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LETTER TO THE EDITOR

Cloning of observables

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Abstract
We introduce the concept of cloning for *classes of observables* and classify cloning machines for qubit systems according to the number of parameters needed to describe the class under investigation. A no-cloning theorem for observables is derived and the connections between cloning of observables and joint measurements of noncommuting observables are elucidated. Relationships with cloning of states and non-demolition measurements are also analysed.

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1. Introduction

Information may be effectively manipulated and transmitted by encoding symbols into quantum states. However, besides several advantages, the quantum nature of the transmitted signals entails some drawback, the most relevant owing to the so-called *no-cloning* theorem: Quantum information encoded in a set of nonorthogonal states cannot be copied [1–3]. In order to overcome this limitation, an orthogonal coding may be devised, which however requires the additional control of the quantum channel since, in general, propagation degrades orthogonality of any set of input quantum signals.

A different scenario arises by addressing transmission of information encoded in the statistics of a set of observables, independently on the quantum state at the input. In a network of this sort there is no need of a precise control of the coding stage whereas, on the other hand, each gate should be *transparent*, i.e. should preserve the statistics of the transmitted observables. In this letter, we address the problem of copying information that has been encoded in the statistics of a set of observables. For this purpose, we introduce the concept of cloning machine for classes of observables and analyse in details the constraint imposed by quantum mechanics to this kind of devices. Two forms of a no-cloning theorem for observables will be derived, and the connections with cloning of states and joint measurements will be discussed.

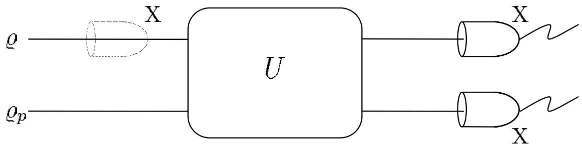

Figure 1. A schematic diagram of a cloning machine for observables $(U, \varrho_p, \mathbf{X})$: a signal qubit prepared in the unknown state ϱ interacts, via a given unitary U, with a probe qubit prepared in the known state ϱ_p. The class of observables $\mathbf{X}$ is cloned if a measurement of any $\mathrm{X} \in \mathbf{X}$ on either of the two qubits at the output gives the same statistics as it was measured on the input signal qubit, *independently* on the initial qubit preparation ϱ.

In the following we assume that information is encoded in the statistics of a set of qubit observables. A cloning machine for the given set is a device in which a signal qubit carrying the information interacts with a probe qubit via a given unitary with the aim of reproducing the statistics of each observable on *both* the qubits at the output. The chance of achieving this task, besides the choice of a suitable interaction, depends on the class of observables under investigation. In this letter we provide a full classification of cloning machines, based on the number of parameters needed to specify the class.

The letter is structured as follows: in section 2 we introduce the concept of cloning machine for a class of observables and illustrate some basic properties. Then, the possibility of realizing a cloning machines for a given class is analysed, depending on the number of parameters individuating the class. The results are summarized in two forms of a no-cloning theorem for observables. In section 3 we analyse the connections between cloning of observables and joint measurements of noncommuting observables. Section 4 closes the letter with some concluding remarks.

2. Cloning of observables

We consider a device in which a signal qubit (say, qubit '1') is prepared in the (unknown) state ϱ and then interacts, via a given unitary U, with a probe qubit ('2') prepared in the known state ϱ_p. For a given class of qubit observables $\mathbf{X} \equiv \{\mathrm{X}(j)\}_{j\in\mathcal{J}}$, where $\mathcal{J}$ is a subset of the real axis and $\mathrm{X}(j) \in \mathcal{L}[\mathbb{C}^2]$, we introduce the concept of cloning as follows. A cloning machine for the class of observables $\mathbf{X}$ is a triple $(U, \varrho_p, \mathbf{X})$, such that

$$\overline{\mathrm{X}}_1 = \overline{\mathrm{X}} \quad \overline{\mathrm{X}}_2 = \overline{\mathrm{X}} \qquad \forall \varrho \quad \forall\, \mathrm{X} \in \mathbf{X},$$

where $\overline{\mathrm{X}} \equiv \mathrm{Tr}_1[\varrho \mathrm{X}]$ is the mean value of the observable X at the input and

$$\overline{\mathrm{X}}_1 \equiv \mathrm{Tr}_{12}[R\mathrm{X} \otimes I], \qquad \overline{\mathrm{X}}_2 \equiv \mathrm{Tr}_{12}[RI \otimes \mathrm{X}] \tag{1}$$

are the mean values of the same observable on the two output qubits (see figure 1). The density matrix $R = U\varrho \otimes \varrho_p U^\dagger$ describes the (generally entangled) state of the two qubits after the interaction, whereas I denotes the identity operator.

The above definition identifies the cloning of an observable with the cloning of its mean value. This is justified by the fact that for any single-qubit observable X, the cloning of the mean value is equivalent to the cloning of the whole statistics. In fact, any $\mathrm{X} \in \mathcal{L}[\mathbb{C}^2]$ has at most two distinct eigenvalues $\{\lambda_0, \lambda_1\}$, occurring with probability p_0, p_1. For a degenerate eigenvalue the statement is trivial. For two distinct eigenvalues we have $\overline{\mathrm{X}} = \lambda_1 p_1 + \lambda_0 p_0$ which, together with the normalization condition $1 = p_0 + p_1$, proves the statement. In other words, we say that the class of observables $\mathbf{X}$ has been cloned if a measurement of any $\mathrm{X} \in \mathbf{X}$

on either of the two qubits at the output gives the same statistics as it was measured on the input signal qubit, *independently* on the initial qubit preparation.

A remark about this choice is in order. In fact, in view of the duality among states and operators on a Hilbert space, one may argue that a proper figure of merit to assess a cloning machine for observables would be a fidelity-like one. This is certainly true for the d-dimensional case, $d > 2$, while for qubit systems a proper assessment can be also made in term of mean-value duplication, which subsumes all the information carried by the signal.

Our goal is now to classify cloning machines according to the number of parameters needed to fully specify the class of observables under investigation. Before beginning our analysis let us illustrate a basic property of cloning machines, which follows from the definition, and which will be extensively used throughout the letter.

Given a cloning machine $(U, \varrho_p, \mathbf{X})$, then $(V, \varrho_p, \mathbf{Y})$ is a cloning machine too, where $V = (W^\dagger \otimes W^\dagger)U(W \otimes I)$ and the class $\mathbf{Y} = W^\dagger \mathbf{X} W$ is formed by the observables $Y(j) = W^\dagger X(j) W$, $j \in \mathcal{J}$. The transformation W may be a generic unitary. We will refer to this property to as *unitary covariance* of cloning machine. The proof proceeds as follows. By definition $\overline{\mathrm{Y}}(j) = \mathrm{Tr}_1[\varrho W^\dagger X(j) W] = \mathrm{Tr}_1[W \varrho W^\dagger \mathrm{X}(j)]$. Then, since $(U, \varrho_p, \mathbf{X})$ is a cloning machine, we have

$$\begin{aligned}
\overline{\mathrm{Y}}(j) &= \mathrm{Tr}_{1,2}[U(W\varrho W^\dagger \otimes \varrho_p)U^\dagger(\mathrm{X}(j) \otimes \mathbb{I})] \\
&= \mathrm{Tr}_{1,2}[U(W \otimes \mathbb{I})(\varrho \otimes \varrho_p)(W^\dagger \otimes \mathbb{I})U^\dagger(W \otimes W)(\mathrm{Y}(j) \otimes \mathbb{I})(W^\dagger \otimes W^\dagger)] \\
&= \mathrm{Tr}_{1,2}[V(\varrho \otimes \varrho_p)V^\dagger(\mathrm{Y}(j) \otimes \mathbb{I})] = \overline{\mathrm{Y}}_1(j).
\end{aligned} \tag{2}$$

The same argument holds for $\overline{\mathrm{Y}}_2(j)$.

Another result which will be used in the following is the parameterization of a two-qubit transformation, which corresponds to a $SU(4)$ matrix, obtained by separating its local and entangling parts. A generic two-qubit gate $SU(4)$ matrix may be factorized as follows [4]:

$$U = L_2 U_{\mathrm{E}} L_1 = L_2 \exp\left[\frac{\mathrm{i}}{2}\sum_{j=1}^{3} \theta_j \sigma_j \otimes \sigma_j\right] L_1, \tag{3}$$

where $\theta_j \in \mathbb{R}$ and the σ_j's are Pauli's matrices. The local transformations L_1 and L_2 belong to the $SU(2) \otimes SU(2)$ group, whereas U_{E} accounts for the entangling part of the transformation U. In our context, decomposition (3), together with unitary covariance of cloning machines, allows us to ignore the local transformations L_1, which corresponds to a different state preparation of signal and probe qubits at the input. On the other hand, as we will see in the following, the degree of freedom offered by the local transformations L_2 will be exploited to design suitable cloning machines for noncommuting observables.

2.1. One-parameter classes of observables

Let us begin our analysis with a class constituted by only one observable A. In this case a cloning machine $(U, \varrho_p, \mathrm{A})$ corresponds to a quantum non-demolition measurement of A itself, i.e. a measurement which introduces no back-action on the measured observable, thus allowing for repeated measurements [5]. As an example, if $\mathrm{A} = \sigma_3$ then $(U_C, |0\rangle\langle 0|, \sigma_3)$ is a cloning machine [6], U_C being the unitary performing the $\mathrm{C}_{\mathrm{not}}$ gate. The proof is straightforward, since

$$\mathrm{Tr}_2\big[(\mathbb{I} \otimes |0\rangle\langle 0|)U_C^\dagger(\sigma_3 \otimes \mathbb{I})U_C\big] = \mathrm{Tr}_2\big[(\mathbb{I} \otimes |0\rangle\langle 0|)U_C^\dagger(\mathbb{I} \otimes \sigma_3)U_C\big] = \sigma_3.$$

The next step is to consider a generic one-parameter class of observables. At first, we note that $(U_C, |0\rangle\langle 0|, \mathbf{X}_3)$ is a cloning machine for the class $\mathbf{X}_3 = \{x_3\sigma_3\}_{x_3 \in \mathbb{R}}$. Using this result, and denoting by σ_0 the identity matrix, we find a cloning machine for the class $\mathbf{X}_\mathrm{A} \equiv \{x\mathrm{A}\}_{x\in\mathbb{R}}$, where $\mathrm{A} = \sum_{k=0}^{3} a_k\sigma_k$ is a generic observable. Explicitly, the triple

$$(U_\mathrm{A}, |0\rangle\langle 0|, \mathbf{X}_\mathrm{A}) \qquad U_\mathrm{A} = \left(W_\mathrm{A}^\dagger \otimes W_\mathrm{A}^\dagger\right) U_C (W_\mathrm{A} \otimes \mathbb{I}) \tag{4}$$

is a cloning machine, with the single-qubit unitary transformation W_A given by $W_\mathrm{A} = \exp\left(\mathrm{i}\phi \sum_{j=1}^{2} n_j\sigma_j\right)$, with $n_2^2 = 1 - n_1^2$,

$$n_1 = \frac{a_2}{\sqrt{a_1^2 + a_2^2}},$$

and

$$\phi = \arccos \frac{a_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}}.$$

In order to prove the statement, one first notes that any observable of the form $\mathrm{A}' = \sum_{j=1}^{3} a_j\sigma_j$ can be obtained from $x_3\sigma_3$ by the unitary transformation $\mathrm{A}' = W^\dagger x_3\sigma_3 W$ with $x_3 = \sqrt{a_1^2 + a_2^2 + a_3^2}$ and $W = W_\mathrm{A}$. The statement then follows from unitary covariance, since $(U_C, |0\rangle\langle 0|, \mathbf{X}_3)$ is a cloning machine, whereas the identity matrix is trivially cloned.

2.2. The set of all qubit observables

Let us now consider the n-parameter class $\mathbf{X}_g = \{x_1\mathrm{X}(1) + \cdots + x_n\mathrm{X}(n)\}_{x_1,\ldots,x_n\in\mathbb{R}}$. Recall that the aim of a cloning machine for observables is to copy the expectation value of a generic linear combination X of the n observables $X(j)$, i.e. $\mathrm{X} = x_1\mathrm{X}(1) + \cdots + x_n\mathrm{X}(n)$. By decomposing each observable $\mathrm{X}(j)$ on the Pauli matrices basis $\mathrm{X}(j) = a_{j,0}\sigma_0 + a_{j,1}\sigma_1 + a_{j,2}\sigma_2 + a_{j,3}\sigma_3$ and reordering one obtains $\mathrm{X} = y_0\sigma_0 + y_1\sigma_1 + y_2\sigma_2 + y_3\sigma_3$, where $y_k = \sum_{j=1}^{n} x_j a_{j,k}$. From the above expression we see that at most a four-parameter class may be of interest, being any other class of observables embodied in that. That being said, the following operator counterpart of the usual no-cloning theorem for states can be formulated.

Theorem 1 (No-cloning of observables I)**.** *A cloning machine* $(U, \varrho_p, \mathbf{X}_g)$ *where* $\mathbf{X}_g$ *is a generic n-parameter class of qubit observables does not exist.*

Proof. Let us reduce to a four-parameter class as above. Then the request $\overline{\mathrm{X}} = \overline{\mathrm{X}}_1 = \overline{\mathrm{X}}_2$ implies

$$\mathrm{Tr}_1[\varrho\sigma_k] = \mathrm{Tr}_{12}[R\sigma_k \otimes I] = \mathrm{Tr}_{12}[RI \otimes \sigma_k] \quad \forall k,$$

which, in turn, violates the no-cloning theorem for quantum states, since it expresses the equality of the input Bloch vector with that of the two partial traces at the output. □

The results of theorem 2.2.1 and the fact that the triple in equation (4) is a cloning machine permit a comparison among cloning machines for observables and for states. Let us write the generic input signal as $\varrho = \frac{1}{2}(\sigma_0 + \mathbf{s}\cdot\boldsymbol{\sigma})$, where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ and $\boldsymbol{s} = (s_1, s_2, s_3)$ is the Bloch vector, and consider the cloning of the single-parameter class $\mathbf{X}_3$ via $(U_C, |0\rangle\langle 0|, \mathbf{X}_3)$. The action of this cloning machine is the perfect copying of the component s_3 of the Bloch vector $\boldsymbol{s}$, whereas the values of s_1 and s_2 are completely disregarded. The same situation occurs with any single-parameter class: a single component of the generalized Bloch vector is copied, upon describing the qubit in a suitable basis. On the other hand, when the whole class of qubit observables is considered, this requirement should be imposed to all the components and cannot be satisfied. Only approximate cloning is allowed for the entire state of a generic qubit, with the whole Bloch vector being shrunk by the same factor with respect to the initial Bloch vector $\boldsymbol{s}$ [7–10]. In the following we analyse intermediate situations between the above two extrema, i.e. perfect copying (one-parameter class) and no-copying (the set of all qubit observables).

2.3. Two-parameter classes of observables

In order to introduce cloning machines for two-parameter classes of observables, let us consider two specific classes: $\mathbf{X}_{c_1} = \{x_0\sigma_0 + x_3\sigma_3\}_{x_0,x_3\in\mathbb{R}}$ and $\mathbf{X}_{nc} = \{x_1\sigma_1 + x_2\sigma_2\}_{x_1,x_2\in\mathbb{R}}$. The first class is constituted by commuting observables, hence one expects no quantum constraints on cloning them. This is indeed the case, and an explicit representative of a cloning machine is given by $(U_C, |0\rangle\langle 0|, \mathbf{X}_{c_1})$. The statement follows by recalling that $(U_C, |0\rangle\langle 0|, \mathbf{X}_3)$ is a cloning machine and by noting that the identity matrix σ_0 is trivially cloned by any cloning machine. As already noted such a cloning machine copies the component s_3 of the input signal Bloch vector $\boldsymbol{s}$. On the other hand, consider the class $\mathbf{X}_{nc}$, constituted by noncommuting observables. If a cloning machine $(U, \varrho_p, \mathbf{X}_{nc})$ existed, then the mean values as well as the statistics of any observable belonging to $\mathbf{X}_{nc}$ would be cloned at its output. As a consequence, one would jointly measure any two noncommuting observables belonging to $\mathbf{X}_{nc}$ (e.g., σ_1 on the output signal and σ_2 on the output probe) without any added noise, thus violating the bounds imposed by quantum mechanics [11–13]. Generalizing this argument to any two-parameter class of noncommuting observables (i.e., to any class $\mathbf{X}_{gnc} = \{c\mathrm{C} + d\mathrm{D}\}_{c,d\in\mathbb{R}}$, with C, D generic noncommuting observables), we then conclude with the following stronger version of theorem 2.2.1.

Theorem 2 (No-cloning of observables II)**.** *A cloning machine for a generic two-parameter class of noncommuting observables does not exist.*

The state-cloning counterpart of the above theorem can be obtained by considering the class of observables $\mathbf{X}_{nc}$: if a cloning machine $(U, \varrho_p, \mathbf{X}_{nc})$ existed, then the components s_1 and s_2 of the Bloch vector $\boldsymbol{s}$ would be cloned for any input signal. The same situation occurs in the case of a two-parameter class generated by any pair of Pauli operators. In other words, it is not possible to simultaneously copy a pair of components of the Bloch vector of a generic state, even completely disregarding the third one [14, 15]. In order to clarify the relationship between cloning of states and cloning of observables, a pictorial view of the action of cloning machines for observables at the level of states is given in figure 2.

In figure 2(*a*), we show the action of cloning machines for the class $\mathbf{X}_3$: any cloning machine for this class should preserve the third component of the Bloch vector, while modifying arbitrarily the other two. As a consequence, given a signal qubit at the input (denoted by a black circle) the Bloch vector of the two output qubits may lie on any point of a plane of fixed latitude. In figure 2(*b*), we show the action of hypothetical cloning machines for the class $\{x_1\sigma_1 + x_3\sigma_3\}$: starting from the input qubit denoted by the black circle, the output qubits would correspond to Bloch vectors having the same first and third components, i.e. lying on the intersection of two planes similar to that of figure 2(*a*). The meaning of theorem 2.3.2 is that of preventing the existence of any such cloning machine. Finally, the no-cloning theorem for states does not allow the existence of any cloning machine for the class $\{x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3\}$, which for the input qubit denoted by the black circle would impose two output states with the same Bloch vector (corresponding to the intersection of the three surfaces).

By theorem 2.3.2, it is also clear that a three-parameter class of observables does not warrant further attention. In fact, a cloning machine for a three-parameter class of noncommuting observables does not exist, whereas a three-parameter class of commuting observables reduces to that of a two-parameter class (of commuting observables).

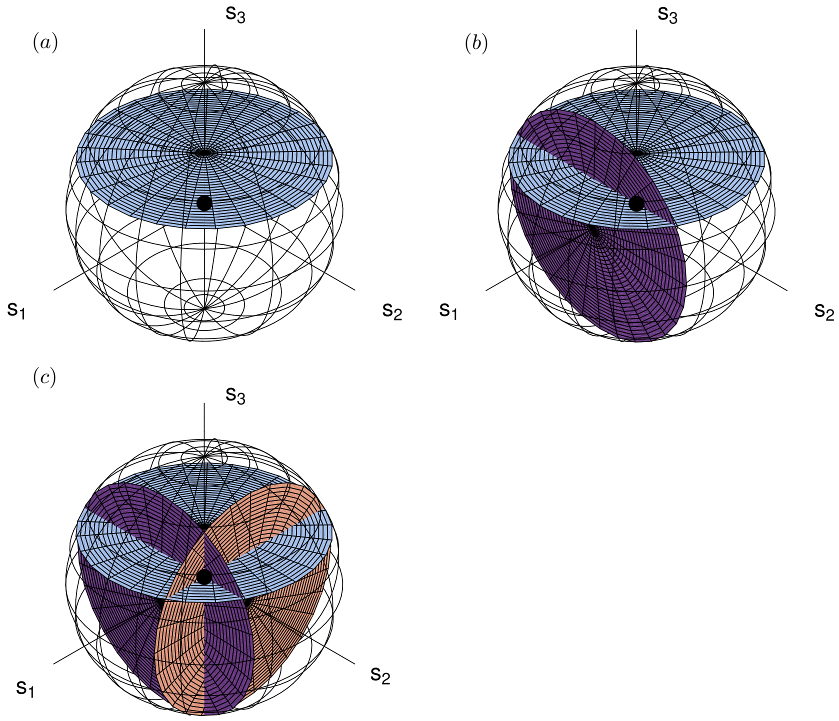

Figure 2. Pictorial representation of the action of cloning machines for observables at the level of the Bloch sphere. In panel (*a*), any cloning machine for the class $\mathbf{X}_3$ should preserve the third component of the Bloch vector, while modifying arbitrarily the other two. As a consequence, given a signal qubit at the input (denoted by a black circle), the Bloch vector of the two output qubits may lie on any point of a plane. In panel (*b*) we show the action of hypothetical cloning machines for the class $\{x_1\sigma_1 + x_3\sigma_3\}$, which would send the input qubit denoted by the black circle to output qubits lying on the intersection of two planes: the meaning of theorem 2.3.2 is that of preventing the existence of any such cloning machine. Finally, the no-cloning theorem for states does not allow the existence of any cloning machine for the class $\{x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3\}$, which for the input qubit denoted by the black circle would impose two output states with the same Bloch vector (corresponding to the intersection of the three surfaces as shown in panel (*c*)).

(This figure is in colour only in the electronic version)

2.4. Class of commuting observables

Concerning commuting observables, it turns out that the cloning machine in equation (4) for a generic single-parameter class of observables also provides a cloning machine for a generic two-parameter class of commuting observables. In order to prove this statement let us first recall the relationship between two generic commuting observables. Given an observable $\mathrm{A} = \sum_{k=0}^{3} a_k\sigma_k$, then a generic observable B commuting with A is given by

$$\mathrm{B} = \sum_{k=0}^{3} b_k\sigma_k \qquad b_1 = a_1b_3/a_3, b_2 = a_2b_3/a_3, \tag{5}$$

whereas b_0 and b_3 are free parameters[4]. Considering now two generic commuting observables A and B, one has that a cloning machine for the class $\mathbf{X}_\mathrm{c} = \{a\mathrm{A} + b\mathrm{B}\}_{a,b\in\mathbb{R}}$ with $[\mathrm{A}, \mathrm{B}] = 0$ is

[4] The roles of $a_{1,2,3}$ can be interchanged to avoid singularities.

given by $(U_{\mathrm{A}}, |0\rangle\langle 0|, \mathbf{X}_{\mathrm{c}})$, where U_{A} is given in (4). The proof starts from the decomposition of A in the Pauli basis, namely $\mathrm{A} = \sum_{k=0}^{3} a_k\sigma_k$, and by defining $\mathrm{A_r} = a_0\sigma_0 + a_3\sigma_3$. From (5), one has that an observable $\mathrm{B_r}$ commuting with $\mathrm{A_r}$ must be of the form $\mathrm{B_r} = b_0\sigma_0 + b_3\sigma_3$. The class of observables defined by $\mathrm{A_r}$ and $\mathrm{B_r}$—i.e., $\mathbf{X}_{\mathrm{c}_2} = \{a\mathrm{A_r} + b\mathrm{B_r}\}_{a,b\in\mathbb{R}}$—coincides with the class $\mathbf{X}_{\mathrm{c}_1}$. As a consequence, since $(U_{\mathrm{C}}, |0\rangle\langle 0|, \mathbf{X}_{\mathrm{c}_1})$ is a cloning machine, one has that $(U_{\mathrm{C}}, |0\rangle\langle 0|, \mathbf{X}_{\mathrm{c}_2})$ is a cloning machine too. Now, following the proof of (4), one can easily show that $\forall$A there exists a unitary W_{A}, such that $\mathrm{A} = W_{\mathrm{A}}^{\dagger}\mathrm{A_r}W_{\mathrm{A}}$. The corresponding transformation on $\mathrm{B_r}$ reads as follows:

$$W_{\mathrm{A}}^{\dagger}\mathrm{B_r}W_{\mathrm{A}} = b_0\sigma_0 + b_3 n_2 \sin\theta\sigma_1 - b_3 n_1 \sin\theta\sigma_2 + b_3\cos\sigma_3. \tag{6}$$

Together with (5), equation (6) explicitly shows that the observable B, defined as $\mathrm{B} = W_{\mathrm{A}}^{\dagger}\mathrm{B_r}W_{\mathrm{A}}$, is the most general observable commuting with A. The proof is thus completed by unitary covariance and recalling that U_{A} is defined as $U_{\mathrm{A}} = \left(W_{\mathrm{A}}^{\dagger}\otimes W_{\mathrm{A}}^{\dagger}\right)U_C(W_{\mathrm{A}}\otimes\mathbb{I})$.

3. Noncommuting observables and joint measurements

As we already pointed out in the previous sections there are no cloning machines for a two-parameter class of noncommuting observables (theorem 2.3.2). Therefore, a question arises on whether, analogously to state-cloning, we may introduce the concept of approximate cloning machines, i.e. cloning of observables involving added noise. Indeed this can be done and optimal approximate cloning machines corresponding to minimum added noise may be found as well.

An approximate cloning machine for the class of observables $\mathbf{X}$ is defined as the triple $(U, \varrho_p, \mathbf{X})_{\mathrm{apx}}$, such that $\overline{\mathrm{X}}_1 = \overline{\mathrm{X}}/g_1$ and $\overline{\mathrm{X}}_2 = \overline{\mathrm{X}}/g_2$, i.e.

$$\mathrm{Tr}_1[\varrho\mathrm{X}] = g_1\mathrm{Tr}_{12}[R\mathrm{X}\otimes I] = g_2\mathrm{Tr}_{12}[RI\otimes\mathrm{X}], \tag{7}$$

for any $\mathrm{X}\in\mathbf{X}$. The quantities g_j, $j = 1, 2$ are independent on the input state and are referred to as the noises added by the cloning process.

Let us begin by again considering the class $\mathbf{X}_{\mathrm{nc}} = \{x_1\sigma_1 + x_2\sigma_2\}_{x_1,x_2\in\mathbb{R}}$. By using the decomposition of a generic $SU(4)$ matrix in equation (3), one may attempt to find an approximate cloning machine considering only the action of the entangling kernel U_{E}. Unfortunately, it can be shown that no U_{E}, g_1 and g_2 exist which realize approximate cloning for $\varrho_p = |0\rangle\langle 0|$. A further single-qubit transformation should be introduced after U_{E}. In particular, the unitary $F = \mathrm{i}/\sqrt{2}(\sigma_1 + \sigma_2)$ flips the Pauli matrices σ_1 and σ_2 (i.e., $F^{\dagger}\sigma_{1,2}F = \sigma_{2,1}$) and permits the realization of an approximate cloning machine. Indeed, the unitary

$$T = (\mathbb{I}\otimes F)U_{\mathrm{nc}}, \qquad U_{\mathrm{nc}} = \exp\left(\mathrm{i}\frac{\theta}{2}(\sigma_1\otimes\sigma_1 - \sigma_2\otimes\sigma_2)\right),$$

realizes the approximate cloning machine $(T, |0\rangle\langle 0|, \mathbf{X}_{\mathrm{nc}})_{\mathrm{apx}}$ with added noises

$$g_1 = \frac{1}{\cos\theta}, \qquad g_2 = \frac{1}{\sin\theta}. \tag{8}$$

In order to prove the cloning properties of T let us start from the unitary $(\mathbb{I}\otimes F)U_{\mathrm{E}}$, where U_{E} is a generic entangling unitary of the form given in equation (3). Then, by imposing approximate cloning for any $\mathrm{X}\in\mathbf{X}_{\mathrm{nc}}$, one obtains the following system of equations:

$$g_1\mathrm{Tr}_2\left[(\mathbb{I}\otimes\varrho_p)U_E^{\dagger}(\sigma_1\otimes\mathbb{I})U_E\right] = \sigma_1 \tag{9a}$$

$$g_1\mathrm{Tr}_2\left[(\mathbb{I}\otimes\varrho_p)U_E^{\dagger}(\sigma_2\otimes\mathbb{I})U_E\right] = \sigma_2 \tag{9b}$$

$$g_2 \mathrm{Tr}_2\big[(\mathbb{I} \otimes \varrho_p) U_E^\dagger (\mathbb{I} \otimes \sigma_2) U_E\big] = \sigma_1 \tag{9c}$$

$$g_2 \mathrm{Tr}_2\big[(\mathbb{I} \otimes \varrho_p) U_E^\dagger (\mathbb{I} \otimes \sigma_1) U_E\big] = \sigma_2. \tag{9d}$$

The system (9) admits the solution $\theta_1 = -\theta_2 = \theta/2, \theta_3 = 0$—i.e., $U_E \equiv U_{\mathrm{nc}}$ with θ being free parameter—with $g_1 = 1/\cos\theta$ and $g_2 = 1/\sin\theta$. Note that other solutions for the $\theta_{1,2,3}$'s parameters may be found, which however give the same added noise as that considered above.

Remarkably, similar cloning machines may be obtained for any class of observables generated by a pair of operators unitarily equivalent to σ_1 and σ_2. In fact, given the two-parameter classes of noncommuting observables defined as $\mathbf{X}_{\mathrm{V}} = \{c\mathrm{C} + d\mathrm{D}\}_{c,d\in\mathbb{R}}$, with $\mathrm{C} = V^\dagger\sigma_1 V$, $\mathrm{D} = V^\dagger\sigma_2 V$ and V generic unitary one has that an approximate cloning machine is given by the triple $(U_V, |0\rangle\langle 0|, \mathbf{X}_{\mathrm{V}})_{\mathrm{apx}}$, with $U_V = (V^\dagger \otimes V^\dagger)(\mathbb{I} \otimes F)U_{\mathrm{nc}}(V \otimes \mathbb{I})$, with added noises $g_1 = 1/\cos\theta$ and $g_2 = 1/\sin\theta$. The statement easily follows from the fact that $(T, |0\rangle\langle 0|, \mathbf{X}_{\mathrm{nc}})_{\mathrm{apx}}$ is a cloning machine and from unitary covariance. Similar results hold for any class of observables unitarily generated by any pair of (noncommuting) Pauli operators.

A question arises about optimality of approximate cloning machines for noncommuting observables. In order to assess the quality and to define optimality of a triple $(U, \varrho_p, \mathbf{X})_{\mathrm{apx}}$, we consider it as a tool to perform a joint measurement of noncommuting qubit observables [16]. For example, consider the cloning machine $(T, |0\rangle\langle 0|, \mathbf{X}_{\mathrm{nc}})_{\mathrm{apx}}$ and suppose to measure σ_1 and σ_2 on the two qubits at the output. We emphasize that the cloning machine $(T, |0\rangle\langle 0|, \mathbf{X}_{\mathrm{nc}})_{\mathrm{apx}}$ clones every observable belonging to $\mathbf{X}_{\mathrm{nc}}$, while we are now considering only the observables σ_1 and σ_2 which, in a sense, generate the class. We have that the measured expectation values of σ_1 and σ_2 at the output are given by $\langle\sigma_h\rangle_{\mathrm{m}} = g_h\langle\sigma_h\rangle$ (with $h = 1, 2$), where the $\langle\sigma_h\rangle$'s are the input mean values. It follows that the measured uncertainties ($\Delta O = \langle O^2\rangle - \langle O\rangle^2$) at the output are given by

$$\Delta_{\mathrm{m}}\sigma_h = g_h^2 \Delta_{\mathrm{i}}\sigma_h,$$

where $\Delta_{\mathrm{i}}\sigma_h$ denote the intrinsic uncertainties for the two quantities at the input. Since for any Pauli operators we have $\sigma_h^2 = \mathbb{I}$, one may rewrite

$$\Delta_{\mathrm{m}}\sigma_1 = \tan^2\theta + \Delta_{\mathrm{i}}\sigma_1 \tag{10}$$

$$\Delta_{\mathrm{m}}\sigma_2 = \cot^2\theta + \Delta_{\mathrm{i}}\sigma_2. \tag{11}$$

As a consequence, the measured uncertainty product is given by

$$\Delta_{\mathrm{m}}\sigma_1\Delta_{\mathrm{m}}\sigma_2 = \Delta_{\mathrm{i}}\sigma_1\Delta_{\mathrm{i}}\sigma_2 + \cot^2\theta\,\Delta_{\mathrm{i}}\sigma_1 + \tan^2\theta\,\Delta_{\mathrm{i}}\sigma_2 + 1.$$

Since the arithmetic mean is bounded from below by the geometric mean, we have

$$\cot^2\theta\,\Delta_{\mathrm{i}}\sigma_1 + \tan^2\theta\,\Delta_{\mathrm{i}}\sigma_2 \geqslant 2\sqrt{\Delta_{\mathrm{i}}\sigma_1\Delta_{\mathrm{i}}\sigma_2},$$

with the equal sign iff $\Delta_{\mathrm{i}}\sigma_1 = \tan^4\theta\,\Delta_{\mathrm{i}}\sigma_2$, then it follows that

$$\Delta_{\mathrm{m}}\sigma_1\Delta_{\mathrm{m}}\sigma_2 \geqslant (\sqrt{\Delta_{\mathrm{i}}\sigma_1\Delta_{\mathrm{i}}\sigma_2} + 1)^2.$$

If the initial signal is a minimum uncertainty state—i.e., $\Delta_{\mathrm{i}}\sigma_1\Delta_{\mathrm{i}}\sigma_2 = 1$—one finally has that the measured uncertainty product is bounded by $\Delta_{\mathrm{m}}\sigma_1\Delta_{\mathrm{m}}\sigma_2 \geqslant 4$. Note that an optimal joint measurement corresponds to have $\Delta_{\mathrm{m}}\sigma_1\Delta_{\mathrm{m}}\sigma_2 = 4$. In our case this is realized when θ is chosen such that $\tan^4\theta = \Delta_{\mathrm{i}}\sigma_1/\Delta_{\mathrm{i}}\sigma_2$. Therefore, since $(T, |0\rangle\langle 0|, \mathbf{X}_{\mathrm{nc}})_{\mathrm{apx}}$ adds the minimum amount of noise in a joint measurement performed on minimum uncertainty states, we conclude that it is an optimal approximate cloning machine for the class under investigation. An optimal approximate cloning machine for the more general class $\mathbf{X}_{\mathrm{gnc}}$ may be also defined, using the concept of joint measurement for noncanonical observables [16].

Let us now consider the comparison with a joint measurement of σ_1 and σ_2 performed with the aid of an optimal universal cloning machine for states [7]. It is easy to show that the best result in this case is given by $\Delta_m\sigma_1\Delta_m\sigma_2 = \frac{9}{2}$, indicating that cloning of observables is more effective than cloning of states to perform joint measurements (for the case of three observables, see [17, 18]). In fact, a symmetric cloning machine for states shrinks the whole Bloch vector $\boldsymbol{s}$ by a factor $\frac{2}{3}$, whereas a cloning machine for observables shrinks the components s_1 and s_2 of $\boldsymbol{s}$ only by a factor $1/\sqrt{2}$ (considering equal noise $g_1 = g_2 = \sqrt{2}$). Note that such a behaviour is different from what happens in the case of continuous variables, for which the optimal covariant cloning of coherent states also provides the optimal joint measurements of two conjugated quadratures [19]. This is due to the fact that coherent states are fully characterized by their complex amplitude, that is by the expectation values of two operators only, whereas the state of a qubit requires the knowledge of the three components of the Bloch vector.

As a final remark, we note that if the requirement of universality is dropped, then cloning machines for states can be found that realize optimal approximate cloning of observables. For example, an approximate cloning machine for the two-parameter class $\mathbf{X}_{\mathrm{nc}}$ can be obtained by considering a phase-covariant cloning for states [20]. In order to clarify this point, let us recall that a phase-covariant cloning machine for states uses a probe in the $|0\rangle$ state and performs the following transformation:

$$|0\rangle|0\rangle \to |0\rangle|0\rangle \qquad |1\rangle|0\rangle \to \cos\theta|1\rangle|0\rangle + \sin\theta|0\rangle|1\rangle, \tag{12}$$

where, in general, $\theta \in [0, 2\pi]$. If we now consider the $\mathbf{X}_{\mathrm{nc}}$ class, it is straightforward to show that equations (7) are satisfied for any $\mathrm{X} \in \mathbf{X}_{\mathrm{nc}}$ using the machine (12), with the optimal added noises given by equation (8). This can be intuitively understood by considering that a phase-covariant cloning machine extracts the optimal information about states lying on the equatorial plane of the Bloch sphere, which in turn include the eigenstates of σ_1 and σ_2.

4. Conclusions

We addressed the encoding of information in the statistics of a set of observables, independently on the quantum state at the input. In a network of this kind there is no need of a precise control of the coding stage whereas, on the other hand, each gate should be *transparent*, i.e. should preserve the statistics of the transmitted observables. To this aim, the concepts of exact and approximate cloning for a class of observables have been introduced and developed. Explicit realizations of cloning machines have been found for classes of commuting observables, which in turn realize quantum non-demolition measurements of each observable belonging to the class. Two no-cloning theorems for observables have been derived which subsumes both the no-cloning for states and the impossibility of joint measurement of noncommuting observables without added noise. In addition, approximate cloning machines for classes of noncommuting observables have been also found, which realize optimal joint measurements. We found that cloning of observables is more effective than universal cloning of states to perform joint measurements of σ_1 and σ_2, since a symmetric cloning machine for states shrinks the whole Bloch vector whereas a cloning machine for observables shrinks only two components by a smaller factor. On the other hand, if one restricts attention to non-universal cloning of states, then approximate cloning of observables is equivalent to phase-covariant cloning of states.

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