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On the Discrimination Between Classical and Quantum States

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Abstract With the purpose of introducing a useful tool for researches concerning foundations of quantum mechanics and applications to quantum technologies, here

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we address three quantumness quantifiers for bipartite optical systems: one is based on sub-shot-noise correlations, one is related to antibunching and one springs from entanglement determination. The specific cases of parametric downconversion seeded by thermal, coherent and squeezed states are discussed in detail.

Keywords Entanglement · Macro-objectification

1 Introduction

The discrimination between quantum and classical states [1–6], besides its very important and deep conceptual relevance, has also recently received much attention due to the development of quantum technologies. On the one hand, it represents a fundamental point for the studies concerning the transition between quantum and classical world, one of the most intriguing research sectors in the foundations of physics; on the other hand it is a tool of the utmost importance when comparing results that can be achieved with quantum and classical protocols.

These studies have concerned various physical systems [1, 2] ranging from quantum optical states [5] to mesons [7] or solid state devices [8] and, recently, have pointed to the need of an operational approach linked to measurements schemes [9–28]. Considering the experimental interest in the frame of quantum optics [28–32], in a recent paper [33] we have considered three different "quantumness" quantifiers applied to the study of quantum-classical transition in a thermal seeded parametric downconversion (PDC): a work that, a part its specific application, has a more general interest since it presents an idea that can find a generalization to various physical systems.

Here we want to extend this first study by considering, as a further example, the application of these three quantifiers to PDC seeded by coherent and squeezed vacuum states, comparing them with what we achieved for thermal seeded case. This allows a more general understanding of the hierarchy of these three quantifiers that emerged in [33] and, due to the easy realizability of these states, paves the way toward a general experimental test of these theoretical results.

The structure of the paper is the following. In the next section we review the phenomenon of seeded PDC. In the following sections we analyze in details the quantumness quantifier coming from sub-shot noise measurement (Sect. 3), Lee's criterion (Sect. 4) and entanglement (Sect. 5). Finally, a general discussion of the results is presented in Sect. 6.

2 Seeded Parametric Downconversion

The evolution of a quantum system induced by the interaction Hamiltonian describing the PDC process for a single pair of coupled modes is described by the unitary operator $U = \exp(i\kappa a_A a_B + h.c.)$, where $\kappa = |\kappa| e^{i\varphi}$ is the coupling constant and a_A and a_B are the annihilation operators for photons on modes A and B, respectively. We consider the PDC process seeded by two single mode input states $\rho_{in} = \rho_A \otimes \rho_B$.

In particular as seed fields on both A- and B-modes we consider the three simplest Gaussian states, namely thermal states, coherent states, and vacuum squeezed states. For the thermal case (T) the input state of the single mode is a mixed incoherent superposition.

$$\rho_j^{(T)} = \sum_{n=0}^{\infty} P_j(n)\ |n\rangle_j \langle n|, \tag{1}$$

where $j = A, B$ and $|n\rangle_j$ denotes the Fock number basis for the single mode of the j-arm, the thermal probability distribution of the input being $P_j(n) = \mu_j^n(1+\mu_j)^{-n-1}$, where μ_j is the average photon number.

In the case of coherent seeding (C) the state in the single mode is obtained by the action of the displacement operator $D(\alpha_j) = \exp[i(\alpha_j a_j + \alpha_j^* a_j^\dagger)]$ on the vacuum state

$$\rho_j^{(C)} = D(\alpha_j)|0\rangle_j \langle 0|D^\dagger(\alpha_j) \tag{2}$$

with $\alpha_j = \sqrt{M_j}\, e^{i\gamma_j}$ is the complex displacement parameters, where M_j represents the mean number of photon of the state, and γ_j is the phase of the j-th coherent field.

For the squeezed vacuum seeds (S) the input state of the single mode is given by

$$\rho_j^{(S)} = S(\xi_j)|0\rangle_j \langle 0|S^\dagger(\xi_j) \tag{3}$$

where the squeezing operator is $S(\xi_j) = \exp[i(\xi_j a_j^2 + \xi_j^* a_j^{\dagger 2})]$, $\xi_j = |\xi_j|e^{i\zeta_j}$ being complex parameters. ζ_j the phase of the squeezed field, and $|\xi_j|$ is connected to its mean photon number through $N_j = \sinh^2|\xi_j|$.

The density matrix at the output of the PDC interaction is given by

$$\rho_{out} = U\rho_{in}U^\dagger. \tag{4}$$

Conversely in the interaction picture the output field modes are given by $A_j = U^\dagger a_j U$, i.e.

$$A_j = \sqrt{N+1}\, a_j + e^{i\varphi}\sqrt{N}\, a_{j'}^\dagger \quad (j, j' = A, B, j \neq j'), \tag{5}$$

where $N = \sinh^2|\kappa|$ is the mean number of photons of the PDC spontaneous emission, and φ is its phase.

The first moments of the photon number distribution in the case of thermal seeds are [35]

$$\begin{aligned} \langle n_A^{(T)}\rangle &= \mu_A + N(1+\mu_A+\mu_B), \\ \langle n_B^{(T)}\rangle &= \mu_B + N(1+\mu_A+\mu_B), \end{aligned} \tag{6}$$

where, $n_j = a_j^\dagger a_j$, $\langle O\rangle = \mathrm{Tr}[O\rho_{out}]$.

When the process is seeded by coherent fields we have:

$$\begin{aligned} \langle n_A^{(C)}\rangle &= M_A + N(1+M_A+M_B) + 2\sqrt{N(N+1)}\sqrt{M_A M_B}\cos(\gamma_A+\gamma_B-\varphi), \\ \langle n_B^{(C)}\rangle &= M_B + N(1+M_A+M_B) + 2\sqrt{N(N+1)}\sqrt{M_A M_B}\cos(\gamma_A+\gamma_B-\varphi). \end{aligned} \tag{7}$$

Contrary to the thermal case, here the intensity of the fields is partially modulated by the phase value of the seeding fields with respect to the phase induced by the PDC process when both the seeds are nonzero (i.e. $M_A, M_B \neq 0$). Eventually when vacuum squeezed input beams are considered, we obtain

$$\begin{aligned} \langle n_A^{(S)} \rangle &= N_A + N(1 + N_A + N_B), \\ \langle n_B^{(S)} \rangle &= N_B + N(1 + N_A + N_B), \end{aligned} \tag{8}$$

where $N_j = \sinh^2 |\xi_j|$ is the average number of photons of the input state in the single mode j, whereas, surprisingly, the phases of the squeezing operators do not play any role in the photon number. Notice that the case of vacuum inputs, $\rho_{in} = |0\rangle\langle 0|_A \otimes |0\rangle\langle 0|_B$, corresponds to spontaneous PDC, i.e. to the generation of twin-beam.

3 Sub-shot Noise Measurement

The shot-noise limit (SNL) in any photodetection process is defined as the lowest level of noise that can be obtained by using semiclassical states of light that is, Glauber coherent states. If one measures the photon numbers in two beams and evaluates their difference, the SNL is the lower limit of noise that can be reached when the beams are classically correlated. On the other hand, when intensity correlations below the SNL are observed, we have a genuine nonclassical effect. We consider a simple measurement scheme where A and B single mode beams at the output of the PDC interaction are individually measured by direct detection (considering in this section ideal detectors with unitary efficiency [34]). The resulting A and B photon counts, which are correlated, are subtracted from each other to demonstrate quantum noise reduction in the difference of photon counts. In order to observe a violation of the SNL we must have

$$\langle [\Delta(n_A - n_B)]^2 \rangle < \langle n_A \rangle + \langle n_B \rangle, \tag{9}$$

where $\langle [\Delta(n_A - n_B)]^2 \rangle$ is the variance of this difference, and $\langle n_A \rangle + \langle n_B \rangle$ is the SNL, i.e. the quantity that would be obtained for uncorrelated coherent beams.

In particular, the condition in (9) reduces to

$$N > \frac{(\mu_A^2 + \mu_B^2)}{2(1 + \mu_A + \mu_B)} \tag{10}$$

for the thermal seeds.

In the case of coherent seeds if the phases satisfy $\cos(\gamma_A + \gamma_B - \varphi) \geq 0$ the condition in (9) is always fulfilled irrespective of the value of N, M_A, M_B, while if $\cos(\gamma_A + \gamma_B - \varphi) < 0$ the condition is fulfilled only when

$$N > \frac{4 M_A M_B \cos^2(\gamma_A + \gamma_B - \varphi)}{1 + 2(M_A + M_B) + (M_A + M_B)^2 - 4 M_A M_B \cos^2(\gamma_A + \gamma_B - \varphi)}. \tag{11}$$

In the case of squeezed vacuum seeds the condition is

$$N > \frac{N_A(1+2N_A) + N_B(1+2N_B)}{2(1+N_A+N_B)}. \tag{12}$$

It is interesting to notice that, in the case of thermal and squeezed input state, there always exists a threshold between the sub-shot-noise and the classical regime, which can be explored by controlling the intensities of the seeds. The behavior of the coherent case is different because, upon properly adjusting the phases, the sub-shot noise condition holds whatever the intensities of the seeds. In an experiment in which the phases of seeds γ_A, γ_B and that of PDC process φ are not locked, one expects, on average, a null value of the cosine in (11) and therefore a permanent sub-shot noise (SSN) condition. It is helpful to define a parameter, $\mathcal{P}_{\text{SSN}}$ quantifying the amount of violation of the SNL

$$\mathcal{P}_{\text{SSN}} = 1 - \frac{\langle[\Delta(n_A - n_B)]^2\rangle}{\langle n_A\rangle + \langle n_B\rangle}. \tag{13}$$

$\mathcal{P}_{\text{SSN}} = 0$ corresponds to the SNL, and the sub-shot noise condition corresponds to $0 < P_{\text{SSN}} \leq 1$. For the state ρ_{out}, in the case of thermal seeds we have

$$\mathcal{P}^{(T)}_{\text{SSN}} = \frac{2\mu_{\text{PDC}}(1+\mu_A+\mu_B) - \mu_A^2 - \mu_B^2}{2\mu_{\text{PDC}}(1+\mu_A+\mu_B) + \mu_A + \mu_B}, \tag{14}$$

thus the maximal violation of SNL is achieved by the twin-beam ($\mu_A = \mu_B = 0$), and by increasing the magnitude of, at least, one of the seeding field the SNL is reached.

For coherent input beams the amount of violation is

$$\mathcal{P}^{(C)}_{\text{SSN}} = \frac{2N(1+M_A+M_B) + 4\sqrt{N(N+1)}\sqrt{M_A M_B}\cos(\gamma_A+\gamma_B-\varphi)}{2N(1+M_A+M_B) + 4\sqrt{N(N+1)}\sqrt{M_A M_B}\cos(\gamma_A+\gamma_B-\varphi) + M_A + M_B}, \tag{15}$$

and also in this case the limit value of 1 is reached again by the twin-beam in the spontaneous emission ($M_A = M_B = 0$). The SNL threshold $\mathcal{P}^{(C)}_{\text{SSN}} = 0$ is obtained when the numerator of (15) is zero, leading to the solutions presented in (11).

Finally, for the squeezed beams the parameter is

$$\mathcal{P}^{(S)}_{\text{SSN}} = \frac{2N(1+N_A+N_B) - 2N_A(1+N_A) - 2N_B(1+N_B)}{2N(1+N_A+N_B) + N_A + N_B}. \tag{16}$$

We notice that for all the cases, it can be shown that $\langle[\Delta(n_A - n_B)]^2\rangle$ is equal to the sum of the mean fluctuation of the two input seeding states. Therefore, $\mathcal{P}_{\text{SSN}}$ always assumes the form

$$\mathcal{P}_{\text{SSN}} \equiv 1 - \frac{\langle[\Delta(n_A - n_B)]^2\rangle}{\langle n_A\rangle + \langle n_B\rangle} = \frac{\langle n_A\rangle + \langle n_B\rangle - \langle[\Delta n_A]^2\rangle_{\rho_{in}} - \langle[\Delta n_B]^2\rangle_{\rho_{in}}}{\langle n_A\rangle + \langle n_B\rangle}, \tag{17}$$

where, with obvious notation, $\langle O\rangle_{\rho_{in}} = \text{Tr}[O\rho_{in}]$.

4 Lee's Criterion

Another interesting criterion of nonclassicality was derived by Lee [36, 37], and it is the two-mode generalization of the well known nonclassicality criterion for single mode beam $\langle n(n-1)\rangle - \langle n\rangle^2 < 0$ [38]. The Lee's criterion states that a bipartite system is nonclassical if the inequality

$$\langle n_A(n_A-1)\rangle + \langle n_B(n_B-1)\rangle - 2\langle n_A n_B\rangle < 0 \tag{18}$$

is satisfied. It is noteworthy to observe that the "Lee's nonclassicality" corresponding to (18), implies the negativity of the Glauber-Sudarshan P-function [36, 37]. Once we consider the state ρ_{out}, the condition in (18) for seeded PDC is achieved when

$$N > \frac{\mu_A^2 + \mu_B^2 - \mu_A\mu_B}{(1+\mu_A+\mu_B)}, \tag{19}$$

while in the case of coherent seeds, we have that, when the phases satisfy $\cos(\gamma_A + \gamma_B - \varphi) \geq 0$, $N > N_-$, while when $\cos(\gamma_A+\gamma_B-\varphi) < 0$, $N > N_+$ with

$$N_\pm = \frac{4g\cos^2 r + ab \pm 2[g\cos^2 r(4g\cos^2 r + 2ab + a^2)]^{1/2}}{2[b^2 - 4g\cos^2 r]} \tag{20}$$

and $g = M_A M_B$, $a = (M_A - M_B)^2$, $b = 1 + M_A + M_B$, $r = \gamma_A + \gamma_B - \varphi$.

For the squeezed thermal seeds we obtain

$$N > \frac{N_A + N_B + 3N_A^2 + 3N_B^2 - 2N_A N_B}{2(1+N_A+N_B)}. \tag{21}$$

It is noteworthy to observe that the Lee condition is stricter than the sub-SNL: always exists a threshold between classicality and nonclassicality for the Lee's criterion.

Analogously to the sub-SNL case we define a parameter quantifying the amount of violation of the classicality bound as

$$\mathcal{P}_{\text{Lee}} = 1 - \frac{\langle[\Delta(n_A - n_B)]^2\rangle + (\langle n_A\rangle - \langle n_B\rangle)^2}{\langle n_A\rangle + \langle n_B\rangle}. \tag{22}$$

$\mathcal{P}_{\text{SSN}} = 0$ corresponds to the bound of the Lee nonclassicality region, and the Lee nonclassicality condition corresponds to $0 < \mathcal{P}_{\text{Lee}} \leq 1$. For the thermal seeds we obtain

$$\mathcal{P}_{\text{Lee}}^{(T)} = 2\,\frac{N(1+\mu_A+\mu_B) - \mu_A^2 - \mu_B^2 + \mu_A\mu_B}{2N(1+\mu_A+\mu_B) + \mu_A + \mu_B}. \tag{23}$$

Thus, the maximal violation of Lee's criterion ($\mathcal{P}_{\text{Lee}} = 1$) is achieved by the twin-beam ($\mu_A = \mu_B = 0$), and by increasing the magnitude of, at least, one of the seeding field the classicality bound is reached.

For coherent input beams, the Lee parameter is

$$\mathcal{P}_{\text{Lee}}^{(C)} = \frac{2N(1+M_A+M_B) + 4\sqrt{N(N+1)}\sqrt{M_A M_B}\cos(\gamma_A+\gamma_B-\varphi) - (M_A - M_B)^2}{2N(1+M_A+M_B) + 4\sqrt{N(N+1)}\sqrt{M_A M_B}\cos(\gamma_A+\gamma_B-\varphi) + M_A + M_B}. \tag{24}$$

It shows a maximum non-classical violation of Lee criterion ($\mathcal{P}^{(C)}_{\text{Lee}} = 1$) when the intensities of seeds are null. Eventually, in the case of squeezed seeding one obtains

$$\mathcal{P}^{(S)}_{\text{Lee}} = \frac{2N(1+N_A+N_B) - 2(N_A^2+N_B^2) - (N_A-N_B)^2 - N_A - N_B}{2N(1+N_A+N_B)+N_A+N_B}. \tag{25}$$

5 Entanglement

The downconversion process is known to provide pairwise entanglement between A- and B-beams. In the spontaneous process, as well as in the case of coherent and vacuum squeezed seeds, the output state is entangled for any value of $N \neq 0$ whereas in the case of a thermally seeded PDC the degree of entanglement crucially depends on the intensity of the seeds, as shown in [35]. In fact, as ρ_{out} is a Gaussian state (since thermal states are Gaussian and the PDC Hamiltonian is bilinear in the field modes) its entanglement properties can be evaluated by checking the positivity of the partial transpose (PPT condition), which represents a sufficient and necessary condition for separability for Gaussian pairwise mode entanglement [39].

In order to check whether and when the state ρ_{out} is entangled we apply the PPT criteria for Gaussian entanglement [39]. For instance, we apply the positive map $\mathcal{L}_B$ to the state ρ_{out}, $\mathcal{L}_B(\rho_{out})$ being the transposition (complex conjugation) only of the subspace B.

Gaussian states are completely characterized by their covariance matrix. In particular the covariance matrix of ρ_{out} is $\mathbf{V}$, with $\mathrm{V}_{\alpha\beta} = \frac{1}{2}\langle\{\Delta w_\alpha, \Delta w_\beta\}\rangle$, where the vector of operators $\mathbf{w} = (X_A, Y_A, X_B, Y_B)^T$ is built with the position and momentum operators $X_j = \frac{a_j + a_j^\dagger}{\sqrt{2}}$ and $Y_j = \frac{a_j - a_j^\dagger}{i\sqrt{2}}$, $j = A, B$. Thus, the separability properties of ρ_{out} can be obtained from the positivity of $\mathcal{L}_B(\rho_{out})$, which can be expressed in terms of its covariance matrix $\widetilde{\mathbf{V}}$ as

$$\widetilde{\mathbf{V}} + \frac{i}{2}\Omega \geq 0, \tag{26}$$

where $\Omega = \omega \oplus \omega$ and $\omega = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

Simon showed that $\widetilde{\mathbf{V}}$ can be calculated exactly as $\mathbf{V}$ with a sign change in the B momentum variable ($Y_B \to -Y_B$), while the other momentum and position variables remain unchanged [39]. Thus, we obtain

$$\widetilde{\mathbf{V}} = \begin{pmatrix} \mathcal{A}_1 & \mathcal{D} & \mathcal{C}_1 & \mathcal{G}_1 \\ \mathcal{D} & \mathcal{A}_2 & \mathcal{G}_2 & \mathcal{C}_2 \\ \mathcal{C}_1 & \mathcal{G}_2 & \mathcal{B}_1 & \mathcal{F} \\ \mathcal{G}_1 & \mathcal{C}_2 & \mathcal{F} & \mathcal{B}_2 \end{pmatrix}, \tag{27}$$

where in the thermal case

$$\begin{aligned} \mathcal{A}_1 &= \mathcal{A}_2 = 1/2 + \mu_A + N(1+\mu_A+\mu_B), \\ \mathcal{B}_1 &= \mathcal{B}_2 = 1/2 + \mu_B + N(1+\mu_A+\mu_B), \\ \mathcal{C}_1 &= \mathcal{C}_2 = \sqrt{N(N+1)}(1+\mu_A+\mu_B), \\ \mathcal{G}_1 &= \mathcal{G}_2 = \mathcal{D} = \mathcal{F} = 0. \end{aligned} \tag{28}$$

In the case of laser seeds the results are the same as in the case of spontaneous PDC, i.e. the same of (28) with $\mu_A = \mu_B = 0$. In the case of squeezed seeds we have

$$
\begin{aligned}
\mathcal{A}_1 &= 1/2 + N(2 + N_A + N_B) \\
&\quad + \sqrt{N_A(1+N_A)}(1+N) + \sqrt{N_B(1+N_B)}N\cos(2\Delta\varphi), \\
\mathcal{A}_2 &= 1/2 + N(2 + N_A + N_B) - \sqrt{N_A(1+N_A)}(1+N) \\
&\quad - \sqrt{N_B(1+N_B)}N\cos(2\Delta\varphi), \\
\mathcal{B}_1 &= 1/2 + N(2 + N_A + N_B) + \sqrt{N_B(1+N_B)}(1+N) \\
&\quad + \sqrt{N_A(1+N_A)}N\cos(2\Delta\varphi), \\
\mathcal{B}_2 &= 1/2 + N(2 + N_A + N_B) - \sqrt{N_B(1+N_B)}(1+N) \\
&\quad - \sqrt{N_A(1+N_A)}N\cos(2\Delta\varphi), \\
\mathcal{D} &= \sqrt{N_B(1+N_B)}N\sin(2\Delta\varphi), \\
\mathcal{F} &= \sqrt{N_A(1+N_A)}N\sin(2\Delta\varphi), \\
\mathcal{C}_1 &= \left[1 + N_A + N_B + \sqrt{N_A(1+N_A)} + \sqrt{N_B(1+N_B)}\right]\sqrt{N(1+N)}\cos(\Delta\varphi), \\
\mathcal{C}_2 &= \left[-1 - N_A - N_B + \sqrt{N_A(1+N_A)} + \sqrt{N_B(1+N_B)}\right]\sqrt{N(1+N)}\cos(\Delta\varphi), \\
\mathcal{G}_1 &= \left[1 + N_A + N_B + \sqrt{N_A(1+N_A)} - \sqrt{N_B(1+N_B)}\right]\sqrt{N(1+N)}\cos(\Delta\varphi), \\
\mathcal{G}_1 &= \left[1 + N_A + N_B - \sqrt{N_A(1+N_A)} + \sqrt{N_B(1+N_B)}\right]\sqrt{N(1+N)}\cos(\Delta\varphi),
\end{aligned}
\tag{29}
$$

with $\Delta\varphi = \zeta_A/2 + \zeta_B/2 - \varphi$.

The PPT criterion of (26) can be rewritten in terms of the smallest partially transposed symplectic eigenvalue $\widetilde{d}_-$ as $\widetilde{d} \geq 2^{-1}$. This condition is never satisfied in the case of coherent and vacuum squeezed seeds, while in the case of thermal seeds we obtain that

$$
\widetilde{d}_- = \frac{1}{\sqrt{2}}\sqrt{\mathcal{A}_1^2 + \mathcal{B}_1^2 + 2\mathcal{C}_1^2 - \sqrt{(\mathcal{A}_1 + \mathcal{B}_1)^2[(\mathcal{A}_1 - \mathcal{B}_1)^2 + 4\mathcal{C}_1^2]}}, \tag{30}
$$

thus the condition $\widetilde{d} \geq 2^{-1}$ is satisfied when [35]

$$
\mu_A\mu_B - N(1 + \mu_A + \mu_B) \geq 0. \tag{31}
$$

It is noteworthy to observe that the separability/entanglement properties of the state ρ_{out} with thermal seeds can be highlighted by the direct photon counting on A- and B-arms. In fact, with an ideal detection system, the inequality

$$
\langle[\Delta(n_A - n_B)]^2\rangle - (\langle n_A\rangle - \langle n_B\rangle)^2 \leq \langle n_A\rangle + \langle n_B\rangle. \tag{32}
$$

exactly corresponds to (31).

Thus, as in the two previous cases we can define a parameter quantifying the amount of the violation of the separability bound $\mathcal{P}_{Ent}$, only in the case of thermal seeds

$$\mathcal{P}_{Ent} = 1 - \frac{\langle[\Delta(n_A - n_B)]^2\rangle - (\langle n_A\rangle - \langle n_B\rangle)^2}{\langle n_A\rangle + \langle n_B\rangle}. \tag{33}$$

$\mathcal{P}_{Ent} = 0$ corresponds to the boundary between the separability and the entanglement regions, in fact for the state ρ_{out} with thermal seeds we obtain

$$\mathcal{P}_{Ent} = 2\frac{N(1+\mu_A+\mu_B) - \mu_A\mu_B}{2\,N(1+\mu_A+\mu_B) + \mu_A + \mu_B}. \tag{34}$$

According to (34) we observe that ρ_{out} is entangled when $0 < \mathcal{P}_{Ent} \leq 1$, and that the maximal violation of the separability bound (corresponding to $\mathcal{P}_{Ent} = 1$) is achieved by the spontaneous PDC ($\mu_A = \mu_B = 0$), while, if one of the two arms is seeded by the vacuum, irrespective of the magnitude of the thermal seed on the other arm, the state is always entangled.

We underline that the parameter $\mathcal{P}_{Ent}$ cannot be considered an entanglement measure (as it does not have the correct properties) [40]. A quantification of entanglement which can be computed for general two-mode Gaussian states is provided by, e.g., the logarithmic negativity.

6 Discussion and Conclusion

After having introduced the classicality quantifiers in the previous sections, here we compare them directly: Fig. 1 shows the regions of nonclassicality of the three quantities considered, namely, the entanglement, the sub-shot noise, and the Lee's nonclassicality. The space of parameter is given by the value of seed mean number of photons (μ_A, μ_B for the thermal seeds, M_A, M_B for the coherent seeds, N_A, N_B for the squeezed vacuum seeds), and spontaneous PDC mean number of photons N. In particular, in Fig. 1(a) the three regions are plotted in the case of thermal seeds, in Fig. 1(b) and (c) are in the case of coherent seeds with $\cos(\gamma_A + \gamma_B - \varphi) = 1$ and $\cos(\gamma_A + \gamma_B - \varphi) = -1$ respectively, and Fig. 1(d) in the case of squeezed vacuum seeds. Clearly, for the three seed states considered, there is a common hierarchy among the criteria. The strictest is always the Lee's criterion, followed by the sub-shot-noise one and then by the entanglement condition. In fact, (9) and (18) show that Lee's criterion is a sufficient condition to be sub-shot noise. At the same time we note that for coherent and vacuum squeezed seeds the output state is always entangled, while in the thermal case, according to (9) and to (31), being sub-shot noise limited is a sufficient condition for being entangled (Fig. 1(a)).

Furthermore, it is interesting to note that in the thermal case the three conditions coincide when $\mu_A = \mu_B$. Analogously, for the vacuum squeezed fields the SNL limit and the Lee's criterion limit converge when $N_A = N_B$, as in can be observed in Fig. 1(d). The same happens in the case of coherent seeds. In this case it is interesting to note that, as the output state always violate the SNL limit when

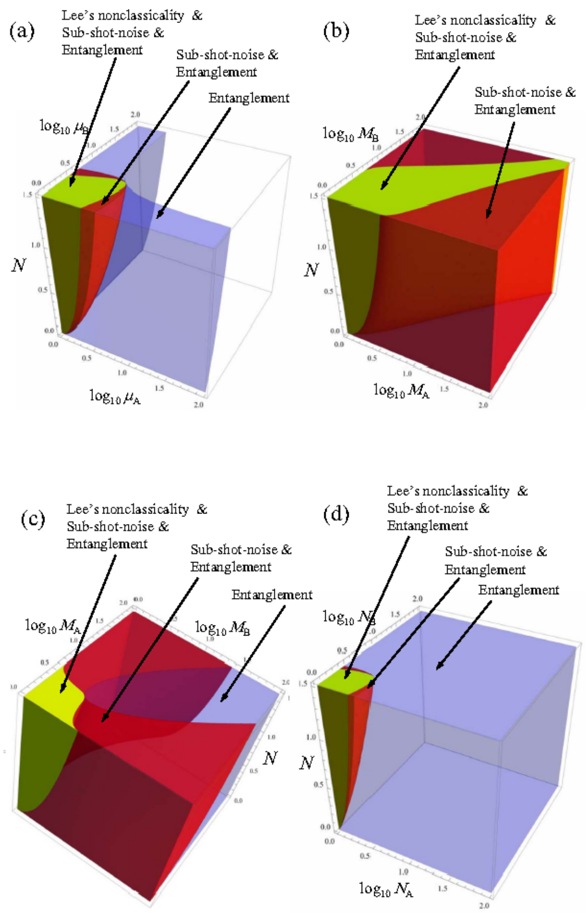

Fig. 1 Regions of nonclassicality (entanglement, SNL violation, Lee's classicality criterion violation) plotted as a function of the mean number of photons of the seeding fields, and of the spontaneous PDC (N) for the different state of the seeding fields considered. In particular (**a**) thermal seeds, (**b**) coherent seeds with $\cos(\gamma_A+\gamma_B-\varphi)=1$, (**c**) coherent seeds with $\cos(\gamma_A+\gamma_B-\varphi)=-1$, (**d**) vacuum squeezed seeds

$\cos(\gamma_A+\gamma_B-\varphi)\geq 0$, when $M_A=M_B$ also the Lee's criterion is always satisfied (see Fig. 1(b)).

It is important to discuss the multimode case because traveling-wave pumped PDC, used in typical quantum optical experiments, is naturally multimode and it is a very hard task to select properly two coupled single modes.

As the quantum correlations induced in the PDC process intrinsically couples modes pairwise, we expect that, in general, no qualitative differences with respect to the single mode case may raise when the multimode case is considered.

Furthermore, the multimode case for thermal seeds has been already discussed in [33], and the calculations for the case of multimode coherent and vacuum squeezed fields can be performed following the same guidelines.

In particular, the sub-shot-noise condition, for all the three possible seeding fields, becomes

$$\sum_q \langle[\Delta(n_{A,q}-n_{B,q})]^2\rangle < \sum_q(\langle n_{A,q}\rangle + \langle n_{B,q}\rangle), \tag{35}$$

where q labels the paired mode. The entanglement condition is always fulfilled by the multimode coherent and vacuum squeezed fields, while the case of the thermal seeds has been already discussed, as mentioned, in [33]. Concerning the Lee's criterion, the extension to the multimode case is not possible, since its derivation is explicitly based on the assumption of a single pair of downconverted modes [36, 37].

In conclusion, we have shown that there is a well defined hierarchy among the considered nonclassicality criteria (entanglement, SNL violation, Lee's classicality criterion violation) when PDC is seeded by simple single mode Gaussian states (such as thermal, coherent and vacuum squeezed states) summarized in Fig. 1. Despite the general beliefs that entanglement is a fragile property of quantum systems here we found evidence that in our particular system entanglement is robust compared to other quantum signatures as for example the sub-shot-noiseness of intensity correlations, which itself is a resource for quantum imaging [41, 42]. Our analysis also poses the question on whether, upon changing the initial conditions, it is possible to generate sub-shot-noise states that are not entangled. The natural extension of this work is the investigation of such a hierarchy in the presence of the most general single mode Gaussian state as seeding fields. Work along these lines is in progress and results will be presented elsewhere [43].

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