ELSEVIER

1 March 1996

Optics Communications 124 (1996) 277-282

OPTICS COMMUNICATIONS

On density matrix reconstruction from measured distributions

Matteo G.A. Paris [1,2]

Arbeitsgruppe Nichtklassiche Strahlung der Max-Planck-Gesellschaft an der Humboldt-Universitat zu Berlin, Rudower Chaussee 5, 12489 Berlin, Germany

Received 15 August 1995; revised version received 4 October 1995; accepted 7 December 1995

Abstract

A method to recover the density matrix of radiation states expressed by a finite superpositions of number states is presented. It starts from realistic, not fully efficient, heterodyne or double homodyne detection. Examples are given by means of numerical simulations.

Keywords: Monte Carlo; Frustration; Phase Diagram
PACS: 42.50.Dv; 03.65.Bz; 42.65.Ky

From the quantum mechanical point of view a single mode radiation field is completely characterized by the density operator $\hat{\rho}$. From the knowledge of density operator one can readily evaluate the probability distribution of any quantity of interest, even if it does not correspond to an observables in strict sense, as it happens for the quantum phase [1]. The problem of recovering $\hat{\rho}$ from measurable quantities is therefore a matter of great interest and recent efforts in this direction are largely justified. Until now the attention has been mostly focused on *Quantum Tomography*, namely the reconstruction of density matrix $\langle\hat{\rho}\rangle$ from a set of homodyne measurements of the field quadratures $\hat{x}_{\varphi} = \frac{1}{2}(a e^{-i\varphi} + a^{\dagger} e^{i\varphi})$. The commutation relation $[a, a^{\dagger}] = 1$ imposes to the quantum phase space of the harmonic oscillator a structure quite different relative to the corresponding one from classical statistical mechanics. There is a unique definition for classical distribution function, whereas for a quantum state $\hat{\rho}$ one can define an entire one-real-parameter of quantum distribution functions $W_s(\alpha, \bar{\alpha})$, the so-called generalized Wigner functions

$$W_s(\alpha, \bar{\alpha}) = \int \frac{d^2\lambda}{\pi} \mathrm{Tr}\{\exp(\lambda a^{\dagger} + \bar{\lambda} a + \tfrac{1}{2} s|\lambda|^2)\} \times \exp(\lambda\bar{\alpha} + \bar{\lambda}\alpha). \qquad (1)$$

The existence of the latter is related to the different ordering of the boson operators a, $a^{\dagger}$ as the statistical average $\langle \bar{\alpha}^k \alpha^l \rangle_s$ over $W_s(\alpha, \bar{\alpha})$ provides the quantum expectation value of the operator product $\{a^{\dagger k} a^l\}_s$ in the s-ordered representation. The most important distributions are obtained for $s = 1$, the Glauber P-function, for $s = 0$, the distribution originally proposed by Wigner and for $s = 1$ the Husimi Q-function which are respectively related to normal, symmetric and antinormal ordering of the boson operators. The possibility of recovering generalized Wigner functions $W_s(\alpha, \bar{\alpha})$ from a *continuous* set of homodyne measurements has been shown theoretically by Vogel and Risken [2],

[1] Permanent address: Dipartimento di Fisica 'Alessandro Volta', Università degli Studi di Pavia, via Bassi 6, I-27100 Pavia, Italia.
[2] E-mail Address: PARIS@PV.INFN.IT

PII S0030-4018(96)00019-3

and later it has been experimentally applied to the reconstruction of Wigner function $W_0(\alpha,\bar{\alpha})$ for coherent and squeezed states [3]. The use of a finite set of φ values had been circumvented by means of a filtering procedure on data borrowed by medical tomography. More recently, a fully quantum mechanical method suitable for statistical sampling has been developed for the reconstruction of the density matrix in the Fock representation, working also in the case of non efficient homodyne detectors [4].

Only little attention was devoted to the reconstruction problem starting from heterodyne or double homodyne detection [5]. This is quite strange as quantum tomography requires the detection of many field quadratures $\hat{x}_\varphi$ and thus a lot of repeated measurements on the state under examination. On the contrary heterodyne detection [6] and double homodyne detection [7] involve the measurement of only two conjugated field quadratures $\hat{x}_\varphi$ and $\hat{x}_{\varphi+\pi/2}$ and their outcomes probability distribution represents a sampling of a Wigner function $W_s(\alpha,\bar{\alpha})$ with $s \leq -1$ [3]. The precise value of the parameter s depends on the quantum efficiency of the involved photodetectors as $s = 1-2\eta^{-1}$ [9]. This lack of interest is probably due to the fact that Wigner distributions $W_s(\alpha,\bar{\alpha})$ are very smoothed functions for $s \leq -1$ and thus they do not seem to provide information on non classical features of the field mode states. This phenomenon is well illustrated in Fig. 1 where I report the Wigner function $W_0(\alpha,\bar{\alpha})$ and the Husimi Q-function $Q(\alpha,\bar{\alpha}) = W_{-1}(\alpha,\bar{\alpha})$ for a single number state $\hat{\rho} = |3\rangle\langle 3|$. The Q-function is positive definite and does not exhibit oscillations, thus it seems that crucial informations are washed out.

The task of this paper is to show that, on the contrary, quantum features in the Wigner distribution $W_s(\alpha,\bar{\alpha})$ for $s \leq -1$ are *not* destroyed, even though they are highly suppressed. I will show that starting from realistic heterodyne or double homodyne detection the *exact* reconstruction of the density matrix in the Fock representation is possible for the relevant class of states expressed by a finite superposition of number states.

[3] The equivalence of heterodyne and heterodyne detection has been shown in Ref. [8]. Throughout the paper any reference to the heterodyne detectors is also valid for the double homodyne detectors.

The ordering expansion of an operator $\hat{O}$ is an important tool in the treatment of optical systems. This arises from the fact that the expectation value $\langle\hat{O}\rangle = \mathrm{Tr}\{\hat{\rho}\hat{O}\}$ can be expressed as a statistical average over $W_s(\alpha,\bar{\alpha})$ of certain non operatorial function $\mathcal{F}_s[\hat{O}](\alpha,\bar{\alpha})$ associated with the s-ordered form of the operator $\hat{O}$

$$\langle\hat{O}\rangle = \int \frac{d^2\alpha}{\pi}\, W_s(\alpha,\bar{\alpha})\, \mathcal{F}_s[\hat{O}](\alpha,\bar{\alpha})\,. \tag{2}$$

For a fixed operator $\hat{O}$ the quantity $\mathcal{F}_s[\hat{O}](\alpha,\bar{\alpha})$ can be both an ordinary function or a tempered distribution [10] depending on the value of s. However, the analytical properties of the Wigner functions assure the integral in Eq. (2) to be well defined for any value of s.

The matrix elements $\rho_{n,m}$ are the expectation values of the generalized projectors $\hat{P}_{n,m} = |n\rangle\langle m|$ and thus their ordering is of interest in the reconstruction problem. The density matrix is Hermitian and thus I will consider only the case $m \geq n$. Using the definition of Fock states and the Louisell expansion of the vacuum [11]

$$|n\rangle = \frac{a^{\dagger n}}{\sqrt{n!}}|0\rangle,$$
$$|0\rangle\langle 0| = \lim_{\varepsilon\to 1^-}\sum_p \frac{(-\varepsilon)^p}{p!} a^{\dagger p}a^p, \tag{3}$$

it is straightforward to obtain the normal ordered form of the projectors

$$\hat{P}_{n,n+k} = \frac{1}{\sqrt{n!(n+k)!}}\lim_{\varepsilon\to 1^-}\sum_p \frac{(-\varepsilon)^p}{p!} a^{\dagger n+p}a^{n+p+k}. \tag{4}$$

An arbitrary t-ordered product $\{a^{\dagger m}a^{m+q}\}_t$ can be expressed in terms of a finite number of s-ordered ones $\{a^{\dagger r}a^{r+q}\}_s$ using the formula [10]

$$\{a^{\dagger m}a^{m+q}\}_t = \sum_{r=0}^{m} \frac{(q+m)!}{(q+r)!}\binom{m}{r}\left(\frac{s-t}{2}\right)^{m-r} \times \{a^{\dagger r}a^{r+q}\}_s, \tag{5}$$

where also s is arbitrary. When t is chosen to be $t=1$ Eq. (5) expresses normal ordered moments in terms of arbitrary ordered ones. Upon inserting Eq. (5) in Eq.

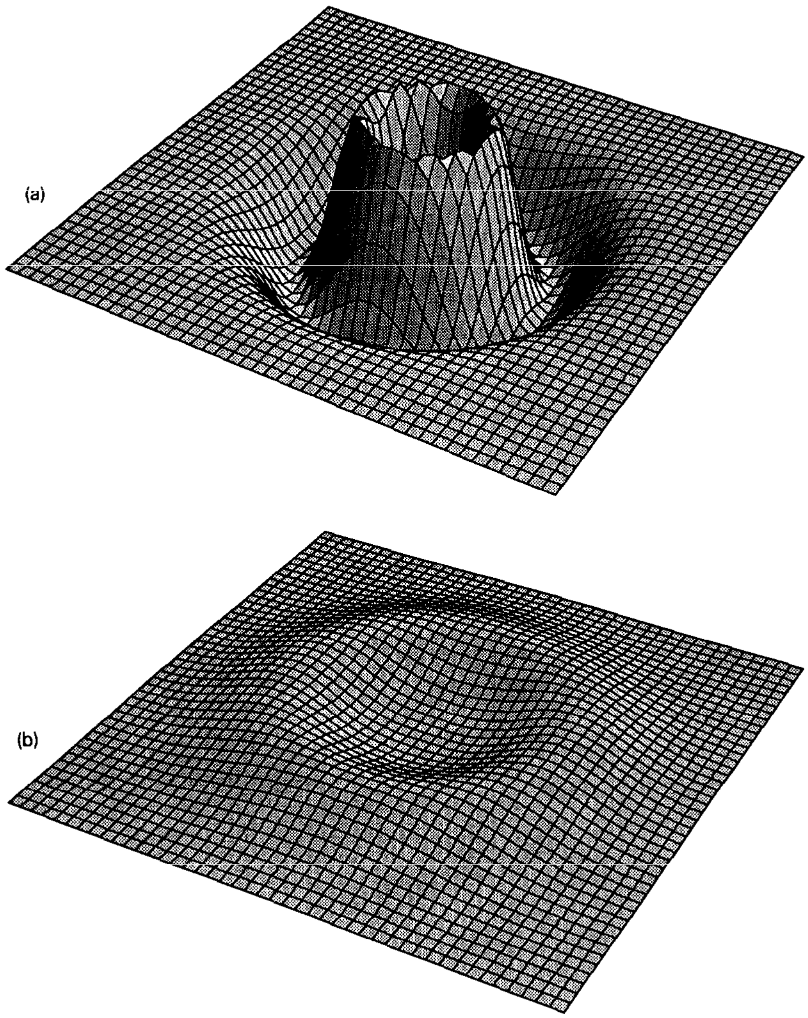

Fig. 1. (a) Wigner function $W_0(\alpha,\bar{\alpha})$ and (b) Husimi function $W_{-1}(\alpha,\bar{\alpha})$ for a single Fock state $\hat{\rho}=|3\rangle\langle 3|$.

(4) one obtains the s-ordered form of the generalized projectors

$$\hat{P}_{n,n+k}=\frac{1}{\sqrt{n!(n+k)!}}\times\lim_{\varepsilon\to 1^-}\sum_{p=0}^{\infty}\frac{(-\varepsilon)^p}{p!}\sum_{r=0}^{p+n}\frac{(p+n+k)!}{(r+k)!}\times\binom{p+n}{r}\left(\frac{s-t}{2}\right)^{p+n-r}\{a^{\dagger r}a^{r+k}\}_s. \qquad (6)$$

The function $\mathcal{F}_s[\hat{P}_{n,n+k}](\alpha,\bar{\alpha})$ is thus expressed by

$$\mathcal{F}_s[\hat{P}_{n,n+k}](\alpha,\bar{\alpha})=\frac{(-)^n}{\sqrt{n!(n+k)!}}\times\lim_{\varepsilon\to 1^-}\frac{\partial^n}{\partial\varepsilon^n}\sum_{p=0}^{\infty}\left(\frac{\varepsilon(1-s)}{2}\right)^p\alpha^k L_p^k\left(\frac{2|\alpha|^2}{1-s}\right), \qquad (7)$$

where $L_p^k(x)$ denotes Laguerre polynomials. Eq. (7) shows the singularity of $\mathcal{F}_s[\hat{P}_{n,n+k}](\alpha,\bar{\alpha})$, which belong to the class of tempered distributions. In the case of $s=-1$, namely for average over Q-function, Eq. (7) is the series representation of the explicit expression

$$\mathcal{F}_s[\hat{P}_{n,n+k}](\alpha,\bar{\alpha}) = \frac{1}{\sqrt{n!(n+k)!}} \mathrm{e}^{|\alpha|^2} \frac{\partial^{2n+k}}{\partial\alpha^{n+k}\partial\bar{\alpha}^n}\delta(\alpha,\bar{\alpha}), \tag{8}$$

where $\delta(\alpha,\bar{\alpha})$ denotes the Dirac distribution in the complex plane. When one has at disposal the *analytical* expression of the considered Wigner distribution function $W_s(\alpha,\bar{\alpha})$ the singularity of $\mathcal{F}_s[\hat{P}_{n,n+k}](\alpha,\bar{\alpha})$ does not prevent the evaluation of $\rho_{n+k,n} = \langle\hat{P}_{n,n+k}\rangle$. However, we are interested in the density matrix reconstruction starting from *measured* Wigner distribution function, namely by statistical sampling of Eq. (2)

$$\int \frac{\mathrm{d}^2\alpha}{\pi} W_s(\alpha,\bar{\alpha})\mathcal{F}_s[\hat{P}_{n,n+k}](\alpha,\bar{\alpha}) \longrightarrow \sum_{j\in\mathrm{data}} W_s(\alpha_j,\bar{\alpha}_j)\mathcal{F}_s[\hat{P}_{n,n+k}](\alpha_j,\bar{\alpha}_j), \tag{9}$$

which cannot be accomplished for $\mathcal{F}_s[\hat{P}_{n,n+k}](\alpha,\bar{\alpha})$ in Eq. (7). Some filtering or smoothing procedure is thus needed on experimental data and this biases the reliability of the reconstruction, as the details of the distribution are crucial and any smoothing unavoidably introduces some *a priori* hypothesis on the measured state.

However, Eq. (2) is far from being a purely formal tool if we restrict our attention on special classes of states.

The exact reconstruction of the entire density matrix (in the Fock representation) is, in fact, possible for states with a finite number of moments $\langle a^{\dagger m}a^n\rangle$ different from zero, in particular for finite superpositions of number states

$$|\psi\rangle = \sum_{n=0}^{M} \psi_n|n\rangle. \tag{10}$$

The latter are extensively investigated in the framework of cavity electrodynamics. They can be produced in different ways in a high-Q cavity, substantially by a careful control on the resonant interaction of the field with the injected two-level atoms [12]. Very recently, it has been also suggested that special nonlinear interactions lead to the generation of Fock states and their superposition [13]. For these states the problem of recovering the density matrix can be solved exactly, leading to a reconstruction formula suitable for statistical sampling. In fact, if the moments $\langle a^{\dagger m}a^n\rangle$ vanish for n or m beyond a certain value the sum over p in Eqs. (6) and (7) is truncated *by definition*. Derivatives and limiting procedures can now be readily carried out leading to the reconstruction formula

$$\rho_{n+k,n} = \left(\frac{s-1}{2}\right)^n \sqrt{\frac{n!}{(n+k)!}} \int \frac{d^2\alpha}{\pi} W_s(\alpha,\bar{\alpha})\alpha^k \times \sum_{p=0}^{N-n-k} \binom{p+n}{p} \left(\frac{1-s}{2}\right)^p L^k_{p+n}\left(\frac{2|\alpha|^2}{1-s}\right). \tag{11}$$

I would stress the fact that Eq. (11) is *exact* as the truncation in Eqs. (6) and (7) is not the result of any approximation but it comes just from considering the finite superposition of number states (10).

The value of N in Eq. (11) has to be chosen large enough to ensure the cancellation of any moment $a^{\dagger N+j}a^{N+i}$, $i,j = 0,1,\ldots$. In practice one can start with a large value of N and then optimize it by means of some stability criterion. I will show in the following the effectiveness of this procedure.

A complete check of the present reconstruction method can be performed by means of numerical simulation of realistic heterodyne (double homodyne) detection. I consider the state

$$|\psi\rangle = \frac{1}{\sqrt{6}}\left[|0\rangle + (2+i)|2\rangle\right], \tag{12}$$

and apply Eq. (11) with a sample of 10^6 experimental (simulated) heterodyne data. The results for unit quantum efficiency $\eta = 1.0$ and for $\eta = 0.8$ are reported in Tables 1 and 2. The reliability of the method is apparent also for non unit quantum efficiency. Confidence intervals are evaluated as usual [14], by dividing the entire sample in subensembles and then calculating the rms deviation relative to the global average. Obviously to obtain the same confidence level with a smaller value of η a larger sample is needed. Tables 1 and 2 have been obtained fixing $N = 2$ in Eq. (11) but, as it has been mentioned above, this information is not needed by the algorithm. In Fig. 2, I report the simulated determination of the matrix elements $\rho_{0,0}$ for the state (12) carried out with different values of N. It is apparent that beyond the critical value $N = 2$ the reconstruction is insensitive to the value of N which slightly affects only the confidence interval. It

Table 1
Reconstructed density matrix for the superposition state in Eq. (12). Simulated experiment with unit quantum efficiency of the photodetectors and a sample of 10^6 data. Theoretical values are given in parentheses.

0.1623 ± 0.0089	$-0.0009 + i0.0033 \pm 0.0085 + i0.0076$	$0.3335 - i0.1630 \pm 0.0050 + i0.0051$	...
(0.1666)	$(0.0 + i0.0)$	$(0.3333 - i0.1666)$	
$-0.0009 - i0.0033 \pm 0.0085 + i0.0076$	0.0090 ± 0.0208	$0.0005 - i0.0035 \pm 0.0074 + i0.0068$	...
$(0.0 + i0.0)$	$(0.0 + i0.0)$	$(0.0 + i0.0)$	
$0.3335 + i0.1630 \pm 0.0050 + i0.0051$	$0.0005 + i0.0035 \pm 0.0074 + i0.0068$	0.8286 ± 0.0123	...
$(0.3333 + i0.1666)$	$(0.0 + i0.0)$	(0.8333)	
...	...	...	0.0089 ± 0.00172

Table 2
Reconstructed density matrix for the superposition state in Eq. (12). Simulated experiment with quantum efficiency of the photodetectors equal to $\eta = 0.8$ and a sample of 10^6 data. Theoretical values are given in parentheses.

0.1823 ± 0.0360	$-0.0601 - i0.0373 \pm 0.1104 + i0.1001$	$0.3327 - i0.1758 \pm 0.0566 + i0.0571$	...
(0.1666)	$(0.0 + i0.0)$	$(0.3333 - i0.1666)$	
$-0.0601 + i0.0373 \pm 0.1104 + i0.1001$	0.0814 ± 0.1061	$0.0597 - i0.0308 \pm 0.0922 + i0.0842$	...
$(0.0 + i0.0)$	$(0.0 + i0.0)$	$(0.0 + i0.0)$	
$0.3327 + i0.1758 \pm 0.0566 + i0.0571$	$0.0597 + i0.0308 \pm 0.0922 + i0.0842$	0.8790 ± 0.0752	...
$(0.3333 + i0.1666)$	$(0.0 + i0.0)$	(0.8333)	
...	...	...	$0.0989 \pm 0.00.1077$

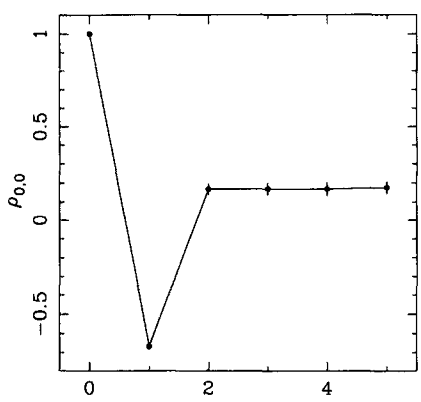

Fig. 2. Determination of the vacuum matrix elements $\rho_{0,0}$ for the superposition state in Eq. (12) with quantum efficiency $\eta = 0.7$ and different value of the parameter N.

is worth noting also the ability of the algorithm in identifying the vanishing matrix elements, as it could be of special interest to know whether or not a Fock component is present in the superposition.

In conclusion, a method for recovering the density matrix in the Fock representation from realistic, not fully efficient, heterodyne or double homodyne measurements has been presented. It works exactly for the relevant class of states expressed as a finite superposition of number states. The reconstruction algorithm is, in this case, very effective and statistically reliable. The application in the general case requires some filtering on experimental data and thus could be useful only when partial information on the measured state is at disposal in advance.

Acknowledgements

The author would thank Professor G.M. D'Ariano and Dr. C. Macchiavello for valuable suggestions and Dr. Valentina De Renzi and Prof. Harry Paul for useful discussions. This work was supported by 'Angelo Della Riccia Foundation'.

References

[1] About the quantum phase problem see, for example, the recent review R. Lynch, Phys. Rep. 256 (1995) 367.
[2] K. Vogel and H.Risken, Phys. Rev. A 40 (1989) 2847.
[3] D.T. Smithey, M. Beck, M.G. Raymer and A. Faridani, Phys. Rev. Lett. 70 (1993) 1244.
[4] G.M. D'Ariano, C. Macchiavello and M.G.A. Paris, Phys. Rev. A 50 4298 (1994);
G.M. D'Ariano, U. Leonhardt and H. Paul, Phys. Rev. A 52 (1995) R1801.

[5] H. Kühn, D.G. Welsch and W. Vogel, J. Mod. Optics 41 (1994) 1607.
[6] J.H. Shapiro and S.S. Wagner, IEEE J. Quantum Electron. QE-20 (1984) 803;
H.P. Yuen and J.H. Shapiro, IEEE Trans. Inform. Theory IT26 (1980) 78.
[7] N.G. Walker, J. Mod. Optics 34 (1987) 15;
J. Noh, A. Fougeres and L. Mandel, Phys. Rev. Lett. 67 (1991) 1426; Phys. Rev. A 45 (1992) 424.
[8] G.M. D'Ariano and M.G.A. Paris, Phys. Rev. A49 (1994) 3022.
[9] U. Leonhardt and H. Paul, Phys. Rev. 48 (1993) 4598;
G.M. D'Ariano, C. Macchiavello and M.G.A. Paris, Phys. Lett. A 198 (1994) 286.
[10] K.E. Cahill and R.J. Glauber, Phys. Rev. 177 (1969) 1857, 1882.
[11] W.H. Louisell, Quantum Statistical properties of Radiation (Wiley, 1973).
[12] M.J. Holland, D. Walls and P. Zoller, Phys. Rev. Lett. 67 (1991) 1716;
K. Vogel, V.M. Akulin and W.P. Schleich, Phys. Rev. Lett. 71 (1993) 1816;
A.S. Parkins, P. Marte, P. Zoller and H.J. Kimble, Phys. Rev. Lett. 71 (1993) 3095.
[13] S.Ya. Kilin and D.B. Horoshko, Phys. Rev. Lett. 74 (1995) 5206.
[14] C.W. Gardiner, Handbook of Stochastic Methods (Springer, Berlin, 1990).