

Local versus nonlocal cloning in a noisy environment

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Abstract

We address the distribution of quantum information among many parties in the presence of noise. In particular, we consider how to optimally send to m receivers the information encoded into an unknown coherent state. On one hand, a local strategy is considered, consisting in a local cloning process followed by direct transmission. On the other hand, a telecloning protocol based on nonlocal quantum correlations is analysed. Both the strategies are optimized to minimize the detrimental effects due to losses and thermal noise during the propagation. The comparison between the local and the nonlocal protocol shows that telecloning is more effective than local cloning for a wide range of noise parameters. Our results indicate that nonlocal strategies can be more robust against noise than local ones, thus being suitable candidates for playing a major role in quantum information networks.

Keywords: cloning, telecloning, quantum communication, entanglement, continuous variables

1. Introduction

One of the aims in the burgeoning field of quantum information with continuous variables (CV) [1, 2] is to replace, for some particularly crucial purposes, the communications technology currently in use. Major developments in this field are, for instance, the experimental realizations of quantum teleportation and cryptography [3]. They involve, in general, only two parties, a sender and a receiver, and are based either on nonlocal quantum correlations, or on protocols involving local manipulation and direct transmission of quantum states. For example, the issue of sending an unknown state to a single receiver has been addressed, comparing the performances of teleportation and direct transmission [4–6]. Of particular interest is the case in which the channel supporting the transfer of quantum states is affected by thermal noise and losses, as in real experiments. As regards the transfer of nonclassical features, it has been shown that entanglement is necessary in a teleportation scenario [4] and that teleportation is preferable to direct transmission in certain regimes [5]. Furthermore, a communication protocol in which information is encoded into the field amplitude (amplitude-modulated communication) of a set of Gaussian pure states has been analysed [6]; the indication is that teleportation can be more effective also in this case.

A further natural step is to consider more complex communication scenarios, where more than two parties are involved in what is called a quantum information network.

The recent experimental realizations of CV dense coding [7] and quantum teleportation networks [8] involve in fact three distinct parties. In this work, we consider the problem of distributing the information encoded in an unknown coherent state of a CV system to m parties, in a multipartite amplitude-modulated communication scenario. As for the single-receiver case, one may ask whether it is better to perform the distribution using a local or a nonlocal strategy. On one hand, one may in fact consider cloning the original state somewhere along the noisy transmission line by means of an optimal local cloning machine. In this case both the signal and the clones are directly coupled with the environment and, then, the fidelity is affected by the unavoidable degradation of the signal and the clones themselves. On the other hand, a pre-shared multipartite entangled state may be used to support a telecloning protocol. In this case, the performance of the protocol is affected by the degradation of the nonlocal correlations of the support. The fact that optimal telecloning, as opposed to teleportation, does not need an infinite amount of entanglement leads to the hypothesis that the degradation of entanglement is not too dramatic in affecting the fidelity of the clones.

In order to face the effects of decoherence, both the local and the nonlocal strategy have to be optimized. In particular, we will outline the role of the location of the cloning machine and of the multimode state source, in the local and nonlocal protocols respectively.

The paper is organized as follows. In section 2 we will introduce the multimode states that will be used as support for the optimized telecloning protocol described in section 3. The optimization of the local strategy will be outlined in section 4. Section 5 will be devoted to the comparison between the two strategies, whereas the main results will be summarized in section 6.

2. Multimode entangled states

Let us begin by introducing the multimode entangled states that will provide the support for the telecloning protocol. Multimode entanglement of Gaussian states, that is states with Gaussian Wigner functions, has attracted much attention recently, both from theoretical and experimental viewpoints [2]. A particularly interesting class of multimode Gaussian states are the coherent states of the group $SU(m, 1)$ [9, 10]. Indeed, these states can be experimentally generated by multimode parametric processes in second-order nonlinear crystals, with Hamiltonians that are at most bilinear in the fields [8, 11]. In particular, these processes involve $m+1$ modes of the field a_0, a_1, \dots, a_m , with mode a_0 that interacts through a parametric amplifier-like Hamiltonian with the other modes, whereas the rest interact with one another only, via a beam splitter-like Hamiltonian. The Hamiltonian of the system is thus given by

$$H_m = \sum_{l < k=1}^m \gamma_{kl}^{(1)} a_k a_l^\dagger + \sum_{k=1}^m \gamma_k^{(2)} a_k a_0 + \text{h.c.}, \quad (1)$$

where $[a_k, a_l] = 0$, $[a_k, a_l^\dagger] = \delta_{k,l}$ ($k, l = 0, \dots, m$) are independent bosonic modes, whereas $\gamma_{kl}^{(1)}$ and $\gamma_k^{(2)}$ are coupling constants. The states generated by H_m from the vacuum are the coherent states of the group $SU(m, 1)$, namely

$$|\Psi_m\rangle = \sqrt{\mathcal{Z}_m} \sum_{\{n\}} \frac{C_1^{n_1} C_2^{n_2} \dots C_m^{n_m} \sqrt{(n_1 + n_2 + \dots + n_m)!}}{\sqrt{n_1! n_2! \dots n_m!}} \times \left| \sum_{k=1}^m n_k; \{n\} \right\rangle, \quad (2)$$

where $\{n\} = \{n_1, n_2, \dots, n_m\}$. The sums over n are extended over natural numbers and, upon introducing the mean values of the number operators $N_k = \langle a_k^\dagger a_k \rangle$, we have defined

$$C_k = \left(\frac{N_k}{1 + N_0} \right)^{1/2}, \quad \mathcal{Z}_m = \frac{1}{1 + N_0} \quad (k = 1, \dots, m). \quad (3)$$

The relevant constant of motion, in this context, is the difference between the mean photon number of mode a_0 and the total mean photon number of the other modes. Since we start from the vacuum, we have

$$N_0 = \sum_{k=1}^m N_k. \quad (4)$$

We notice from equation (2) that for $m = 1$ the twin-beam state is recovered. Having evolved from the vacuum with a quadratic Hamiltonian, the states $|\Psi_m\rangle$ are Gaussian. They are

completely characterized by the covariance matrix σ , whose entries are defined as

$$[\sigma]_{kl} = \frac{1}{2} \langle \{R_k, R_l\} \rangle - \langle R_l \rangle \langle R_k \rangle, \quad (5)$$

where $\{A, B\} = AB + BA$ denotes the anticommutator, $\mathbf{R} = (q_0, p_0, \dots, q_m, p_m)^\top$ and the position and momentum operator are defined as $q_k = (a_k + a_k^\dagger)/\sqrt{2}$ and $p_k = (a_k - a_k^\dagger)/i\sqrt{2}$. The covariance matrix for the states $|\Psi_m\rangle$ reads as follows:

$$\sigma_m = \begin{pmatrix} \mathcal{N}_0 & \mathcal{A}_1 & \mathcal{A}_2 & \dots & \mathcal{A}_m \\ \mathcal{A}_1 & \mathcal{N}_1 & \mathcal{B}_{1,2} & \dots & \mathcal{B}_{1,m} \\ \mathcal{A}_2 & \mathcal{B}_{1,2} & \mathcal{N}_2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \mathcal{B}_{m-1,m} \\ \mathcal{A}_m & \mathcal{B}_{1,m} & \dots & \mathcal{B}_{m-1,m} & \mathcal{N}_m \end{pmatrix}, \quad (6)$$

where the entries are given by the following 2×2 matrices ($k = 0, \dots, m, h = 1, \dots, m, j = 2, \dots, m$ and $0 < i < j$):

$$\begin{aligned} \mathcal{N}_k &= (N_k + \frac{1}{2}) \mathbb{I} & \mathcal{A}_h &= \sqrt{N_h(N_0 + 1)} \mathbb{P} \\ \mathcal{B}_{i,j} &= \sqrt{N_i N_j} \mathbb{I}, \end{aligned} \quad (7)$$

with $\mathbb{I} = \text{Diag}(1, 1)$ and $\mathbb{P} = \text{Diag}(1, -1)$. The basic property of the states $|\Psi_m\rangle$ is that they are fully inseparable, i.e., they are inseparable for any grouping of the modes [12]. By virtue of this property, the states $|\Psi_m\rangle$ can provide the support for a telecloning protocol, as we will see in the next section.

3. Telecloning in a noisy environment

As already mentioned, one of the main results in CV quantum communication is the realization of the teleportation protocol. The natural generalization of standard teleportation to many parties corresponds to the so-called telecloning protocol [13], i.e. a protocol that provides at a distance many imperfect copies of the original input state. Teleportation is based on the coherent states of $SU(1, 1)$, which provide the shared entangled states supporting the protocol. Thus, in order to implement a multipartite version of this protocol, one is naturally led to consider as shared entangled state the coherent states of $SU(m, 1)$ introduced in the previous section. Actually, it has already been shown in [12] that these states permit one to achieve optimal symmetric and asymmetric telecloning of pure Gaussian states. The telecloning protocol is depicted schematically in figure 1. After being prepared, the state $|\Psi_m\rangle$ propagates thorough $m+1$ noisy channels. In particular, we can consider that modes a_1, \dots, a_m propagate in noisy channels characterized by the same losses Γ_c . We may then define an effective propagation time $\tau_c = \Gamma_c t$ equal for all the modes a_1, \dots, a_m , while the effective propagation time $\tau_0 = \Gamma_0 t$ for mode a_0 is left different from τ_c . Consider in fact a scenario in which one has two distant locations (see figure 1). The distance between the two stations can be viewed as a total effective propagation time τ_T which can be written as $\tau_T = \tau_0 + \tau_c$. Then, the choice made above corresponds to the possibility of choosing at will, for a given τ_T , which modes (a_1, \dots, a_m or a_0) will be affected by the unavoidable noise that separates the sending and the receiving station and to what extent. With a slight abuse of language, we may say

that one can choose whether to put the source of the entangled state $|\Psi_m\rangle$ close to the sending station ($\tau_T = \tau_c$), close to the receiving one ($\tau_T = \tau_0$) or somewhere in between. A similar strategy has been pursued in [14] for optimizing the CV teleportation protocol in a noisy environment. In the following, the optimal location and the optimal $|\Psi_m\rangle$ for a given amount of noise will be given. For simplicity, all the noisy channels will be characterized by the same effective temperature, that is the mean thermal photons μ will be taken equal for all the channels. As a consequence, the covariance matrix of the evolved state is given by (see, e.g., [2])

$$\sigma_{m,n} = \mathbb{G}^{1/2} \sigma_m \mathbb{G}^{1/2} + (1 - \mathbb{G}) \sigma_{\infty,m}, \quad (8)$$

where σ_m is the initial covariance matrix of equation (6) and we have defined

$$\mathbb{G} = e^{-\tau_0} \mathbb{I} \left(\bigoplus_{j=1}^m e^{-\tau_c} \mathbb{I} \right) \quad \sigma_{\infty,m} = (\mu + \frac{1}{2}) \mathbb{I}_{2m}. \quad (9)$$

Performing the calculation explicitly, upon defining $\kappa = \mu + \frac{1}{2}$, we obtain

$$\sigma_{m,n} = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}, \quad (10)$$

where $A = e^{-\tau_0} \mathcal{N}_0 + \kappa(1 - e^{-\tau_0}) \mathbb{I}$, $C = \sqrt{e^{-\tau_T}} (\mathcal{A}_1, \dots, \mathcal{A}_m)$ and

$$B = \begin{pmatrix} e^{-\tau_c} \mathcal{N}_1 + \kappa(1 - e^{-\tau_c}) \mathbb{I} & e^{-\tau_c} \mathcal{B}_{1,2} & & & \\ e^{-\tau_c} \mathcal{B}_{1,2} & e^{-\tau_c} \mathcal{N}_2 + \kappa(1 - e^{-\tau_c}) \mathbb{I} & & & \\ \vdots & & \ddots & & \\ e^{-\tau_c} \mathcal{B}_{1,m} & & & e^{-\tau_c} \mathcal{B}_{m-1,m} & \\ \dots & & & & e^{-\tau_c} \mathcal{B}_{m-1,m} \\ \vdots & & & & \vdots \\ \vdots & & & & e^{-\tau_c} \mathcal{B}_{m-1,m} \\ e^{-\tau_c} \mathcal{B}_{m-1,m} & e^{-\tau_c} \mathcal{N}_m + \kappa(1 - e^{-\tau_c}) \mathbb{I} & & & \end{pmatrix}. \quad (11)$$

As in the case of standard teleportation, the telecloning protocol now proceeds by performing a joint measurement on modes a_0 and b , which is excited in the unknown coherent state $|\alpha\rangle$ that we want to teleport and clone. The measurement corresponds to an ideal double-homodyne detection of the complex photocurrent $Z = b + a_0^\dagger$, described by the following Gaussian characteristic function:

$$\chi[M, X](\Lambda) = \exp\{-\frac{1}{2} \Lambda^T M \Lambda - i \Lambda^T X\}. \quad (12)$$

In equation (12) the covariance matrix M and the vector of first moments X are given by

$$M = \mathbb{P} \sigma_{\text{in}} \mathbb{P}, \quad X = \mathbb{P} \bar{X} + Z, \quad (13)$$

where $Z = \{\text{Re}[z], \text{Im}[z]\}$ is the measurement result, $\sigma_{\text{in}} = \frac{1}{2} \mathbb{I}$ and $\bar{X} = \{\text{Re}[\alpha], \text{Im}[\alpha]\}$ are the covariance matrix and the vector of first moments of the input coherent state $|\alpha\rangle$. Then, the state ϱ_c , conditioned to the result Z , is a Gaussian state with covariance matrix [15]

$$\sigma_c = B - C^T (A + M)^{-1} C \quad (14)$$

and vector of first moments $H = C^T (A + M)^{-1} X$. The probability $P(Z)$ of the outcome Z is given by

$$P(Z) = \frac{1}{\sqrt{\det(A + M)}} \exp\left\{-\frac{1}{2} X^T (A + M)^{-1} X\right\}. \quad (15)$$

After the measurement, the conditional state should be transformed by a further unitary operation, depending on the outcome of the measurement. In our case, this is an m -mode product displacement $U_z = \bigotimes_{h=1}^m D_h^\dagger(z)$. This is a local transformation, which generalizes to m modes the procedure already used in the original CV teleportation protocol. The characteristic function of the modes a_1, \dots, a_m is now given by

$$\chi[U_z \varrho_c U_z^\dagger](\Lambda) = \chi[\varrho_c](\Lambda) \exp\{i \Lambda^T \mathbb{J}^T Z^*\}, \quad (16)$$

where Λ is the usual $2m$ -component column vector spanning the reciprocal phase space of modes a_1, \dots, a_m , whereas $*$ indicates complex conjugation and \mathbb{J} is given by the $2 \times 2m$ matrix $\mathbb{J} = (\mathbb{I}, \dots, \mathbb{I})$. The characteristic function of the overall output state ϱ_{out} is obtained by averaging over all the possible outcomes:

$$\chi_{\text{out}}(\Lambda) = \int d^{2m} Z P(Z) \chi[\varrho_c](\Lambda) \exp\{i \Lambda^T \mathbb{J}^T Z^*\} \quad (17)$$

$$= \exp\left\{-\frac{1}{2} \Lambda^T [B + \mathbb{J}^T \mathbb{P} (A + M) \mathbb{P} \mathbb{J} - \mathbb{J}^T \mathbb{P} C - C^T \mathbb{P} \mathbb{J}] \Lambda - i \Lambda^T \mathbb{J}^T \bar{X}\right\}, \quad (18)$$

which, in turn, gives the following covariance matrix for the h th clone $\varrho_h = \text{Tr}_{i \neq h}[\varrho_{\text{out}}]$:

$$\sigma_h = \left(\frac{1}{F_h} - \frac{1}{2}\right) \mathbb{I}. \quad (19)$$

In the equation above, F_h represents the fidelity $F_h = \langle \alpha | \varrho_h | \alpha \rangle$ between the h th clone and the original coherent state, i.e.,

$$F_h = \{\det[\sigma_h + \frac{1}{2} \mathbb{I}]\}^{-1/2} \\ = \left\{2 + 2\mu + \left[e^{-\tau_0} (N_0 - \mu) + e^{-\tau_T + \tau_0} (N_h - \mu) - 2\sqrt{e^{-\tau_T} N_h (N_0 + 1)}\right]\right\}^{-1}, \quad (20)$$

where we have reintroduced the mean thermal photons μ . Notice that the fidelity does not depend on the amplitude α of the input state. Remarkably, from equation (20) it follows that the present telecloning scheme is able to perform both symmetric and asymmetric distributions of information [12]. However, being interested in the comparison between the performances of telecloning and of the local strategy that will be outlined in the next section, from now on we will consider only the symmetric instance. That is, we set $N_1 = \dots = N_m = N$ and $N_0 = mN$, from which it follows that all the clones are equal to one another ($F_1 = \dots = F_m = F$).

The next step is now to optimize, for a fixed amount of noise, the shared state $|\Psi_m\rangle$ and the location of its source between the sending and the receiving station. That is, relying upon the fidelity as the relevant figure of merit, one has to find the optimal N and τ_0 which maximize F for τ_T and μ fixed. The results of the optimization are summarized in table 1 [12], where we have defined the following quantities:

$$F^a = \frac{m}{m[2 + \mu(1 - e^{-\tau_T})] - 1}, \quad (21)$$

$$F^b = [2 + \mu - (1 + \mu)e^{-\tau_T}]^{-1}, \quad (22)$$

$$F^c = \left\{2 + 2\mu - \sqrt{e^{-\tau_T}/m} [1 + \mu(1 + m)]\right\}^{-1}. \quad (23)$$

The most interesting feature which emerges from an inspection of table 1 is that telecloning saturates the bounds

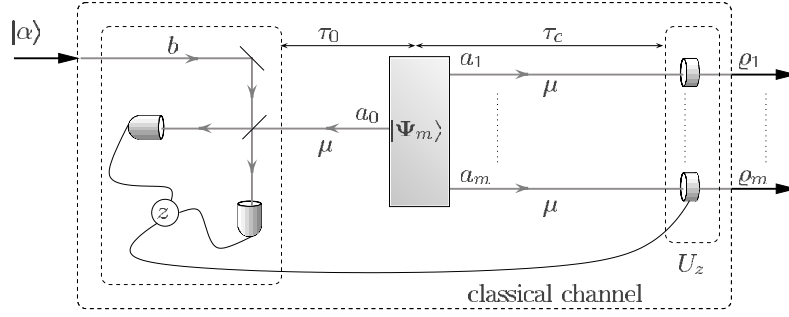


Figure 1. Schematic diagram of the telecloning scheme. After the preparation of the state $|\Psi_m\rangle$, a conditional measurement is made on the mode a_0 , which corresponds to the joint measurement of the sum and difference quadratures on two modes: mode a_0 itself and another reference mode b , which is excited in a coherent state $|\alpha\rangle$, to be teleported and cloned. The result z of the measurement is classically sent to the parties who want to prepare approximate clones, where suitable displacement operations (see the text) on modes a_1, \dots, a_m are performed. We indicated with μ the mean thermal photons in the propagation channels. The effective propagation times τ_0 and τ_c (see the text) are related to the losses during propagation.

Table 1. Values of the optimized N^{opt} and τ_0^{opt} for fixed values of τ_T and μ . The value reached by the fidelity F^{max} for these optimal choices is given in the last column.

μ	τ_T	τ_0^{opt}	N^{opt}	F^{max}
$\forall \mu$	$0 < \tau_T < \ln m$	τ_T	$\frac{1}{m(me^{-\tau_T} - 1)}$	F^a
$\mu < 1/(m-1)$	$\ln m < \tau_T < \ln \left[\frac{(1+\mu)^2}{m\mu^2} \right]$	$\frac{1}{2}(\tau_T + \ln m)$	$N \rightarrow \infty$	F^c
$\mu < \frac{1}{m-1}$	$\tau_T > \ln \frac{(1+\mu)^2}{m\mu^2}$	τ_T	$\frac{e^{-\tau_T}}{1 - me^{-\tau_T}}$	F^b
$\mu > \frac{1}{m-1}$	$\tau_T > \ln m$	τ_T	$\frac{e^{-\tau_T}}{1 - me^{-\tau_T}}$	F^b

for optimal cloning [16] even in the presence of losses, for propagation times $\tau_T < \ln m$, i.e. propagation times which diverge as the number of modes increases. More specifically, consider the first row in table 1 and set $\mu = 0$. Then, one has that for $\tau_T < \ln m$ the maximum fidelity is given by $F^{\text{max}} = m/(2m-1)$. That is, the optimal fidelity for a symmetric cloning can still be attained, carefully choosing N and τ_0 . This is due to the fact that multimode entanglement is robust against this type of noise and, even if decreased along the transmission line, it is still sufficient to provide optimal cloning. Actually, as we already mentioned, there is no need for an infinite amount of entanglement to perform an optimal telecloning process [10]. Notice that when $F^{\text{max}} = F_b$ in table 1, telecloning ceases to be interesting, because F_b is lower than the so-called classical limit $F = \frac{1}{2}$ [17]. It is in fact immediate that $F_b > \frac{1}{2}$ only when

$$\tau_T < \ln \left(1 + \frac{1}{\mu} \right). \quad (24)$$

However, from the third row of table 1 one has that $\tau_T > \ln \frac{(1+\mu)^2}{m\mu^2}$ and $\mu < \frac{1}{m-1}$, which implies that inequality (24) cannot be satisfied. The same conclusion holds also if one considers the fourth row of table 1. The comparison of the results in table 1 with the performances of a local distribution of information will be given in section 5.

4. Local cloning plus direct transmission (LCDT)

Let us now consider the situation in which the distribution of quantum information does not rely upon sharing any

entanglement between the parties involved. We refer to such protocols as local cloning + direct transmission (LCDT) schemes. In the notation introduced in section 3, one has that the sending and the receiving stations are separated by an effective time τ_T . The input coherent state, characterized by the covariance matrix $\sigma_{\text{in}} = \frac{1}{2}\mathbb{I}$ and the amplitude \bar{X} , propagates through a noisy channel for time τ_0 , after which it is cloned by a suitably chosen local optimal symmetric cloning machine. Then, the clones are sent to the receiving station, via m noisy channels for a propagation time τ_c . Before calculating the fidelity of such an LCDT strategy, it is necessary to identify the proper cloning machine to use. The natural requirement for a coherent state cloning machine is its covariance with respect to displacements in the phase space [18]. This implies that the cloning map is a Gaussian noise map of the form

$$\mathcal{Q}_{\text{clo}} = \frac{1}{\pi \bar{n}} \int d^2\beta e^{-|\beta|^2/\bar{n}} D(\beta) \mathcal{Q}_{\text{in}} D^\dagger(\beta), \quad (25)$$

\mathcal{Q}_{in} being the density matrix of the state at the input of the cloning machine and \bar{n} is the noise added by the cloning process. The density matrix of the clones \mathcal{Q}_{clo} is the partial trace over all the modes except the one of the overall state R at the output of the cloning machine, namely $\mathcal{Q}_{\text{clo}} = \text{Tr}_{a_2, \dots, a_m}[R]$ (recall that we are considering the case in which the clones are all equal). Actually, the overall state R plays no role in our analysis. Indeed, once the partial traces \mathcal{Q}_{clo} are fixed by the requirement in equation (25), the overall state R has no influence on the clones propagating through the noisy channels. In fact, the m noisy channels are independent, and the overall

Liouvillian superoperator \mathcal{L} factorizes into the single-channel superoperators \mathcal{L}_h . As a consequence, one can easily show, considering the Kraus decomposition of each \mathcal{L}_h , that

$$\varrho_l = \text{Tr}_{a_2, \dots, a_m} [\mathcal{L}(R)] = \mathcal{L}_1(\text{Tr}_{a_2, \dots, a_m} [R]) = \mathcal{L}_1(\varrho_{\text{clo}}), \quad (26)$$

where ϱ_l is the final state of the clones. The remaining step to be performed is now the optimization of the location of the cloning machine. To this end, let us calculate the fidelity between the clones at the end of the transmission line and the input state. After propagating for a time τ_0 , the input state covariance matrix and amplitude are given by

$$\sigma_{\text{clo}}^{\text{in}} = [\tfrac{1}{2} + (1 - e^{-\tau_0})\mu]\mathbb{I}, \quad X_{\text{clo}}^{\text{in}} = e^{-\tau_0/2}\overline{X}. \quad (27)$$

Then, the cloning machine produces m optimal clones ($\bar{n} = (m - 1)/m$) accordingly to equation (25), i.e.,

$$\sigma_{\text{clo}}^{\text{out}} = [\tfrac{1}{2} + (1 - e^{-\tau_0})\mu + \bar{n}]\mathbb{I}, \quad X_{\text{clo}}^{\text{out}} = e^{-\tau_0/2}\overline{X}. \quad (28)$$

Letting the latter propagate one finally has

$$\sigma_l = [\tfrac{1}{2} + (1 - e^{-\tau_l})\mu + \bar{n}e^{-\tau_l}]\mathbb{I}, \quad X_l = e^{-\tau_l/2}\overline{X}, \quad (29)$$

from which the fidelity F_d follows:

$$F_d = \frac{1}{1 + \bar{n}e^{-\tau_c} + (1 - e^{-\tau_l})\mu} \times \exp\left\{-\frac{1}{2} \frac{(1 - e^{-\tau_l/2})^2 |\alpha|^2}{1 + \bar{n}e^{-\tau_c} + (1 - e^{-\tau_l})\mu}\right\}. \quad (30)$$

The maximum of F_d is given by

$$F_d^{\text{max}} = \frac{2e^{\tau_l-1}}{|\alpha|^2(e^{\tau_l/2} - 1)^2} \quad (31)$$

and it is attained for

$$\tau_c^{\text{opt}} = \tau_l - \ln\left\{\frac{1}{2\bar{n}}[|\alpha|^2(e^{\tau_l/2} - 1)^2 + 2\mu - 2e^{\tau_l}(1 + \mu)]\right\}. \quad (32)$$

However, from equation (32) it follows that τ_c^{opt} is admissible (namely, $\tau_c^{\text{opt}} < \tau_l$) only when

$$|\alpha|^2 > |\tilde{\alpha}|^2 = \frac{2[\bar{n} - \mu + e^{\tau_l}(1 + \mu)]}{(e^{\tau_l/2} - 1)^2}. \quad (33)$$

The equation above, in turn, implies that F_d^{max} is always lower than the classical bound $F = \frac{1}{2}$. In fact, one has that

$$F_d^{\text{max}} \leq \frac{2e^{\tau_l-1}}{|\tilde{\alpha}|^2(e^{\tau_l/2} - 1)^2} = \frac{1}{e^{\tau_l} + [\bar{n} + \mu(e^{\tau_l} - 1)]} < \frac{1}{e}. \quad (34)$$

In other words, we have that, when the LCDT strategy is useful, the best location of the cloning machine is at the sending station ($\tau_c = \tau_l$). This is due to the fact that the noisy propagation after the cloning machine, besides degrading the signal, decreases the noise added by the cloning machine, as we can see from the term $\bar{n}e^{-\tau_c}$ in equation (29). The fidelity F_d of the clones produced by the optimal local strategy is thus given by

$$F_d = \frac{m}{m[1 - \mu + e^{\tau_l}(1 + \mu)] - 1} \times \exp\left\{\tau_l - \frac{1}{2} \frac{m(1 - e^{\tau_l/2})^2 |\alpha|^2}{m[1 - \mu + e^{\tau_l}(1 + \mu)] - 1}\right\}, \quad (35)$$

which shows that the fidelity depends on the original input state. In a communication scenario, in which the information is amplitude modulated, it is thus necessary to introduce a fidelity averaged over all the possible input. Let us suppose that the message we want to transmit is encoded in an alphabet distributed according to a Gaussian probability density function of variance Ω^2 :

$$G_\Omega(\alpha) = \frac{1}{\pi\Omega^2} e^{-|\alpha|^2/\Omega^2}. \quad (36)$$

The averaged fidelity $F_{\text{LCDT}} = \int d^2\alpha G_\Omega(\alpha) F_d$ of the clones is thus given by

$$F_{\text{LCDT}} = \frac{me^{\tau_l}}{m[1 - \mu + \Omega^2(1 - 2e^{\tau_l/2}) + e^{\tau_l}(1 + \mu + \Omega^2)]}. \quad (37)$$

The result above will be compared with the telecloning fidelity in the next section.

5. Comparison between local and nonlocal strategy

We are now in a position to compare the performances of the LCDT strategy and telecloning. First, notice that the fidelity F_{LCDT} in equation (37) goes to zero as Ω increases. This means that for a truly random distributed coherent state the local strategy is not useful at all. On the other hand, equations (21)–(23) explicitly show that the performances of telecloning do not depend on the value of the coherent amplitude, as follows from the covariance of the process. Hence, as one may expect, telecloning is undoubtedly more effective than the LCDT strategy in the case of a generic unknown coherent state. Let us now consider the case of finite Ω in the absence of thermal noise ($\mu = 0$). Then the fidelity is given by F_a and F_c in equations (21), (23), which specialize as follows:

$$F_a = \frac{m}{2m - 1}, \quad F_c = \frac{1}{2 - \sqrt{e^{-\tau_l}/m}}. \quad (38)$$

Equation (38) implies, as already pointed out, that optimality is still achieved for a time $\tau_l < \ln m$, and also that the fidelity is greater than the classical bound at any time. The comparison between equations (38) and (37) is given in figure 2 for $m = 2, 5$. The figure shows that, even for small values of the width $\Omega \simeq 2$, telecloning is more effective than the LCDT strategy. A similar behaviour is found also when thermal noise is considered ($\mu \neq 0$), as figure 3 shows. Notice that in the latter case, as one may expect, telecloning may also not give a better fidelity than the classical limit. This happens for propagation time τ_l larger than the threshold

$$\tau_l^{\text{a,th}} = \ln\left[\frac{(1 + \mu + m\mu)^2}{4m\mu^2}\right] \quad (39)$$

for $\mu < \frac{1}{m-1}$, and larger than

$$\tau_l^{\text{c,th}} = -\ln\left[1 - \frac{1}{m\mu}\right] \quad (40)$$

otherwise.

From a quantum communication viewpoint it is interesting to consider the threshold for Ω above which telecloning

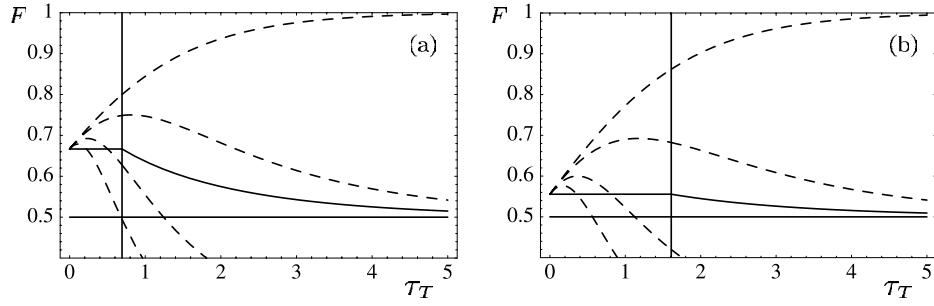


Figure 2. Comparison for $\mu = 0$ and $m = 2$ (a) ($m = 5$ (b)) between the telecloning fidelity given in equation (38) (solid line) and the fidelity of the LCDT strategy given in equation (37) (dotted lines). The latter refer to the cases of $\Omega = 0, 1, 2, 3$ from top to bottom. The vertical line corresponds to $\tau_T = \ln 2$ (a) ($\tau_T = \ln 5$ (b)).

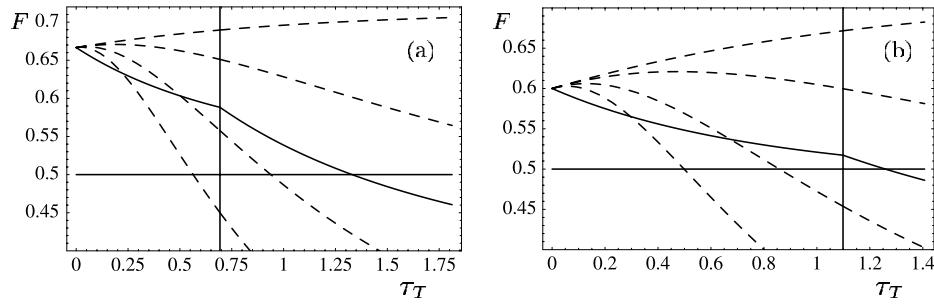


Figure 3. Comparison for $\mu = 0.4$ and $m = 2$ (a) ($m = 3$ (b)) between the telecloning fidelity given in equations (21), (23) (solid line) and the fidelity of the LCDT strategy given in equation (37) (dotted lines). The latter refer to the cases of $\Omega = 0, 1, 2, 3$ from top to bottom. The vertical line corresponds to $\tau_T = \ln 2$ (a) ($\tau_T = \ln 3$ (b)).

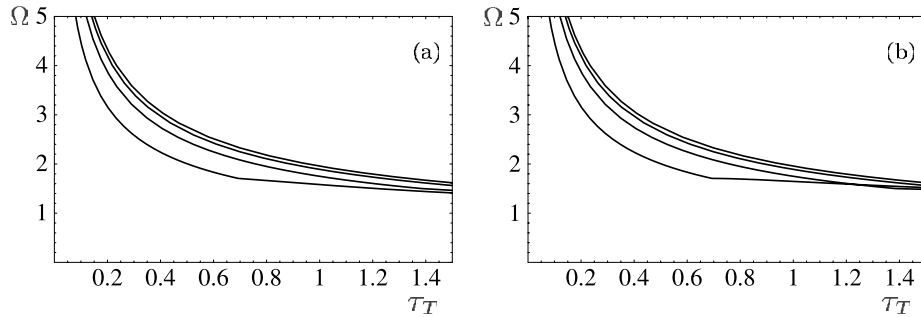


Figure 4. Plot of the thresholds $\Omega_{a,th}$ in equation (41) and $\Omega_{c,th}$ in equation (42) for different values of the number of clones. We fixed $\mu = 0$ (a) ($\mu = 0.4$ (b)) and, from bottom to top, we set $m = 2, 4, 8, 16$. The region above the lines refers to the case in which telecloning is more effective than the LCDT strategy.

becomes more effective than LCDT strategy. The latter can be analytically retrieved and one has that F_a in equation (21) is greater than F_{LCDT} when $\Omega > \Omega_{a,th}$, with

$$\Omega_{a,th}^2 = \frac{(1 + e^{\tau_T/2})(m - 1)}{(e^{\tau_T/2} - 1)m}, \quad (41)$$

whereas F_c in equation (23) is greater than F_{LCDT} when $\Omega > \Omega_{c,th}$, with

$$\Omega_{c,th}^2 = \frac{1 + m(\mu - 1) + me^{\tau_T}(1 + \mu) - \sqrt{m}e^{\tau_T/2}(1 + \mu + m\mu)}{m(e^{\tau_T/2} - 1)^2}. \quad (42)$$

Notice that $\Omega_{a,th}$ does not depend on the thermal photons μ . In figure 4 the thresholds $\Omega_{a,th}$ and $\Omega_{c,th}$ are plotted for different

values of m and μ (see the caption for details). The regions below the thresholds refer to the regimes for which the local strategy is more effective than telecloning. We notice that the benefits of telecloning become slightly less effective as the number of modes m and the thermal noise μ increase. This is due to the fact that in this case the performance of ideal telecloning decreases too.

6. Conclusions

In this paper we considered the application of CV cloning in a quantum communication scenario. In particular, we analysed an amplitude-modulated channel in which a coherent signal has to be distributed among m parties. On the basis of the fidelity of the clones as a figure of merit, we compared two strategies for

performing the distribution task in a noisy environment: on one hand the optimized LCDT scheme, where no entanglement is present; on the other hand the optimized telecloning protocol. Since the noise acts differently in the two protocols, we found that telecloning is more effective than the LCDT scheme for a wide range of noise parameters. This result shows that entanglement, besides being recognized as a valuable resource for a variety of two-party protocols, is also a *robust* resource when many parties are involved. Furthermore, the high fidelity obtained by telecloning suggests that entanglement may be a resource for enhancing the exchanged information in a multiparty communication network. Work along these lines is in progress and results will be reported elsewhere.

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