



Squeezed vacuum as a universal quantum probe

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ARTICLE INFO

Article history:

Received 10 December 2008

Received in revised form 8 January 2009

Accepted 19 January 2009

Available online 22 January 2009

Communicated by P.R. Holland

ABSTRACT

We address local quantum estimation of bilinear Hamiltonians probed by Gaussian states. We evaluate the relevant quantum Fisher information (QFI) and derive the ultimate bound on precision. Upon maximizing the QFI we found that single- and two-mode squeezed vacuum represent an optimal and universal class of probe states, achieving the so-called Heisenberg limit to precision in terms of the overall energy of the probe. We explicitly obtain the optimal observable based on the symmetric logarithmic derivative and also found that homodyne detection assisted by Bayesian analysis may achieve estimation of squeezing with near-optimal sensitivity in any working regime.

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1. Introduction

In this Letter we address quantum estimation of unitary operations for continuous variable systems. In particular we analyze the estimation of the interaction parameter θ for unitaries of the form $U_\theta = \exp(-i\theta G)$ where G is a linear or bilinear bosonic Hamiltonian of the form $G = a^\dagger b + ab^\dagger$, $G = a^\dagger b^\dagger + ab$, or $G = a^{\dagger 2} + a^2$, $[a, a^\dagger] = 1$ and $[b, b^\dagger] = 1$ being mode operators. We are interested in evaluating the ultimate bound on precision (sensitivity), i.e. the smallest value of the parameter that can be discriminated, and to determine the optimal measurement achieving those bounds.

As a matter of fact, linear and bilinear interactions for bosonic systems are a key ingredient for the development of continuous variable quantum information processing [1–4]. They are usually realized by means of parametric processes, as single- and two-mode squeezing, or by linear optical elements such as phase-shifting and two-mode mixing. The precise characterization of linear optical gates is also of interest in interferometry [6–8], absorption measurement [9] and characterization of detectors [10].

In general, interaction parameters cannot be directly accessed experimentally, and the estimation process consists in *probing* the interaction by a known quantum signal ϱ_0 , which is measured after the interaction (see Fig. 1). The relevant constraint in the optimization of those schemes concerns the total energy of the probe, which should be kept as low as possible to avoid any possible modification or degradation of the gate itself. Overall, the problem we are facing is that of devising the optimal measurement,

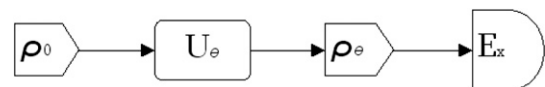


Fig. 1. General scheme for the indirect estimation of the unitary U_θ probed by the signal ϱ_0 .

i.e. a positive operator-valued measure (POVM) $\{E_x\}_{x \in \mathcal{X}}$, to be performed on the probe $\varrho_\theta = U_\theta \varrho_0 U_\theta^\dagger$ after the interaction, at fixed energy $N = \text{Tr}[\varrho_0 \sum_j n_j]$ of the incoming signal, $\sum_j n_j$ being the total number operator of the involved modes.

The above problem may be properly addressed in the framework of quantum estimation theory (QET) [11–13], which provides analytical tools to find the optimal measurement according to some given criterion. In turn, there are two main paradigms in QET: Global QET looks for the POVM minimizing a suitable cost functional, averaged over all possible values of the parameter to be estimated. The result of a global optimization is thus a single POVM, independent on the value of the parameter. On the other hand, *local* QET looks for the POVM maximizing the Fisher information, thus minimizing the variance of the estimator, at a fixed value of the parameter [14,15]. Roughly speaking, one may expect local QET to provide better performances since the optimization concerns a specific value of the parameter, with some adaptive or feedback mechanism assuring the achievability of the ultimate bound [16]. Global QET has been mostly applied to find optimal measurements and to evaluate lower bounds on precision for the estimation of parameters imposed by unitary transformations. For bosonic systems these include single-mode phase [17,18], displacement [19], squeezing [20,21] as well as two-mode transformations, e.g. bilinear coupling [9]. Local QET has been applied to the estimation of quantum phase [24] and to estimation problems with

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open quantum systems and non unitary processes [25]; to finite dimensional systems [26], to optimally estimate the noise parameter of depolarizing [27] or amplitude-damping [28], and for continuous variable systems to estimate the loss parameter of a quantum channel [29].

In this Letter we consider the estimation the interaction parameters of bilinear bosonic Hamiltonians from the perspective of local QET. In particular, we focus our attention to measurement schemes as in Fig. 1 with the probe state chosen within the set of Gaussian states [1–3,5,30], which represents a class of signals achievable with current technology. We evaluate the relevant quantum Fisher information (QFI) and derive the ultimate bound on precision. Upon maximizing the QFI we found that single- and two-mode squeezed vacuum represents an optimal and universal class of probe states, achieving the so-called Heisenberg limit to precision in terms of the overall energy of the probe. Remarkably, by comparison with results coming from global optimization of the measurement [9,20,21] we found that Gaussian states are effective resources, which allow to achieve the ultimate bound on precision. Besides, we found that homodyne detection assisted by Bayesian analysis may achieve near-optimal sensitivity in any working regime.

The Letter is structured as follows: in the next section we briefly review local quantum estimation theory with some remarks on the implementation of the optimal measurements. In Section 3 we evaluate the optimal measurements and the corresponding bounds on precision for the local estimation of bilinear couplings using Gaussian probes. In Section 4 we address estimation of squeezing using homodyne detection and Bayesian analysis and show that near-optimal precision may be achieved in any working regime. In Section 5 we compare our results with those coming from global estimation and close the Letter with some concluding remarks.

2. Local quantum estimation theory

In this section we review some concepts of local quantum estimation theory [22,23] which will be used in the rest of the Letter. As a matter of fact, many quantities of interest in different branches of physics cannot be directly accessed experimentally, either in principle, as in the case of field measurement [31], or due to experimental impediments. In these cases, one has to indirectly estimate the value of those physical parameters by measuring a different observable, somehow related to the quantity of interest. This indirect procedure of parameter estimation implies an additional uncertainty for the measured value, that cannot be avoided even in optimal conditions. The aim of quantum estimation theory is to optimize the inference procedure by minimizing this additional uncertainty. In the classical theory of parameter estimation the Cramer–Rao bound [32] establishes a lower bound for the variance of any unbiased estimator $\hat{\theta}$ of the parameter θ . This lower bound is given by the inverse of the so-called Fisher information (FI):

$$\Delta\theta^2 \geq \frac{1}{F(\theta)} \quad (1)$$

where the Fisher information is defined as

$$F(\theta) = \sum_x p(x|\theta) \left(\frac{\partial \ln p(x|\theta)}{\partial \theta} \right)^2. \quad (2)$$

Here θ is the parameter to be estimated, and x denotes the outcome of the measurement of the quantity X related to θ . The notation $p(x|\theta)$ indicates the conditional probability of obtaining the value x when the parameter has the value θ .

A quantum analogue to Eq. (2) may be found starting from the Born rule

$$p(x|\theta) = \text{tr}[E_x \rho_\theta] \quad (3)$$

where E_x are the elements of a positive operator-valued measure (POVM) and ρ_θ is the density operator, parametrized by the quantity of interest, describing the quantum state of the measured system. The Fisher information is then rewritten as

$$F(\theta) = \sum_x \frac{\text{Re Tr}[\rho_\theta E_x \Lambda_\theta]^2}{\text{Tr}[E_x \rho_\theta]} \quad (4)$$

where we introduced the Symmetric Logarithmic Derivative (SLD) Λ_θ , which is the self-adjoint operator defined as

$$\frac{\Lambda_\theta \rho_\theta + \rho_\theta \Lambda_\theta}{2} \equiv \frac{\partial \rho_\theta}{\partial \theta}. \quad (5)$$

It can then be shown [14,15] that the Fisher information (4) is upper bounded by the so-called *quantum Fisher information* (QFI):

$$F \leq H \equiv \text{Tr}[\rho_\theta \Lambda_\theta^2]. \quad (6)$$

In turn, the quantity $1/H$ represents an ultimate lower bound on precision for any quantum measurement (followed by any classical data processing) aimed to estimate the parameter θ . The SLD is itself an optimal measurement, that is, using the POVM E_x obtained from the projectors over the eigenbasis of Λ_θ we saturate the inequality (6).

In this work we will focus on systems where the dependence of ρ_θ from the parameter θ is generated by a family of unitary transformations: $\rho_\theta = U_\theta \rho_0 U_\theta^\dagger$ where $U_\theta = \exp(-i\theta G)$, G is the Hamiltonian that generates the transformation and ρ_0 is a given quantum state used to probe the Hamiltonian process. In this case it is possible to obtain an explicit formula for the SLD operator and the QFI. At first we take the eigenbasis of ρ_0 : $\rho_0 = \sum_k p_k |\psi_k\rangle\langle\psi_k|$. From (5) we can rewrite Λ_θ in this basis as follows

$$\Lambda_\theta = 2i \sum_{jk} G_{jk} \frac{p_j - p_k}{p_j + p_k} U_\theta |\psi_j\rangle\langle\psi_k| U_\theta^\dagger \quad (7)$$

where $G_{jk} = \langle\psi_j|G|\psi_k\rangle$ are the matrix elements of the generator G . Eq. (7) shows that Λ_θ depends on θ only through the unitary transformation U_θ . As a consequence it is possible to define the operator Λ_0 , independent from θ , such that $\Lambda_\theta = U_\theta \Lambda_0 U_\theta^\dagger$. It also follows that the quantum Fisher information is independent from θ . In fact, $H = \text{Tr}[\rho_\theta \Lambda_\theta^2] = \text{Tr}[U_\theta \rho_0 U_\theta^\dagger U_\theta \Lambda_0^2 U_\theta^\dagger] = \text{Tr}[\rho_0 \Lambda_0^2]$. Explicit formulas to calculate H may be given in the eigenbasis of ρ_0

$$H = 4 \sum_{nk} p_n \frac{p_n - p_k}{p_n + p_k} G_{nk}^2 \quad (8)$$

$$= 4\langle G^2 \rangle - 8 \sum_{nk} \frac{p_k p_n}{p_n + p_k} G_{nk} G_{kn}. \quad (9)$$

As it may be seen from the above formulas a probe described by a pure state $\rho_0 = |\psi_0\rangle\langle\psi_0|$ maximizes the QFI. In those cases the QFI reduces to the variance of the generating Hamiltonian G , i.e. $H = 4\Delta G^2$. In addition, for a pure state we have $\rho_\theta^2 = \rho_\theta$ and thus $\Lambda_0 = 2i[\rho_0, G]$, i.e.

$$\Lambda_0 = 2i \sum_k (G_{0k} |\psi_0\rangle\langle\psi_k| - G_{k0} |\psi_k\rangle\langle\psi_0|). \quad (10)$$

3. Estimation of bilinear couplings

In this section we address the case of local estimation of various bilinear couplings (single- and two-mode squeezing, two-mode mixing) using Gaussian probes at fixed energy.

3.1. Single-mode squeezing

Here we consider the estimation of the parameter θ imposed by the unitary transformation $\exp(-i\theta G)$, where G is the generating Hamiltonian

$$G = \frac{1}{2}(a^{\dagger 2} + a^2). \quad (11)$$

We analyze the precision achievable in the estimation of θ by using different classes of (Gaussian) probe states. The measurement aimed to estimate θ is made on the transformed state

$$\rho_{\theta} = \exp(-i\theta G)\rho_0 \exp(-i\theta G). \quad (12)$$

At first we analyze the case of a Gaussian pure probe, i.e. a squeezed coherent state of the form $\rho_0 = |\psi_0\rangle\langle\psi_0|$ with $|\psi_0\rangle = S(r)D(\alpha)|0\rangle$, where

$$D(\alpha) = \exp[\alpha a^{\dagger} - \alpha^* a], \quad (13)$$

$$S(r) = \exp\left[\frac{r}{2}(a^{\dagger 2} - a^2)\right] \quad (14)$$

and where, without loss of generality, we have chosen a real squeezing parameter r and a complex displacement $\alpha = xe^{i\phi}$. Since ρ_0 is a pure state, the QFI will be given by

$$H = 4\Delta G^2 = \langle (a^{\dagger 2} + a^2)^2 \rangle - \langle a^{\dagger 2} + a^2 \rangle^2. \quad (15)$$

Upon evaluating all the expectation values we obtain:

$$\Delta G^2 = -x^2 \cos 2\phi \sinh 2r + (2N + 1) \sinh^2 r + N + \frac{1}{2} \quad (16)$$

where $N \equiv \langle a^{\dagger} a \rangle = x^2 + \sinh^2 r$ denotes the overall energy of the probe signal. The signal optimization corresponds to the maximization of H over the state parameter with the constraint of fixed N . The phase ϕ is a free parameter since it does not influence the total energy. The choice $\cos 2\phi = -1$ maximizes H leading to

$$H = 4(N - \sinh^2 r) \sinh 2r + 4(2N + 1) \sinh^2 r + 4N + 2 \quad (17)$$

which grows monotonically with $\sinh^2 r$ and achieve its maximum

$$H_{\max} = 8N^2 + 8N + 2 \quad (18)$$

for $\sinh^2 r = N$ and $\alpha = 0$, corresponding to a squeezed vacuum probe. Thus, to obtain the maximum accuracy in the estimation of θ it is more efficient to use all the energy in squeezing rather than field amplitude. Since using mixed states cannot improve the quality of estimation we conclude that squeezed vacuum is the optimal Gaussian probe. Summarizing, the most convenient way of estimating a squeezing parameter is to probe the transformation by a squeezed vacuum probe. The corresponding QFI scales as $H \simeq 8N^2$ in terms of the overall energy of the probe.

3.2. Two-mode mixing

Here we consider the case where the generator G is the two-mode mixing Hamiltonian:

$$G = a^{\dagger} b + ab^{\dagger}. \quad (19)$$

The optimal probe is made by a pure state. However, in order to see explicitly the effects of thermal contribution we consider a probe state made by factorized squeezed thermal states:

$$\rho_0 = [S_a(r) \otimes S_b(s)] \nu_a \otimes \nu_b [S_a^{\dagger}(r) \otimes S_b^{\dagger}(s)] \quad (20)$$

where $\nu_{a,b}$ are the density matrices of thermal states:

$$\nu_k = \frac{1}{(\bar{n}_k + 1)} \sum_n \left(\frac{\bar{n}_k}{\bar{n}_k + 1} \right)^n |n\rangle\langle n|. \quad (21)$$

Eq. (8) rewrites as

$$H = 4 \sum_{jkmn} p_{jk} \frac{p_{jk} - p_{mn}}{p_{jk} + p_{mn}} G_{jkmn} G_{mnjk} \quad (22)$$

where $p_{kn} = p_k p_n$ are the thermal coefficients. The Heisenberg evolution of the mode operators allows to calculate the matrix elements of G

$$\begin{aligned} G_{jkmn} &= \langle j, k | S_a^{\dagger}(r) S_b^{\dagger}(s) (a^{\dagger} b + ab^{\dagger}) S_a(r) S_b(s) | m, n \rangle \\ &= \cosh(r+s) (\sqrt{(m+1)(k+1)} \delta_{j=m+1} \delta_{n=k+1} \\ &\quad + \sqrt{(j+1)(n+1)} \delta_{m=j+1} \delta_{k=n+1}) \\ &\quad + \sinh(r+s) (\sqrt{(j+1)(k+1)} \delta_{m=j+1} \delta_{n=k+1} \\ &\quad + \sqrt{(m+1)(n+1)} \delta_{j=m+1} \delta_{k=n+1}). \end{aligned} \quad (23)$$

The resulting QFI reads as follows

$$\begin{aligned} H &= 4 \left[\sinh^2(r+s) \left(\frac{(\bar{n}_a - \bar{n}_b)^2}{2\bar{n}_a \bar{n}_b + \bar{n}_a + \bar{n}_b} + \frac{(\bar{n}_a + \bar{n}_b + 1)^2}{2\bar{n}_a \bar{n}_b + \bar{n}_a + \bar{n}_b + 1} \right) \right. \\ &\quad \left. + \frac{(\bar{n}_a - \bar{n}_b)^2}{2\bar{n}_a \bar{n}_b + \bar{n}_a + \bar{n}_b} \right]. \end{aligned} \quad (24)$$

The total photon number of the system is given by the sum

$$N = \bar{n}_a + \bar{n}_b + (2\bar{n}_a + 1) \sinh^2 r + (2\bar{n}_b + 1) \sinh^2 s. \quad (25)$$

The QFI (24) has no point of gradient zero that is compatible with the energy bound (25). Since it is a continuous function, to find its maximum we need to investigate its value at the borders of its domain. Let us first consider the case $\bar{n}_a = \bar{n}_b = 0$, i.e. a probe made by two disentangled squeezed vacuums. The energy writes as $N = \sinh^2 r + \sinh^2 s$ and the QFI becomes $H = 4 \sinh^2(r+s)$. The maximum is reached when $r = s$, which gives

$$H_1 = 4N^2 + 8N. \quad (26)$$

The second case is given by two thermal states, when $r = s = 0$. The QFI achieves its maximum $H_2 = 4N$ for $\bar{n}_a = 0$ $\bar{n}_b = 0$, i.e. when one of the states is at zero temperature. In this case the scaling is dramatically degraded and the corresponding precision is at the so-called shot-noise limit. The last possible combination is given by a thermal state and a squeezed vacuum, i.e. $r = 0$ and $\bar{n}_b = 0$. The total energy is now given by $N = \bar{n}_a + \sinh^2 s$, and the QFI by $H = 4[N + 2\bar{n}_a(N - \bar{n}_a)]$. The maximum is obtained when the energy is equally distributed between the thermal state and the squeezed state, $\bar{n}_a = \sinh^2 s = \frac{N}{2}$:

$$H_3 = 2N^2 + 4N. \quad (27)$$

The Fisher information is smaller than the case of two squeezed vacuum though the scaling as N^2 still holds. Overall, we see that the maximum Fisher information is obtained using two equally squeezed vacuums. Since this is the combination of two pure states, we can use (10) to obtain the SLD that realizes the optimal measurement:

$$\Lambda_0 = 2i\sqrt{N(N+2)} S_a S_b (|0, 0\rangle\langle 1, 1| - |1, 1\rangle\langle 0, 0|) S_a^{\dagger} S_b^{\dagger}. \quad (28)$$

In order to investigate the role of entanglement in the estimation procedure we consider the probe prepared the state

$$\rho_0 = |\psi_{00}\rangle\langle\psi_{00}|, \quad (29)$$

$$|\psi_{jk}\rangle \equiv |\psi_{jk}(\phi, \lambda)\rangle = U(\phi) T(\lambda) |j, k\rangle \quad (30)$$

where

$$U(\phi) = \exp[-i\phi(ab^{\dagger} + a^{\dagger}b)], \quad (31)$$

$$T(\lambda) = \exp[-i\lambda(ab + a^{\dagger}b^{\dagger})]. \quad (32)$$

The probe is transformed into $\rho_\theta = e^{-i\theta G} \rho_0 e^{i\theta G}$ where again we are using the generator (19). Since we are dealing with a pure state, the QFI $H = 4\Delta G^2$ is given by

$$H = 16 \cosh^2 |\lambda| \sinh^2 |\lambda| (1 - 4 \cos^2 \phi \sin^2 \phi) \quad (33)$$

where we have used the Heisenberg evolution of the mode operators. The energy constraint is given by

$$N = \langle a^\dagger a \rangle + \langle b^\dagger b \rangle = 2 \sinh^2 |\lambda|, \quad (34)$$

thus the QFI can be rewritten as

$$H = (4N^2 + 8N)(1 - 4 \cos^2 \phi \sin^2 \phi). \quad (35)$$

Since $\partial_\phi N = 0$, we can freely choose a value for ϕ , in order to maximize H . The maximum Fisher information is obtained for $\cos 4\phi = 1$ and corresponds to $H = 4N^2 + 8N$, i.e. no improvement is obtained using an entangled probe. The SLD operator that realizes the optimal measurement is found using (10):

$$\begin{aligned} A_0 = 2i\sqrt{2N(N+1)} \{ & |\psi_{00}\rangle\langle\psi_{20}| + |\psi_{00}\rangle\langle\psi_{02}| \\ & - |\psi_{20}\rangle\langle\psi_{00}| - |\psi_{02}\rangle\langle\psi_{00}| \}. \end{aligned} \quad (36)$$

3.3. Two-mode squeezing

The procedure used for the case of two-mode mixing may be analogously applied when the generator G is given by the two-mode squeezing Hamiltonian:

$$G = ab + a^\dagger b^\dagger. \quad (37)$$

First we analyze the case of an initial density matrix, see (20), that describes two disentangled squeezed thermal states. The same steps done to obtain (24) can be repeated, using the Hamiltonian (37) instead of (19). The maximum of the QFI is again achieved by a probe made by two equally squeezed vacuum states and corresponds to

$$H_{\max} = 4(2N + 1)^2 \quad (38)$$

where $N = 2 \sinh^2 r$. The SLD reads as follows

$$A_0 = 2i(N + 1) S_a S_b (|0, 0\rangle\langle 1, 1| - |1, 1\rangle\langle 0, 0|) S_a^\dagger S_b^\dagger. \quad (39)$$

The same can be done for the case of a probe such as (29). The corresponding QFI is given by

$$\begin{aligned} H = 8 \cosh^2 |\lambda| [& (\cos^2 \phi - \sin^2 \phi)^2 \cos(2 \arg \lambda) \sinh^2 |\lambda| \\ & + 2 \sinh^2 |\lambda| + 1]. \end{aligned} \quad (40)$$

The maximum Fisher information $H_{\max} = 4N^2 + 8N$ is achieved when $\cos(\arg 2\lambda) = 1$ and $\cos 2\phi = 1$ and using the SLD

$$A_0 = 2i(2N + 1)(|\psi_{00}\rangle\langle\psi_{11}| - |\psi_{11}\rangle\langle\psi_{00}|). \quad (41)$$

4. Estimation of squeezing by homodyne detection

In Section 3 we have shown that squeezed vacuum is the optimal reference Gaussian state to estimate the parameter of a squeezing transformation. However, the optimal measurement maximizing the QFI, that is the SLD, is not realizable with current technology. It is thus of interest to investigate whether a feasible measure may be used to effectively probe the perturbed squeezed vacuum. We focus to the case of single-mode squeezing estimation; an analogue analysis may be performed for two-mode operations. Our approach is to exploit homodyne detection to measure field-quadrature:

$$x_\alpha = \frac{1}{2}(ae^{-i\alpha} + a^\dagger e^{i\alpha}) \quad (42)$$

and inferring the squeezing parameter through the results obtained with multiple homodyne measurements. The homodyne probability $p(x|\theta)$ is given by

$$p(x|\theta) = \text{Tr}[\rho_\theta \Pi_x(\theta)], \quad (43)$$

$\Pi_x = |x\rangle_\theta \langle x|$ being the spectral measure of the quadrature (42). The resulting distribution for a squeezed vacuum to which an unknown squeezing has been applied, is a zero mean ($\text{Tr}[\rho_\theta x_\alpha] = 0$) Gaussian distribution

$$p(x|\theta) = \frac{1}{\sqrt{2\pi \Sigma_\theta^2}} \exp\left\{-\frac{x^2}{2\Sigma_\theta^2}\right\} \quad (44)$$

with variance

$$\begin{aligned} \Sigma_\theta^2 = \cos(2\alpha)\sqrt{N(N+1)} \\ + \left(N + \frac{1}{2}\right) [\cosh(2\theta) + \sin(2\alpha) \sinh(2\theta)]. \end{aligned} \quad (45)$$

The reason to choose homodyne detection is that the classical Fisher information (2) of the homodyne distribution $p_\alpha(x|\theta)$ may be optimized over α in order to achieve the same scaling as the QFI versus the energy of the probe. Being $\alpha_1 = \arg \max_\alpha F_\alpha(\theta)$ we have

$$\cos \alpha_1 = \left[-\sqrt{\frac{1}{2} - \frac{\sqrt{N(N+1)}}{(1+2N) \cosh \theta - \sinh \theta}} \right], \quad (46)$$

$$F_{\alpha_1}(\theta) \stackrel{N \gg 1}{\simeq} 8N^2. \quad (47)$$

This means that homodyne detection with optimized phase α is a good candidate to achieve ultimate bounds to precision, as far as it saturates the classical Cramer–Rao bound. Indeed, Von Mises–Bernstein–Laplace theorem ensures that Bayesian *a posteriori* distribution $p(\theta|\{x\}_M)$, representing the probability of the squeezing to be θ given the homodyne sample $\{x\}_M$, converges asymptotically to a Gaussian distribution, centered in the true value with variance saturating the Cramer–Rao bound. In other words, Bayesian estimators are asymptotically unbiased and efficient. In the following, we thus discuss in some details estimation of squeezing by homodyne detection and Bayesian analysis. We consider a large number M of homodyne measurements on repeated preparations of the same system. Since the measurements are independent, the *a posteriori* distribution is proportional to the product of the single data distribution

$$p(\theta|\{x\}_M) \propto \prod_{k=1}^M p(\theta|x_k) = \prod_{k=1}^M \frac{p(x_k|\theta)p(\theta)}{p(x_k)} \quad (48)$$

where we repeatedly used the Bayes Theorem. $p(\theta)$ is the *a priori* distribution of the parameter, $p(x)$ the overall probability of the outcome x , while $p(x|\theta)$ is the probability to obtain the outcome x when the squeezing parameter is θ . The probability $p(\theta|\{x\}_M)$ has to be normalized, Eq. (48) thus rewrites as

$$p(\theta|\{x\}_M) = \frac{1}{A} p(\theta)^M \prod_{k=1}^M \frac{p(x_k|\theta)}{p(x_k)} \quad (49)$$

where A is the normalization constant given by

$$A = \int_{-\infty}^{+\infty} p(\theta)^M \prod_{k=1}^M \frac{p(x_k|\theta)}{p(x_k)} d\theta. \quad (50)$$

We assume to have no *a priori* information on the squeezing θ , i.e. we take $p(\theta)$ as a uniform function. Notice also that the product of the distributions $p(x_k)$ does not depend on θ and it cancels out due to normalization. Finally, since we wish to perform a large

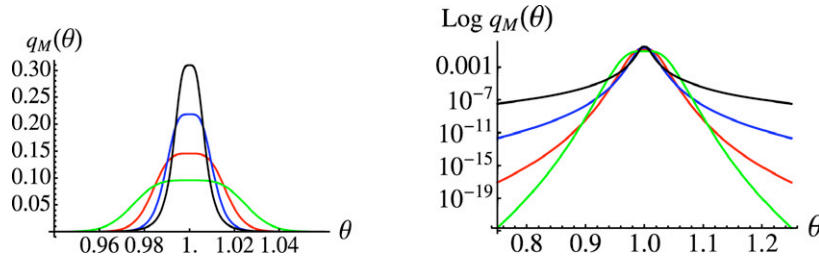


Fig. 2. (Color online.) Left: Rescaled a posteriori distribution $q_M(\theta)$ for $M = 5$, $N = 40$ (black), $M = 10$, $N = 20$ (blue), $M = 20$, $N = 10$ (red), $M = 40$, $N = 5$ (green). Right: LogPlot of the rescaled a posteriori distribution for the same values of the parameters.

number $M \gg 1$ of measurements, the product in (49) will contain many repeated elements: each outcome x is obtained a number of times proportional to its probability $p(x|\theta^*)$, being θ^* the true (and unknown) value of the squeezing parameter. We can then re-order the product so that its index now runs through all possible values of x :

$$p(\theta|\{x\}_M) \simeq \frac{1}{A} \prod_x p(x|\theta)^{Mp(x|\theta^*)} = \frac{1}{A} \exp \left\{ M \int p(x|\theta^*) \ln p(x|\theta) dx \right\} \quad (51)$$

where we have taken a limit to the continuum for the variable x . The integral in (51) can be solved leading to

$$\int_{-\infty}^{+\infty} p(x|\theta^*) \ln p(x|\theta) dx = -\frac{1}{2} \left[\frac{\Sigma_*^2}{\Sigma_\theta^2} + \ln(2\pi \Sigma_\theta^2) \right] \quad (52)$$

where we have introduced the short notation $\Sigma_*^2 \equiv \Sigma_{\theta^*}^2$. Overall, we obtain

$$p(\theta|\{x\}_M) = \frac{1}{A} \left[\Sigma_\theta^2 \exp \left(\frac{\Sigma_*^2}{\Sigma_\theta^2} \right) \right]^{-M/2} \quad (53)$$

where we have redefined A so to include all terms independent from θ . The mean $\bar{\theta}$ of the *a posteriori* distribution $p(\theta|\{x\}_M)$ is our estimator and the variance $\Delta\theta^2$ the corresponding confidence interval

$$\bar{\theta} = \int_{-\infty}^{+\infty} d\theta \theta p(\theta|\{x\}_M), \quad (54)$$

$$\Delta\theta^2 = \int_{-\infty}^{+\infty} d\theta (\theta - \bar{\theta})^2 p(\theta|\{x\}_M). \quad (55)$$

An optimal value for the homodyne phase α is obtained upon minimizing the variance of the *a posteriori* distribution. Besides the value α_1 reported above we found that optimal scaling ($\propto M^{-1}N^{-2}$) of the variance may be achieved also for the phase value

$$\alpha_2 = -\text{sign}(\theta^*) \arccos \left[\sqrt{\text{sech}(2\theta^*) \sinh^2 \theta^*} \right],$$

which, remarkably, is independent on the probe energy N (indeed, we have $\alpha_1 = \alpha_2 + O(1/N)$).

In Fig. 2 we report the rescaled distribution $q_M(\theta) = p(\theta|\{x\}_M)/(MN)$ for different values of the probe energy and the number of measurements, we also report $p(\theta|\{x\}_M)/(MN)$ in a logarithmic scale to enlighten the differences in the distribution tails. As it is apparent from the plots the relevant parameter is the energy of the probe. For highly excited probes, i.e. for $N \gg 1$, we expand Σ_θ^2 as

$$\Sigma_\theta^2 = \left(N + \frac{1}{2} \right) [\cos(2\alpha) + \cosh(2\theta) + \sin(2\alpha) \sinh(2\theta)] - \frac{\cos(2\alpha)}{8N} + O\left(\frac{1}{N^2}\right) \quad (56)$$

and neglect all orders scaling as N^{-2} or higher. Upon choosing the homodyne phase α_2 we have

$$\Sigma_*^2 \simeq \frac{\text{sech}(2\theta^*)}{8N}, \quad (57)$$

$$\Sigma_\theta^2 \simeq \text{sech}(2\theta^*) \left[(2N+1) \sinh^2(\theta - \theta^*) + \frac{1}{8N} \right]. \quad (58)$$

Upon substituting (57) and (58) into (53) we see explicitly that $p(\theta|\{x\}_M) = p(\theta - \theta^*|\{x\}_M)$ and that the estimator is indeed unbiased, i.e. $\bar{\theta} = \theta^*$. We also found that the variance is independent from the true value of the squeezing θ^* : Numerical computation shows that the variance $\Delta\theta^2$ scales as $\sim \frac{1}{4MN^2}$ for large N , that is, apart from a factor two, the same scaling of the inverse of the QFI (18). Notice that the optimal phase α_2 , depends on θ^* , which is the unknown parameter that we are trying to estimate. This is consistent with the local nature of the estimator procedure. From a practical point of view this means that some kind of feedback mechanism or adaptive technique should be employed to adjust the phase of the homodyne detector [16,33]. We conclude that homodyne detection with Bayesian analysis is a robust and accurate estimation technique for the squeezing parameter. Remarkably, this scheme may be implemented with current technology.

5. Conclusions

In this Letter we have addressed local quantum estimation of bilinear Hamiltonians probed by Gaussian states. We evaluated the relevant quantum Fisher information (QFI) thus obtaining the ultimate bound on precision. Upon maximizing the QFI we found that single- and two-mode squeezed vacuum represent an optimal and universal class of probe states, achieving the so-called Heisenberg limit to precision in terms of the overall energy of the probe. For two-mode operations no improvement may be obtained using two-mode entangled probes. Notice, however, that we have not investigated the use of probes made by states of the system under investigation (either single- or two-mode) entangled with auxiliary modes. This may improve estimation of the parameter [34]; work along these lines is in progress and results will be reported elsewhere.

It is worth noting that the Heisenberg scaling $\Delta\theta \sim N^{-1}$ in terms of the overall energy of the probe may be achieved also using global quantum estimation techniques (see e.g. [9] for the case of two-mode mixing). In that case, however, optimization of the probe have been performed over the whole set of quantum states, not focusing on Gaussian states. In turn, this means that Gaussian states are effective resources, which allow to achieve the ultimate bound on precision imposed by quantum mechanics using measurement schemes feasible with current technology.

This has been confirmed by a Bayesian analysis applied to the estimation of squeezing by homodyne detection, which achieves near-optimal sensitivity in any working regime, i.e. for any (true) value of the squeezing parameter. For the estimation of squeezing, Heisenberg scaling for Gaussian probes has been also found exploiting global strategies [21]. In that case, however, though the measurement does not depend on the value of the parameter, there is a strong dependence on the probe states. We have also explicitly obtained the optimal observables based on the symmetric logarithmic derivative, which however do not correspond, in general, to a feasible detection scheme.

Our results confirm, in term of statistical distinguishability, that Gaussian states and Gaussian measurements assisted by Bayesian analysis represent robust and accurate resources for the estimation of unitary operations of interest in continuous variable quantum information.

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