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Quantum steering with Gaussian states: A tutorial

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ABSTRACT

Quantum steering refers to the apparent possibility of exploiting quantum correlations to remotely influence the quantum state of a subsystem, by measuring local degrees of freedom. In continuous-variable (CV) quantum information, this notion is strongly linked to the possibility of demonstrating the EPR paradox, whence the name *EPR steering*. Recently, another type of steering with CV Gaussian states was proposed under the name of *nonclassical steering*, stemming from the idea of remotely generating Glauber P-nonclassicality by conditional Gaussian measurements on two-mode Gaussian states. In this tutorial, we thoroughly illustrate these phenomena, firstly introducing quantum steering in its most general setting, and then focusing on Gaussian states and the connection with P-nonclassicality. We discuss the strong and weak forms of nonclassical steering, their relation with entanglement and how to formulate them in an invariant form with respect to local Gaussian unitary operations. For two-mode squeezed thermal states (TMST), we show that EPR steering coincides with nonclassical steering, and in particular this implies that a single type of Gaussian measurements is sufficient to check steerability for this class, unlike for the more general situation which usually requires the choice of distinct measurements.

1. Introduction

The classification of quantum correlations is a very active front of research since the early days of quantum mechanics. In this article, we investigate *quantum steering* and *nonclassical steering*, a class of asymmetric quantum correlations stronger than entanglement [1], but weaker than violation of Bell's inequality [2,3], that was introduced in relation to the EPR argument [4,5], to indicate the possibility of one party to collapse (or *steer*) the wavefunction of the other party into different quantum states by means of suitable measurements. Despite this early appearance, steering received firm mathematical bases only recently [6,7], and we refer to this definition as *EPR steering*, particularly in the context of continuous-variable (CV) systems [8]. The central idea of EPR steering is to use the influence of the measurements performed by one party (say Alice) to convince the other party (say Bob) that the shared state was entangled: if the initial correlated state allows for such a task, it is called *steerable* by Alice. Steering is now widely considered a fundamental resource for quantum communication tasks [9–14] and many criteria for its detection have been explored [15–18].

Independently of quantum correlations, a variety of other concepts of nonclassicality have been put forward [19]; in particular, the negativity of the Wigner quasiprobability distribution [20,21] received considerable attention recently, because it is believed to be necessary for universal quantum computation [22,23]. Gaussian states are defined precisely by their Gaussian Wigner distributions, therefore they cannot exhibit Wigner negativity as a form of nonclassicality. However, it is widely believed that Gaussian squeezed states do possess nonclassical features [24,25], which can be captured by the more general notion of *P-nonclassicality*. This is also the most widespread notion of nonclassicality for CV quantum states and it relies upon the negativity of the Glauber P-function [26–30], i.e. the expansion of a CV quantum state $\hat{\rho}$ onto coherent states $|\alpha\rangle$ $(\alpha \in \mathbb{C})$ according to:

$$\hat{\rho} = \int_{\mathbb{C}} d^2\alpha \, P\left[\hat{\rho}\right](\alpha) \, |\alpha\rangle\langle\alpha| \tag{1}$$

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$\hat{\boldsymbol{\rho}}$ is considered *nonclassical* whenever the *distribution* $P\left[\hat{\boldsymbol{\rho}}\right](\alpha)$ cannot be interpreted as a valid probability density, either because it can be represented by a function that attains negative values, or because a regularization of it does [31–37,26,38,30]. The main reason for the wide use of the P-function is that it leads to the most *physically inspired* notion of nonclassicality. It has direct empirical consequences, for example in quantum optics, where it is known to be necessary for antibunching and sub-Poissonian photon statistics [31]. Viceversa, classicality according to the P-function implies the empirical adequacy of Maxwell's Equations for the phenomenological description of the corresponding state of light. Moreover, P-nonclassical states are usually harder to fabricate [39,40], thereby giving a *resource* character to this type of nonclassicality [41,42]. Despite their different origins, insightful connections between nonclassicality and quantum correlations are known. Most importantly, the multimode extension of Eq. (1) entails that all P-classical states are separable or, conversely, that all entangled states are P-nonclassical. In Gaussian quantum optics this fact implies, for example, that the two output modes of a beam-splitter can be entangled only if there is some P-nonclassicality in the input modes. Recently, an attempt was advanced to bridge the standard notion of quantum steering with that of P-nonclassicality through the notion of *nonclassical steering* [43] for two-mode Gaussian states. It was shown that for a relevant subclass of them, the possibility of generating nonclassicality on one mode by conditioning upon Gaussian measurements on the other mode is equivalent to the condition of standard quantum steering, while for more generic states *nonclassical steering* splits in two distinct forms: a weak type that has little interest as a quantum correlation, and a strong type which is sufficient to imply quantum steering, provided that tighter conditions are met.

In this tutorial, we provide a step-by-step introduction to quantum steering, first focusing on Gaussian states, and then to nonclassical steering for two-mode Gaussian states, with a special emphasis on their similarities, as needed to develop a thorough understanding of the hierarchy of quantum correlations for this simple and paradigmatic family of quantum states. In Section 2 we introduce the standard notion of quantum steering in all generality through a thought experiment in which two parties have to check that a black box is able to prepare entangled states, but only one party can perform trustworthy, generic quantum measurements. We strive to clarify all hidden assumptions. Next, we shortly review Gaussian states in Section 3 and we discuss the steering inequality for the covariance matrix; in this continuous-variable context, we refer to the standard notion as EPR steering. Nonclassicality and the related notion of nonclassical steering are described in Section 4, where the weak and strong type are discussed and brought together with the idea of EPR steering on a common background. In Section 5, the relevant class of two-mode squeezed thermal states is introduced to show that EPR steering can be demonstrated with a single measurement, and it coincides with both strong and weak nonclassical steering for these states.

2. The notion of quantum steering: checking an untrusty device

To illustrate the current definition of quantum steering, we will construct a step-by-step scenario in which Alice and Bob share a purported quantum state $\hat{\boldsymbol{\rho}}_{AB}$ and Bob's measurement apparatus is deemed untrustable. In the literature on the subject, all the assumptions necessary to make this construction solid are seldom spelled out in full detail as we are instead about to do.

The first assumption is that we have a black box with two outputs, A and B, that should emit repeatedly a *declared, entangled quantum state* $\hat{\boldsymbol{\rho}}_{AB}$, such that subsystem A goes to Alice and subsystem B goes to Bob. They both know the declared state $\hat{\boldsymbol{\rho}}_{AB}$, but they do not trust the black box. Moreover, they both have "universal" measuring devices that should be able to perform any quantum measurement on their subsystems, but Bob's device cannot be trusted for whatever reason. Their task is to check that the actual states outputted by the black box are indeed entangled, for example because they would like to later use them for a quantum cryptography protocol.

Suppose that Bob sets his device to measure the observable X on his subsystem. Then, if everything is right, he must get the possible results b according to a probability distribution given by:

$$p_X(b) \;=\; \mathrm{Tr}_{AB}\left[\left(\mathbb{I}_A\otimes\hat{\boldsymbol{\Pi}}_b^X\right)\hat{\boldsymbol{\rho}}_{AB}\right] \tag{2}$$

where $\hat{\boldsymbol{\Pi}}_b^X$ is the POVM element corresponding to a measurement of the observable X resulting in the outcome b. If he doesn't get the expected probabilities, for any choice of observables, then he will know that either the black box, his device, or both are corrupted. Let us suppose, therefore, that this first test passes, that is, all probabilities Bob computes with Eq. (2) are correct. Of course, this is not enough to trust the "black box plus Bob's measurement device" assembly, because we are only accessing local degrees of freedom of subsystem B of the purported state $\hat{\boldsymbol{\rho}}_{AB}$. Alice can do the same with her subsystem and, since her device is trustworthy, she would conclude that her subsystem is correctly described by $\hat{\boldsymbol{\rho}}_A=\mathrm{Tr}_B\left[\hat{\boldsymbol{\rho}}_{AB}\right]$. Still, many states besides $\hat{\boldsymbol{\rho}}_{AB}$ can have the correct partial trace, without actually being equivalent to $\hat{\boldsymbol{\rho}}_{AB}$: Alice and Bob must now check *correlations* between their subsystems. If also Bob's device were trustworthy, they could simply do a full tomography and verify if the black box is actually able to prepare the purported state. However, with our assumptions, they should find another way. After a little thought, they come up with the following idea: if the state outputted by the black box is really entangled, and if Bob's device works fine, then a measurement on subsystem B should change the quantum state of subsystem A conditioned upon the outcome of the measurement. Alice cannot immediately check that this influence took place, by the no-signaling theorem, but she can do that if Bob tells her the measurement outcome he found; indeed, if he measures X and gets b, Alice predicts that her quantum state will be:

$$\hat{\boldsymbol{\rho}}_{A|b,X} \;=\; \frac{1}{p_X(b)}\,\mathrm{Tr}_B\left[\left(\mathbb{I}_A\otimes\hat{\boldsymbol{\Pi}}_b^X\right)\hat{\boldsymbol{\rho}}_{AB}\right] \tag{3}$$

where $p_X(b)$ is correctly given by Eq. (2), as previously certified. Now, if $\hat{\boldsymbol{\rho}}_{AB}$ is separable, we could write it as:

$$\hat{\boldsymbol{\rho}}_{AB} \;=\; \int d\lambda\; p(\lambda)\,\hat{\boldsymbol{\rho}}_{A,\lambda}\otimes\hat{\boldsymbol{\rho}}_{B,\lambda} \tag{4}$$

where the integral sign is symbolic and possibly includes discrete sums and $p(\lambda)$ is a probability distribution. In that case, we can define $p_X(b|\lambda)=\mathrm{Tr}\left[\hat{\boldsymbol{\Pi}}_b^X\hat{\boldsymbol{\rho}}_{B,\lambda}\right]$ and we can write:

$$\begin{aligned}\hat{\boldsymbol{\rho}}_{A|b,X} \;&=\; \frac{1}{p_X(b)}\int d\lambda\; p(\lambda)p_X(b|\lambda)\,\hat{\boldsymbol{\rho}}_{A,\lambda}\\ &=\; \int d\lambda\; p(\lambda|b,X)\,\hat{\boldsymbol{\rho}}_{A,\lambda}\end{aligned} \tag{5}$$

The last step was derived by applying Bayes theorem and *defining* $p(\lambda|b,X)$ to be the probability that a particular value of λ described the initial factorized state, upon measuring X on Bob's subsystem and getting the result b. Eq. (5) is a local hidden state (LHS) model on Alice's side, since it implies that Alice gets states $\hat{\boldsymbol{\rho}}_{A,\lambda}$ at each run, according to the probability distribution $p(\lambda)$, and then she makes a Bayesian update to $\hat{\boldsymbol{\rho}}_{A|b,X}$ after being reported the outcome b from Bob: as far as she is concerned, there was no "action at a distance" but simply an updated knowledge of the hidden variable λ that specifies the quantum state of his subsystem. It must be stressed that Bayes theorem not always applies in quantum mechanics, and the operative meaning of the probability $p(\lambda|b,X)$ can be obscure with our assumptions. However, this is not problematic for us: we are defining that probability from measurable quantities and we only care that it *is* a probability distribution, which is immediately checked. We now come to the final, key point: there exist entangled quantum states $\hat{\boldsymbol{\rho}}_{AB}$ such that the corresponding *conditional* states $\hat{\boldsymbol{\rho}}_{A|b,X}$ on Alice's side cannot be described by a single LHS model as in Eq. (5) for all possible choices of measurements X on Bob's side. With such states, even if they cannot trust neither the black box that prepares them nor Bob's measurement apparatus, they can check that the shared state is entangled by picking appropriate choice of measurements X on Bob's side and checking that Alice's conditional states are the expected $\hat{\boldsymbol{\rho}}_{A|b,X}$. These states are called *steerable by Alice*. Notice that we proved that separable states cannot be steerable, therefore also that steering implies entanglement. However, it turns out that a generic entangled state $\hat{\boldsymbol{\rho}}_{AB}$ *could* have a LHS model for Bob's conditional states and for all possible measurements on Alice's side: not all entangled states are steerable [7]. Notice also that the LHS model does not require us to trust Bob's apparatus: we simply ask if there can be *any* two probability distributions $p_X(b|\lambda)$ and $p(\lambda)$ and ensemble $\hat{\boldsymbol{\rho}}_{A,\lambda}$ of density operators making Eq. (5) to hold, with the constraint that $\int d\lambda\, p_X(b|\lambda) = p_X(b)$ but with no further requests on how the stochastic map $p_X(b|\lambda)$ was derived.

There are still some hidden assumptions in this picture. A crucial one is that the probability $p(\lambda)$ according to which the black box prepares the factorized state $\hat{\boldsymbol{\rho}}_{A,\lambda} \otimes \hat{\boldsymbol{\rho}}_{B,\lambda}$ at each run *does not depend on the choice of measurement* X on Bob's side. If the black box were controlled by Bob, for example, this would require that it is Alice to decide what measurement she would like him to perform, and that she makes her decision only after she got her part of the state. If Alice does not believe any assumption about what kind of states $\hat{\boldsymbol{\rho}}_{AB}$ can be prepared by the black box, it is simple to argue that it is indeed *necessary* for Alice to ask Bob to perform different kind of measurements. To show that, it is sufficient to consider a pure entangled state which can be expanded in two different orthonormal, factorized bases:

$$\sum_n c_n |\phi_n\rangle_A \otimes |u_n\rangle_B \;=\; \sum_m d_m |\psi_m\rangle_A \otimes |v_m\rangle_B \tag{6}$$

If Bob were to measure his state in the same $\{u_n\}$ basis at each run, Alice would simply get a random state from the ensemble $\sum_n c_n^2 |\phi_n\rangle\langle\phi_n|$, and Bob's outcome would tell her the actual value of n for that particular run. Clearly, these observations can be explained also by classical correlations, and they would not convince Alice that the original state was indeed entangled. On the other hand, if Alice can ask him to measure in the $\{|v_m\rangle\}$ basis on some of the runs, she could check that in those cases she gets different conditional states, drawn from the same ensemble (since the partial trace must be the same) being resolved in a different way by Bob's outcomes. More in general, it has been shown that two POVMs can be used to show steering if and only if they are *not jointly measurable* [44]. In the following, we will see that if Alice accepts some assumptions about what kind of states $\hat{\boldsymbol{\rho}}_{AB}$ can be generated by the black box, it can become unnecessary for her to choose Bob's measurement: Bob can choose a single observable to be measured on his subsystem at each run to convince her that the output state of the black box, among those she considers possible, is an entangled one.

While we decided to spell out all the details and hidden assumptions to introduce the definition of quantum steering, it is worth mentioning that the notion of "assemblage" can be exploited to concisely describe the same phenomenon [45]. In this framework, an assemblage $\{\hat{\boldsymbol{\sigma}}_{b,X}\}_b := \{p_X(b)\hat{\boldsymbol{\rho}}_{A|b,X}\}_b$ is provided by the set of *unnormalized* quantum states of Alice's system that can be prepared by conditional measurement of X on Bob's one and considering all possible outcomes labeled by b. By fiat, they have the property that $\mathrm{Tr}_A\, \hat{\boldsymbol{\sigma}}_{b,X} = p_X(b)$, so that they also encode the respective probabilities. Moreover, recalling once again that the *unconditional* state of Alice after Bob's measurement cannot depend on his choice because of the no-signaling theorem, we have that for two distinct measurement choices X and X', the assemblages associated with them satisfy:

$$\sum_b \hat{\boldsymbol{\sigma}}_{b,X} = \sum_{b'} \hat{\boldsymbol{\sigma}}_{b',X'} = \hat{\boldsymbol{\rho}}_A \tag{7}$$

The question of steerability of $\hat{\boldsymbol{\rho}}_{AB}$ by Bob then is equivalent to asking whether there exist a set of assemblages that cannot be simulated by classical strategies enacted by Bob; if they exist, those assemblages would be called *steerable*. Given an assemblage, its steerability can be judged via semidefinite programming [45,46].

3. Gaussian states and conditions for quantum steering

A continuous variable quantum system of n (bosonic) modes relies on a Hilber space $\mathcal{H}$ which is isomorphic to $L^2(\mathbb{R}^n)$. However, it should not be interpreted as a quantum state of n distinguishable particles, but rather as a tensor product of n Fock spaces, one for each mode. A mode represents a possible state in which a particle can be created, and for bosonic systems each mode can host any number of particles; for example, given a wave vector $\underline{k}$ in a free quantum field theory, there is a mode associated with it corresponding to particle-like excitations of the field having a well-defined momentum equal to $\underline{k}$. A convenient description of such systems can be provided by introducing a vector $\hat{\mathbb{R}} = (\hat{x}_1, \hat{p}_1, \ldots, \hat{x}_n, \hat{p}_n)^T$ is of canonical operators (or quadrature operators), T denoting transposition, related to the mode creation and annihilation operators by:

$$\hat{x}_j = \frac{\hat{a}_j + \hat{a}_j^\dagger}{\sqrt{2}}\,, \qquad \hat{p}_j = -i\frac{\hat{a}_j - \hat{a}_j^\dagger}{\sqrt{2}}$$

but it is crucial to stress that quadrature operators do *not* have the meaning of position and momentum operators for distinguishable particles.

Given a density operator $\hat{\boldsymbol{\rho}}$, i.e. a quantum state of such a system, it is convenient to associate with it a function on n-dimensional phase space, known as the *characteristic function* [47,48], which is defined as:

$$\chi\left[\hat{\boldsymbol{\rho}}\right](\Lambda) = \mathrm{Tr}[\hat{\boldsymbol{\rho}}\, \mathrm{e}^{-i\Lambda^T\hat{\mathbb{R}}}] \tag{8}$$

where Λ is the vector of the Fourier-conjugate variables of phase space. $\hat{\boldsymbol{\rho}}$ is said to be a *Gaussian state* if, apart possibly for a phase-space-dependent phase factor, it is a Gaussian function of the phase space variables.

Let us now consider a bipartite Gaussian state $\hat{\boldsymbol{\rho}}_{AB}$, having $n_A \geq 1$ modes controlled by Alice and the remaining $n_B \geq 1$ modes pertaining to Bob, so that $n = n_A + n_B$ is the total number of modes of $\hat{\boldsymbol{\rho}}_{AB}$. We write its characteristic function as [47–49]:

$$\chi\left[\hat{\boldsymbol{\rho}}_{AB}\right](\Lambda) \;=\; \exp\left\{-\frac{1}{2}\Lambda^T\boldsymbol{\sigma}\Lambda \;-\; i\Lambda^T\langle\hat{\mathbb{R}}\rangle\right\} \tag{9}$$

where we introduced the *covariance matrix* (CM):

$$\sigma_{jk} = \frac{1}{2}\langle\{\hat{R}_j, \hat{R}_k\}\rangle - \langle\hat{R}_j\rangle\langle\hat{R}_k\rangle \tag{10}$$

with $\langle\hat{\mathbb{R}}\rangle = \mathrm{Tr}_{AB}[\hat{\boldsymbol{\rho}}_{AB}\hat{\mathbb{R}}]$ and $\hat{R}_j$ being the j-th entry of the operators vector $\hat{\mathbb{R}}$. It is clear from these definitions that a Gaussian state is fully characterized by the vector $\langle\hat{\mathbb{R}}\rangle \in \mathbb{R}^{2n}$ and the $2n \times 2n$ real symmetric matrix $\boldsymbol{\sigma}$. The uncertainty relations (UR) for canonically-conjugate variables may be recast into a constraint on the CM associated with physical states [50]:

$$\boldsymbol{\sigma} + \frac{i}{2}\boldsymbol{\Omega} \geq 0 \tag{11}$$

where $\boldsymbol{\Omega} = \oplus_{j=1}^{n}\boldsymbol{\omega}$ (for n modes) and $\boldsymbol{\omega} = i\sigma_y$ is the standard symplectic form on $\mathbb{R}^2$, σ_y being the second Pauli matrix. The inequality in (11) is to be read as a matrix inequality stating that the matrix on the left-hand side is a positive-semidefinite matrix. If we order the couple of canonically conjugated quadratures in such a way that the first n_A of them refer to Alice's modes and the remaining ones to Bob's modes, we can write the CM $\boldsymbol{\sigma}$ of $\hat{\boldsymbol{\rho}}_{AB}$ in block form:

$$\boldsymbol{\sigma} = \begin{pmatrix} \mathbb{A} & \mathbb{C} \\ \mathbb{C}^T & \mathbb{B} \end{pmatrix} \tag{12}$$

where $\mathbb{A} \geq 0$ (resp. $\mathbb{B} \geq 0$) is still a valid CM that can be interpreted as the covariance matrix of the *unconditional* state of Alice's modes $\hat{\boldsymbol{\rho}}_A = \mathrm{Tr}_B\left[\hat{\boldsymbol{\rho}}_{AB}\right]$ (resp. of Bob's modes $\hat{\boldsymbol{\rho}}_B = \mathrm{Tr}_A\left[\hat{\boldsymbol{\rho}}_{AB}\right]$), while the $2n_A \times 2n_B$ matrix $\mathbb{C}$ encodes the correlations between the two collections of modes. When exploring the steering properties of $\hat{\boldsymbol{\rho}}_{AB}$, we shall limit ourselves to Gaussian measurements. This limitation is appropriate for a number of reasons: first of all, proving steerability with Gaussian measurements naturally implies steerability in the general sense. Secondly, from an experimental viewpoint, Gaussian measurements are standard in quantum optics. Furthermore, this allows a clear and analytical characterization, as we shall briefly discuss.

We should begin by reviewing the theory of Gaussian measurements on Gaussian states. A positive operator-valued measure (POVM) $\{\hat{\boldsymbol{\Pi}}_{\underline{\alpha}}\}$ for an n-modes continuous variable quantum system defines a Gaussian measurement if all of its operators have a Gaussian Wigner function, meaning that to each $\hat{\boldsymbol{\Pi}}_{\underline{\alpha}}$ one can associate an ordinary Gaussian state (albeit not normalized to 1). Usually the outcomes $\underline{\alpha}$ of an n-modes Gaussian POVM are $2n$-dimensional real vectors of expectation values for the Gaussian state associated with the measurement operator, while the CM of the POVM is constant for all the operators and characterizes the type of measurement. Now if Bob were to measure a Gaussian POVM with CM $\boldsymbol{\sigma}_M$ on his n_B modes, he would get outcomes $\underline{\alpha} \in \mathbb{R}^{2n_B}$ with a Gaussian distribution $p_{\underline{\alpha}} = \mathrm{Tr}_{AB}[\hat{\boldsymbol{\rho}}_{AB}(\mathbb{I}_A \otimes \hat{\boldsymbol{\Pi}}_{\underline{\alpha}})]$ whose covariance matrix is $\mathbb{B} + \boldsymbol{\sigma}_M$, where $\mathbb{B}$ is the CM of $\hat{\boldsymbol{\rho}}_B$ according to Eq. (12), while the mean value vector is given by $\langle\mathbb{R}_B\rangle = \mathrm{Tr}_B\left[\hat{\boldsymbol{\rho}}_B\hat{\mathbb{R}}_B\right]$, where $\hat{\mathbb{R}}_B$ is the vector of his quadratures operators. The *conditional* state of Alice's modes after Bob got his outcome $\underline{\alpha}$ is still Gaussian, and we shall denote it by:

$$\hat{\boldsymbol{\rho}}_{A,\underline{\alpha}} = \frac{1}{p(\underline{\alpha})}\mathrm{Tr}_B\left[\hat{\boldsymbol{\rho}}_{AB}\left(\mathbb{I}_A \otimes \hat{\boldsymbol{\Pi}}_{\underline{\alpha}}\right)\right] \tag{13}$$

Now comes a crucial point about Gaussian quantum steering. Bob has to report to Alice his outcome in order for her to check that an influence took place on her modes, i.e. that they are actually described by the conditional state $\hat{\boldsymbol{\rho}}_{A,\underline{\alpha}}$ after she measured. Otherwise, she would find her modes in the *unconditional* state $\hat{\boldsymbol{\rho}}_A$ no matter what he did, by the no signaling theorem. However, Bob's outcome will *only* affect the mean value vector of Alice's conditional state $\hat{\boldsymbol{\rho}}_{A,\underline{\alpha}}$. But the mean value vector can be modified arbitrarily by *local* unitary operations,[1] therefore it cannot encode quantum correlations between Alice and Bob. The covariance matrix $\boldsymbol{\sigma}_{A,c}$ of the conditional state $\hat{\boldsymbol{\rho}}_{A,\underline{\alpha}}$, on the other hand, *does not* depend on Bob's outcome, but just on the initial CM $\boldsymbol{\sigma}$ and on the CM $\boldsymbol{\sigma}_M$ of Bob's Gaussian POVM. Specifically, leveraging on the trace rules for the characteristic functions of positive operators, it can be shown that [49,51,52]:

$$\boldsymbol{\sigma}_{A,c} = \mathbb{A} - \mathbb{C}^T\left(\mathbb{B} + \boldsymbol{\sigma}_M\right)^{-1}\mathbb{C} \tag{14}$$

which takes the form of a Schur complement [53]. To state it yet in another way, whenever Bob performs a Gaussian measurement with CM $\boldsymbol{\sigma}_M$ on his modes, Alice's state will have the same CM $\boldsymbol{\sigma}_{A,c}$ independently of the outcome he got, but it will be centered at different mean value vectors which are Gaussian distributed so that the ensemble of her conditional states reproduces the unconditional state $\hat{\boldsymbol{\rho}}_A$, as it must. If she comes to know Bob's outcome at each run, however, she can resolve the ensemble and distinguish between the different conditional states $\hat{\boldsymbol{\rho}}_{A,\underline{\alpha}}$.

From this line of reasoning it should be clear that whether the initial state $\hat{\boldsymbol{\rho}}_{AB}$ is steerable by Bob or not is entirely determined by its CM $\boldsymbol{\sigma}$. Indeed, it has been shown [6,7] that a necessary and sufficient condition for steering by Gaussian measurements on Bob's modes is the *violation* of the following matrix inequality:

$$\boldsymbol{\sigma} + \frac{i}{2}\left(\boldsymbol{\Omega}_A \oplus \mathbb{0}_B\right) \geq 0 \tag{15}$$

The idea of the proof is that Ineq. (15) is both necessary and sufficient for the existence of a $2n_A \times 2n_A$ CM for Alice's modes, $\boldsymbol{\sigma}_U$, which is physical ($\boldsymbol{\sigma}_U + i\Omega_A/2 \geq 0$) and such that $\boldsymbol{\sigma}_{A,c} - \boldsymbol{\sigma}_U \geq 0$ for all possible conditional CMs $\boldsymbol{\sigma}_{A,c}$, that is for all possible measurement's covariance matrices $\boldsymbol{\sigma}_M$. Then $\boldsymbol{\sigma}_U$ could be exploited by Bob to concoct an ensemble $\{p_{\underline{\beta}}, \hat{\boldsymbol{\rho}}_U^{\beta}\}$ of Gaussian states to be sent to Alice to simulate his ability to influence her modes at a distance. Therefore, if such a matrix $\boldsymbol{\sigma}_U$ exists, no matter what kind of Gaussian measurement is performed on Bob's modes, Alice won't be convinced that the initial state must have been entangled and, by definition, $\hat{\boldsymbol{\rho}}_{AB}$ is *not steerable* (with Gaussian measurements) by Bob. Conversely, whenever Ineq. (15) is *violated* (meaning that the matrix on the left-hand side has at least one negative eigenvalue) there will exist Gaussian measurements such that the corresponding conditional states could not have been simulated by a predetermined ensemble chosen by Bob.

Further intuition on Gaussian steering can be gained by considering the simplest scenario, where $\hat{\boldsymbol{\rho}}_{AB}$ is a two-mode state with mode A sent to Alice and mode B controlled by Bob. Then $\hat{\boldsymbol{\rho}}_{AB}$ is steerable with Gaussian measurements by Bob if and only if there is no physical, unique single-mode covariance matrix $\boldsymbol{\sigma}_U$ which is *narrower*, with respect to all possible quadrature combinations of Alice's mode, than all the conceivable conditional covariance matrices. In other words, if any Gaussian function in Alice's phase space which is narrower than the Wigner functions of all possible conditional states *at the same time in* $\hat{x}_A$ *and* $\hat{p}_A$ is necessarily *unphysical*, then the state is steerable. Since this implies that there exist two (incompatible) Gaussian measurements on Bob's mode such that the combined minimal variances of Alice's conditional states would violate the UR, it can be appreciated that Gaussian steering for a two-mode state is equivalent to the possibility of showing the EPR

[1] Here by local we mean acting on the collection of modes pertaining to one party only.

paradox with it. Partly due to this connection, and also to clearly distinguish the different steering notions by their names, from now on we will refer to steering with CV quantum systems as *EPR steering*.

4. Steering notion based on P-nonclassicality

In this section, after reviewing the notion of P-nonclassicality, we will exploit it to define nonclassical steering for two-mode Gaussian states and then compare it to the standard notion of quantum steering that was discussed above. The goal is to provide an intuition for the relations between these different notions and also to argue for the application of nonclassicality as a witness for quantum steering in continuous-variable systems. Indeed, the necessity for experimentally accessible properties that would allow to demonstrate steering was already pointed out in the original paper [6,7] and was later addressed by a number of protocols (see [2] and references therein).

The concept of P-nonclassicality of a quantum state relies on its Glauber P-function, a member of a continuous family of phase space quasiprobability distributions, known as s-ordered Wigner functions, that are defined according to [26]:

$$W_s\left[\hat{\boldsymbol{\rho}}\right](X) = \int_{\mathbb{R}^n} \frac{d^{2n}\Lambda}{(2\pi^2)^n}\, \mathrm{e}^{\frac{1}{4}s|\Lambda|^2 + i\Lambda^T \boldsymbol{X}} \chi\left[\hat{\boldsymbol{\rho}}\right](\Lambda) \tag{16}$$

for $s \in [-1, 1]$ and $\chi\left[\hat{\boldsymbol{\rho}}\right](\Lambda)$ is the characteristic function introduced before. For $s = 0$ one recovers the standard Wigner function, which is simply the Fourier transform of the characteristic function (9), while for $s = -1$ Eq. (16) yields the Husimi Q function. The case $s = 1$ corresponds instead to the Glauber P-function, which is therefore the most singular of the family and can behave even more singularly than a tempered distribution. When the P-function (or a regularization of it [26,38,30]) of a CV quantum state $\hat{\boldsymbol{\rho}}$ attains negative values in some regions of phase space [54], the state is termed *nonclassical* [31–35]. The so-called *nonclassical depth* of a CV state $\hat{\boldsymbol{\rho}}$ is then the quantity $\mathrm{t} = \frac{1}{2}(1 - s_m)$, where s_m is the largest real number such that $W_s\left[\hat{\boldsymbol{\rho}}\right](X)$ is nonsingular $\forall s < s_m$. Thus $\hat{\boldsymbol{\rho}}$ is nonclassical if $\mathrm{t} > 0$ and classical if $\mathrm{t} = 0$. It should be emphasized that several protocols have been devised to experimentally and theoretically certify the nonclassicality of light states (see [55,56] and references therein).

Going back to the characteristic function in Eq. (9) and assuming a two-mode Gaussian state $\hat{\boldsymbol{\rho}}_{AB}$, it is straightforward to conclude from Eq. (16) that the state is nonclassical if and only if the least eigenvalue of $\boldsymbol{\sigma}$ is smaller than $\frac{1}{2}$. Examples of classical Gaussian states are coherent and thermal states, while squeezed vacuum states are always nonclassical. In the following, we will be interested in characterizing how quantum correlations in the joint Gaussian quantum state $\hat{\boldsymbol{\rho}}_{AB}$ may be exploited to influence the nonclassicality of one mode (say B) by Gaussian measurements on the other one (mode A). In doing so, Local Gaussian Unitary Transformations (LGUTs) do not affect these correlations, and therefore we may freely perform LGUTs on the two modes to bring $\hat{\boldsymbol{\rho}}_{AB}$ into a simpler form. In particular, by means of LGUTs a two-mode Gaussian state can always be brought into the so-called *canonical form* [57,58,49], for which the CM $\boldsymbol{\sigma}$ can be decomposed in 2×2 diagonal blocks $\boldsymbol{\sigma} = \begin{pmatrix} \mathbb{A} & \mathbb{C} \\ \mathbb{C}^T & \mathbb{B} \end{pmatrix}$ with:

$$\mathbb{A} = a \cdot \mathbb{I}_2\,, \quad \mathbb{B} = b \cdot \mathbb{I}_2\,, \quad \mathbb{C} = \mathrm{diag}(c_1, c_2) \tag{17}$$

while $a, b, c_1, c_2 \in \mathbb{R}$. We now note that the *unconditional state* of mode A, defined either as the state that Alice uses to describe her mode without knowing anything about Bob's mode *or* as the state she assigns to her mode by assuming that Bob has performed some measurement on his mode without letting her know the outcome, is given by $\hat{\boldsymbol{\rho}}_A = \mathrm{Tr}_B\left[\hat{\boldsymbol{\rho}}_{AB}\right]$ and has a CM $\boldsymbol{\sigma}_A = \mathbb{A}$. Since the UR imply that $a \geq \frac{1}{2}$ this means that $\hat{\boldsymbol{\rho}}_A$ must be classical. The same holds true for mode B, thus we may say that given a two-mode Gaussian state $\hat{\boldsymbol{\rho}}_{AB}$ in canonical form, neither of the two modes has any intrinsic nonclassicality. Based on this observation, we can ask whether a two-mode Gaussian state in canonical form can be used to prepare a nonclassical state of one mode by performing a generalized Gaussian measurement on the other mode. We are thereby led to the following definition [43]:

Definition 1. A two-mode Gaussian state $\hat{\boldsymbol{\rho}}_{AB}$ in canonical form is called *weakly nonclassically steerable* (WNS) from mode B to mode A ($B \to A$) if there exists a Gaussian positive operator-valued measure (POVM) $\{\hat{\boldsymbol{\Pi}}_\alpha\}_{\alpha\in\mathbb{C}}$ on mode B such that the *conditional state of mode A*

$$\hat{\boldsymbol{\rho}}_{c,\alpha} = \frac{1}{p_\alpha} \mathrm{Tr}_B\left[\hat{\boldsymbol{\rho}}_{AB}\left(\mathbb{I}_A \otimes \hat{\boldsymbol{\Pi}}_\alpha\right)\right] \tag{18}$$

is *nonclassical*, where $p_\alpha = \mathrm{Tr}_{AB}[\hat{\boldsymbol{\rho}}_{AB}(\mathbb{I}_A \otimes \hat{\boldsymbol{\Pi}}_\alpha)]$ is the probability of observing the outcome $\alpha \in \mathbb{C}$.

One must bear in mind that EPR steering is already a *nonclassical phenomenon*, since it involves quantum correlations whose effects on the steered side cannot be reproduced by any local hidden variable model [6,7]. Notwithstanding this, we introduce here the term *nonclassical steering* because we are bringing in the additional notion of P-nonclassicality, which is, by its definition, conceptually disjoint from the nonclassical character of quantum correlations. Therefore one should not interpret nonclassical steering as a somehow *less classical* variant of EPR steering, but rather as a different phenomenon that aims at capturing the ability to remotely influence P-nonclassicality within correlated CV quantum states. Having settled their definition, let us now deduce a simple criterion to discern weakly nonclassically steerable states, starting with the following proposition:

Proposition 1. *The least classical (i.e. with highest possible nonclassical depth) conditional state $\hat{\boldsymbol{\rho}}_{c,\alpha}$ of mode A attainable with Gaussian measurements on mode B of a two-mode Gaussian state $\hat{\boldsymbol{\rho}}_{AB}$ in canonical form is reached by quadrature detection on mode B, either of the $\hat{x}_B$ quadrature if $|c_2| \geq |c_1|$, or of the $\hat{p}_B$ quadrature otherwise.*

Proof. Let us denote by $\boldsymbol{\sigma}_c$ the CM of the conditional state, which does not depend on the outcome α as discussed previously. Therefore, $\hat{\boldsymbol{\rho}}_{AB}$ in canonical form is WNS if and only if there exists a Gaussian POVM such that the least eigenvalue of $\boldsymbol{\sigma}_c$ is smaller than $\frac{1}{2}$. The effects of the most general Gaussian POVM on a single mode may be written as $\hat{\boldsymbol{\Pi}}_\alpha = D(\alpha)\hat{\boldsymbol{\rho}}_G D^\dagger(\alpha)/\pi$ where $D(\alpha) = \exp\{\alpha\hat{a}^\dagger - \alpha^*\hat{a}\}$ is the displacement operator and $\hat{\boldsymbol{\rho}}_G$ is a single-mode Gaussian state with $\langle\hat{\mathbb{R}}\rangle = 0$. Furthermore, we may choose the following convenient parametrization for the CM $\boldsymbol{\sigma}_M$ of $\hat{\boldsymbol{\rho}}_G$:

$$\boldsymbol{\sigma}_M = \frac{1}{2\mu\mu_s}\begin{pmatrix} 1 + \kappa_s\cos\phi & -\kappa_s\sin\phi \\ -\kappa_s\sin\phi & 1 - \kappa_s\cos\phi \end{pmatrix} \tag{19}$$

where $\mu = \mathrm{Tr}[\hat{\boldsymbol{\rho}}_G^2] \in [0, 1]$ is the purity of $\hat{\boldsymbol{\rho}}_G$, $\mu_s = [1 + 2\sinh^2 r_m]^{-1}$, $\kappa_s = \sqrt{1 - \mu_s{}^2}$, r_m being the squeezing parameter of the state, and $\phi \in [0, 2\pi)$ is a phase. As stated by Eq. (14), the conditional CM is given by $\boldsymbol{\sigma}_c = \mathbb{A} - \mathbb{C}^T\left(\mathbb{B} + \boldsymbol{\sigma}_M\right)^{-1}\mathbb{C}$. Since $\mathbb{A}$ is diagonal, the minimum λ_m (over all possible CMs $\boldsymbol{\sigma}_M$) of the smallest eigenvalue of $\boldsymbol{\sigma}_c$ is attained for the supremum of the greatest eigenvalue of $\mathbb{C}^T(\mathbb{B}+\boldsymbol{\sigma}_M)^{-1}\mathbb{C}$, which is positive semidefinite. By explicit calculation, this supremum requires $\phi=0$ if $|c_2|\geq|c_1|$, and $\phi=\pi$ otherwise. The resulting expression is a monotonic decreasing function of μ_s, since one can see by inspection that its first derivative with respect to μ_s is always nonpositive. Therefore, one needs to set $\mu_s=0$ in order to attain the supremum and in this limit the value of μ becomes irrelevant. The limit $\mu_s\to 0$ makes the Gaussian POVM $\hat{\mathbf{\Pi}}_\alpha$ to collapse into the spectral measure of the $\hat{x}$ (resp. $\hat{p}$) quadrature for $\phi=0$ (resp. $=\pi$). □

This result immediately leads us to the aforementioned criterion:

Proposition 2. *A two-mode Gaussian state $\hat{\boldsymbol{\rho}}_{AB}$ in canonical form is WNS ($B\to A$) if and only if the parameters of its CM, as defined by Eq. (17), satisfy:*

$$a-\frac{c^2}{b}<\frac{1}{2}, \quad c=\max\{|c_1|,|c_2|\} \tag{20}$$

Proof. Let us suppose that $c=|c_2|\geq|c_1|$, so that we can fix $\phi=0$ in Eq. (19). Then, for $\mu_s\to 0$, one can explicitly compute $\lambda_m=a-c^2/b$. But the initial state $\hat{\boldsymbol{\rho}}_{AB}$ is WNS if and only if the least classical conditional state *is nonclassical*, which amounts to $\lambda_m<1/2$, as stated by Eq. (20). Otherwise, if $c=|c_1|>|c_2|$, one should choose $\phi=\pi$ to arrive at the same conclusion. □

We call this property *weak* nonclassical steering because it does not imply entanglement. Indeed, there are (non isolated) choices for the values of a,b,c_1,c_2 that correspond to physical states ($\boldsymbol{\sigma}>0$ and fulfilling UR) that are separable and WNS, e.g. $a=b=13.9$, $c_1=4.6$, $c_2=-13.7$. Besides, there exist WNS states with $c_1c_2>0$, which is a sufficient condition for separability. Intuitively, the issue is that a single, optimal Gaussian measurement is sufficient for Bob to show Alice that a state is WNS, but the assumption of a canonical two-mode Gaussian state is too generic and a single choice of measurement cannot be sufficient to demonstrate EPR steering, which in turn implies that it cannot prove entanglement. In fact, it has been shown [43] that WNS can be achieved with Gaussian states with arbitrarily low Gaussian Quantum Discord.

Motivated by these examples, we seek a stronger condition than the one provided in Definition 1 to explore other quantum correlations that can arise while studying the remote generation of P-nonclassicality with Gaussian states in canonical form. Recalling that Proposition 1 identifies quadrature measurements as the best Gaussian measurements to induce nonclassicality, we introduce the following more stringent notion of nonclassical steering:

Definition 2. A two-mode Gaussian state $\hat{\boldsymbol{\rho}}_{AB}$ in canonical form is called *strongly nonclassically steerable* (SNS) ($B\to A$) if the measurement of *any* quadrature on mode B generates a nonclassical conditional state of mode A.

Following the proof of Proposition 2, we immediately conclude:

Proposition 3. *A two-mode Gaussian state $\hat{\boldsymbol{\rho}}_{AB}$ in canonical form is SNS ($B\to A$) if and only if the parameters of its CM, as defined by Eq. (17), satisfy:*

$$a-\frac{c'^2}{b}<\frac{1}{2}, \quad c'=\min\{|c_1|,|c_2|\} \tag{21}$$

Proof. The least nonclassical conditional state is reached, among all quadrature measurements, by the "wrong" choice of phase ($\phi=\pi$ for $|c_2|\geq|c_1|$ and $\phi=0$ otherwise). Therefore, it is sufficient to demand that the minimum eigenvalue of $\boldsymbol{\sigma}_c$ is less than $\frac{1}{2}$ also in this case, thereby arriving at Ineq. (21). □

In order to generalize these definitions from two-mode Gaussian states in canonical form to *all* Gaussian states of two modes, we should take into account (local) single-mode squeezing transformations, which may alter the nonclassicality of each mode independently of their quantum correlations. However, since any two-mode Gaussian state can be brought to its *unique* canonical form through LGUTs without altering the correlations, we can extend the definitions in the following way:

Definition 3. A generic two-mode Gaussian state $\hat{\boldsymbol{\rho}}_{AB}$ is called weakly (strongly) nonclassically steerable if the *unique* Gaussian state $\hat{\boldsymbol{\rho}}'_{AB}$ *in canonical form* related to $\hat{\boldsymbol{\rho}}_{AB}$ by LGUTs is weakly (strongly) nonclassically steerable.

The reader might wonder whether a tensor product of two single-mode Gaussian states might end up being (weakly or strongly) nonclassically steerable according to the above definition, a clearly undesirable circumstance. However, this is not possible: since we defined weak and strong nonclassical steerability for generic two-mode Gaussian states by referring to the properties of the corresponding state in canonical form, any two-mode Gaussian state of the form $\hat{\boldsymbol{\rho}}_A\otimes\hat{\boldsymbol{\rho}}_B$ will be associated with a state $\hat{\boldsymbol{\rho}}'_A\otimes\hat{\boldsymbol{\rho}}'_B$ in canonical form, where $\hat{\boldsymbol{\rho}}'_A$ and $\hat{\boldsymbol{\rho}}'_B$ are necessarily classical, as was argued to arrive at Definition 3. Since conditioning upon measurements on one mode of a factorized state cannot affect the state of the other mode, $\hat{\boldsymbol{\rho}}'_A\otimes\hat{\boldsymbol{\rho}}'_B$ cannot be (weakly nor strongly) nonclassically steerable, and by Definition 3 also $\hat{\boldsymbol{\rho}}_A\otimes\hat{\boldsymbol{\rho}}_B$ will not be.

In order to extend also the results regarding the necessary and sufficient conditions for WNS/SNS, we need to specify the effect of LGUTs on $\boldsymbol{\sigma}_c$. Any Gaussian unitary transformation is implemented by a symplectic linear transformation in the phase space formalism, and viceversa. Therefore a LGUT on a two-mode system is described by an element $S_A\oplus S_B$ acting on quantum phase space, where $S_{A(B)}\in \mathrm{SL}_{A(B)}(2)$. The 2×2 blocks of a generic $\boldsymbol{\sigma}$ are transformed according to:

$$\mathbb{A}' = S_A\mathbb{A}S_A^T \quad \mathbb{B}' = S_B\mathbb{A}S_B^T \quad \mathbb{C}' = S_A\mathbb{C}S_B^T \tag{22}$$

Let us now suppose that $S_A\oplus S_B$ brings the initial $\boldsymbol{\sigma}$ in canonical form, so that $\mathbb{A}'=a'\mathbb{I}_2$, $\mathbb{B}'=b'\mathbb{I}_2$ and $\mathbb{C}'=\mathrm{diag}(c'_1,c'_2)$. The conditional CM $\boldsymbol{\sigma}_c$ resulting from a Gaussian measurement with CM $\boldsymbol{\sigma}_M$ on the initial state with CM $\boldsymbol{\sigma}$ can be rearranged as:

$$\boldsymbol{\sigma}_c = S_A^T\left[\mathbb{A}'-\mathbb{C}'\left(\mathbb{B}'+\boldsymbol{\sigma}'_M\right)^{-1}\mathbb{C}'^T\right]S_A \tag{23}$$

where the CM of the measurement has been redefined according to $\boldsymbol{\sigma}'_M=S_B^T\boldsymbol{\sigma}_M S_B$. We see that performing the measurement (associated with) $\boldsymbol{\sigma}_M$ on the two-mode state with CM $\boldsymbol{\sigma}$ is equivalent to perform the modified measurement $\boldsymbol{\sigma}'_M$ on the canonical form state related to $\boldsymbol{\sigma}$ and then performing the transformation induced by S_A on the resulting conditional CM. This means that we can simply factor out the action of S_A because it doesn't interfere with the steering process. Meanwhile, as long as S_B does not introduce infinite squeezing, we can still approach the desired limit of $\boldsymbol{\sigma}'_M$, acting on the state in canonical form, by taking a limit of $\boldsymbol{\sigma}_M$ with a suitable phase. Finally, to get the necessary and sufficient conditions for WNS and SNS in the general case, we can now rewrite Ineq. (20) and Ineq. (21), replacing a,b,c_1,c_2 with their expressions in terms of symplectic invariants [57] $I_1=a^2$, $I_2=b^2$, $I_3=c_1c_2$, and $I_4=(ab-c_1{}^2)(ab-c_2{}^2)$, which are indeed invariant under LGUTs.

Proposition 4. *A generic two-mode Gaussian state $\hat{\boldsymbol{\rho}}_{AB}$ is* weakly nonclassically steerable *from mode $B \to A$ if and only if its symplectic invariants satisfy the inequality:*

$$\frac{I_1 I_2 - I_3{}^2 + I_4 - \sqrt{\left(I_1 I_2 - I_3{}^2 + I_4\right)^2 - 4 I_1 I_2 I_4}}{2 I_2 \sqrt{I_1}} < \frac{1}{2}, \tag{24}$$

while it is strongly nonclassically steerable *if they satisfy the stronger inequality:*

$$\frac{I_1 I_2 - I_3{}^2 + I_4 + \sqrt{\left(I_1 I_2 - I_3{}^2 + I_4\right)^2 - 4 I_1 I_2 I_4}}{2 I_2 \sqrt{I_1}} < \frac{1}{2}. \tag{25}$$

Strong nonclassical steering obviously implies weak nonclassical steering, but it also implies entanglement. We will show this implicitly by proving a stronger result:

Theorem 1. *A two-mode Gaussian state $\hat{\boldsymbol{\rho}}_{AB}$ that is SNS $B \to A$ is also EPR-steerable in the same direction, therefore also entangled.*

Proof. As stated in Ineq. (15), EPR steerability $B \to A$ of a Gaussian state by Gaussian measurements amounts to the *violation* of the inequality $\boldsymbol{\sigma} + \frac{i}{2}\boldsymbol{\omega}_A \oplus \mathbb{0}_B \geq 0$ by its CM. Exploiting LGUT-invariance, we can restrict the comparison between EPR steerability and SNS to Gaussian states in canonical form. In this case, keeping in mind that $a > \frac{1}{2}$, violation of the above inequality reduces to [7,59] $(a - c_1^2/b)(a - c_2^2/b) < 1/4$, which is certainly true under the SNS Ineq. (21). □

Notice that, by combining Ineq. (24) and Ineq. (25) with the EPR steerability $B \to A$ criterion for a two-mode Gaussian state in canonical form, $(a - c_1^2/b)(a - c_2^2/b) < 1/4$, we can readily express also Ineq. (15) in terms of symplectic invariants:

$$4 I_4 < I_2 \tag{26}$$

which most clearly shows the asymmetric nature of this quantum correlation, since $I_2 = b^2$ appears on the right-hand side. Again, the related inequality for EPR steerability $A \to B$ can be obtained by replacing I_2 with I_1 in Ineq. (26).

To sum up, we may say that strong nonclassical steering is the strongest quantum correlation considered in this paper for two-mode Gaussian states: a strongly nonclassically steerable state is necessarily also weakly nonclassically steerable and EPR steerable in the same direction, therefore also entangled. On the converse, weak nonclassical steering is a very weak form of nonclassical correlation: it does not require entanglement, nor a minimal amount of Gaussian quantum discord.

It is also remarkable that the quantities on the left sides of (20) and (21) are precisely conditional variances appearing in the Reid EPR-criterion [60,61], whose test is already experimentally accessible [62,63]. The idea is that, when Bob measures either the $\hat{x}_B$ or the $\hat{p}_B$ quadrature of his mode, the conditional states of Alice will have minimal variances along orthogonal quadratures; the EPR paradox arises if the product of these minimal quadratures is less than $\frac{1}{4}$, which naively seems to contradict the Uncertainty Relations.[2] As already noted, this is precisely the condition of EPR steering (see the proof of Theorem 1). This is also in agreement with the well-known result stating that quadrature measurements are the best choice for Gaussian EPR steering [64].

5. Gaussian steering triangoloids

Let us now focus on the relevant class of two-mode squeezed thermal states (TMST) [65–67]. The parameters of their CMs are given by ($r \in \mathbb{R}^+$):

$$\begin{aligned}
a &= \frac{1}{2}(1 + N_A + N_B)\cosh 2r \, + \, \frac{1}{2}(N_A - N_B) \\
b &= \frac{1}{2}(1 + N_A + N_B)\cosh 2r \, - \, \frac{1}{2}(N_A - N_B) \\
c &= c_1 = -c_2 = \frac{1}{2}(1 + N_A + N_B)\sinh 2r
\end{aligned} \tag{27}$$

where N_i $(i = A, B)$ denotes the average number of thermal photons in each mode. Since TMST are all and only those states whose CM is in canonical form with the additional constraint that $c_1 = -c_2 = c$, evidently they are also precisely those states in canonical form for which the conditions for WNS and SNS coincide, by Eq. (20) and Eq. (21): the most nonclassical conditional state on mode A is obtained by *any* quadrature measurement on mode B. From the proof of Theorem 1, it is also clear that TMST states are EPR-steerable from one mode to the other *if and only if* they are nonclassically steerable (strongly and therefore also weakly) in the same direction. This observation provides a new, somehow surprising, role for the notion of P-nonclassicality: it is *the* property that Alice should check, after Bob's measurement on his mode, to certify that the shared TMST state is indeed entangled; we note that this fact could find applications in one-sided device-independent quantum key distribution [9]. Note that the *universal* steerability condition for TMST states becomes:

$$\cosh 2r \; > \; 1 \, + \, \frac{2N_A(1 + 2N_B)}{1 + N_A + N_B} \tag{28}$$

which is readily interpreted as a lower bound on the two-mode squeezing needed to make the TMST steerable $B \to A$.

In order to illustrate nonclassical steering for TMST states, we employ plots of *triangoloids*. Consider the conditional CM of mode A parametrized by $(\mu_c, \mu_{sc}, \phi_c)$ as in Eq. (19). For TMST it is possible to compute the functional dependence of these parameters on the initial TMST parameters N_A, N_B, r and the POVM's parameters μ, μ_s, ϕ. In particular $\phi_c = \phi$, thus the phase may be discarded, while for the remaining parameters [43]:

$$\begin{aligned}
\mu_c \;&=\; \frac{1}{2}\sqrt{\frac{\alpha^2 - \beta^2}{(c^2 - a\alpha)^2 - a^2\beta^2}} \\
\mu_{sc} \;&=\; \frac{\sqrt{(\alpha^2 - \beta^2)\left[(c^2 - a\alpha)^2 - a^2\beta^2\right]}}{a(\alpha^2 - \beta^2) - \alpha c^2}
\end{aligned} \tag{29}$$

where $\alpha = b + \frac{1}{2\mu\mu_s}$, $\beta = \frac{\kappa_s}{2\mu\mu_s}$ and $\kappa_s = \sqrt{1 - \mu_s^2}$. For a fixed TMST state, we can thus plot the region of achievable conditional states in the (μ_c, μ_{sc})-space, as obtained by considering all the POVM's parameters μ and μ_s. These are the curvilinear triangles (triangoloids) in Fig. 1, where we also displayed the nonclassical region (light-brown region), i.e. those parameters corresponding to nonclassical states of mode A according to [43]:

$$\mu_{sc} \; < \; \frac{2\mu_c}{1 + \mu_c{}^2} \tag{30}$$

It follows that the TMST state associated with a given triangoloid is nonclassically steerable $B \to A$ when the parameters N_A, N_B and r of the state are such that the triangoloid intersects the nonclassical region, as in the right panel of Fig. 1, because only in that scenario

[2] The contradiction is not real, of course, since the two variances result from distinct, incompatible measurements on Bob's mode.

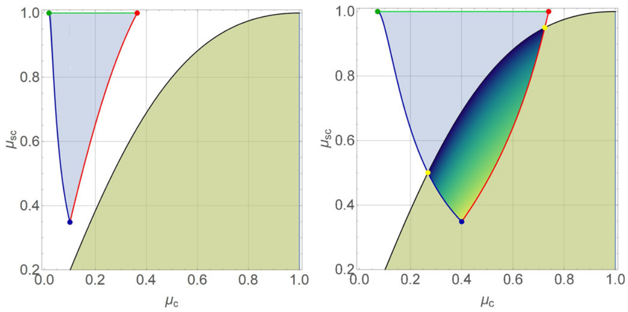

Fig. 1. (Left): Triangoloid for TMST state with $N_A = N_B = 4.5$ and $r = 1.2$, μ_c is the purity of the conditional state, while $\mu_{sc} = (1 + 2\sinh^2 r_c)^{-1}$ quantifies squeezing of the conditional state. The light-brown region contains all nonclassical conditional states. (Right): triangoloid for $N_A = N_B = 0.75$ and $r = 1.2$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

there will be some Gaussian POVM on mode B yielding a nonclassical state of mode A. We shaded the intersection area according to the nonclassical depths, with lighter regions for higher $\mathfrak{t}$. As it may be also appreciated graphically, the decisive point for nonclassical steering of a TMST is the blue, lower vertex of the triangoloid, attained by quadrature detection on mode B: if this point is outside the nonclassical region, all other points of the triangoloid are outside too and an intersection between the nonclassical region and the triangoloid will no longer be possible. Notice that the equivalence of EPR steering and nonclassical steering for TMSTs has a neat graphical interpretation: the light-brown nonclassical region is *the largest* region such that a TMST whose triangoloid intersects it is necessarily entangled. Indeed, suppose that a larger region in the (μ_c, μ_{sc}) square exists such that whenever the triangoloid intersects it, one can be sure that the state is entangled; since these diagrams can be drawn by Bob from the data gathered by measurements on his mode only, it would mean that the intersection with such a larger region would *convince* Bob that the state was entangled, so it must have been steerable. However, we proved that the largest region such that an intersection of the triangoloid implies steering is precisely the nonclassical region.

The equivalence between nonclassical steering (of the weak and/or strong type) with EPR steering for TMST states is insightful yet from another aspect: it shows that EPR steering is possible even with a single type of measurement, if Alice (i.e. the steered party) is willing to accept some hypotheses about the shared quantum states. In the present case, if she believes that the original two-mode state was a TMST Gaussian state and that Bob's measurement apparatus, as unreliable as it could be, is only capable of performing Gaussian measurements, then she will be convinced that the TMST state was also entangled if and only if, upon getting the outcome from Bob, she finds out that her conditional state is nonclassical. To achieve this, it is sufficient for Bob to do a single type of projective measurement of a (generic) quadrature on his mode and report the outcome: there is no need for Alice to ask in advance what kind of measurement he should perform in each run.

A thorough description of triangoloids and their role in the study of nonclassical steering can also be found in [43], where the propagation of a TMST state through a thermal environment and its detrimental effect on these quantum correlations was also explored.

6. Conclusions

In this tutorial, we reviewed the general notion of quantum steering, highlighting the implicit assumptions in the thought experiment that is usually brought out to motivate this type of quantum correlations, i.e. proving entanglement of shared bipartite states to another party who just trusts her own local measurements. We then specialized this notion to Gaussian states and Gaussian measurements, providing intuitive motivation for the EPR steering condition for them. Subsequently, we introduced P-nonclassicality and the related concept of nonclassical steering, both of the weak and strong type, which stem from the intuition of exploiting P-nonclassicality as a witness of EPR steering in two-mode Gaussian states. We went on by showing that WNS amounts to ask that at least one of such variances is smaller than the vacuum value, whereas SNS requires the same to be true *for both* these variances separately. EPR-steerability instead asks that *the product* of them is smaller than the value attained by the same quantity on the vacuum [68]. After discussing LGUT-invariance and providing general criteria for two-mode states, we explored in greater detail the relations between the three steering notions and the Reid's criterion to test the EPR paradox for two-mode states. Finally, we applied the concepts to two-mode squeezed thermal states, which can be conveniently treated with Gaussian triangoloids diagrams. This example showcases the intimate connection between nonclassical and EPR steering, and also provides an instance where a single type of measurement is sufficient to demonstrate EPR steering and therefore to prove to the other party that the shared TMST state is entangled, a possibility that could find applications in one-sided device-independent quantum key distribution [9] with CV protocols and quantum optics, especially in light of the quickly developing strategies to test P-nonclassicality (see [69] and references therein).

CRediT authorship contribution statement

Massimo Frigerio: Conceptualization, Formal analysis, Investigation, Writing – original draft, Writing – review & editing. **Claudio Destri:** Writing – review & editing, Validation. **Stefano Olivares:** Formal analysis, Investigation, Methodology, Supervision, Writing – review & editing, Conceptualization. **Matteo G.A. Paris:** Writing – review & editing, Validation, Supervision, Methodology, Investigation, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

[1] R.F. Werner, Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model, Phys. Rev. A 40 (1989) 4277–4281, https://doi.org/10.1103/PhysRevA.40.4277.
[2] R. Uola, A.C.S. Costa, H.C. Nguyen, O. Gühne, Quantum steering, Rev. Mod. Phys. 92 (2020) 015001, https://doi.org/10.1103/RevModPhys.92.015001.
[3] M.T. Quintino, T. Vértesi, D. Cavalcanti, R. Augusiak, M. Demianowicz, A. Acín, N. Brunner, Inequivalence of entanglement, steering, and bell nonlocality for general measurements, Phys. Rev. A 92 (2015) 032107, https://doi.org/10.1103/PhysRevA.92.032107.
[4] E. Schrödinger, Discussion of probability relations between separated systems, Math. Proc. Camb. Philos. Soc. 31 (1935) 555–563, https://doi.org/10.1017/S0305004100013554.
[5] A. Einstein, B. Podolsky, N. Rosen, Can quantum-mechanical description of physical reality be considered complete?, Phys. Rev. 47 (1935) 777–780, https://doi.org/10.1103/PhysRev.47.777.
[6] H.M. Wiseman, S.J. Jones, A.C. Doherty, Steering, entanglement, nonlocality, and the Einstein-Podolsky-Rosen paradox, Phys. Rev. Lett. 98 (2007) 140402, https://doi.org/10.1103/PhysRevLett.98.140402.
[7] S.J. Jones, H.M. Wiseman, A.C. Doherty, Entanglement, Einstein-Podolsky-Rosen correlations, bell nonlocality, and steering, Phys. Rev. A 76 (2007) 052116, https://doi.org/10.1103/PhysRevA.76.052116.
[8] Y. Xiang, B. Xu, L. Mišta, T. Tufarelli, Q. He, G. Adesso, Investigating Einstein-Podolsky-Rosen steering of continuous-variable bipartite states by

non-Gaussian pseudospin measurements, Phys. Rev. A 96 (2017) 042326, https://doi.org/10.1103/PhysRevA.96.042326.
[9] C. Branciard, E.G. Cavalcanti, S.P. Walborn, V. Scarani, H.M. Wiseman, One-sided device-independent quantum key distribution: security, feasibility, and the connection with steering, Phys. Rev. A 85 (2012) 010301, https://doi.org/10.1103/PhysRevA.85.010301.
[10] Q. He, L. Rosales-Zárate, G. Adesso, M.D. Reid, Secure continuous variable teleportation and Einstein-Podolsky-Rosen steering, Phys. Rev. Lett. 115 (2015) 180502, https://doi.org/10.1103/PhysRevLett.115.180502.
[11] E.S. Gómez, G. Cañas, E. Acuña, W.A.T. Nogueira, G. Lima, Non-Gaussian-state generation certified using the Einstein-Podolsky-Rosen-steering inequality, Phys. Rev. A 91 (2015) 013801, https://doi.org/10.1103/PhysRevA.91.013801.
[12] S. Wollmann, N. Walk, A.J. Bennet, H.M. Wiseman, G.J. Pryde, Observation of genuine one-way Einstein-Podolsky-Rosen steering, Phys. Rev. Lett. 116 (2016) 160403, https://doi.org/10.1103/PhysRevLett.116.160403.
[13] Y. Xiang, I. Kogias, G. Adesso, Q. He, Multipartite Gaussian steering: monogamy constraints and quantum cryptography applications, Phys. Rev. A 95 (2017) 010101, https://doi.org/10.1103/PhysRevA.95.010101.
[14] X. Deng, Y. Xiang, C. Tian, G. Adesso, Q. He, Q. Gong, X. Su, C. Xie, K. Peng, Demonstration of monogamy relations for Einstein-Podolsky-Rosen steering in Gaussian cluster states, Phys. Rev. Lett. 118 (2017) 230501, https://doi.org/10.1103/PhysRevLett.118.230501.
[15] I. Kogias, P. Skrzypczyk, D. Cavalcanti, A. Acín, G. Adesso, Hierarchy of steering criteria based on moments for all bipartite quantum systems, Phys. Rev. Lett. 115 (2015) 210401, https://doi.org/10.1103/PhysRevLett.115.210401.
[16] J. Schneeloch, P.B. Dixon, G.A. Howland, C.J. Broadbent, J.C. Howell, Violation of continuous-variable Einstein-Podolsky-Rosen steering with discrete measurements, Phys. Rev. Lett. 110 (2013) 130407, https://doi.org/10.1103/PhysRevLett.110.130407.
[17] C.-W. Lee, S.-W. Ji, H. Nha, Quantum steering for continuous-variable states, J. Opt. Soc. Am. B 30 (2013) 2483, https://doi.org/10.1364/JOSAB.30.002483.
[18] S.-W. Ji, J. Lee, J. Park, H. Nha, Quantum steering of Gaussian states via non-Gaussian measurements, Sci. Rep. 6 (2016) 29729, https://doi.org/10.1038/srep29729.
[19] A. Ferraro, M.G.A. Paris, Nonclassicality criteria from phase-space representations and information-theoretical constraints are maximally inequivalent, Phys. Rev. Lett. 108 (2012) 260403, https://doi.org/10.1103/PhysRevLett.108.260403.
[20] A. Kenfack, K. yczkowski, Negativity of the Wigner function as an indicator of non-classicality, J. Opt. B, Quantum Semiclass. Opt. 6 (2004) 396–404, https://doi.org/10.1088/1464-4266/6/10/003.
[21] F. Albarelli, M.G. Genoni, M.G.A. Paris, A. Ferraro, Resource theory of quantum non-Gaussianity and Wigner negativity, Phys. Rev. A 98 (2018) 052350, https://doi.org/10.1103/PhysRevA.98.052350.
[22] A. Mari, J. Eisert, Positive Wigner functions render classical simulation of quantum computation efficient, Phys. Rev. Lett. 109 (2012) 230503, https://doi.org/10.1103/PhysRevLett.109.230503.
[23] S. Rahimi-Keshari, T.C. Ralph, C.M. Caves, Sufficient conditions for efficient classical simulation of quantum optics, Phys. Rev. X 6 (2016) 021039, https://doi.org/10.1103/PhysRevX.6.021039.
[24] M. Hillery, Total noise and nonclassical states, Phys. Rev. A 39 (1989) 2994–3002, https://doi.org/10.1103/PhysRevA.39.2994.
[25] C.T. Lee, Nonclassical photon statistics of two-mode squeezed states, Phys. Rev. A 42 (1990) 1608–1616, https://doi.org/10.1103/PhysRevA.42.1608.
[26] K.E. Cahill, R.J. Glauber, Ordered expansions in boson amplitude operators, Phys. Rev. 177 (1969) 1857–1881, https://doi.org/10.1103/PhysRev.177.1857.
[27] K.E. Cahill, R.J. Glauber, Density operators and quasiprobability distributions, Phys. Rev. 177 (1969) 1882–1902, https://doi.org/10.1103/PhysRev.177.1882.
[28] H.-W. Lee, Theory and application of the quantum phase-space distribution functions, Phys. Rep. 259 (1995) 147–211, https://doi.org/10.1016/0370-1573(95)00007-4, https://www.sciencedirect.com/science/article/pii/0370157395000074.
[29] J. Sperling, Characterizing maximally singular phase-space distributions, Phys. Rev. A 94 (2016) 013814, https://doi.org/10.1103/PhysRevA.94.013814.
[30] S. Ryl, J. Sperling, E. Agudelo, M. Mraz, S. Köhnke, B. Hage, W. Vogel, Unified nonclassicality criteria, Phys. Rev. A 92 (2015) 011801, https://doi.org/10.1103/PhysRevA.92.011801.
[31] L. Mandel, Non-classical states of the electromagnetic field, Phys. Scr. T 12 (1986) 34–42, https://doi.org/10.1088/0031-8949/1986/t12/005.
[32] C.T. Lee, Measure of the nonclassicality of nonclassical states, Phys. Rev. A 44 (1991) R2775–R2778, https://doi.org/10.1103/PhysRevA.44.R2775.
[33] N. Lütkenhaus, S.M. Barnett, Nonclassical effects in phase space, Phys. Rev. A 51 (1995) 3340–3342, https://doi.org/10.1103/PhysRevA.51.3340.
[34] G.S. Agarwal, K. Tara, Nonclassical character of states exhibiting no squeezing or sub-Poissonian statistics, Phys. Rev. A 46 (1992) 485–488, https://doi.org/10.1103/PhysRevA.46.485.
[35] T. Kiesel, W. Vogel, V. Parigi, A. Zavatta, M. Bellini, Experimental determination of a nonclassical Glauber-Sudarshan p function, Phys. Rev. A 78 (2008) 021804, https://doi.org/10.1103/PhysRevA.78.021804.
[36] U.M. Titulaer, R.J. Glauber, Correlation functions for coherent fields, Phys. Rev. 140 (1965) B676–B682, https://doi.org/10.1103/PhysRev.140.B676.
[37] W. Vogel, Nonclassical states: an observable criterion, Phys. Rev. Lett. 84 (2000) 1849–1852, https://doi.org/10.1103/PhysRevLett.84.1849.
[38] T. Kiesel, W. Vogel, Nonclassicality filters and quasiprobabilities, Phys. Rev. A 82 (2010) 032107, https://doi.org/10.1103/PhysRevA.82.032107.
[39] S.L. Braunstein, Squeezing as an irreducible resource, Phys. Rev. A 71 (2005) 055801, https://doi.org/10.1103/PhysRevA.71.055801.
[40] F. Albarelli, A. Ferraro, M. Paternostro, M.G.A. Paris, Nonlinearity as a resource for nonclassicality in anharmonic systems, Phys. Rev. A 93 (2016) 032112, https://doi.org/10.1103/PhysRevA.93.032112.
[41] B. Yadin, F.C. Binder, J. Thompson, V. Narasimhachar, M. Gu, M.S. Kim, Operational resource theory of continuous-variable nonclassicality, Phys. Rev. X 8 (2018) 041038, https://doi.org/10.1103/PhysRevX.8.041038.
[42] H. Kwon, K.C. Tan, T. Volkoff, H. Jeong, Nonclassicality as a quantifiable resource for quantum metrology, Phys. Rev. Lett. 122 (2019) 040503, https://doi.org/10.1103/PhysRevLett.122.040503.
[43] M. Frigerio, S. Olivares, M.G.A. Paris, Steering nonclassicality of Gaussian states, Phys. Rev. A 103 (2021) 022209, https://doi.org/10.1103/PhysRevA.103.022209.
[44] M.T. Quintino, T. Vértesi, N. Brunner, Joint measurability, Einstein-Podolsky-Rosen steering, and bell nonlocality, Phys. Rev. Lett. 113 (2014) 160402, https://doi.org/10.1103/PhysRevLett.113.160402.
[45] P. Skrzypczyk, M. Navascués, D. Cavalcanti, Quantifying Einstein-Podolsky-Rosen steering, Phys. Rev. Lett. 112 (2014) 180404, https://doi.org/10.1103/PhysRevLett.112.180404.
[46] D. Cavalcanti, P. Skrzypczyk, Quantum steering: a review with focus on semidefinite programming, Rep. Prog. Phys. 80 (2016) 024001, https://doi.org/10.1088/1361-6633/80/2/024001.
[47] A. Ferraro, S. Olivares, M.G.A. Paris, Gaussian States in Quantum Information, Bibliopolis, Napoli, 2005.
[48] A. Serafini, Quantum Continuous Variables: A Primer of Theoretical Methods, Taylor & Francis Group, Boca Raton, 2017.
[49] S. Olivares, Quantum optics in the phase space, Eur. Phys. J. Spec. Top. 203 (2012) 3–24.
[50] A. Serafini, Detecting entanglement by symplectic uncertainty relations, J. Opt. Soc. Am. B 24 (2007) 347–354, https://doi.org/10.1364/JOSAB.24.000347.
[51] J. Eisert, S. Scheel, M.B. Plenio, Distilling Gaussian states with Gaussian operations is impossible, Phys. Rev. Lett. 89 (2002) 137903, https://doi.org/10.1103/PhysRevLett.89.137903.
[52] G. Giedke, J. Ignacio Cirac, Characterization of Gaussian operations and distillation of Gaussian states, Phys. Rev. A 66 (2002) 032316, https://doi.org/10.1103/PhysRevA.66.032316.
[53] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, 1985.
[54] F.m.c. Damanet, J. Kübler, J. Martin, D. Braun, Nonclassical states of light with a smooth p function, Phys. Rev. A 97 (2018) 023832, https://doi.org/10.1103/PhysRevA.97.023832.
[55] N. Biagi, M. Bohmann, E. Agudelo, M. Bellini, A. Zavatta, Experimental certification of nonclassicality via phase-space inequalities, Phys. Rev. Lett. 126 (2021) 023605, https://doi.org/10.1103/PhysRevLett.126.023605.
[56] M. Bohmann, E. Agudelo, J. Sperling, Probing nonclassicality with matrices of phase-space distributions, Quantum 4 (2020) 343, https://doi.org/10.22331/q-2020-10-15-343.
[57] A. Serafini, F. Illuminati, S.D. Siena, Symplectic invariants, entropic measures and correlations of Gaussian states, J. Phys. B, At. Mol. Opt. Phys. 37 (2003) L21–L28, https://doi.org/10.1088/0953-4075/37/2/l02.
[58] L.-M. Duan, G. Giedke, J.I. Cirac, P. Zoller, Inseparability criterion for continuous variable systems, Phys. Rev. Lett. 84 (2000) 2722–2725, https://doi.org/10.1103/PhysRevLett.84.2722.
[59] I. Kogias, A.R. Lee, S. Ragy, G. Adesso, Quantification of Gaussian quantum steering, Phys. Rev. Lett. 114 (2015) 060403, https://doi.org/10.1103/PhysRevLett.114.060403.
[60] M.D. Reid, Demonstration of the Einstein-Podolsky-Rosen paradox using nondegenerate parametric amplification, Phys. Rev. A 40 (1989) 913–923, https://doi.org/10.1103/PhysRevA.40.913.
[61] M.D. Reid, P.D. Drummond, W.P. Bowen, E.G. Cavalcanti, P.K. Lam, H.A. Bachor, U.L. Andersen, G. Leuchs, Colloquium: the Einstein-Podolsky-Rosen paradox: from concepts to applications, Rev. Mod. Phys. 81 (2009) 1727–1751, https://doi.org/10.1103/RevModPhys.81.1727.
[62] Z.Y. Ou, S.F. Pereira, H.J. Kimble, K.C. Peng, Realization of the Einstein-Podolsky-Rosen paradox for continuous variables, Phys. Rev. Lett. 68 (1992) 3663–3666, https://doi.org/10.1103/PhysRevLett.68.3663.
[63] S.L.W. Midgley, A.J. Ferris, M.K. Olsen, Asymmetric Gaussian steering: when Alice and Bob disagree, Phys. Rev. A 81 (2010) 022101, https://doi.org/10.1103/PhysRevA.81.022101.
[64] J. Kiukas, C. Budroni, R. Uola, J.-P. Pellonpää, Continuous-variable steering and incompatibility via state-channel duality, Phys. Rev. A 96 (2017) 042331, https://doi.org/10.1103/PhysRevA.96.042331.
[65] V. D'Auria, S. Fornaro, A. Porzio, S. Solimeno, S. Olivares, M.G.A. Paris, Full characterization of Gaussian bipartite entangled states by a single homodyne detector, Phys. Rev. Lett. 102 (2009) 020502, https://doi.org/10.1103/PhysRevLett.102.020502.

[66] S. Cialdi, C. Porto, D. Cipriani, S. Olivares, M.G.A. Paris, Full quantum state reconstruction of symmetric two-mode squeezed thermal states via spectral homodyne detection and a state-balancing detector, Phys. Rev. A 93 (2016) 043805, https://doi.org/10.1103/PhysRevA.93.043805.
[67] S. Olivares, Introduction to generation, manipulation and characterization of optical quantum states, Phys. Lett. A 418 (2021) 127720, https://doi.org/10.1016/j.physleta.2021.127720, https://www.sciencedirect.com/science/article/pii/S0375960121005843.
[68] E.G. Cavalcanti, S.J. Jones, H.M. Wiseman, M.D. Reid, Experimental criteria for steering and the Einstein-Podolsky-Rosen paradox, Phys. Rev. A 80 (2009) 032112, https://doi.org/10.1103/PhysRevA.80.032112.
[69] V. Gebhart, M. Bohmann, Neural-network approach for identifying nonclassicality from click-counting data, Phys. Rev. Res. 2 (2020) 023150, https://doi.org/10.1103/PhysRevResearch.2.023150.