 0 & 0 & -\mathcal{S} & \mathcal{T} & -\mathcal{S} \\ 0 & 0 & 0 & -\mathcal{S} & -\mathcal{S} & \mathcal{T} \end{pmatrix}, \tag{17}$$

where $\mathcal{R} = \cosh 2r + \frac{1}{3}\sinh 2r$, $\mathcal{T} = \cosh 2r - \frac{1}{3}\sinh 2r$, $\mathcal{S} = -\frac{4}{3}\cosh r \sinh r$ and r is the squeezing parameter (with equal squeezing in all initial modes). The second class of tripartite entangled states is generated in a single nonlinear crystal through the following interaction Hamiltonian:

$$H_{\rm int} = \gamma_1 a_1^\dagger a_3^\dagger + \gamma_2 a_2^\dagger a_3 + \text{H.c.} \tag{18}$$

$H_{\rm int}$ describes two interlinked bilinear interactions taking place among three modes of the radiation field coupled with the support of two parametric pumps. It can be realized in $\chi^{(2)}$ media by a suitable configuration exposed in [23]. Notice that the same dynamics can be implemented in different physical systems, including optomechanical couplers and Bose–Einstein condensates in the linear regime [24–26]. The effective coupling constants γ_j, $j = 1, 2$, of the two parametric processes are proportional to the nonlinear susceptibilities and the pump intensities. If we take the vacuum $|\mathbf{0}\rangle \equiv |0\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3$ as the initial state, the evolved state $|\mathbf{T}\rangle = \mathrm{e}^{-\mathrm{i}H_{\rm int}t}|\mathbf{0}\rangle$ belongs to the class of the coherent states of $SU(2,1)$ and it reads [24, 27]

$$|\mathbf{T}\rangle = \frac{1}{\sqrt{1+N_1}} \sum_{pq} \left(\frac{N_2}{1+N_1}\right)^{p/2} \left(\frac{N_3}{1+N_1}\right)^{q/2} \times \mathrm{e}^{-\mathrm{i}(p\phi_2 + q\phi_3)} \sqrt{\frac{(p+q)!}{p!q!}}\,|p+q, p, q\rangle, \tag{19}$$

where $N_j(t) = \langle a_j^\dagger(t) a(t)\rangle$ represents the average number of photons in the jth mode and ϕ_j are phase factors. The explicit expressions of $N_j(t)$ are

$$N_2 = \frac{|\gamma_1|^2|\gamma_2|^2}{\Omega^4}[\cos\Omega t - 1]^2, \qquad N_3 = \frac{|\gamma_1|^2}{\Omega^2}\sin^2(\Omega t), \tag{20}$$

with $\Omega = \sqrt{|\gamma_2|^2 - |\gamma_1|^2}$ and $N_1 = N_2 + N_3$. For this second class, the initial state Gaussian and the Hamiltonian being quadratic, the evolved states will also be Gaussian. The Wigner function reads as follows [21, 28]:

$$W_T(\mathbf{x}, \mathbf{y}) = \frac{1}{\pi^3} \exp\left[-(\mathbf{x}, \mathbf{y})\mathbf{V}^{-1}\begin{pmatrix}\mathbf{x} \\ \mathbf{y}\end{pmatrix}\right], \tag{21}$$

where $\mathbf{V}^{-1}$ is the inverse of the covariance matrix, whose explicit expression is

$$\mathbf{V} = \begin{pmatrix} \mathcal{F} & \mathcal{A} & \mathcal{B} & 0 & -\mathcal{D} & -\mathcal{E} \\ \mathcal{A} & \mathcal{G} & \mathcal{C} & -\mathcal{D} & 0 & \mathcal{L} \\ \mathcal{B} & \mathcal{C} & \mathcal{H} & -\mathcal{E} & -\mathcal{L} & 0 \\ 0 & -\mathcal{D} & -\mathcal{E} & \mathcal{F} & -\mathcal{A} & -\mathcal{B} \\ -\mathcal{D} & 0 & -\mathcal{L} & -\mathcal{A} & \mathcal{G} & \mathcal{C} \\ -\mathcal{E} & \mathcal{L} & 0 & -\mathcal{B} & \mathcal{C} & \mathcal{H} \end{pmatrix}, \tag{22}$$

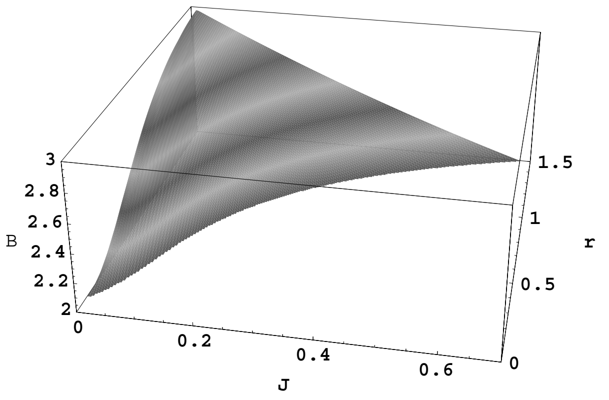

Figure 1. Plot of the Bell combination (24). Only values violating Bell inequality (4) are shown.

where

$$\mathcal{A} = 2\sqrt{N_2(1+N_1)}\cos\phi_2 \qquad \mathcal{D} = 2\sqrt{N_2(1+N_1)}\sin\phi_2$$
$$\mathcal{F} = 2N_1 + 1 \qquad \mathcal{B} = 2\sqrt{N_3(1+N_1)}\cos\phi_3$$
$$\mathcal{E} = 2\sqrt{N_3(1+N_1)}\sin\phi_3 \qquad \mathcal{G} = 2N_2 + 1 \tag{23}$$
$$\mathcal{C} = 2\sqrt{N_2N_3}\cos(\phi_2 - \phi_3) \qquad \mathcal{L} = 2\sqrt{N_2N_3}\sin(\phi_2 - \phi_3)$$
$$\mathcal{H} = 2N_3 + 1.$$

Both classes of states are fully inseparable for any value of the coupling constants; namely, they cannot be written as a factorized state for any grouping of the modes. Therefore, they are good candidates to reveal true tripartite nonlocality. Being Gaussian states, however, nonlocality cannot be revealed by homodyne detection. In the following we analyse the results for displaced parity and pseudospin tests.

3.1. Displaced parity test

Let us start the study of tripartite system nonlocality using the 'displaced parity test'. Considering the correlation function $E_{\rm DP}(\alpha)$ given by equation (10), the state (16) was found in [13] to give a maximal violation of $\mathcal{B}_{3,\rm DP} \simeq 2.32$ in the limit of large squeezing and small displacement. The study in [13] however was performed for a particular choice of displacement parameters: $\alpha_1 = \alpha_2 = \alpha_3 = 0$ and $\alpha_1' = \alpha_2' = \alpha_3' = \mathrm{i}\sqrt{\mathcal{J}}$. A numerical optimization of the displacement parameters led us to identify a number of parametrizations that allow a significantly higher violation of Bell's inequality. As an example, consider the one given by $\alpha_1 = \alpha_2 = \alpha_3 = \mathrm{i}\sqrt{\mathcal{J}}$ and $\alpha_1' = \alpha_2' = \alpha_3' = -2\mathrm{i}\sqrt{\mathcal{J}}$ from which it follows that

$$\mathcal{B}_{3,\rm DP} = 3\exp(-12\mathrm{e}^{-2r}\mathcal{J}) - \exp(24\mathrm{e}^{2r}\mathcal{J}), \tag{24}$$

hence the remarkably high asymptotic value of $\mathcal{B}_{3,\rm DP} = 3$ is found for large r and $\mathcal{J} \neq 0$ (see figure 1). The importance of a suitable choice of the displacement parameters is apparent if this asymptotic value is compared to the violations obtained in the nonlocality study performed in [13]. In that work in fact generalizations to more than three modes of state (16) were also considered, giving an increasing violation of Bell inequality as the number of modes increases, but never finding a violation greater than 2.8. Determining the optimal choice of the displacement parameters for a given state is in general

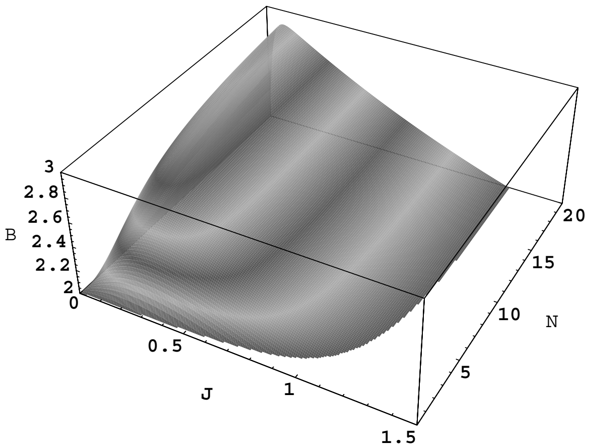

Figure 2. Bell combination obtained choosing optimized displacement parameters for state (19) (see text for details). Only values violating Bell inequality (4) are shown.

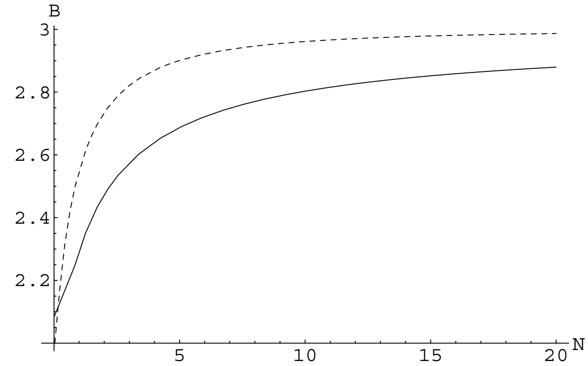

Figure 3. Bell combination $\mathcal{B}_{3,\mathrm{DP}}$ for state (16), dotted curve, and (19), solid curve. The displacement parameters $\mathcal{J}$ have been optimized to give the maximum value of $\mathcal{B}$ for N fixed.

a challenging task. To our knowledge indeed there exists no general prescription to find it, and ultimately one must rely onto a numerical analysis (see, e.g. [29]). Nevertheless, a careful inspection of the symmetries of the state under consideration may be helpful. In order to clarify this observation let us consider the explicit expression of the correlation function $E_{\mathrm{DP}}(\alpha_1, \alpha_2, \alpha_3)$ for state (16):

$$E(\alpha_1, \alpha_2, \alpha_3) = \exp\Big\{-\tfrac{2}{3}\mathrm{e}^{2r}[(y_1 + y_2 + y_3)^2 + (x_2 - x_3)^2 + (x_2 - x_1)^2 + (x_1 - x_3)^2] - \tfrac{2}{3}\mathrm{e}^{-2r}[(x_1 + x_2 + x_3)^2 + (y_2 - y_3)^2 + (y_2 - y_1)^2 + (y_1 - y_3)^2]\Big\}. \qquad (25)$$

Hence, from equation (4) it follows that the Bell combination $\mathcal{B}_{3,\mathrm{DP}}$ is given by the sum of three positive terms and one negative one. It is reasonable to expect that the maximal violation of nonlocality will be achieved for large r. We see from equation (25) that, in this limit, all the correlation functions in $\mathcal{B}_{3,\mathrm{DP}}$ vanish for nonzero displacements, unless the coefficients of e^{2r} are zero. Hence we impose the following system of equations, which allows the three positive terms in $\mathcal{B}_{3,\mathrm{DP}}$ not to vanish (we consider $\alpha_k = x_k + \mathrm{i}y_k$ and $\alpha'_k = x'_k + \mathrm{i}y'_k$ for $k = 1, 2, 3$):

$$\begin{aligned}
&(y'_1 + y_2 + y_3)^2 + (x_2 - x_3)^2 + (x_2 - x'_1)^2 + (x'_1 - x_3)^2 = 0\\
&(y_1 + y'_2 + y_3)^2 + (x'_2 - x_3)^2 + (x'_2 - x_1)^2 + (x_1 - x_3)^2 = 0\\
&(y_1 + y_2 + y'_3)^2 + (x_2 - x'_3)^2 + (x_2 - x_1)^2 + (x_1 - x'_3)^2 = 0.
\end{aligned} \qquad (26)$$

We see that the parametrization used to obtain equation (24) is a solution of this system. Clearly, any other solution will give, in the limit of large r, the same violation given by equation (24), namely $\mathcal{B}_{3,\mathrm{DP}} \to 3$. In order to compare the violation of Bell's inequality admitted by the state (16) with the one that will be obtained below considering the state (19), it is useful to rewrite equation (24) as a function of total mean photon number $N = N_1 + N_2 + N_3$. Given that $N = 3\sinh^2 r$ and optimizing the displacement $\mathcal{J}$ we obtain the result shown in figure 3. The asymptotic expression of the optimized displacement as a function of N is $\mathcal{J} = \frac{1}{8N}\mathrm{arcsinh}\Big(\sqrt{\frac{N}{3}}\Big)$, hence very small angles are required.

Consider now the tripartite state generated by the Hamiltonian (18). The correlation function is now given by equation (10) through the Wigner function (21). The symmetry of the state suggests a maximum violation of the Bell inequality for $N_2 = N_3 = N/4$ (recalling equation (20)), while the fact that the separability of the state does not depend on the phases ϕ_2 and ϕ_3 [21] suggests that they are not crucial for the nonlocality test. If we consider the same parametrization that led us to equation (24) and fix $\phi_2 = \phi_3 = \pi$, we find

$$\mathcal{B}_{3,\mathrm{DP}} = \frac{-1 + \mathrm{e}^{6\mathcal{J}\left(1+N+2\sqrt{2}\sqrt{N(2+N)}\right)} + 2\mathrm{e}^{\frac{3}{2}\mathcal{J}\left(4+7N+6\sqrt{2}\sqrt{N(2+N)}\right)}}{\mathrm{e}^{4\mathcal{J}\left(3+3N+2\sqrt{2}\sqrt{N(2+N)}\right)}}, \qquad (27)$$

from which follows an asymptotic violation of Bell's inequalities of $\mathcal{B}_{3,\mathrm{DP}} \simeq 2.89$, for large N and small $\mathcal{J}$. A slightly better result is found if a parametrization more suitable and numerically optimized for state (19) is considered: $\alpha_1 = \frac{2}{3}\sqrt{\mathcal{J}}, \alpha_2 = \alpha_3 = \alpha'_1 = 0, \alpha'_2 = -\sqrt{\mathcal{J}}, \alpha'_3 = \sqrt{\mathcal{J}}, \phi_2 = 0$ and $\phi_2 = \pi$. The Bell combination $\mathcal{B}_{3,\mathrm{DP}}$ for this choice of parameters is depicted in figure 2. We note that in this case a larger choice of angles allows the violation of the Bell inequality if compared with figure 1. As before, optimizing the displacement $\mathcal{J}$ for each N, it is possible to find the maximum violation of the Bell inequality as a function of N. We find that the asymptotic relation between the optimized displacement and the total photon number is now $\mathcal{J}N \simeq 3.21$, confirming that not too small displacements are required. The asymptotic violation of Bell's inequality is now $\mathcal{B}_{3,\mathrm{DP}} \simeq 2.99$. To compare the results obtained for the two states (16) and (19) we have plotted in figure 3 the Bell combination $\mathcal{B}_{3,\mathrm{DP}}$ as a function of the mean total energy N, while the displacement $\mathcal{J}$ has been chosen in order to maximize $\mathcal{B}_{3,\mathrm{DP}}$ at fixed energy. Notice that even if the two states have exactly the same asymptotic violation, state (16) reaches it for lower energies.

3.2. Pseudospin test

Consider now the pseudospin nonlocality test. Let us calculate the expectation value of the correlation function (14) for the state $|T\rangle$ (for simplicity we consider $\phi_2 = \phi_3 = 0$). The only non-vanishing contributions are given by

$$
\begin{aligned}
c_1 &= \langle s_z^1 \otimes s_x^2 \otimes s_x^3 \rangle = \langle s_z^1 \otimes s_y^2 \otimes s_y^3 \rangle \\
&= -\frac{\sqrt{N_2 N_3}}{2(1+N_1)^2} \sum_{s,t} \left(\frac{N_2}{1+N_1} \right)^{2s} \left(\frac{N_3}{1+N_1} \right)^{2t} \\
&\quad \times \frac{(2s+2t+1)!}{(2s)!(2t)!\sqrt{(2s+1)(2t+1)}}, \\
c_2 &= \langle s_x^1 \otimes s_z^2 \otimes s_x^3 \rangle = -\langle s_y^1 \otimes s_z^2 \otimes s_y^3 \rangle \\
&= \frac{\sqrt{N_3}}{2(1+N_1)^{3/2}} \sum_{s,t} \left(\frac{N_2}{1+N_1} \right)^{2s} \left(\frac{N_3}{1+N_1} \right)^{2t} \\
&\quad \times \frac{(2s+2t)!}{(2s)!(2t)!} \sqrt{\frac{2s+2t+1}{2t+1}}, \\
c_3 &= \langle s_x^1 \otimes s_x^2 \otimes s_z^3 \rangle = -\langle s_y^1 \otimes s_x^2 \otimes s_z^3 \rangle \\
&= \frac{\sqrt{N_2}}{2(1+N_1)^{3/2}} \sum_{s,t} \left(\frac{N_2}{1+N_1} \right)^{2s} \left(\frac{N_3}{1+N_1} \right)^{2t} \\
&\quad \times \frac{(2s+2t)!}{(2s)!(2t)!} \sqrt{\frac{2s+2t+1}{2s+1}},
\end{aligned} \tag{28}
$$

and by $\langle s_z^1 \otimes s_z^2 \otimes s_z^3 \rangle = 1$. The correlation function then, according to equations (13) and (14), reads as follows:

$$
\begin{aligned}
E_{\mathrm{PS}}(\mathbf{d}) &= \cos\theta^1 \cos\theta^2 \cos\theta^3 \\
&\quad + c_1 \cos\theta^1 \sin\theta^2 \sin\theta^3 (\cos\varphi^2 \cos\varphi^3 + \sin\varphi^2 \sin\varphi^3) \\
&\quad + c_2 \cos\theta^2 \sin\theta^1 \sin\theta^3 (\cos\varphi^1 \cos\varphi^3 - \sin\varphi^1 \sin\varphi^3) \\
&\quad + c_3 \cos\theta^3 \sin\theta^1 \sin\theta^2 (\cos\varphi^1 \cos\varphi^2 + \sin\varphi^1 \sin\varphi^2).
\end{aligned} \tag{29}
$$

Hence, without loss of generality, we can fix for example $\varphi^1 = 0$ and $\varphi^2 = \varphi^3 = \pi$ and look for the maximum violation of Bell inequality (4) constructed from equation (29). Notice that if the coefficients c_i $(i = 1, 2, 3)$ were equal to one then the maximum violation admitted, $\mathcal{B}_{3,\mathrm{PS}} = 4$, should be reached. Considering equation (28) two limiting cases can be studied. First, for large N_2 and small N_3 (or vice versa) a numerical evaluation of the coefficients c_i shows that $c_3 \to 1$, while the other two vanish. Hence, considering $\theta^3 = 0$, the correlation function (29) reduces to that of a TWB subjected to a pseudospin nonlocality test (see equation (39) below), hence allowing an asymptotic violation of $\mathcal{B}_{3,\mathrm{PS}} = 2\sqrt{2}$. This result should be expected, since in this limiting case state (19) reduces to a TWB for modes a_1 and a_2, while mode a_3 remains in the vacuum state and factors out. Consider now the case in which $N_2 = N_3 = N/4$. A numerical evaluation shows that the coefficients $c_i \to \frac{1}{2}$ for large N, hence also in this case the maximum violation cannot be attained. The asymptotic violation turns out to be $\mathcal{B}_{3,\mathrm{PS}} \simeq 2.63$.

As already mentioned in section 2 other representations for the pseudospin operators can be considered. Using equations (15) and (21) it is possible to calculate the correlation function $E'_{\mathrm{PS}}(\mathbf{d})$. Setting again the azimuthal angles $\varphi^i = 0$, the latter shows the same structure as $E_{\mathrm{PS}}(\mathbf{d})$ where now the coefficients c_i are replaced by

$$
c'_1 = \frac{2\arctan\left(\frac{N}{2\sqrt{1+N}}\right)}{\pi(1+N)} \qquad c'_2 = c'_3 = \frac{2\arctan\sqrt{N}}{\pi(1+\frac{N}{2})}. \tag{30}
$$

An appropriate choice of angles leads to a maximal violation of Bell's inequality given by $\mathcal{B}_{3,\mathrm{PS}} \simeq 2.22$ (see figure 4), which is now reached for $N \simeq 1$, a value for which the coefficients c'_i are approximately near their maxima. As already pointed out, we see that different representations of the pseudospin operators give rise to different expectation values for the Bell operator.

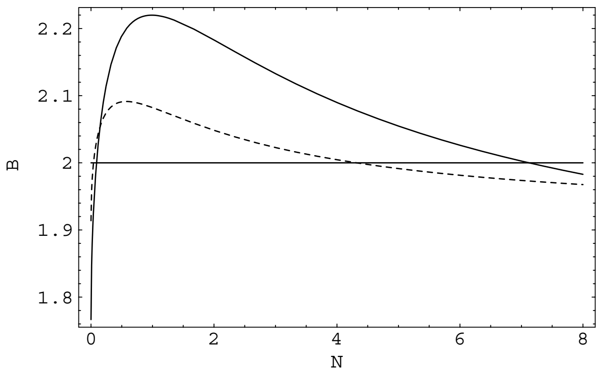

Figure 4. Plot of Bell combination $\mathcal{B}_{3,\mathrm{PS}}$ for states (21) and (16), solid and dashed curves respectively. N is the total number of photons as in figure 3.

Applying now the same procedure to state (16) we find the same structure for the correlation function E'_{PS}, where the coefficients are now given by

$$
c'_1 = c'_2 = c'_3 = \frac{-6\arctan\left(\frac{4\cosh(r)\sinh(r)}{\sqrt{3(2+\mathrm{e}^{4r})}}\right)}{\pi\sqrt{5+4\cosh(4r)}}. \tag{31}
$$

After an optimization of the angles θ^i we obtain a maximal violation of $\mathcal{B}_{3,\mathrm{PS}} \simeq 2.09$ (see figure 4), for $r \simeq 0.42$ $(N \simeq 0.56)$ that maximizes the coefficients c_i.

4. Degaussified state and two-mode nonlocality

In this section we consider the tripartite state (19) as a source of two-mode states. In particular, we study the nonlocality of a two-mode non-Gaussian state obtained by a conditional measurement performed on state (19).

Gaussian states are at the heart of quantum information processing with continuous variables. The reason for this is that the character of the vacuum state of quantum electrodynamics is Gaussian. This observation, in combination with the fact that the quantum evolutions achievable with current technology are described by Hamiltonian operators at most bilinear in the quantum fields, accounts for the fact that the states commonly produced in laboratories are Gaussian. In fact, bilinear evolutions preserve the Gaussian character of the vacuum state. In addition, it is worth noticing that the operation of tracing out a mode from a multipartite Gaussian state preserves the Gaussian character too, and the same observation is valid when the evolution of a state in a standard noisy channel is considered. Indeed, the only feasible way to 'degaussify' a state is through a conditional measurement, or by statistically mixing it with another Gaussian state. The reason to study non-Gaussian states is that when the Gaussian character is lost, then immediately the Wigner function of the state becomes negative, for pure states, hence stronger nonclassical properties should emerge. Actually, various authors have recently investigated the nonlocality properties of two-mode non-Gaussian states. In particular, a twin-beam state subjected to inconclusive photon subtraction (IPS state) has been considered in [30] and [31], while in [17] it has been pointed out that if the entangled coherent states [32] could be produced experimentally they would allow for the maximal violation (i.e., $\mathcal{B}_2 = 2\sqrt{2}$) both in the case of a DP test as well as a PS test.

The most natural way to obtain a non-Gaussian state from a Gaussian one is by elimination of its vacuum component. In fact, such a state is necessarily described by a negative Wigner function (in fact $\langle 0|\varrho|0\rangle \propto \int d^{2n}\boldsymbol{\alpha}\, W(\boldsymbol{\alpha})e^{-2|\alpha|^2}$). Due to the structure of state (19) its vacuum component can be subtracted by a conditional measurement on mode a_3, the same observation being valid for mode a_2. Consider a photodetector able to distinguish only the presence or the lack of photons, i.e., an ON/OFF photodetector, and the state ϱ_1 conditioned to the presence of at least one photon. The probability operator measure (POVM) is two valued, $\{\Pi_0, \Pi_1\}$, $\Pi_0 + \Pi_1 = \mathbf{I}$, with the element associated with the 'no photon' result given by

$$\Pi_0 = \mathbf{I}_1 \otimes \mathbf{I}_2 \otimes \sum_n (1-\eta)^n |n\rangle_{33}\langle n|, \tag{32}$$

where η is the efficiency of the photodetector. The probability of the outcome is given by

$$P_1 = \mathrm{Tr}_{123}[|\mathbf{T}\rangle\langle\mathbf{T}|\Pi_1] = \frac{\eta N_3}{(1+\eta N_3)}, \tag{33}$$

while the conditional output state reads as follows:

$$\begin{aligned}\varrho_1 &= \frac{1}{P_1}\mathrm{Tr}_3[|\mathbf{T}\rangle\langle\mathbf{T}|\Pi_1] \\ &= \frac{1+\eta N_3}{(1+N_1)\eta N_3}\sum_{p=1}^{\infty}\left(\frac{N_3}{1+N_1}\right)^p \frac{1-(1-\eta)^p}{p!}(a^\dagger)^p \\ &\quad\times \sum_{n,n'}\left(\frac{N_2}{1+N_1}\right)^{n+n'} |n\,n\rangle\langle n'\,n'|a^p \\ &= \frac{1+\eta N_3}{(1+N_1+N_2)\eta N_3}\sum_{p=1}^{\infty}\left(\frac{N_3}{1+N_1}\right)^p \\ &\quad\times \frac{1-(1-\eta)^p}{p!}(a^\dagger)^p|X\rangle\langle X|a^p, \end{aligned} \tag{34}$$

where we have identified the TWB with $X = \sqrt{\frac{N_2}{1+N_1}}$. To calculate the Wigner function associated with state ρ_1, consider that the characteristic function of the POVM Π_1 reads as follows:

$$\chi[\Pi_1](\mu) = \pi\delta^2(\mu) - \frac{1}{\eta}\exp\left[-|\mu|^2\frac{2-\eta}{2\eta}\right], \tag{35}$$

hence the characteristic function of ϱ_1 is given by

$$\begin{aligned}\chi[\varrho_1](\lambda_1,\lambda_2) &= \frac{1}{P_1}\Bigg\{\chi[|T\rangle\langle T|](\lambda_1,\lambda_2,0) \\ &\quad - \frac{1}{\eta}\int\frac{d^2\mu}{\pi}\chi[|T\rangle\langle T|](\lambda_1,\lambda_2,\mu)\exp\left[-|\mu|^2\frac{2-\eta}{2\eta}\right]\Bigg\}. \end{aligned} \tag{36}$$

After some algebra the Wigner function associated with state ρ_1 can now be calculated. It reads as follows:

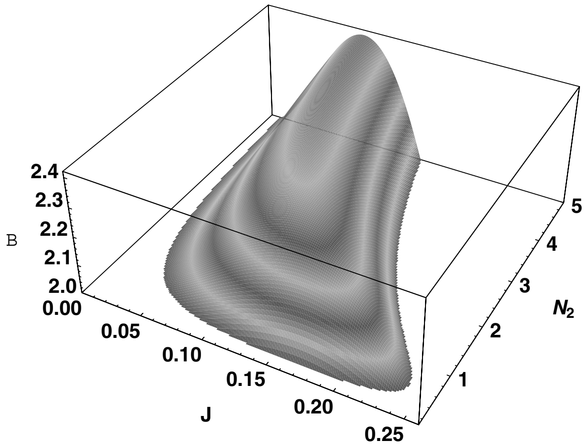

Figure 5. Bell combination obtained choosing optimized displacement parameters for state ϱ_1 (see text for details). Only values violating inequality (2) are shown.

$$\begin{aligned}&W_1(\mathbf{x},\mathbf{y}) \\ &= \frac{1+\eta N_3}{4\eta N_3}\Bigg\{\left(\frac{2}{\pi}\right)^2\frac{1}{\sqrt{\det V'}}\exp\left[-(\mathbf{x},\mathbf{y})(\mathbf{V}')^{-1}\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix}\right] \\ &\quad - \frac{1}{\eta}\left(\frac{2}{\pi}\right)^2\frac{2}{\sqrt{\det D}}\exp\left[-(\mathbf{x},\mathbf{y})(\mathbf{D}^{-1})'\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix}\right]\Bigg\}, \end{aligned} \tag{37}$$

where, from now on, $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$, and $\mathbf{D} = \mathbf{V} + \mathrm{diag}\left(0, 0, \frac{2-\eta}{\eta}, 0, 0, \frac{2-\eta}{\eta}\right)$. In order to simplify the notation we have indicated with $\mathbf{O}'$ the 4×4 matrix obtained from the 6×6 matrix $\mathbf{O}$ deleting the elements corresponding to the third mode (third row/column and sixth row/column), due to the trace over the third mode. Of course, the easiest way to obtain a bipartite state from state (19) is to discard a mode, say the third one, by tracing over it. The state ϱ_{Tr} then obtained is simply given by the following Wigner function:

$$W_{\mathrm{Tr}}(\mathbf{x},\mathbf{y}) = \left(\frac{2}{\pi}\right)^2\frac{1}{4\sqrt{\det V'}}\exp\left[-(\mathbf{x},\mathbf{y})(\mathbf{V}')^{-1}\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix}\right]. \tag{38}$$

The Wigner function W_{Tr} being Gaussian, we expect that this state will exhibit weaker nonlocality with respect to state (34). In the rest of the section the nonlocal properties of the usual TWB state and of the states (34) and (38) will be compared. Notice that state (16) can be considered as an extension to three modes of the TWB. All of the three nonlocality tests introduced in section 2 will be taken into account.

4.1. Displaced parity test

We first study the violation of inequality (2) in the case of a 'displaced parity test'. As already mentioned, in [6] Banaszek and Wodkiewicz found for the first time that the TWB state exhibits a violation of local realism. They obtained the following asymptotic violation for infinite energy: $\mathcal{B}_{2,\mathrm{DP}} \simeq 2.19$. Generalizing their procedure this result can be improved, yielding to a maximum asymptotic violation of $\mathcal{B}_{2,\mathrm{DP}} \simeq 2.32$ [17]. We have considered the following parametrization to obtain the maximum violation for a TWB: $\alpha_1 = -\alpha_2 = i\sqrt{\mathcal{J}}$ and $\alpha'_1 = -\alpha'_2 = -3i\sqrt{\mathcal{J}}$. The asymptotic relation between the squeezing parameter and the displacement angles is $e^{2r}\mathcal{J} = \frac{\log 3}{32}$. Using the same parametrization and considering the Bell combination $\mathcal{B}_{2,\mathrm{DP}}$ for the state ϱ_{Tr}, it turns out that the same asymptotic value of the TWB is reached for large N_2 and small N_3. In fact, as already noticed, when this limit is considered the original tripartite state (19) reduces to a factorized state composed by a TWB and the vacuum state. Consider now the conditional state ρ_1 and again the case of large N_2 and small N_3, say $N_3 = 10^{-2}\frac{1}{N_2}$. As in the tripartite case the phase coefficients ϕ_2 and ϕ_3 play no rule in the characterization of nonlocality. A stronger violation of Bell inequality is then found and it is depicted in figure 5, where the parametrization $\alpha_1 = \frac{1}{2}\alpha_2 = \frac{1}{3}\alpha_1' = \mathrm{i}\sqrt{\mathcal{J}}$ and $\alpha_2' = 0$ has been adopted. Indeed the asymptotic violation is higher then the previous, namely $\mathcal{B}_{2\mathrm{DP}} \simeq 2.41$. It can be found, for large N_2, when $\mathcal{J}N_2 \simeq 0.042$. A comparison with the violation of nonlocality attained with a IPS state shows an identical asymptotic behaviour [31]. Nevertheless, the scheme proposed here offers the advantage that the production rate of state ϱ_1 [i.e., the conditional probability (33)] is much greater then the production rate of the IPS state [see [33], equation (14)]. This is due to the fact that only a single ON/OFF detection is required to produce ϱ_1, rather than the coincidence of two ON/OFF detections for the case of the IPS state. This could be useful from a practical viewpoint.

4.2. Pseudospin test

Let us now focus on the 'pseudospin nonlocality test'. Considering a TWB state, it is known that the correlation function (14) has the following expression (setting to zero the azimuthal angles) [7]:

$$E_{\mathrm{PS}}(\theta_1, \theta_2) = \cos\theta_1\cos\theta_2 + f_{\mathrm{TWB}}\sin\theta_1\sin\theta_2, \tag{39}$$

where, denoting by N the total photon number,

$$f_{\mathrm{TWB}} = \frac{\sqrt{N(N+2)}}{1+N}. \tag{40}$$

It turns out that the violation of Bell inequality in this context increases monotonically to the maximum value of $2\sqrt{2}$ as the function f_{TWB} goes to unity. A straightforward calculation shows that an expression identical in form to equation (39) can be found both in case of state ϱ_1 and ϱ_{Tr}, where the following functions f_1 and f_{Tr} can be identified:

$$\begin{aligned} f_1 &= 2\sqrt{\frac{N_2}{1+N_1}\frac{(1+N_3\eta)}{N_3(1+N_1)\eta}}\sum_{k,p=0}^{\infty}\frac{(2k+p)!}{(2k)!p!} \\ &\quad\times\sqrt{\frac{2k+p+1}{2k+1}}(1-(1-\eta)^p)\left(\frac{N_3}{1+N_1}\right)^p\left(\frac{N_2}{1+N_1}\right)^{2k}, \\ f_{\mathrm{Tr}} &= 2\sqrt{\frac{N_2}{1+N_1}}\frac{1}{1+N_1}\sum_{p,q=0}^{\infty}\left(\frac{N_2}{1+N_1}\right)^{2p}\left(\frac{N_3}{1+N_1}\right)^{2q} \\ &\quad\times\frac{(2p+2q)!}{(2p)!(2q)!}\sqrt{\frac{2q+2p+1}{2p+1}}. \end{aligned} \tag{41}$$

In order to compare the violations in the three different cases, let us fix as in the previous subsection a small value for N_3. A plot of the functions f_{TWB}, f_1 and f_{Tr} versus the total number of photons of the TWB for the former and of the initial three-partite state for the latter two is given in figure (6). It can be seen that state ϱ_1 achieves large violations for smaller energies with respect to the other two states. Finally, a comparison with the violation attained with the IPS state may be found in [34].

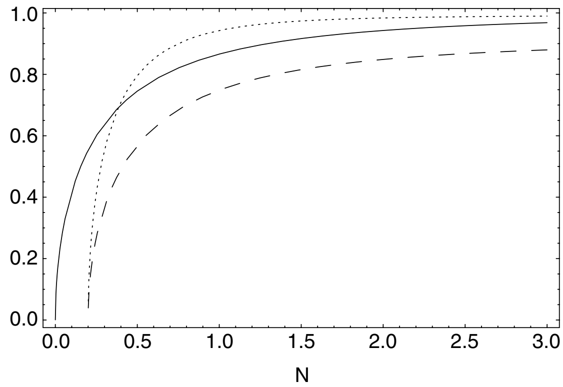

Figure 6. Comparison between the values of the functions f_{TWB} (solid curve), $f1$ (dotted curve) and f_{Tr} (dashed curve) defined in the text (the summation has been numerically performed for $\eta = 0.8$ and $N_3 = 0.1$).

4.3. Homodyne detection

The negativity of the Wigner function (37) may suggest performing a nonlocality test based upon a homodyne detection scheme. While the positivity of a Wigner function avoids violating the Bell inequality (2) with such a test, its negativity is yet not sufficient in general to ensure a violation. Quantum states with a negative Wigner function that does not violate local realism with a homodyne test are given for example in [35]. Considering state (34) it is necessary to calculate the correlation function (8). Substituting the Wigner function (37) into equation (8) and performing the integral we obtain the following result:

$$\begin{aligned} E_H(\psi) &= \frac{1+\eta N_3}{4\eta N_3}\Bigg\{-\left(\frac{2}{\pi}\right)^2\frac{1}{\sqrt{\det V'}}\Bigg[2(1+2N_3)\pi \\ &\quad\times\arctan\left(\frac{2\cos\psi}{\sqrt{\frac{(1+2N_1)(1+2N_2)}{(1+N_1)N_2}-4\cos^2\psi}}\right)\Bigg] \\ &\quad-\frac{1}{\eta}\left(\frac{2}{\pi}\right)^2\frac{2}{\sqrt{\det D}}\Bigg[\frac{2\pi(-1+N_3(-2+\eta))}{1+N_3\eta} \\ &\quad\times\arctan\left(\frac{2\cos\psi}{\sqrt{\frac{(1+2N_1-N_3\eta)(1+2N_2+N_3\eta)}{(1+N_1)N_2}-4\cos^2\psi}}\right)\Bigg]\Bigg\}, \end{aligned} \tag{42}$$

where $\psi = \theta+\varphi+\phi_2$. A plot of the correlation function (42) is depicted in figure 7 for unitary efficiency η, together with the classical correlation function of two spin-$\frac{1}{2}$ particles [36] (see the caption for details). Unfortunately, the comparison shows clearly that the correlation given by equation (42) is always lower then the classical one, hence despite the negativity of Wigner function (37) no violation of the Bell inequality is achievable with this scheme.

5. Conclusions

A detailed analysis of the nonlocality properties of multipartite continuous variable systems obtained by parametric optical systems has been presented, using the more recent approaches developed with this aim. We have considered in particular two classes of tripartite Gaussian states that seem promising for quantum communication purposes in order to implement multipartite quantum protocols. The results show that for

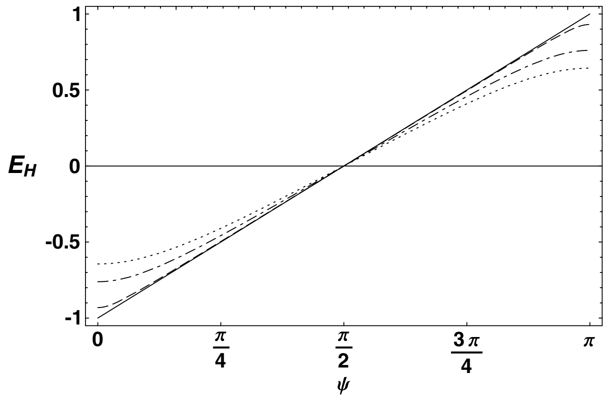

Figure 7. Comparison between the correlation functions obtained from two spin-$\frac{1}{2}$ particles classically correlated (solid line) and from equation (42): $N_2 = 0.5$ (dotted curve), $N_2 = 1$ (dot–dashed curve), $N_2 = 5$ (dashed curve). In all cases we have fixed $N_3 = 0.5$ and $\eta = 1$.

these states a nonlocality test based on displaced parity measurements is more suitable to reveal violation of local realism than one based on pseudospin operators. These results are just the opposite of what have been obtained for the bipartite case. Notice, however, that a systematic approach to pseudospin operators for continuous variables has not been developed yet, hence we have only used the two inequivalent configurational parametrizations more suitable for calculations. For the displaced parity test we obtained a remarkably high asymptotic value for the Bell parameter, $\mathcal{B}_{3,\mathrm{DP}} \simeq 3$. In this case the choice of a proper parametrization, suitable for the state under investigation, has been revealed to be crucial.

We have also explored the possibility of enhancing nonlocality in bipartite systems considering states endowed with a nonpositive Wigner function. We investigated a method to conditionally produce such a state from the tripartite systems considered above. As expected, the Bell parameter reaches a value higher than for a TWB, namely $\mathcal{B}_{2,\mathrm{DP}} \simeq 2.41$. In the case of a pseudospin test an enhancement of nonlocality has also been demonstrated, while a violation of local realism using a dichotomic quadrature measurement cannot be achieved.

Acknowledgments

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