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ПРОБЛЕМА РЕГИСТРАЦИИ ОДИНОЧНЫХ ФОТОНОВ, РЕКОНСТРУКЦИЯ (ТОМОГРАФИЯ) СОСТОЯНИЙ ОПТИЧЕСКИХ ПОЛЕЙ

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ON RECONSTRUCTING PHOTON STATISTICS BY ON/OFF DETECTORS: TOWARD MULTI-PARTITE CASE

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Abstract—Reconstruction of photon statistics of optical states provides fundamental information on the nature of any optical field and finds various relevant applications. Nevertheless, no detector that can well discriminate the number of incident photons is available, whereas reconstruction by quantum tomography is not an easily implementable technique suited for a diffuse use. Even if on/off detectors, as usual avalanche photodiodes operating in Geiger mode, seem useless as photocounters, recently it was shown how reconstruction of photon statistics is possible by considering a variable quantum efficiency. We have experimentally demonstrated the potentialities of this technique by reconstructing photon statistics for various optical fields both in cw and pulsed regime ranging from heralded photon states to multi-thermal and coherent ones. In this paper, after having introduced the method, a review of the results obtained for the mono-partite case is given and the extension to bi-partite case is presented in detail.

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INTRODUCTION

The statistics of the radiation field plays a crucial role in fundamental quantum optics [1, 2] and finds relevant applications in quantum communication [3], imaging [4], and spectroscopy [5]. Nevertheless, at the moment, photodetectors well suited for this purpose are not available, since the few existing examples [6–9] still have very severe limitations. On the other hand, the alternative of reconstructing density matrix by quantum tomography [10–12] leads to various technical difficulties that are particular severe in the pulsed regime (where mode matching between signal an local oscillator is very challenging) and it is not suited for a widespread use.

Recently [13], a maximum-likelihood method based on on/off detection performed with different quantum efficiencies [14–16] has been developed and experimentally demonstrated for mono-partite quantum optical states. The results are reliable and accurate also for detectors with relatively low quantum efficiency.

In this paper, after a general review of this scheme and of previous results, we present our recent experimental work addressed on one hand to an accurate estimate of uncertainties on reconstruction of diagonal elements of density matrix and on the other hand to extend this reconstruction to multi-partite case. Various experimental data, showing the validity of the method also in this last case, will be presented and discussed.

MONO-PARTITE CASE: THEORY

In the mono-partite case, the procedure presented in Ref. [15, 16] consists in measuring a given signal by on/off detection using different values η_ν ($\nu = 1, \ldots, K$) of the quantum efficiency. The information provided by experimental data is contained in the collection of frequencies $f_\nu = f_0(\eta_\nu) = n_{0\nu}/n_\nu$ where $n_{0\nu}$ is the number of "no click" events and n_ν the total number of runs with quantum efficiency η_ν.

Then we consider the expression of the probability of no click

$$p_0(\eta) = \sum_n (1-\eta)^n \varrho_n \qquad (1)$$

as a statistical model for the parameters ϱ_n (diagonal element of the density matrix $\varrho_n = \langle n|\varrho|n\rangle$) to be solved by maximum-likelihood (ML) estimation.

Upon defining $p_\nu \equiv p_0(\eta_\nu)$ and $A_{\nu n} = (1-\eta_\nu)^n$, we rewrite expression (1) as $p_\nu = \sum_n A_{\nu n} \varrho_n$. Since the model is linear and the parameters to be estimated are positive (LINPOS problem), the solution can be obtained by using the Expectation–Maximization algorithm (EM)

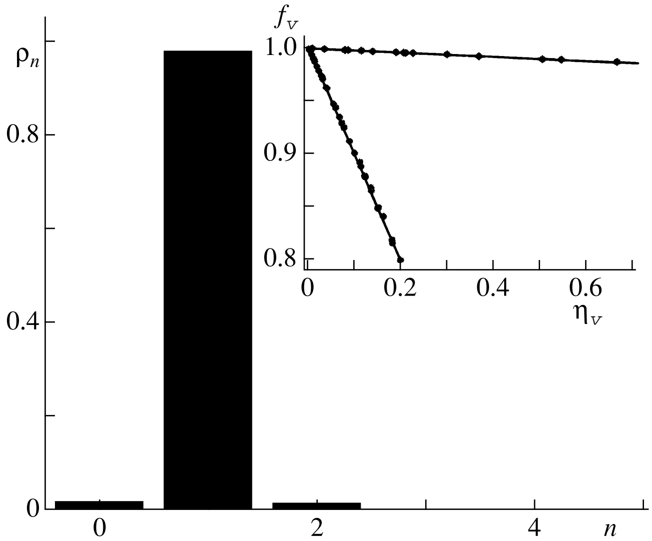

Fig. 1. Reconstruction of the photon distribution for the heralded single-photon state produced in type II PDC. Inset: The steepest curve corresponds to experimental frequencies f_ν of no-click events as a function of the quantum efficiency η_ν for a PDC heralded photon state compared with the theoretical curve $p_\nu = 1 - \eta_\nu$. For the sake of completeness, the curve for data on a weak coherent state are shown as well (higher one).

[17]. By imposing the restriction $\sum_n \varrho_n = 1$, we obtain the iterative solution

$$\varrho_n^{(i+1)} = \varrho_n^{(i)} \sum_{\nu=1}^{K} \frac{A_{\nu n}}{\sum_m A_{\nu m}} \frac{f_\nu}{p_\nu[\{\varrho_n^{(i)}\}]}, \tag{2}$$

where $p_\nu[\{\varrho_n^{(i)}\}]$ are the probabilities p_ν, as calculated by using the reconstructed distribution $\{\varrho_n^{(i)}\}$ at the ith iteration. As a measure of convergence we use the total absolute error at the ith iteration $\varepsilon^{(i)} = \sum_{\nu=0}^{K} |f_\nu - p_\nu[\{\varrho_n^{(i)}\}]|$ and stop the algorithm as soon as $\varepsilon^{(i)}$ reaches a minimum or goes below a given threshold.

Since the reconstruction of the photon statistics is based on the ML algorithm, which by an iterative method reaches the ϱ_n best reproducing the experimental data (i.e., the statistics of 'no-clicks' f_ν), the confidence interval on the ML determinations may be evaluated according to the following argument. As ML methods are unbiased at convergence we have $p_\nu\{\varrho_n^{\infty}\} \equiv f_\nu$. If in practice we stop at a given iteration L, we have fluctuations in the reconstructed $\{\varrho_n\}$ and, in turn, $p_\nu\{\varrho_n^{L}\} \neq f_\nu$. The fluctuations in the single components ϱ_n leads to

$$p_\nu\{\varrho_n^{L}\} = f_\nu + \frac{\partial p_\nu}{\partial \varrho_n}(\varrho_n - \varrho_n^{*}), \tag{3}$$

where we have denoted by ϱ_n^{*} the true value of the distribution. Thus the uncertainty can be estimated as $\delta\varrho_n^{\nu} = |p_\nu\{\varrho_n^{L}\} - f_\nu|/A_{\nu n}$ and, averaging over the different values of the quantum efficiency,

$$\delta\varrho_n \simeq \frac{1}{K}\sum_{\nu=1}^{K} \frac{|p_\nu\{\varrho_n^{L}\} - f_\nu|}{A_{\nu n}}. \tag{4}$$

MONO-PARTITE CASE: EXPERIMENT

As an example of the application of this scheme to monopartite case we present here the reconstruction of a heralded single photon state. Other examples, ranging form coherent states to multithermal ones can be found in Ref. [13].

In little more detail, a pair of correlated photons has been generated by pumping a type II β-Barium–Borate (β-BaB_2O_4, BBO) crystal with a cw Argon ion laser beam (351 nm) in collinear geometry. Since the correlated photons have orthogonal polarization, we have split them by means of a polarizing beam splitter. The detection of one of the two by a silicon avalanche photodiode detector (SPCM-AQR-15, Perkin Elmer) was then used as an indication of the presence of the second photon in the other channel, i.e., a window of 4.9 ns for detection in arm 2 was opened in correspondence to the detection of a photon in arm 1. This "heralded photon" was then measured by a silicon avalanche photodiode detector preceded by an iris and an interference filter (IF) at 702 nm, 4 nm FWHM, inserted with the purpose of reducing the stray light. The quantum efficiency of the detection apparatus (including IF and iris) was measured to be 20% by using the PDC calibration scheme (see [18]). Lower quantum efficiencies were simulated by inserting calibrated neutral optical filters on the optical path.

A comparison of the observed frequencies f_ν with the theoretical curve $(1 - \eta_\nu)$ is presented in the inset of Fig. 1. The photon distribution has been reconstructed using $K = 34$ different values of the quantum efficiency ranging from $\eta_\nu \simeq 0$ to $\eta_\nu \simeq 20\%$ with $n_\nu = 10^6$ runs for each η_ν. Results at iteration $i = 10^6$ are shown in Fig. 1. As expected the PDC heralded photon state largely agrees with a single photon Fock state. However, also a small two photons component and a vacuum one are observed. The ρ_2 contribution is expected, by estimating the probability that a second photon randomly enters the detection window, to be 1.85% of ρ_1, in agreement, within uncertainties, with what observed. A non zero ρ_0 is also expected due to background. This quantity can be evaluated to correspond to $(2.7 \pm 0.2)\%$ by measuring the counts when the polarization of the pump beam is rotated to avoid generation of parametric fluorescence. Also this estimate is in good agreement

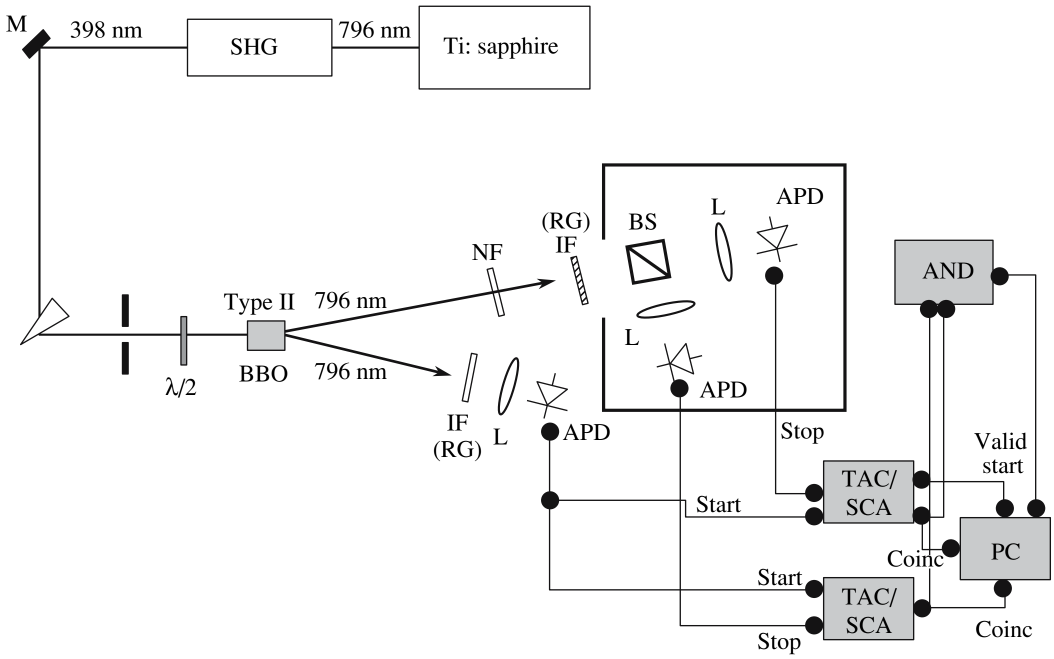

Fig. 2. Setup for the generation and measurement of split heralded photons: the type II PDC signal beam is collected by D1, to herald the presence of idler photons sent to the beam splitter and, from it, to D2 or D3; in the idler arm, neutral filters were inserted to change step by step the system's quantum efficiency.

with the reconstructed ρ_0. The relative uncertainty of the reconstructed state is 0.1%.

THEORY OF THE RECONSTRUCTION OF BIPARTITE STATES

For many applications, it would be important to extend this method to multipartite states. In the following we present in detail the bipartite case; the generalization to a larger number of modes directly follows, *mutate mutandis*, from analogous considerations.

Let us now consider independent on/off photodetection on two (spatially separated) modes of radiation. The joint statistics is given by

$$P_{00}(\eta) = \sum_{nk} A_{\eta n} A_{\eta k} \varrho_{nk}, \tag{5}$$

$$P_{01}(\eta) = \sum_{nk} A_{\eta n}(1 - A_{\eta k}) \varrho_{nk}, \tag{6}$$

$$P_{10}(\eta) = \sum_{nk} (1 - A_{\eta n}) A_{\eta k} \varrho_{nk}, \tag{7}$$

$$P_{11}(\eta) = 1 - P_{00}(\eta) - P_{10}(\eta) - P_{01}(\eta), \tag{8}$$

where $\varrho_{nk} = \langle n|_1 \langle k|_2 \varrho |k\rangle_2 |n\rangle_1$ is the joint photon distribution of the two modes (1), (2). In analogy with monopartite case we adopt the following strategy: by placing in front of the detector K filters with different transmissions, we may perform the detection with K different values η_ν, $\nu = 1\ldots K$, ranging from zero to a maximum value $\eta_K = \eta_{max}$ (equal to the nominal quantum efficiency of the detection apparatus).

By introducing the vectors

$$\begin{aligned} \mathbf{g} = (&p_{00}(\eta_1), \ldots, p_{00}(\eta_K), p_{01}(\eta_1), \ldots, \\ &p_{01}(\eta_K), p_{10}(\eta_1), \ldots, p_{10}(\eta_K)) \end{aligned} \tag{9}$$

and

$$\mathbf{q} = (\varrho_{00}, \varrho_{01}, \varrho_{10}, \ldots), \tag{10}$$

$\varrho_{nk} \longrightarrow q_p$ with $p = 1 + k + n(1 + N)$, i.e., $k = (p - 1)\mathrm{mod}(1 + N)$ and $n = (p - 1 - k)/(1 + N)$ we can rewrite Eqs. (5)–(8) as

$$g_\mu = \sum_p B_{\mu p} q_p, \tag{11}$$

where $\mu = 1, \ldots 3K$, $p = 1, \ldots, (1 + N)^2$ and where we have introduced the matrix

$$B_{\mu p} = \begin{cases} A_{\mu n} A_{\mu k}, & \mu = 1, \ldots, K; \\ A_{\mu n}(1 - A_{\mu k}), & \mu = K + 1, \ldots, 2K; \\ (1 - A_{\mu n}) A_{\mu k}, & \mu = 2K + 1, \ldots, 3K. \end{cases} \tag{12}$$

Let us now suppose that the ϱ_n's are negligible for $n > N$ and that the η_μ's are known, then Eq. (11) represents a statistical linear model for the positive unknowns q_p.

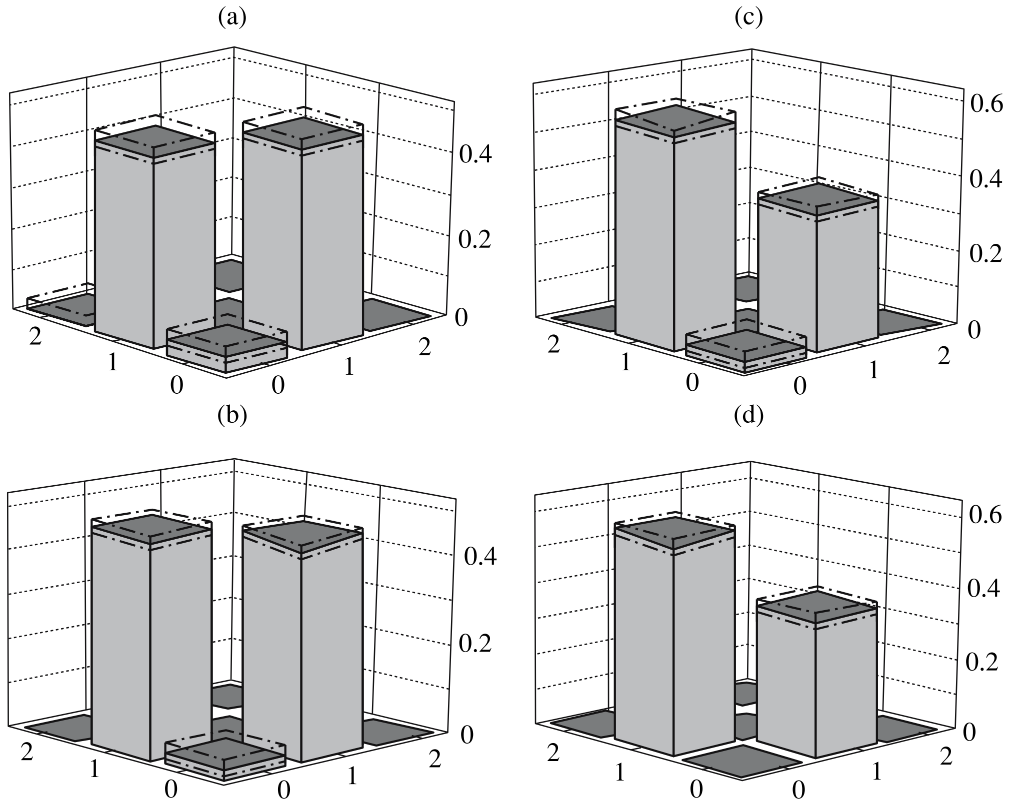

Fig. 3. Reconstruction of diagonal elements of density matrix for heralded photons. 50-50 BS, with IF (a) and RG (b). 40–60 BS, with IF (c) and RG (d). Dashed lines indicate confidence intervals.

The solution of this LINPOS problem may be well approximated by the iterative algorithm

$$q_p^{(i+1)} = q_p^{(i)} \left(\sum_{\mu=1}^{3K} B_{\mu p} \right)^{-1} \sum_{\mu=1}^{3K} B_{\mu p} \frac{h_\mu}{g[\{q_p^{(i)}\}]}, \qquad (13)$$

where index p summarizes indeces nk following the rule given before.

In Eq. (13) $q_p^{(i)}$ denotes the p-th element of reconstructed statistics at the i-th step, $g_\mu[\{q_p^{(i)}\}]$ the theoretical probabilities as calculated from Eq. (11) at the i-th step, whereas h_μ denotes the frequency of the events with quantum efficiency η_μ, i.e., $h_\mu = n_{00\mu}/n_\mu$ for $\mu = 1, \ldots, K$, $h_\mu = n_{01\mu}/n_\mu$ for $\mu = K + 1, \ldots, 2K$ and $h_\mu = n_{10\mu}/n_\mu$ for $\mu = 2K + 1, \ldots, 3K$, being n_μ the total number of runs performed with $\eta = \eta_\mu$. In the following we assume all n_μ to be equal.

The evaluation of uncertainties follows the same arguments presented for the mono-partite case.

EXPERIMENTAL DATA: SPLITTED HERALDED PHOTON

In order to test the method of reconstruction of photon statistics in the bipartite case, we begin by applying it to split heralded photon states generated by Parametric Down Conversion.

In our set-up (see Fig. 2) a pulsed (with 200 fs pulses) laser beam pumped a type II BBO crystal. The beam, at 398 nm, was generated by second harmonic of a titanium-sapphire beam at 796 nm and had a 0.2 W power. The detection by a photo-detector (D1) of one photon on one of two correlated branches of degenerated PDC emission was then used as trigger for heralding the presence of the correlated photon in the other direction. The optical path of this second photon was then separated by a beam splitter (BS), generating a bi-partite state. The BS was followed on both output arms by a detection apparatus, D2 and D3, respectively. All the detectors were APD silicon photo-detectors (SPCM-AQR-15, Perkin Elmer).

In correspondence to the detection of a photon in arm 1, a coincidence window was opened on both detectors on arm 2. This was obtained by addressing the output of the first detector as a start of two Time to Amplitude Converters (TAC) that received the detector signal of D2 and D3 as stops. The 20 ns window was set such to not include spourious coincidences with PDC photons of the following pulse (the repetition rate of the laser was 70 MHz). The TAC outputs were then addressed both to counters and to an AND logical gate for measuring coincidences between them. These outputs

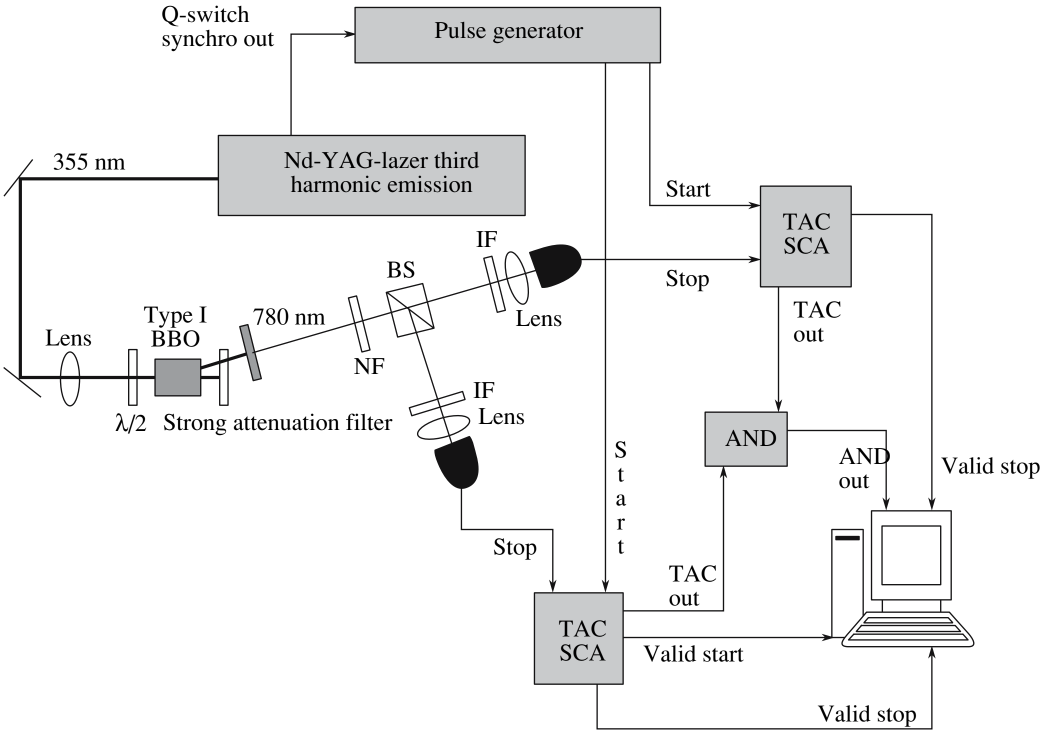

Fig. 4. Setup for the generation and measurement of split multithermal states: a beam of type I PDC (eventually strongly attenuated) goes through the BS and, then, each branch is filtered by an IF and collected by an APD.

allowed to evaluate the probabilities P_{00}, P_{01}, P_{10}, P_{11} needed for reconstructing the statistics of the bi-partite state with the method described in the previous section.

The background was evaluated and subtracted by measuring the TAC and AND outputs out of the window triggered by D1 detection. The ratio between the sum of coincidences in D2 and D3 and the counting on D1 in absence of neutral filters allowed to estimate the detector quantum efficiency [20].

With the purpose to verify the method in different cases, we considered 4 different alternatives given by the combination of a balanced (50–50%) or unbalanced (40–60%) BS with either a large band, red glass filter (RG) with cut-off wave length at 750 nm, or an interference filter (IF), 10 nm Full Width Half Maximum.

The different quantum efficiencies were obtained by inserting (before the BS) Schott neutral filters, whose calibration was obtained by measuring the ratio between the number of photons on D2 with the filter and without it.

The reconstructed statistics for these four situations are shown in Fig. 3. The uncertainties have been evaluated as described previously.

As evident from the figure the state corresponds well to a single photon. Only probabilities P_{01}, P_{10} are different (within uncertainties) from zero and their ratio is the value expected for the ratio of the output ports of the BS (unity for the balanced one, 2/3 for the unbalanced one). As expected in this regime, no multi-photon component is observed: i.e., P_{11}, P_{20}, P_{02}, et cetera are compatible (within uncertainties) with zero.

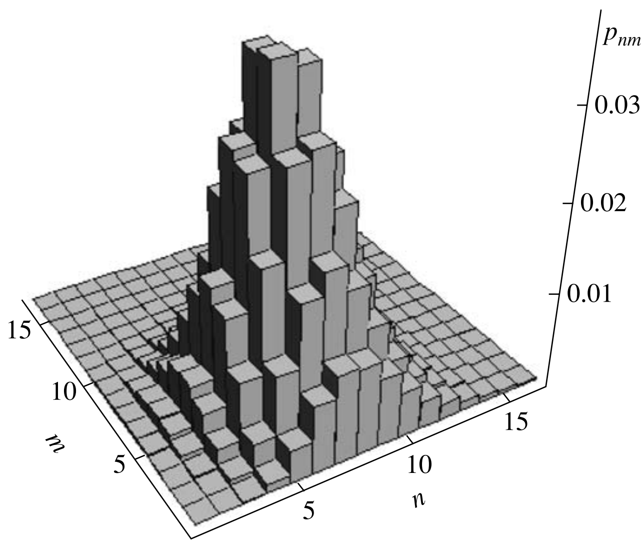

Fig. 5. Reconstructed diagonal elements of the density matrix of a split (40–60%) multithermal state.

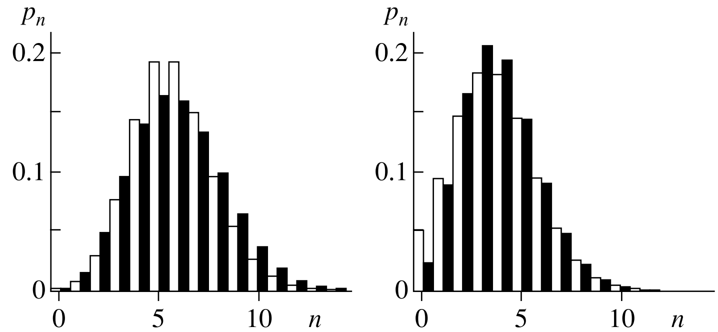

Fig. 6. Marginals of the reconstructed diagonal elements of the density matrix of a split multithermal state compared with theoretical predictions.

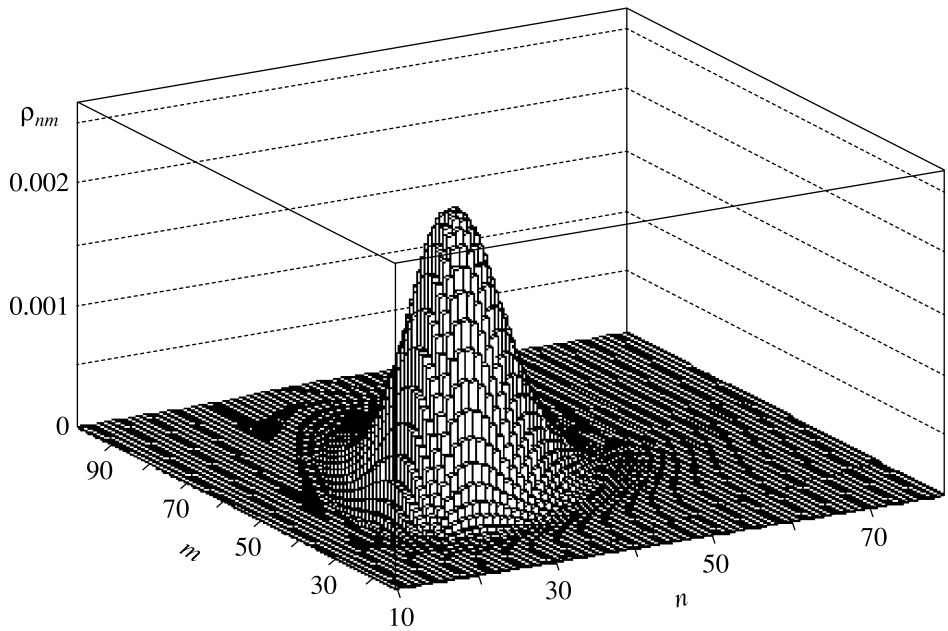

Fig. 7. Reconstruction of diagonal elements of the density matrix for a bipartite state: unattenuated PDC multithermal distribution split by a beam splitter 40–60%.

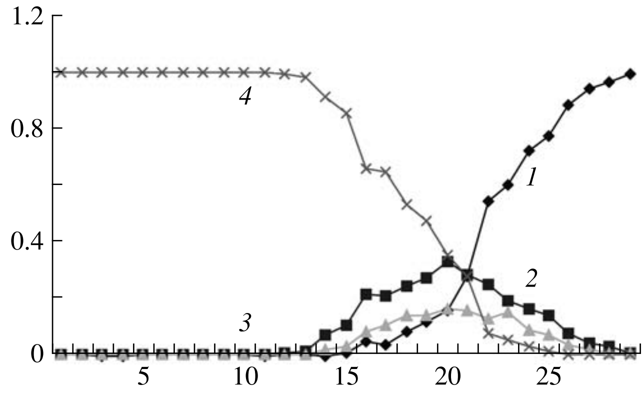

Fig. 8. Joint detection probabilities for unattenuated PDC multithermal distribution split by a beam splitter 40–60%: P_{00} (curve *1*, ◆), P_{01} (*2*, ■), P_{10} (*3*, △), and P_{11} (*4*, ×).

EXPERIMENTAL DATA: MULTITHERMAL STATISTICS

As a second example, we consider a single branch of PDC emission without triggering, which corresponds to a multi-thermal statistics [1]. The bi-partite state is again generated by separating the two paths through a BS.

Before the BS, we have a multithermal distribution with μ modes and N total (average) photons

$$p_n(\mu, N) = \binom{n+\mu-1}{\mu-1}\frac{1}{(1+N/\mu)^{\mu}(1+\mu/N)^{n}}. \quad (14)$$

As a consequence, the joint distribution at the output is

$$p_{nm}(\mu, N) = \\ = \frac{(n+m+\mu-1)!}{n!m!(\mu-1)!}\frac{1}{(1+N/\mu)^{\mu}(1+\mu/N)^{n+m}}, \quad (15)$$

whereas the resulting two marginals are multithermal distributions with the same number of modes and average photons given by $N\tau$ and $N(1-\tau)$, respectively. Thus, the expected statistics of clicks are

$$p_{00} = \sum_{nm} p_{nm}(\mu, N)(1-\eta)^n(1-\eta)^m = \\ = \left(1+\frac{\eta N}{\mu}\right)^{-\mu}, \quad (16)$$

$$p_{01} = \sum_{nm} p_{nm}(\mu, N)(1-\eta)^n[1-(1-\eta)^m] = \\ = \left(1+\frac{\eta N\tau}{\mu}\right)^{-\mu} - \left(1+\frac{\eta N}{\mu}\right)^{-\mu}, \quad (17)$$

$$p_{10} = \sum_{nm} p_{nm}(\mu, N)[1-(1-\eta)^n](1-\eta)^m = \\ = \left(1+\frac{\eta N(1-\tau)}{\mu}\right)^{-\mu} - \left(1+\frac{\eta N}{\mu}\right)^{-\mu}. \quad (18)$$

For what concerns the setup (see Fig. 4), the state has been produced by pumping a $5 \times 5 \times 5$ mm type I BBO crystal with a beam of a Q-switched triplicated (to 355 nm) Neodimium-YAG laser with pulses of 5 ns, power up to 200 mJ per pulse, and 10 Hz repetition rate.

Due to the very high power of the pump beam, a state with a relatively large number of photon was generated every each pulse. We have therefore attenuated (by 1nm FWHM IF and neutral filters) the multithermal state before detection.

The calibration of neutral filters was realized by measuring the transmission of a diode laser beam with the same direction and frequency of the PDC emission to be analyzed.

The result of this reconstruction is shown in Fig. 5. Also in Fig. 6, we show the two marginal reconstructions compared with the theoretical expectation. Again the statistics is rather well reconstructed, as certified by a smaller than unity value of the reduced χ^2 for both the marginal distributions. Also the fidelity $F = \sum_n \sqrt{\varrho_n \varrho_n^{mth}}$ of the reconstructed distribution to the expected multithermal $\{\varrho_n^{mth}\}$ is larger than 99% for both the marginal distributions. Furthermore, the optimal number of iterations (i.e., leading to maximum average fidelity of the two marginal distributions) corresponds to the minimum of ϵ, thus confirming the good convergence properties of the algorithm.

Finally, we present some preliminary data addressed to investigate the case of a larger number of photons. The experimental scheme is essentially the same of the previous example, but without attenuating the PDC emission. The BS was 60–40%.

The result of this reconstruction is shown in Fig. 7, while the joint detection probabilities used for this reconstruction are shown in Fig. 8. Qualitatively a multithermal distribution (with on average about 40 photons) is reconstructed, certifying the potentiality of the scheme to work even in such regime. Nevertheless, a precise quantitative comparison between reconstructed state and theoretical predictions still requires both a more accurate analysis of the convergence of the reconstruction procedure and a larger statistics of data (now limited by the low repetition rate of the laser) for reducing their statistical uncertainties. A new setup overcoming these limits is under realization.

CONCLUSIONS

In this paper we have discussed the possibility of reconstructing photon statistics by using on/off detectors.

After having reviewed the results for the mono-partite case, which show the interesting potentialities of this technique, we have presented new data (some preliminary) on reconstruction of bipartite states. These results clearly demonstrate the possibility to successfully extend the method to multipartite case.

ACKNOWLEDGMENTS

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