Fidelity Matters: The Birth of Entanglement in the Mixing of Gaussian States

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We address the interaction of two Gaussian states through bilinear exchange Hamiltonians and analyze the correlations exhibited by the resulting bipartite systems. We demonstrate that entanglement arises if and only if the fidelity between the two input Gaussian states falls under a threshold value depending only on their purities, first moments, and the strength of the coupling. Our result clarifies the role of quantum fluctuations (squeezing) as a prerequisite for entanglement generation and provides a tool to optimize the generation of entanglement in linear systems of interest for quantum technology.

Gaussian states (GS), that is, quantum states with Gaussian Wigner functions, play a leading role in continuous variable quantum technology [1] for their extremal properties [2] and because they may be generated with current technology, in particular, in the quantum optics context [3–5]. As a consequence, much attention has been dedicated to the characterization of Gaussian entanglement [6–14]. Among the possible mechanisms to generate Gaussian entanglement, the one consisting in mixing squeezed states [15–21] is of special interest in view of its feasibility, which indeed had been crucial to achieve continuous variable teleportation [22]. The entangling power of bilinear interactions has been widely analyzed, either to optimize the generation of entanglement [23,24] or to find relations between their entanglement and purities [25] or teleportation fidelity [26,27].

In this Letter, we address bilinear, energy-conserving, i.e., exchange, interactions described by Hamiltonians of the form $H_I = g(a^\dagger b + ab^\dagger)$, where a and b are bosonic annihilation operators, $[a, a^\dagger] = 1$ and $[b, b^\dagger] = 1$, and g the coupling constant. These Hamiltonians are suitable to describe very different kinds of quantum systems, such as, e.g., two light modes in a beam splitter or a frequency converter, collective modes in gases of cold atoms [28], atom-light nondemolition measurements [29], optomechanical oscillators [27,30], nanomechanical oscillators [31], and superconducting resonators [32], all of which are of interest for quantum technology. Our analysis can be applied to all these systems and lead to very general results about the resources needed for Gaussian entanglement generation.

The bilinear Hamiltonians H_I generally describe the action of simple passive interactions, and, in view of this simplicity, their fundamental quantum properties are often overlooked. Actually, the exchange amplitudes for the quanta of one of the systems strongly depend on the statistics of the quanta of the other one and on the particle indistinguishability. This mechanism gives rise to

interference and, thus, to the birth of correlations in the output bipartite system. A question arises about the nature of these correlations, depending on the parameters of the input signals and coupling constant. In this Letter, motivated by recent results on the dynamics of bipartite GS through bilinear interactions [33,34] and by their experimental demonstration [35], we investigate the relation between the properties of two input GS and the correlations exhibited by the output state. Our main result is that entanglement arises if and only if the fidelity between the two input states falls under a threshold value depending only on their purities, first-moment values, and the strength of the coupling. Our analysis provides a direct link between the mismatch in the quantum properties of the input signals and the creation of entanglement, thus providing a better understanding of the process leading to the generation of nonclassical correlations. In fact, if, on the one hand, it is well known that squeezing is a necessary resource to create entanglement [16,17,24], on the other hand, in this Letter we show what is the actual role played by the squeezing that is making the two input GS different enough to entangle the output system.

The most general single-mode Gaussian state can be written as $\varrho = \varrho(\alpha, \xi, N) = D(\alpha)S(\xi)\nu_{\mathrm{th}}(N)S^\dagger(\xi)D^\dagger(\alpha)$, where $S(r) = \exp[\frac{1}{2}(\xi a^{\dagger 2} - \xi^* a^2)]$ and $D(\alpha) = \exp[\alpha a^\dagger - \alpha^* a)]$ are the squeezing operator and the displacement operator, respectively, and $\nu_{\mathrm{th}}(N) = (N)^{a^\dagger a}/(1 + N)^{a^\dagger a + 1}$ is a thermal equilibrium state with N average number of quanta, a being the annihilation operator. Up to introducing the vector of operator $\mathbf{R}^T = (R_1, R_2) \equiv (q, p)$, where $q = (a + a^\dagger)/\sqrt{2}$ and $p = (a - a^\dagger)/(i\sqrt{2})$ are the so-called quadrature operators, we can fully characterize ϱ by means of the first-moment vector $\bar{\mathbf{X}}^T = \langle \mathbf{R}^T \rangle = \sqrt{2}(\mathrm{Re}[\alpha], \mathrm{Im}[\alpha])$, with $\langle A \rangle = \mathrm{Tr}[A\varrho]$, and of the 2×2 covariance matrix (CM) $\boldsymbol{\sigma}$, with $[\boldsymbol{\sigma}]_{hk} = \frac{1}{2}\langle R_h R_k + R_k R_h \rangle - \langle R_h \rangle \langle R_k \rangle$, $k = 1, 2$, which explicitly reads $[\boldsymbol{\sigma}]_{kk} = (2\mu)^{-1}[\cosh(2r) - (-1)^k \cos(\psi) \sinh(2r)]$ for $k = 1, 2$ and $[\boldsymbol{\sigma}]_{12} = [\boldsymbol{\sigma}]_{21} = -(2\mu)^{-1} \sin(\psi) \sinh(2r)$, where we

put $\xi = re^{i\psi}$, $r, \psi \in \mathbb{R}$ and introduced the purity of the state $\mu = \mathrm{Tr}[\varrho^2] = (1 + 2N)^{-1}$. Since we are interested in the dynamics of the correlations, which are not affected by the first moment, we start addressing GS with zero first moments ($\alpha = 0$). The general case will be considered later on in this Letter.

When two uncorrelated, single-mode GS ϱ_k with CMs $\boldsymbol{\sigma}_k$, $k = 1, 2$, interact through the bilinear Hamiltonian H_I, the 4×4 CM $\boldsymbol{\Sigma}$ of the evolved bipartite state $\varrho_{12} = U_g(t)\varrho_1 \otimes \varrho_2 U_g^\dagger(t)$, $U_g(t) = \exp\{-iH_I t\}$ being the evolution operator, can be written in the following block-matrix form [1]:

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_1 & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{12} & \boldsymbol{\Sigma}_2 \end{pmatrix}, \quad \begin{aligned} \boldsymbol{\Sigma}_1 &= \tau\boldsymbol{\sigma}_1 + (1 - \tau)\boldsymbol{\sigma}_2, \\ \boldsymbol{\Sigma}_2 &= \tau\boldsymbol{\sigma}_2 + (1 - \tau)\boldsymbol{\sigma}_1, \\ \boldsymbol{\Sigma}_{12} &= \tau(1 - \tau)(\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1), \end{aligned} \quad (1)$$

$\tau = \cos^2(gt)$ being an effective coupling parameter, and where the presence of a nonzero covariance term $\boldsymbol{\Sigma}_{12}$ suggests the emergence of quantum or classical correlations between the two systems. Since $\boldsymbol{\Sigma}_{12}$ depends on the difference between the input state CMs, a question naturally arises about the relation between the "similarity" of the inputs and the birth of (nonlocal) correlations. In this Letter, we answer this question and demonstrate that entanglement arises if and only if the fidelity between the two input GS falls under a threshold value, which depends only on their purities, the value of the first moments, and the coupling τ.

Let us now consider the pair of uncorrelated, single-mode GS $\varrho_k = \varrho(\xi_k, N_k)$, $k = 1, 2$, and assume, without loss of generality, $\xi_1 = r_1$ and $\xi_2 = r_2 e^{i\psi}$, with $r_k, \psi \in \mathbb{R}$. After the interaction, we found that the presence of entanglement at the output is governed by the sole fidelity $F(\varrho_1, \varrho_2) = [\mathrm{Tr}(\sqrt{\sqrt{\varrho_1}\varrho_2\sqrt{\varrho_1}})]^2$ between the inputs. Our results may be summarized by the following theorem.

Theorem 1.—The state $\varrho_{12} = U_g(t)\varrho_1 \otimes \varrho_2 U_g^\dagger(t)$, resulting from the mixing of two GS with zero first moments, $\varrho_1(r_1, N_1)$ and $\varrho_2(r_2 e^{i\psi}, N_2)$, is entangled if and only if the fidelity $F(\varrho_1, \varrho_2)$ between the inputs falls below a threshold value $F_e(\mu_1, \mu_2; \tau)$, which depends only on their purities $\mu_k = \mathrm{Tr}[\varrho_k^2] = (1 + 2N_k)^{-1}$, $k = 1, 2$, and on the effective coupling parameter $\tau = \cos^2(gt)$.

Proof.—In order to prove the theorem, we recall that a bipartite Gaussian state ϱ_{12} is entangled if and only if the minimum symplectic eigenvalue $\tilde{\lambda}$ of CM associated with the partially transposed state is $\tilde{\lambda} < 1/2$ [6]. Moreover, without loss of generality, we can address the scenario in which r_k and N_k, $k = 1, 2$, are fixed and we let ψ vary in the interval $[0, 2\pi]$.

First of all, we prove that $\tilde{\lambda} < 1/2 \Rightarrow F(\varrho_1, \varrho_2) < F_e(\mu_1, \mu_2; \tau)$ (necessary condition). As we will see, this will allow us to find the analytic expression of the threshold $F_e(\mu_1, \mu_2; \tau)$, which will be used to prove the sufficient

condition, i.e., $F(\varrho_1, \varrho_2) < F_e(\mu_1, \mu_2; \tau) \Rightarrow \tilde{\lambda} < 1/2$. Figure 1 shows the typical behavior of $\tilde{\lambda}$ and of the fidelity F as a function of the squeezing phase ψ for fixed r_k and N_k, $k = 1, 2$ (here we do not report their analytic expressions since they are quite cumbersome). As one can see, both $\tilde{\lambda}$ and F are monotonic, decreasing (increasing) functions of ψ in the interval $[0, \pi]$ ($[\pi, 2\pi]$, respectively) and have a minimum at $\psi = \pi$, whose actual value depends on both r_k and N_k but not on τ. In our case, one finds that, for fixed r_k and N_k, $k = 1, 2$, if $\min_\psi \tilde{\lambda} < 1/2$, then there exists a threshold value $\psi_e \equiv \psi_e(r_1, \mu_1, r_2, \mu_2, \tau)$:

$$\psi_e = \arccos\left\{\frac{\cosh(2r_1)\cosh(2r_2) - f(\mu_1, \mu_2, \tau)}{\sinh(2r_1)\sinh(2r_2)}\right\},$$

where we introduced

$$f(\mu_1, \mu_2, \tau) = \frac{1 + \mu_1^2\mu_2^2 - (\mu_1^2 + \mu_2^2)(1 - 2\tau)^2}{8\mu_1\mu_2\tau(1 - \tau)},$$

and $\mu_k = \mathrm{Tr}[\varrho_k^2] = (1 + 2N_k)^{-1}$, $k = 1, 2$, are the purities of the inputs, such that if $\psi \in (\psi_e, 2\pi - \psi_e)$, then $\tilde{\lambda} < 1/2$; i.e., ϱ_{12} is entangled. Since the fidelity between the two GS ϱ_k, characterized by the CMs $\boldsymbol{\sigma}_k$, $k = 1, 2$ (and zero first moments), is given by [36] $F(\varrho_1, \varrho_2) = (\sqrt{\Delta + \delta} - \sqrt{\delta})^{-1}$, where $\Delta = \det[\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2]$ and $\delta = 4\prod_{k=1}^2(\det[\boldsymbol{\sigma}_k] - \frac{1}{4})$, the threshold value $F_e \equiv F_e(\mu_1, \mu_2; \tau)$ of the fidelity is thus obtained by setting $\psi = \psi_e$ and explicitly reads

$$F_e = \frac{4\mu_1\mu_2\sqrt{\tau(1 - \tau)}}{\sqrt{g_- + 4\tau(1 - \tau)g_+} - \sqrt{4\tau(1 - \tau)g_-}}, \quad (2)$$

where $g_\pm \equiv g_\pm(\mu_1, \mu_2) = \prod_{k=1,2}(1 \pm \mu_k^2)$. The threshold depends only on τ and on the purities μ_k of the input GS and is independent of the squeezing parameters r_k, despite the fact that ψ_e does. Finally, if $\tilde{\lambda} < 1/2$, i.e., ϱ_{12}
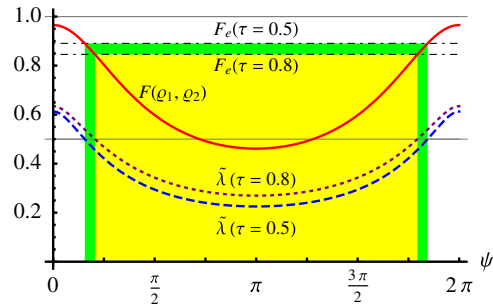

FIG. 1 (color online). Plot of the fidelity $F(\varrho_1, \varrho_2)$ (red, solid line) between the two input states and of the minimum symplectic eigenvalue $\tilde{\lambda}$ as a function of ψ for $\tau = 0.5$ (blue, dashed line) and $\tau = 0.8$ (purple, dotted line). The other involved parameters are $\xi_1 = 0.5$, $N_1 = 0.2$, $\xi_2 = 0.7e^{i\psi}$, and $N_2 = 0.3$. The colored regions denote the ranges of ψ leading to an entangled state for the given τ, while the horizontal dot-dashed lines refer to the corresponding thresholds F_e.

is entangled, then $F(\varrho_1, \varrho_2) < F_e(\mu_1, \mu_2; \tau)$. This concludes the first part of the proof.

Now we focus on the sufficient condition, i.e., $F(\varrho_1, \varrho_2) < F_e(\mu_1, \mu_2; \tau) \Rightarrow \tilde{\lambda} < 1/2$. Thanks to the first part of the theorem and since both F and $\tilde{\lambda}$ are continuous functions of ψ, for fixed r_k and N_k, $k = 1, 2$, which have a minimum in $\psi = \pi$, it is enough to show that $F_{\min} \equiv \min_\psi F(\varrho_1, \varrho_2) < F_e(\mu_1, \mu_2; \tau) \Rightarrow \lambda_{\min} \equiv \min_\psi \tilde{\lambda} < 1/2$. We have

$$F_{\min} = \frac{2\mu_1 \mu_2}{\sqrt{1 + \mu_1^2 \mu_2^2 + 2\mu_1 \mu_2 \cosh[2(r_1 + r_2)]} - \sqrt{g_-}},$$

where g_- is the same as in Eq. (2), and

$$\tilde{\lambda}_{\min} = \frac{1}{2} \frac{[\gamma - \sqrt{\gamma^2 - (2\mu_1 \mu_2)^2}]^{1/2}}{\sqrt{2}\mu_1 \mu_2},$$

with $\gamma = (\mu_1^2 + \mu_2^2)(1 - 2\tau)^2 + 8\mu_1 \mu_2 \tau(1 - \tau)\cosh[2(r_1 + r_2)]$, respectively. The inequality $F_{\min} < F_e(\mu_1, \mu_2; \tau)$, where $F_e(\mu_1, \mu_2; \tau)$ is given in Eq. (2), is satisfied if $\gamma > 1 + \mu_1^2 \mu_2^2$, which leads to $\tilde{\lambda}_{\min} < 1/2$, as one may verify after a straightforward calculation. Now, since $\tilde{\lambda}$ is a continuous function of ψ, there exists a range of values centered at $\psi = \pi$, where the minimum occurs, in which $\tilde{\lambda} < 1/2$ and, thus, $F(\varrho_1, \varrho_2) < F_e(\mu_1, \mu_2; \tau)$, because of the first part of the theorem (necessary condition). This concludes the proof of the theorem. ∎

As a matter of fact, the presence of nonzero first moments does not affect the nonclassical correlations exhibited by a bipartite Gaussian state, which depend only on the CM [1]. Thus, we can state the following straightforward corollary.

Corollary 1.—If $\bar{\mathbf{X}}_k^T = \mathrm{Tr}[(q_k, p_k)\varrho_k] \neq 0$, where $q_k = (a_k + a_k^\dagger)/\sqrt{2}$ and $p = (a_k^\dagger - a_k)/(i\sqrt{2})$ are the quadrature operators of the system $k = 1, 2$, then the state $\varrho_{12} = U_g(t)\varrho_1 \otimes \varrho_2 U_g^\dagger(t)$ is entangled if and only if

$$F(\varrho_1, \varrho_2) < \Gamma(\bar{\mathbf{X}}_1, \bar{\mathbf{X}}_2)F_e(\mu_1, \mu_2; \tau), \qquad (3)$$

where $F_e(\mu_1, \mu_2; \tau)$ is still given in Eq. (2) and

$$\Gamma(\bar{\mathbf{X}}_1, \bar{\mathbf{X}}_2) = \exp[-\tfrac{1}{2}\bar{\mathbf{X}}_{12}^T(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)^{-1}\bar{\mathbf{X}}_{12}], \qquad (4)$$

where $\bar{\mathbf{X}}_{12} = (\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2)$.

Proof.—The proof follows from Theorem 1 by noting that the presence of nonzero first moments does not modify the evolution of the CM, whereas the expression of the fidelity becomes [36] $F(\varrho_1, \varrho_2) = \Gamma(\bar{\mathbf{X}}_1, \bar{\mathbf{X}}_2) \times (\sqrt{\Delta + \delta} - \sqrt{\delta})^{-1}$, where Δ and δ have been defined above. ∎

Theorem 1 states that if the two Gaussian inputs are "too similar," the correlations induced by the interaction are local, i.e., may be mimicked by local operations performed on each of the systems. The extreme case corresponds to mixing a pair of identical GS: In this case the interaction

produces no effect, since the output state is identical to the input one [33,34], i.e., a factorized state made of two copies of the same input states, and we have no correlations at all at the output. Notice that for pure (zero mean) states the threshold on fidelity reduces to $F_e(1, 1; \tau) = 1 \ \forall \ \tau$; namely, any pair of not identical (zero mean) pure GS gives rise to entanglement at the output. On the contrary, two thermal states $\nu_k \equiv \nu_{\mathrm{th}}(N_k)$, $k = 1, 2$, as inputs, i.e., the most classical GS, lead to $F(\nu_1, \nu_2) > F_e(\mu_1, \mu_2; \tau)$: This fact, thanks to the Theorem 1, shows that we need to squeeze one or both of the classical inputs in order to make the states different enough to give rise to entanglement. Notice, finally, that the thresholds in Eqs. (2) and (3) involve strict inequalities and when the fidelity between the inputs is exactly equal to the threshold the output state is separable.

The threshold $F_e(\mu_1, \mu_2; \tau)$ is symmetric under the exchange $\mu_1 \leftrightarrow \mu_2$ and if one of the two state is pure, i.e., if $\mu_k = 1$, then $F_e(\tau) = \sqrt{2}\mu_h/\sqrt{1 + \mu_h^2}$, with $h \neq k$; i.e., the threshold no longer depends on τ.

For what concerns Gaussian entanglement, i.e., the resource characterized by the violation of Simon's condition on CM [6], our results also apply to the case of non-Gaussian input signals, upon evaluating the fidelity between the GS with the same CMs of the non-Gaussian ones. In fact, violation of Simon's condition is governed only by the behavior of the CM independently on the Gaussian character of the inputs states. On the other hand, identical non-Gaussian states may give rise to entangled output, the mixing of two single-photon states in quantum optical systems being the paradigmatic example [37]. In other words, the entanglement raising from the mixing of two identical non-Gaussian states cannot be detected by Simon's condition on CM.

Up to now we have considered the correlation properties of the output states with respect to the fidelity between the input ones. However, similar relations may be found for the fidelities $F(\varrho_h, \tilde{\varrho}_k)$, $k, h = 1, 2$, between the input and output states, respectively, where $\tilde{\varrho}_h = \mathrm{Tr}_k[\varrho_{12}]$, with $h \neq k$, are the reduced density matrices of the output states taken separately. In this case we found that the output is entangled if and only if $F(\varrho_h, \tilde{\varrho}_k) < F_e(\varrho_h, \tilde{\varrho}_k)$ where all the thresholds $F_e(\varrho_h, \tilde{\varrho}_k)$ still depend only on μ_1, μ_2, and τ (here we put as arguments the density matrices in order to avoid confusion with the previous thresholds). The analytic expressions of $F_e(\varrho_h, \tilde{\varrho}_k)$ are cumbersome and are not reported explicitly, but we plot in Fig. 2 the input-output fidelities and the corresponding thresholds for a particular choice of the involved parameters. If we look at the interaction between the two systems as a quantum noisy channel for one of the two, namely, $\varrho_k \rightarrow \mathcal{E}(\varrho_k) \equiv \mathrm{Tr}_h[\varrho_{12}]$, $h \neq k$, then the birth of the correlations between the outgoing systems corresponds to a reduction of the input-output fidelity: The correlations arise at the expense of the information contained in the
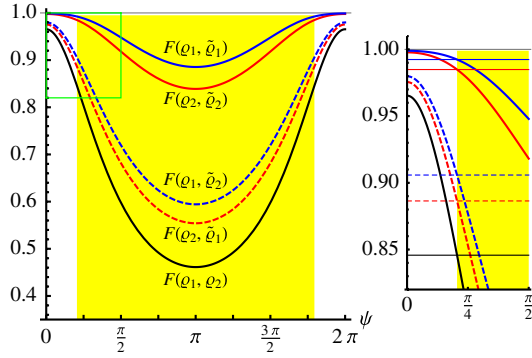
FIG. 2 (color online). Plot of the fidelities $F(\varrho_h, \tilde{\varrho}_k)$ for $\tau = 0.8$ and the same choice of the other involved parameters as in Fig. 1. The yellow region shows the interval of values of ψ leading to an entangled state. The right panel is a magnification of the green, boxed region of the left panel: The horizontal lines refer to the corresponding thresholds $F_e(\varrho_h, \tilde{\varrho}_k)$. See the text for details.

input signals. In turn, this result may be exploited for decoherence control and preservation of entanglement using bath engineering [35].

In conclusion, we have analyzed the correlations exhibited by two initially uncorrelated GS which interact through a bilinear exchange Hamiltonian. We found that entanglement arises if and only if the fidelity between the two inputs falls under a threshold value depending only on their purities, the first moments, and the coupling constant. Similar relations have been obtained for the input-output fidelities. Our theorems clarify the role of squeezing as a prerequisite to obtain entanglement out of bilinear interactions and provide a tool to optimize the generation of entanglement by passive (energy-conserving) devices. Our results represent progress for the fundamental understanding of nonclassical correlations in continuous variable systems and may find practical applications in quantum technology. Because of the recent advancement in the generation and manipulation of GS, we foresee experimental implementations in optomechanical and quantum optical systems.

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