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QUANTUM PHASE PROBLEM FOR HARMONIC AND TIME-DEPENDENT OSCILLATOR SYSTEMS

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We address generalized measurements of linear multimode operators and discuss some aspects relevant to constructing angle operators for arbitrary quadratic Hamiltonian systems via Weyl-ordered expansions in terms of position and momentum operators.

Keywords: quantum angle operator, generalized measurement, harmonic oscillator system, linear multimode operator, heterodyne detection

1. Introduction

Constructing well-behaved quantum phase operators for harmonic systems has been a challenging issue since the early days of quantum mechanics and has been attacked by resorting to very different physical and mathematical perspectives (see, e.g., [1]–[3]). Several questions arise while attempting to satisfactorily define a Hermitian operator that satisfies canonical commutation relations with the Hamiltonian operator (or with an action operator in the general case). Fundamental problems in achieving the goal have been promptly recognized while considering Dirac's proposal to use a polarlike decomposition for the mode operators to define quantum Hermitian amplitude and phase components closely to their classical c-number counterparts through the prescription $\hat{a} = e^{i\hat{\phi}}\widehat{N}^{1/2}$, where $\widehat{N} = \hat{a}^\dagger\hat{a}$ [4]. In addition to the need to take the periodicity of the phase variable into account (a property that would be easily established by choosing the operator domain modulo 2π and introducing δ-functions to let basic canonical commutation relations hold except at one boundary of the domain), we encounter the obstacle that the spectrum of the number operator $\widehat{N}$ does not extend to negative values and $e^{i\hat{\phi}}$ is actually not unitary. Moreover, it is impossible to properly divide both sides of an operator of the type $\hat{a} = \hat{u}\widehat{N}^{1/2}$ by $\widehat{N}^{1/2}$; we might use $(\widehat{N}^-)^{1/2}$ instead, where $\widehat{N}^- = \sum_{n=1}^{\infty} n^{-1}|n\rangle\langle n|$ is the pseudoinverse of $\widehat{N}$.

Many quantum phase concepts have been proposed and investigated while considering ways around Pauli's prohibition (i.e., the situation where there is no self-adjoint operator canonically conjugate to a Hamiltonian if the Hamiltonian spectrum is bounded from below). Among them, a strategy based on enlarging the Hilbert spaces and allowing *negative number states* (inaccessible to physical systems) was developed in [5], where realizing a Hermitian phase operator using the unitary operator $e^{i\hat{\phi}} = \sum_{n=-\infty}^{\infty} |n\rangle\langle n+1|$ was suggested. On the other hand, working with periodic functions of the operator rather than the operator itself is the idea behind the proposal [6] to introduce the ansatz $\widehat{E} \equiv (\widehat{N}+1)^{-1/2}\hat{a}$ defining an operator $\widehat{E}$ playing the role of the analogue of $e^{i\hat{\phi}}$. In this approach, the manifest lack of unitarity of $\widehat{E}$ becomes

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a restriction of the Hilbert space domain. Treating the quantum phase definition within a coherent state representation has also been considered [7].

Here, we present some results obtained by attacking the quantum phase problem from two different, but related, perspectives. We first consider generalized measurements for multiboson linear operators in Sec. 2. We recall that most of the recent progress regarding quantum phase operators has been stimulated by the breakthrough remark that self-adjointness actually imposes an unnecessary restriction on the observable concept: quantum observables are generally positive operator-valued measures [8], and phase observables are defined as positive operator-valued measures that transform covariantly under time translations [9]. In this picture, the postulate of quantum mechanics stating that only observables corresponding to self-adjoint operators can be measured is preserved based on the Naimark dilatation theorem. Whenever we have a quantum variable whose positive operator-valued measure is not reducible to a projection-valued measure, different measuring systems would measure different variable outcomes. This is the case for phase, and we abandon the concept of an *ideal* quantum phase operator to introduce operators associated with *feasible* quantum phases, whose definition is based on using measurement processes.[1]

Section 2 originates from a formulation of the quantum phase problem based on using phase-space distributions, which allows providing quantum averages in a form resembling classical averages. Each Hermitian phase operator $\hat{\phi}$ such that the phase distribution $P(\varphi) = \mathrm{tr}[\delta(\hat{\phi} - \varphi)\hat{\rho}]$ attributes the correct sharp phase to any large-amplitude localized state $\hat{\rho}$ is expressible as the operator obtained from the classical phase by a direct quantization of phase-space variables supported by an ordering rule (e.g., the Weyl ordering). A detailed discussion of applying this framework to modes of the standard harmonic oscillator can be found in [11]. A question thus naturally arises regarding the possibility of using it, for instance, whenever we treat time-dependent oscillators. Section 3 is devoted to some aspects implied by the answer to this question.

2. Generalized measurements and phase operators

A measurement process implies an interaction between the system under consideration and a measurement device. We can measure a variable if it is associated with a self-adjoint operator defined on the Hilbert space $\mathcal{H}_\mathrm{S} \otimes \mathcal{H}_\mathrm{P}$ of the system plus probe of the apparatus. An observer living in $\mathcal{H}_\mathrm{S}$ has access only to observables supported in $\mathcal{H}_\mathrm{S}$ itself. The idea underlying generalized measurements is to extend variable operators in $\mathcal{H}_\mathrm{S}$ to suitable symmetric operators in $\mathcal{H}$: when an orthogonal measurement in $\mathcal{H}$ is performed, the observer in $\mathcal{H}_\mathrm{S}$ knows only about components of states in this space. The operatorial description of the measurement restricted to $\mathcal{H}_\mathrm{S}$ is given in terms of the positive operator-valued measure that follows by taking the trace of the spectral measure of the extended operator over the probe degree of freedom [8].

The simplest and most relevant example is provided just by the measurements of the feasible photon phase in the context of *heterodyne detection*. Heterodyne detection is understood to perform the joint measurement of two conjugate quadratures of the field that would result by mixing a single-mode signal field of nominal frequency ω_1 with a local oscillator field whose frequency ω_L is slightly offset by an amount $\omega_\mathrm{I} \ll \omega_1$ from that of the input signal through a beam splitter (BS) (see, e.g., [12]).[2] A photodetector is placed after the BS, and the output photocurrent is filtered at the frequency ω_I. In standard optical heterodyne detection, measuring the filtered photocurrent corresponds to realizing the quantum measurement of the normal operator $\widehat{Z}_\mathrm{SW} = a_1 + a_2^\dagger$, where $\hat{a}_1$ and $\hat{a}_2^\dagger$ are the respective photon annihilation and creation operators for the input and image signals [13]. Measuring the real and imaginary parts of the output photocurrent provides the simultaneous measurement of the real and imaginary parts of $\widehat{Z}_\mathrm{SW}$.

[1] But a *canonical* phase variable can be defined [10].

[2] For modes of the radiation field, the simplest two-mode interaction is the linear mixing Hamiltonian $\widehat{H} \propto (a_1\hat{a}_2^\dagger + \hat{a}_2 a_1^\dagger)$, corresponding to the interaction in a linear optical medium (BS).

The interaction between the incoming signal and a BS apparatus both enlarges the one-photon signal Hilbert space to the two-photon signal plus image space $\mathcal{H}_{12} = \mathcal{H}_{a_1} \otimes \widehat{H}_{a_2}$ and allows "negative number" photon states. The spectrum of the relative number operator $\widehat{N}_{12} = \hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2$ is in fact unbounded, and a unitary operator $\widehat{D}_{12} = e^{i\hat{\phi}_{12}}$ exists on $\mathcal{H}_{12}$ satisfying $[\widehat{D}_{12}, \widehat{N}] = \widehat{D}_{12}$. It can be written as [14]

$$\widehat{D}_{12} = \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} |n-1, m\rangle\!\rangle \langle\!\langle m, n|, \tag{2.1}$$

where the $|n, m\rangle\!\rangle$ denote the basis elements of the relative number state representation, i.e.,

$$|n, m\rangle\!\rangle = \Theta(n)|m+n\rangle_{a_1}|m\rangle_{a_2} + \Theta(-n-1)|m\rangle_{a_1}|m-n\rangle_{a_2}, \tag{2.2}$$

$$\widehat{N}_{12}|n, m\rangle\!\rangle = n|n, m\rangle\!\rangle, \tag{2.3}$$

with $-\infty < n < \infty$, $m \geq 0$, $\Theta(n) = 1$ for $n \geq 0$, and $\Theta(n) = 0$ for $n < 0$. Because $[\widehat{Z}_{\mathrm{SW}}, \widehat{Z}_{\mathrm{SW}}^\dagger] = 0$, a joint measurement of the real and imaginary part of the operator $\widehat{Z}_{\mathrm{SW}}$ can be performed, thus providing the realization of the Shapiro–Wagner feasible phase operator. The problem of measuring the phase is thus formulated as the problem of estimating the phase shifts experienced by the signal-plus-image quantum states.

2.1. Heterodyne-based phase operators. Quite a different situation arises if the input field frequency is outside the optical regime and the interaction between the signal and the BS results in the nonnormal operator [15]

$$\widehat{Z}_\gamma = a_1 + \gamma a_2^\dagger, \qquad \gamma = \sqrt{\frac{\omega_1 - \omega_{\mathrm{I}}}{\omega_1 + \omega_{\mathrm{I}}}} < 1, \qquad [Z_\gamma, Z_\gamma^\dagger] = 1 - \gamma^2. \tag{2.4}$$

The commutator of the measurement operator quadratures does not vanish, $[\mathrm{Re}\,\widehat{Z}_\gamma, \mathrm{Im}\,\widehat{Z}_\gamma] = i(1-\gamma^2)/2$; another apparatus should therefore be conceived to jointly measure them. For this, we seek a Naimark extension of $\widehat{Z}_\gamma$. The simplest one involves just a single additional bosonic mode, $\hat{a}_3$ for example, and is realized using the operator

$$\widehat{Z}_{\mathrm{N}} = \widehat{Z}_\gamma + \kappa \hat{a}_3^\dagger \tag{2.5}$$

defined in the three-mode Hilbert space $\mathcal{H} = \mathcal{H}_{a_1} \otimes \mathcal{H}_{a_2} \otimes \mathcal{H}_{a_3}$ along with the requirements $\mathrm{Tr}_a(\sigma a_3^\dagger) = 0$ and $\kappa^2 = 1 - \gamma^2$. It can be implemented using bilinear interactions between modes followed by measuring quadratures at the output [16]. The optical devices are associated with evolution operators of the form

$$\widehat{B}_{jk}(\theta_{jk}) = e^{-i\theta_{jk}(a_j a_k^\dagger + a_k a_j^\dagger)} \tag{2.6}$$

with the three-mode splittings defined by the parameters

$$\theta_{12} = \arcsin \frac{\gamma}{\sqrt{1+\gamma^2}}, \qquad \theta_{13} = \arcsin \sqrt{\frac{1-\gamma^2}{2}}, \qquad \theta_{23} = \arcsin \sqrt{\frac{1-\gamma^2}{1+\gamma^2}},$$

followed by a π rotation (see Fig. 1).

It can be seen that in the Caves description of the heterodyne process, a feasible phase $\hat{\theta}_{\mathrm{N}}$ for a quasimonochromatic signal can also be defined if a Naimark mode is introduced to obtain measurement operator (2.5). This phase $\hat{\theta}_{\mathrm{N}}$ is in fact obtained from the unambiguous polar decomposition of operator (2.5). After the three-mode number difference operator $\widehat{N} = \hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2 - \hat{a}_3^\dagger \hat{a}_3$ is introduced, the commutator

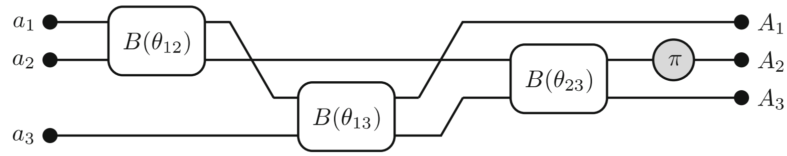

Fig. 1. Schematics of the apparatus for generalized measurement of a nonnormal two-photon linear operator. The quadratures $(\hat{A}_1+\hat{A}_1^\dagger)/\sqrt{2}$ and $i(\hat{A}_2^\dagger-\hat{A}_2)/\sqrt{2}$ of the output modes are to be measured.

$[\hat{\theta}_{\rm N},\widehat{N}]=i$ can be interpreted as the canonical conjugation of the Caves feasible phase $\hat{\theta}_{\rm N}$ with respect to $\widehat{N}$.

Having understood how a generalized measurement of $\widehat{Z}_\gamma$ can be accomplished, we should next analyze the role in the statistics of the measurement played by preparations of states in concrete experiments. Obviously, we may take full advantage of possible freedom in preparing some of the modes in experimental frameworks. Details of evaluating the probability density of the outcomes for a given initial preparation $R_{12}\otimes\rho_3$ can be found in [16]. Here, we simply note that the measurement of $\widehat{Z}_\gamma$ on the class of factored signals described by $R_{12}=\varrho_1\otimes|0\rangle_2\,{}_2\langle 0|$, where ϱ_1 is a generic preparation of the mode a_1 and $|0\rangle_2$ is the ground state of the mode a_2, does not lead to added noise with respect to the measurement of the normal operator $\hat{a}_1+\hat{a}_2$, namely, the measurement of signal field quadratures in the framework of standard optical heterodyne detection.

2.2. Multiboson linear operators. The arguments used above for a single mode can be generalized to general systems, at least in principle. In doing so, we must encode Naimark extensions into experimental settings in order to realize the simultaneous detection of conjugate variables in practice. Interactions that are linear and bilinear in the field modes play a major role in quantum information and can be experimentally realized in optical and condensate systems. It is therefore relevant to obtain a clear picture of the possible experimental schemes that would allow generalized measurements of their quadratures. Generalized measurements of the multimode linear operators

$$\widehat{Z}^{(m_1,m_2)}=\sum_{k_1=1}^{m_1}A_{k_1}\hat{a}_{k_1}+\sum_{k_2=1}^{m_2}B_{k_2}\hat{a}^\dagger_{m_1+k_2}, \tag{2.7}$$

where

$$[\widehat{Z}^{(m_1,m_2)},\widehat{Z}^{(m_1,m_2)\dagger}]=\sum_{k_1=1}^{m_1}|A_{k_1}|^2-\sum_{k_2=1}^{m_2}|B_{k_2}|^2\neq 0, \tag{2.8}$$

were discussed in [17]. Once again, the most economical choice for a valid Naimark extension $\widehat{Z}_{\rm N}^{(m_1,m_2)}$ clearly consists in adding just a single mode to (2.7) through either an annihilation or a creation operator as dictated by the sign of commutator (2.8). The condition $[\widehat{Z}_{\rm N}^{(m_1,m_2)},\widehat{Z}_{\rm N}^{(m_1,m_2)\dagger}]=0$ ensures that a feasible phase observable $\hat{\theta}^{(m_1,m_2)}$ for $\widehat{Z}^{(m_1,m_2)}$ can be safely defined as the phase of the one-mode extended operator $\widehat{Z}_{\rm N}^{(m_1,m_2)}$. The operator $\hat{\theta}^{(m_1,m_2)}$ is canonically conjugate to the (m_1+m_2+1)-mode relative number operator

$$\widehat{N}=\sum_{k=1}^{m_2+1}\widehat{N}_{m_1+k}-\sum_{k=1}^{m_1}\widehat{N}_{m_1+k},\qquad \widehat{N}_r=a_r^\dagger a_r.$$

Similarly to the cases previously discussed, a linear amplifier scheme can be used for generalized measurement by processing all the (input plus Naimark) modes to produce an output signal carried by a different collection of modes that are linearly related to the original modes. The modes $\hat{a}_k$ interact with each other through a unitary operator $\widehat{U}^{(m_1,m_2)}$, which imposes the linear transformation

$$\begin{pmatrix} \hat{A}_1 \\ \vdots \\ \hat{A}_{m_1+m_2} \\ \hat{A}_{m_1+m_2+1} \end{pmatrix} = \widehat{U}^{\dagger}_{(m_1,m_2)} \begin{pmatrix} \hat{a}_1 \\ \vdots \\ \hat{a}_{m_1+m_2} \\ \hat{a}_N \end{pmatrix} \widehat{U}_{(m_1,m_2)} = M^{(m_1,m_2)} \begin{pmatrix} \hat{a}_1 \\ \vdots \\ \hat{a}_{m_1+m_2} \\ \hat{a}_N \end{pmatrix}$$

($[\hat{A}_i, \hat{A}^{\dagger}_j] = \delta_{i,j}\hat{1}$). For the quadratures of output modes to be measured, identifying the sequence of BSs that naturally generalize the optical setting in Fig. 1 proceeds by imposing the decomposition operator $\widehat{U}_{(m_1,m_2)}$ (eventually completed by a π rotation):

$$\widehat{U}_{(m_1,m_2)} = \widehat{\mathfrak{B}}_{m+1,m}\widehat{\mathfrak{B}}_{m+1,m-1}\cdots\widehat{\mathfrak{B}}_{m+1,1}\widehat{\mathfrak{B}}_{m,m-1}\widehat{\mathfrak{B}}_{m,m-2}\cdots\widehat{\mathfrak{B}}_{2,1},$$

where

$$\widehat{\mathfrak{B}}_{j,k} = \left[\widehat{B}_{j,k}(\theta_{j,k})\prod_{s\neq j,k}\otimes\mathbb{I}_s\right], \qquad \widehat{B}_{jk}(\theta_{jk}) = e^{-i\theta_{jk}(\hat{a}_j\hat{a}^{\dagger}_k + \hat{a}_k\hat{a}^{\dagger}_j)}.$$

For instance, the scheme in Fig. 1 can be used for the generalized measurement of the operator

$$\widehat{Z}^{(2,0)} = A_1\hat{a}_1 + A_2\hat{a}_2, \quad A_1, A_2 \in \mathbb{R}_+, \tag{2.9}$$

if the BSs are characterized by

$$\theta^{(2,0)}_{21} = \arcsin\frac{A_2}{\sqrt{A_1^2 + A_2^2}}, \qquad \theta^{(2,0)}_{31} = \frac{\pi}{4}, \qquad \theta^{(2,0)}_{32} = \frac{\pi}{2},$$

but the same sequence of BSs also allows realizing the measurement of quadratures of the $(0,2)$-type operator $\widehat{Z}^{(2,0)\dagger}$ without the final π rotation. Forthcoming studies will consider the search for possibly more effective setups for generalized measurements of phase and also detection schemes for performing generalized measurements of nonlinear amplifiers of interest.

3. Action–angle variables for time-dependent oscillators

As already mentioned in the introduction, different physical grounds have been used to effectively tackle the problem of defining suitable quantum phase operators. Although the basic conceptual problems are still unsolved and further caution is required because the nonlinear canonical transformation from position–momentum to action–angle coordinates are generally nonbijective,[3] the arguments and effective tools developed for the standard harmonic oscillator can be adapted to a more complicated system, at least in principle. Because time-dependent oscillators have widespread applications in physics, we focus on them. Their Hamiltonian is

$$H = \frac{p^2}{2m(t)} + \frac{m(t)\omega^2(t)q^2}{2}. \tag{3.1}$$

Action–angle variables for a system described by this Hamiltonian have the forms

$$J = \frac{m}{2\sqrt{\kappa}\,\sigma^2}q^2 + \frac{1}{2\sqrt{\kappa}}\left[\frac{\sigma}{\sqrt{m}}p - \sqrt{m}\,\sigma q\frac{d}{dt}\left(\log\frac{\sigma}{\sqrt{m}}\right)\right]^2, \tag{3.2}$$

$$\theta = \arctan\left\{\frac{\sigma^2}{\sqrt{\kappa}}\left[\frac{p}{mq} - \frac{d}{dt}\left(\log\frac{\sigma}{\sqrt{m}}\right)\right]\right\}, \tag{3.3}$$

[3] We assume that the operators we consider here have an unambiguous interpretation.

where κ is an arbitrary positive constant and the function $\sigma(t)$ is a solution of the equation

$$\frac{d^2\sigma}{dt^2}+\left[\omega^2-\frac{d}{dt}\left(\frac{1}{2m}\frac{dm}{dt}\right)-\left(\frac{1}{2m}\frac{dm}{dt}\right)^2\right]\sigma=\frac{\kappa}{\sigma^3}. \tag{3.4}$$

An explicit formal representation in terms of the position and momentum operators of a Weyl-ordered quantum angle operator associated with the time-dependent oscillator can be obtained analogously to the procedure used in [18] for the standard harmonic oscillator. A basic angle operator can then be found in the form [19]

$$\hat{\theta}_{\rm W}(\hat{q},\hat{p})=\sum_{k=0}^{\infty}\alpha_{-k,k}(t)\widehat{T}_{-k,k}(\hat{q},\hat{p}),\quad k=0,1,2,\ldots, \tag{3.5}$$

where

$$\begin{aligned}
\alpha_{0,0}&=\frac{\Delta}{J_{pp}}\arctan\tilde{J}_{qp}^{-1},\\
\alpha_{-1-2k,1+2k}&=\frac{(-1)^k}{1+2k}\frac{\Delta}{J_{pp}}\left[-\frac{\tilde{J}_{qq}\Delta}{1+\tilde{J}_{qp}^2}\right]^{1+2k}{}_2F_1\left(-\frac{1}{2}-k,-k,\frac{1}{2},-\tilde{J}_{qp}^2\right),\\
\alpha_{-2-2k,2+2k}&=(-1)^{k+1}\frac{\tilde{J}_{qp}\Delta}{J_{pp}}\left[-\frac{\tilde{J}_{qq}\Delta}{1+\tilde{J}_{qp}^2}\right]^{2+2k}{}_2F_1\left(-\frac{1}{2}-k,-k,\frac{3}{2},-\tilde{J}_{qp}^2\right),\\
\Delta&=\frac{\sqrt{\kappa}}{2\sigma^2\sqrt{\tilde{J}_{qq}-\tilde{J}_{qp}^2}},\qquad J_{pp}=\frac{\sigma^2}{2m\sqrt{k}},\\
\tilde{J}_{qp}&=-m\frac{d}{dt}\left(\log\frac{\sigma}{\sqrt{m}}\right),\qquad \tilde{J}_{qq}=\kappa\frac{m^2}{\sigma^4}+m^2\left(\log\frac{\sigma}{\sqrt{m}}\right)^2,
\end{aligned} \tag{3.6}$$

and

$$\widehat{T}_{-n,n}=\frac{1}{2^n}\sum_{j=0}^{n}\binom{n}{j}\hat{q}^j\hat{p}^{-n}\hat{q}^{n-j}. \tag{3.7}$$

Equations (3.5)–(3.7) provide an expression in terms of position and momentum operators (defined at a given initial time) for the operator associated with the time-dependent oscillator quantum angle. To elucidate the action of the operator $\hat{\theta}_{\rm W}$ on quantum states and how its states are affected by the mechanism of coherence loss, we must obtain a picture of the action of the operators $\widehat{T}_{-n,n}$. To avoid the issue of their non-self-adjointness on the real line, we can require spatial confinement between two points, $q\in[-\ell,\ell]$ for example.[4] This can be discussed in the light of recent revisions of the Pauli theorem and the canonical conjugation concept (see [22]), thus bearing in mind that the self-adjoint operator canonically conjugate to a confined position operator is nonunique in the general case. It is known that for a free confined particle, a consistent choice would be furnished by any operator $\hat{p}^\gamma=-i\partial_q$ whose domain is defined by vectors in the domain of the position operator $\hat{q}$ with square-integrable first derivatives and with norms satisfying a symmetric boundary condition, e.g., $\phi(-\ell)=e^{-2i\gamma}\phi(\ell)$ with $\gamma\in[0,1)$ [22]. The momentum spectral problem is then immediately solved, giving the eigenvectors $e^{ip_{\gamma,k}q}/\sqrt{2}$ associated with the eigenvalues $p_{\gamma,k}=\gamma+k\pi$. For each allowed value of the parameter γ, we obtain a bounded self-adjoint inverse momentum operator (if $\gamma=0$, then a divergent contribution from the zero momentum eigenvalue must be removed by hand, as usual). We can accordingly obtain self-adjoint operators $\widehat{T}^{\gamma}_{-n,n}$ from (3.5). For a

[4]We recall that for a free particle, the Kijowski arrival distribution and the Aharonov–Bohm time operator $\widehat{T}_{-1,1}$ have been formally related by using the concept of a positive operator-valued measure in the unconfined case [20] and as the limit of the distribution obtained by restricting the position domain to a finite real interval [21].

system described by (3.1) and trapped in a finite coordinate range, we can use this result to define a local self-adjoint angle operator in sufficiently small neighborhoods of the origin (see, e.g., [23]) in the presence of either a symmetric ($\gamma = 0$) or an antisymmetric ($\gamma \neq 0$) momentum boundary condition. In this respect, it is interesting to investigate features of the operators $\widehat{T}^{\gamma}_{-n,n}$ that have so far received little attention. In the coordinate representation, each of them takes a Fredholm integral operator form, i.e.,

$$\langle q'|\widehat{T}^{\gamma}_{-n,n}|\psi\rangle = \int_{-\ell}^{\ell} T^{\gamma}_{-n,n}(q'|q)\psi(q)\,dq. \tag{3.8}$$

By virtue of (3.5), direct evaluation of the kernels of $T^{0}_{-n,n}$, for instance, yields

$$T^{0}_{-n,n}(q'|q) = \frac{[i\pi(q+q')]^n}{n!}B_n\left(\frac{q-q'}{2}\right)\Theta(q'-q) + \\ + \frac{[-i\pi(q+q')]^n}{n!}B_n\left(\frac{q'-q}{2}\right)\Theta(q-q'), \tag{3.9}$$

where $B_n(x)$ and $\Theta(x)$ denote the nth-order Bernoulli polynomial and the Heaviside step function.

The spectral problem for operators with kernels (3.9) takes the form of a linear differential equation of the order $2n$, whose general solutions (regardless of the initial/boundary conditions) are

$$\Psi^{0}_{-2,2}(q) = \xi_{2,0} + \sum_{r=1}^{3}\xi_{2,r}q^{r-1}F_{2,r}\left(-\frac{q^4}{16\ell\lambda^{0}_{-2,2}}\right),$$

$$\Psi^{0}_{-3,3}(q) = \xi_{3,0} + \sum_{r=1}^{5}\xi_{3,r}q^{r-1}F_{3,r}\left(-i\frac{q^6}{216\ell\lambda^{0}_{-3,3}}\right), \qquad \text{etc.,}$$

where the $\xi_{n,m}$ are constants, the $F_{n,m}$ are functions of the $(n, 2n-1)$-hypergeometric type, and the $\lambda^{0}_{-n,n}$ are eigenvalues. A general discussion of the spectral problem for the operators $\widehat{T}^{\gamma}_{-n,n}$ will be reported elsewhere.

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