Optimal quantum repeaters for qubits and qudits

Marco G. Genoni and Matteo G. A. Paris*
Dipartimento di Fisica and INFM, Università degli Studi di Milano, I-20133 Milano, Italy

(Received 10 January 2005; published 6 May 2005)

A class of optimal quantum repeaters for qubits is suggested. The schemes are *minimal*, i.e., they involve a single additional *probe* qubit, and *optimal*, i.e., they provide the maximum information adding the minimum amount of noise. Information gain and state disturbance are quantified by fidelities which, for our schemes, saturate the ultimate bound imposed by quantum mechanics for randomly distributed signals. Special classes of signals are also investigated, in order to improve the information-disturbance trade-off. Extension to higher-dimensional signals (qudits) is straightforward.

DOI: 10.1103/PhysRevA.71.052307 PACS number(s): 03.67.Hk, 03.65.Ta, 03.67.−a

I. INTRODUCTION

In a multiuser transmission line, each user should decode the transmitted symbol and leave the carrier for the subsequent user. What they need is an ideal *repeater*, i.e., a device that for each shot retrieves the message without altering the carrier. However, symbols are necessarily encoded in states of a physical system and therefore the ultimate bound on the performances as a repeater is posed by quantum mechanics. Indeed, a perfect quantum repeater cannot be achieved, i.e., quantum information cannot be perfectly copied, neither locally [1] nor at a distance [2]. Any measurement performed to extract information on a quantum state in turn alters the state itself, i.e., produces a disturbance.

The trade-off between information gain and quantum state disturbance can be quantified using fidelities. Let us describe a generic scheme for indirect measurement as a quantum operation, i.e., without referring to any explicit unitary realization. The operation is described by a set of *measurement operators* $\{A_k\}$, with the condition $\Sigma_k A_k^\dagger A_k = \mathbb{I}$. The probability operator-valued measure (POVM) of the measurement is given by $\{\Pi_k \equiv A_k^\dagger A_k\}$, whereas its action on the input state is expressed as $\varrho \to \Sigma_k A_k \varrho A_k^\dagger$. This means that, if ϱ is the initial quantum state of the system under investigation, the probability distribution of the outcomes is given by $p_k = \mathrm{Tr}[\varrho \Pi_k] = \mathrm{Tr}[\varrho A_k^\dagger A_k]$, whereas the conditional output state, after having detected the outcome k, is expressed as $\sigma_k = A_k \varrho A_k^\dagger / p_k$, such that the overall quantum state after the measurement is described by the density matrix $\sigma = \Sigma_k p_k \sigma_k = \Sigma_k A_k \varrho A_k^\dagger$.

Suppose now you have a quantum system prepared in a pure state $|\psi\rangle$. If the outcome k is observed at the output of the repeater, then the estimated signal state is given by $|\phi_k\rangle$ (the typical inference rule being $k \to |\phi_k\rangle$ with $|\phi_k\rangle$ given by the set of eigenstates of the measured observable), whereas the conditional state $|\psi_k\rangle = 1/\sqrt{p_k} A_k |\psi\rangle$ is left for the subsequent user. The amount of disturbance is quantified by evaluating the overlap of the conditional state $|\psi_k\rangle$ to the initial one $|\psi\rangle$, whereas the amount of information extracted by the measurement corresponds to the overlap of the inferred state $|\phi_k\rangle$ to the initial one. The corresponding fidelities, for a given input signal $|\psi\rangle$, are given by

$$F_\psi = \sum_k p_k \frac{|\langle \psi | A_k | \psi \rangle|^2}{p_k} = \sum_k |\langle \psi | A_k | \psi \rangle|^2, \tag{1}$$

$$G_\psi = \sum_k p_k |\langle \psi | \phi_k \rangle|^2, \tag{2}$$

where we have already performed the average over the outcomes. The relevant quantities to assess the repeater are then given by the average fidelities

$$F = \int_{\mathbb{A}} d\psi F_\psi, \quad G = \int_{\mathbb{A}} d\psi G_\psi, \tag{3}$$

which are obtained by averaging F_ψ and G_ψ over the possible input states, i.e., over the alphabet $\mathbb{A}$ of transmittable symbols. F will be referred to as the transmission fidelity and G as the estimation fidelity.

Let us first consider two extreme cases. If nothing is done, the signal is preserved and thus $F=1$. However, at the same time, our estimation has to be random and thus $G=1/d$, where d is the dimension of the Hilbert space. This corresponds to a *blind* quantum repeater [3] which reprepares any quantum state received at the input without gaining any information on it. The opposite case is when the maximum information is gained on the signal, i.e., when the optimal estimation strategy for a single copy is adopted [4–6]. In this case $G=2/(d+1)$, but then the signal after this operation cannot provide anymore information on the initial state and thus $F=2/(d+1)$. Between these two extrema there are intermediate cases, i.e., quantum measurements providing only partial information while partially preserving the quantum state of the signal for subsequent users. These schemes, which correspond to feasible quantum repeaters, may also be viewed as quantum nondemolition measurements [7], which have been widely investigated for continuous variable systems, and recently received attention also for qubits [8].

The fidelities F and G are not independent of each other. Assuming that $\mathbb{A}$ corresponds to the set of *all* possible quantum states, Banaszek [9] has explicitly evaluated the expressions of fidelities in terms of the measurement operators, rewriting Eq. (3) as

$$F = \frac{1}{d(d+1)} \left(d + \sum_k |\mathrm{Tr}[A_k]|^2 \right),$$

*Electronic address: matteo.paris@fisica.unimi.it

$$G = \frac{1}{d(d+1)}\left(d + \sum_k \langle \phi_k|\Pi_k|\phi_k\rangle\right), \qquad (4)$$

where $|\phi_k\rangle$ is the set of states used to estimate the initial signal. Of course, the estimation fidelity is maximized choosing $|\phi_k\rangle$ as the eigenvectors of Π_k corresponding to the maximum eigenvalues.

Using Eqs. (4), it is possible to derive the bound that fidelities should satisfy according to quantum mechanics. For randomly distributed d-dimensional signals, i.e., when the alphabet $\mathbb{A}$ corresponds to the set of *all* quantum states for a qudit, the information-disturbance trade-off reads as follows [9]:

$$(F-F_0)^2 + d^2(G-G_0)^2 + 2(d-2)(F-F_0)(G-G_0) \leqslant \frac{d-1}{(d+1)^2}, \qquad (5)$$

where $F_0=\frac{1}{2}(d+2)/(d+1)$ and $G_0=\frac{1}{2}3/(d+1)$. For randomly distributed qubits, i.e., assuming a two-dimensional Hilbert space, and with the alphabet $\mathbb{A}$ equal to the whole Bloch sphere, the bound (5) reduces to

$$\left(F-\frac{2}{3}\right)^2 + 4\left(G-\frac{1}{2}\right)^2 \leqslant \frac{1}{9}. \qquad (6)$$

From Eq. (5), one knows the maximum transmission fidelity compatible with a given value of the estimation fidelity or, in other words, the minimum unavoidable amount of noise that is added to the knowledge about a set of signals if one wants to achieve a given level of information.

In this paper, we suggest a set of explicit unitary realizations for the indirect estimation of qubits. Our schemes are *minimal*, since they involve a single additional probe qubit, and *optimal*, i.e., the corresponding fidelities saturate the bound (6) with the equal sign. The schemes can be easily generalized to the case of qudits, yet being minimal and saturating the bound (5). Recently [10], similar schemes have been suggested, also with the possibility of obtaining signal-independent fidelities through a twirl operation [11,12].

The paper is structured as follows. In Sec. II, the simplest example of our class of schemes will be described in detail. Its possible generalizations, involving the measurement of a generic spin component, are analyzed in Sec. II A, whereas generalization to dimension d is described in Sec. II B. In Sec. III, we return back to qubits, and consider the transmission of signals with quantum states that do not span the entire Hilbert space, i.e., of alphabets that are proper subsets of the Bloch sphere. It will be shown that discrete alphabets can be used to beat the bound (6), whereas continuous alphabets different from the whole Bloch sphere lead to inferior performances. A general condition on the class of signals to beat the bound will be also derived. Section IV closes the paper with some concluding remarks.

II. MINIMAL IMPLEMENTATION OF OPTIMAL QUANTUM REPEATERS

In this section, we suggest a measurement scheme to estimate the state of a generic qubit without its destruction.

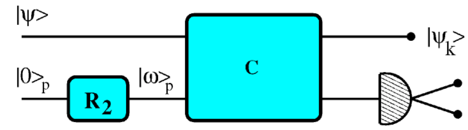

FIG. 1. Minimal implementation of the optimal quantum repeater for qubits.

This scheme is minimal because it involves a single additional probe qubit, and optimal because it saturates the bound (6).

The measurement scheme is shown in Fig. 1. The signal qubit

$$|\psi\rangle = \cos\frac{\theta_1}{2}|0\rangle + e^{i\phi_1}\sin\frac{\theta_1}{2}|1\rangle$$

is coupled with a probe qubit

$$|\omega\rangle_p = \mathbf{R}_2|0\rangle_p = \cos\frac{\theta_2}{2}|0\rangle_p + e^{i\phi_2}\sin\frac{\theta_2}{2}|1\rangle_p$$

by a C_{not} gate (denoted by $\mathbf{C}$). $\mathbf{R}_i$ denotes a qubit rotation by angles (θ_i,ϕ_i) with respect to the z axis. After the interaction, the spin component in the z direction is measured on the probe qubit.

According to Eqs. (1) and (2), the fidelities F_ψ and G_ψ are given by

$$F_\psi = p_0|\langle\psi|\psi_0\rangle|^2 + p_1|\langle\psi|\psi_1\rangle|^2,$$

$$G_\psi = p_0|\langle\psi|0\rangle|^2 + p_1|\langle\psi|1\rangle|^2,$$

where $p_k=\langle\psi|\Pi_k|\psi\rangle$, $k=0,1$ are the probabilities for the two possible outcomes, and

$$|\psi_k\rangle = \frac{A_k|\psi\rangle}{\sqrt{p_k}}.$$

are the corresponding conditional (pure) states, which are left for the subsequent user. Moreover, in writing G_ψ, we assumed the inference rule $k\to|k\rangle$, with $|k\rangle$ eigenstates of the measured observable σ_z.

The measurement operators A_k for our schemes are given by

$$A_k = {}_p\langle k|\mathbf{C}|\omega\rangle_p, \qquad (7)$$

whereas the POVM can be evaluated as follows:

$$\Pi_k = \mathrm{Tr}_p[\mathbf{C}\mathbb{I}\otimes|\omega\rangle_{pp}\langle\omega|\mathbf{C}^\dagger\mathbb{I}\otimes|k\rangle_{pp}\langle k|] = A_k^\dagger A_k, \qquad (8)$$

where $\mathrm{Tr}_p[\cdots]$ denotes partial trace over the probe degrees of freedom. Explicitly, in the standard basis, we have

$$A_0 = \begin{pmatrix} \cos\frac{\theta_2}{2} & 0 \\ 0 & e^{i\phi_2}\sin\frac{\theta_2}{2} \end{pmatrix}, \quad A_1 = \begin{pmatrix} e^{i\phi_2}\sin\frac{\theta_2}{2} & 0 \\ 0 & \cos\frac{\theta_2}{2} \end{pmatrix}.$$

The mean fidelities F and G are obtained by averaging over all the possible input states, i.e., the whole Bloch sphere,

$$F = \frac{1}{4\pi}\int_0^{2\pi} d\phi_1 \int_0^{\pi} d\theta_1 \sin\theta_1 F_\psi ,$$

$$G = \frac{1}{4\pi}\int_0^{2\pi} d\phi_1 \int_0^{\pi} d\theta_1 \sin\theta_1 G_\psi .$$

According to Eq. (4), this corresponds to

$$F = \frac{1}{6}(2 + |\mathrm{Tr}[A_0]|^2 + |\mathrm{Tr}[A_1]|^2), \qquad (9)$$

$$G = \frac{1}{6}(2 + \langle 0|\Pi_0|0\rangle + \langle 1|\Pi_1|1\rangle). \qquad (10)$$

Explicit calculations of formulas (9) and (10) lead to

$$F = \frac{1}{6}\left(2 + 2\left|\cos\frac{\theta}{2} + e^{i\phi_2}\sin\frac{\theta_2}{2}\right|^2\right) = \frac{2}{3}\left(1 + \sin\frac{\theta_2}{2}\cos\frac{\theta_2}{2}\cos\phi_2\right), \qquad (11)$$

$$G = \frac{1}{6}\left(2 + 2\cos^2\frac{\theta_2}{2}\right) = \frac{1}{3}\left(1 + \cos^2\frac{\theta_2}{2}\right). \qquad (12)$$

Equations (11) and (12) say that any (allowed) ratio between the two fidelities may be achieved by a suitable preparation of the probe. At this point, we set $\phi_2 = 0$ and substitute Eq. (12) into Eq. (11) in order to find the explicit dependence $F = F(G)$. We have

$$F = \frac{2}{3}(1 + \sqrt{-9G^2 + 9G - 2}). \qquad (13)$$

The function $F(G)$ in Eq. (13) corresponds to the bound (6) with the equal sign and therefore proves that our scheme is an *optimal* explicit unitary realization of a quantum repeater for qubits. Notice that we have set $\phi_2 = 0$, i.e., this result has been obtained using only one of the two probe degrees of freedom.

A. A more general scheme

We now explore the possibility of generalizing our scheme for the measurement of the spin in a generic direction, i.e., for the measurement of the observable $\sigma_m = \mathbf{R}_m^\dagger \sigma_z \mathbf{R}_m$. Let us consider a scheme similar to that in Fig. 1 with the $\mathbf{C}$ gate replaced by the gate $\mathbf{W} = (\mathbb{I} \otimes \mathbf{R}_m)\ \mathbf{C}$. The corresponding map operators are given by

$$A'_k = {}_{mp}\langle k|(\mathbb{I} \otimes \mathbf{R}_m)\mathbf{C}|\omega\rangle_p = {}_p\langle k|\mathbf{C}|\omega\rangle_p = A_k, \qquad (14)$$

with $|k\rangle_m$ eigenstates of σ_m, whereas the POVM is obtained as

$$\Pi'_k = \mathrm{Tr}_p[\mathbf{W}\mathbb{I} \otimes |\omega\rangle_{pp}\langle\omega|\mathbf{W}^\dagger \mathbb{I} \otimes |k\rangle_{mpmp}\langle k|] = \mathrm{Tr}_p[\mathbf{C}\mathbb{I} \otimes |\omega\rangle_{pp}\langle\omega|\mathbf{C}^\dagger \mathbb{I} \otimes |k\rangle_{pp}\langle k|] = \Pi_k. \qquad (15)$$

The primed operators in Eqs. (14) and (15) are equal to the operators in Eqs. (7) and (8). As a consequence, the probabilities p_k and the conditional states $|\psi_k\rangle$ are the same as in the scheme of the previous section. At this point, we note that the simple inference rule $k \to |k\rangle_m$ cannot be used. In this case, in fact, we would obtain the same fidelity F as in the previous section, but a different fidelity G, and thus our repeater could not be optimal. However, it is straightforward to reestablish the same expression of G using the inference rule $k \to |k\rangle$ (with $|k\rangle$ eigenstates of σ_z). Using this procedure, the fidelities become equal to Eqs. (11) and (12) and the repeater is again optimal.

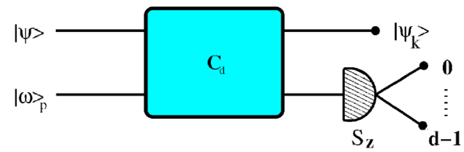

FIG. 2. Minimal implementation of optimal quantum repeater for qudits.

B. Optimal quantum repeaters for qudits

The optimal repeater for qubits described in the previous sections can be generalized to obtain an optimal repeater for qudits. The scheme is depicted in Fig. 2 and is similar to that of Fig. 1 with the C_{not} replaced by its d-dimensional counterpart, i.e., by the gate acting as $\mathbf{C}_d|i\rangle|s\rangle_p = |i\rangle|i \oplus s\rangle_p$, where $\oplus$ denotes sum modulo d [13]. The corresponding matrix elements read ${}_p\langle s|\langle i|\mathbf{C}_d|j\rangle|s'\rangle_p = \delta_{ij}\delta_{s,s'\oplus j}$.

The probe qudit is prepared in the state

$$|\omega\rangle_p = \cos\theta_2|0\rangle_p + \gamma\sin\theta_2\frac{1}{\sqrt{d}}\sum_{s=0}^{d-1}|s\rangle_p, \qquad (16)$$

where

$$\gamma = \frac{\sqrt{1 + d\tan^2\theta_2} - 1}{\sqrt{d}\tan\theta_2} \qquad (17)$$

is a normalization factor. As for qubits, an optimal repeater can be obtained exploiting a single probe degree of freedom. After the interaction, the spin of the probe is measured in a given direction. In the following, having in mind the equivalence already shown for qubits, we refer to a scheme where the spin is measured in the z direction.

The measurement operators are given by

$$A_k = {}_p\langle k|\mathbf{C}_d|\omega\rangle_p = \sum_{ij}(A_k)_{ij}|i\rangle\langle j|, \qquad (18)$$

where

$$(A_k)_{ij} = \delta_{ij}\left[\delta_{kj}\cos\theta_2 + \gamma\sin\theta_2\frac{1}{\sqrt{d}}\sum_s \delta_{k,j\oplus s}\right]. \qquad (19)$$

The fidelities are evaluated using Eqs. (4) and (19), arriving at

$$F = \frac{1}{d+1}[1 + (\cos\theta_2 + \gamma\sqrt{d}\sin\theta_2)^2],$$

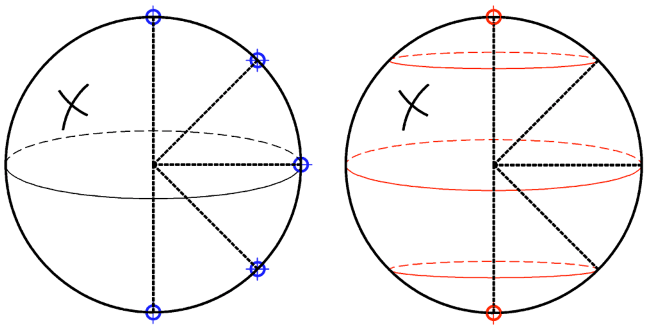

FIG. 3. Left: qubits belonging to discrete class **A** (N=5). Right: qubits belonging to continuous class **B** (N=5).

$$G = \frac{1}{d+1}\left[1 + \left(\cos\theta_2 + \frac{\gamma}{\sqrt{d}}\sin\theta_2\right)^2\right], \qquad (20)$$

which may be tuned by varying the preparation of the probe, i.e., the value of θ_2. Inserting Eq. (20) into Eq. (5), we found that the bound is saturated for γ given by Eq. (17). In other words, the scheme of Fig. 2 with a d-dimensional C_{not} and a probe qudit given by Eq. (16) provides an optimal quantum repeater for qudits.

III. SPECIAL CLASSES OF QUBITS

The bound in Eq. (6) has been derived with the assumption that the incoming signal is chosen at random on the whole Bloch sphere. In this section, we analyze whether a different choice of the alphabet may be used to beat the bound and, in turn, to improve the information-disturbance trade-off. As we will see, this is indeed the case assuming that the input signal is chosen within a discrete set of states, whereas a continuous subset of the Bloch sphere leads to degraded performances.

Let us consider the optimal repeater of Sec. II with the input signal chosen within the following two classes of states.

A. A discrete set made of N states ψ_j equally spaced in θ and with random phase ϕ. Since the fidelities F_ψ are phase-independent, we set, without loss of generality, $\phi=0$,

$$|\psi_j\rangle_A = \cos\frac{\theta_j}{2}|0\rangle + \sin\frac{\theta_j}{2}|1\rangle, \quad j=0,\ldots,(N-1),$$

where $\theta_j = j\pi/(N-1)$.

B. A continuous set of $2\pi \times N$ states, equally spaced in θ and with random phases,

$$|\psi_{j\phi}\rangle_B = \cos\frac{\theta_j}{2}|0\rangle + e^{i\phi}\sin\frac{\theta_j}{2}|1\rangle,$$

$$j=0,\ldots,(N-1),$$

where $\theta_j = j\pi/(N-1)$ and $\phi \in [0,2\pi]$.

The two sets are schematically depicted in Fig. 3.

Using Eqs. (1) and (2), we find that fidelities corresponding to states $|\psi_j\rangle_A$ and $|\psi_{j\phi}\rangle_B$ (with the same θ_j and different phases) are given by

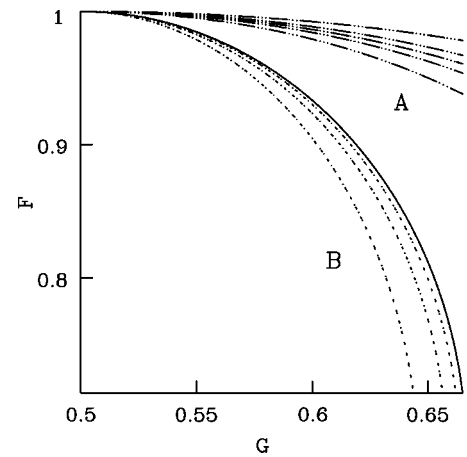

FIG. 4. The functions $F(G)$ for signals of class **A** and **B** for different values of N. The solid line denotes the bound $F(G)$ imposed by Eq. (6). Dot-dashed lines denote $F_{AN}(G_{AN})$, whereas dotted lines are for $F_{BN}(G_{BN})$. We plot curves for $N=4,5,7,11$ and $N=1000$. $F_{AN}(G_{AN})$ is always above the bound (6) and decreases with increasing N. $F_{BN}(G_{BN})$ is always below the bound (6) and increases with increasing N.

$$F_j = \cos^4\frac{\theta_j}{2} + \sin^4\frac{\theta_j}{2} + 4\sin^2\frac{\theta_j}{2}\cos^2\frac{\theta_j}{2}\sin\frac{\theta_2}{2}\cos\frac{\theta_2}{2}$$
$$= \frac{1}{2}[(1+\cos^2\theta_j) + \sin\theta_2(1-\cos^2\theta_j)], \qquad (21)$$

$$G_j = \left(\cos^4\frac{\theta_j}{2} + \sin^4\frac{\theta_j}{2}\right)\cos^2\frac{\theta_2}{2} + 2\sin^2\frac{\theta_j}{2}\cos^2\frac{\theta_j}{2}\sin^2\frac{\theta_2}{2}$$
$$= \frac{1}{2}(1+\cos^2\theta_j\cos\theta_2). \qquad (22)$$

The mean fidelities for class **A** are given by

$$F_{AN} = \frac{1}{N}\sum_{j=0}^{N-1} F_j = \frac{1+3N+(N-1)\sin\theta_2}{4N},$$

$$G_{AN} = \frac{1}{N}\sum_{j=0}^{N-1} G_j = \frac{1+3N+(N-1)\cos\theta_2}{4N}, \qquad (23)$$

from which we obtain

$$F_{AN}(G_{AN}) = \frac{1}{4N}\Bigg(1+3N+\frac{N-1}{N+1} \times\sqrt{(N+1)^2-4N^2(1-2G_{AN})^2}\Bigg). \qquad (24)$$

F_{AN} is a monotonously decreasing function of N and, as can be seen in Fig. 4, is above the bound set by Eq. (6) for any value of N. Therefore, by transmitting a discrete alphabet of symbols, we can beat the bound (6), i.e., the protocol is more convenient than the transmission of the whole class of qubits.

As concerns class **B**, the mean fidelities are evaluated as follows:

$$F_{BN} = \frac{1}{2\pi\sum_{j=0}^{N-1}\sin\theta_j}\sum_{j=0}^{N-1} 2\pi(\sin\theta_j)F_j,$$

$$G_{BN} = \frac{1}{2\pi\sum_{j=0}^{N-1}\sin\theta_j}\sum_{j=0}^{N-1} 2\pi(\sin\theta_j)G_j. \qquad (25)$$

For even N, we obtain

$$F_{BN} = \frac{1}{4}[3 + \sin\theta_2 + 2ie^{[i(3N-1)\pi]/[2(N-1)]}(1 + \sin\theta_2) - ie^{[i(5N-1)\pi]/[2(N-1)]}(3 + \sin\theta_2)]$$

$$G_{BN} = \frac{1}{4}[2 + \cos\theta_2 + 2ie^{[i(3N-1)\pi]/[2(N-1)]} - ie^{[i(5N-1)\pi]/[2(N-1)]}(2 + \cos\theta_2)], \qquad (26)$$

whereas for odd N,

$$F_{BN} = \frac{1 + \sin\theta_2 + \cos\dfrac{\pi}{N-1}(3 + \sin\theta_2)}{2\left(1 + 2\cos\dfrac{\pi}{N-1}\right)},$$

$$G_{BN} = \frac{1 + \cos\dfrac{\pi}{N-1}(2 + \cos\theta_2)}{2\left(1 + 2\cos\dfrac{\pi}{N-1}\right)}. \qquad (27)$$

Using Eqs. (26) and (27), we have calculated the explicit function $F_{BN}(G_{BN})$. The resulting expression is quite cumbersome and will not be reported here. In Fig. 4, we show the function $F_{BN}(G_{BN})$ for different values of N. All the curves are below the bound curve (6), approaching it for $N\rightarrow\infty$. Therefore, if we need to transmit a continuous alphabet, it is more effective to transmit qubits on the whole Bloch sphere rather than on a continuous subset.

A general condition may be found for an alphabet of signals to beat the bound (6). For an unspecified class of states, the fidelities may be evaluated using Eqs. (21) and (22). We have that

$$F = \frac{1}{2}[1 + \overline{\cos^2\theta} + \sin\theta_2(1 - \overline{\cos^2\theta})],$$

$$G = \frac{1}{2}(1 + \cos\theta_2\,\overline{\cos^2\theta}), \qquad (28)$$

where $\overline{(\cdots)}$ denotes the average over the alphabet. Substituting Eqs. (28) in Eq. (6), we found that any class of states violating the inequality

$$\left[\overline{\cos^2\theta} - \frac{1}{3} + (1 - \overline{\cos^2\theta})\sin\theta_2\right]^2 + 4\cos^2\theta_2\,\overline{\cos^2\theta}^2 \leqslant \frac{4}{9} \qquad (29)$$

provides a better information-disturbance trade-off than randomly distributed signals. As an example, for $\theta_2=0$, i.e., for the maximum value of the estimation fidelity G the bound (6) is surpassed for classes of states for which $\overline{\cos^2\theta} > 1/3$.

IV. CONCLUSIONS

In this paper, unitary realizations of quantum repeaters, i.e., nondemolitive estimations of qubits and qudits, have been suggested. The schemes are *minimal*, i.e., they involve a single probe system in addition to the signal, and *optimal*, i.e., they obtain the maximum information with the minimum amount of noise allowed by quantum mechanics for randomly distributed signals.

We then analyzed the performances of optimal repeaters on different classes of qubits, corresponding to alphabets that are subsets of the whole Bloch sphere. We derive a general condition that a class of states should satisfy to beat the bound (6) showing that discrete alphabets can beat this bound, whereas continuous alphabets lead to inferior performances.

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