

## Discording Power of Quantum Evolutions

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(Received 27 August 2012; published 2 January 2013)

We introduce the discording power of a unitary transformation, which assesses its capability to produce quantum discord, and analyze in detail the generation of discord by relevant classes of two-qubit gates. Our measure is based on the Cartan decomposition of two-qubit unitaries and on evaluating the maximum discord achievable by a unitary upon acting on classical-classical states at fixed purity. We find that there exist gates which are perfect discorders for any value of purity  $\mu$ , and that they belong to a class of operators that includes the  $\sqrt{\text{SWAP}}$ . Other gates, even those universal for quantum computation, do not possess the same property: the CNOT, for example, is a perfect disorder only for states with low or unit purity, but not for intermediate values. The discording power of a two-qubit unitary also provides a generalization of the corresponding measure defined for entanglement to any value of the purity.

DOI: 10.1103/PhysRevLett.110.010501

PACS numbers: 03.67.Mn, 03.65.Ud

The primary aim of the science of quantum information is the exploitation of the quantum structure of nature for information processing and communication tasks. Among the quantum features of a physical system, entanglement is usually considered the prominent resource, providing speed-up in various quantum information and communication tasks [1]. In the realm of mixed-state quantum information, however, instances are known where quantum advantages are obtained in the presence of little or no entanglement. In fact, quantum discord [2,3] has been proposed as the source behind this enhancement [4], and some indications in this direction have been given [5–8]. The notion of nonclassicality springing from information theory has been also discussed in comparison with that coming from phase-space constraints [9].

Although introduced in the context of environment induced decoherence, quantum discord has been then related to the performance of quantum and classical Maxwell's demons [10] and to the thermodynamic efficiency of a photo-Carnot engine [11], as well as the total entanglement consumption [12], and the minimum possible increase of quantum communication cost [13] in state merging protocols. Furthermore, its propagation properties have been studied in Ref. [14] in connection with microcausality and in Ref. [15] for quantum spin channels.

Quantum discord can be activated into distillable entanglement [16,17], and has been shown to be a resource in quantum state discrimination [18] and quantum locking [19]. It has been also shown [20,21] that discord quantifies the minimum damage made by a decoherence process to the performance of many quantum information processing protocols, including teleportation, distillation and dense coding.

In continuous variable systems, Gaussian quantum discord [22,23] has been experimentally measured [24–26] and represents a measure of the advantage provided by coherent quantum systems over their classical counterpart. It has been also suggested [27] that the geometric quantum discord [28] is the optimal resource for remote quantum state preparation, though the interpretation of geometric discord as a measure of correlations has been questioned [29].

This body of recent knowledge represents a strong motivation to understand in quantitative terms how well quantum discord may be produced by a given operation. To this aim we focus on two-qubit systems and introduce the discording power of (nonlocal) unitary gates, a quantity which allows us to investigate in detail the controlled production of symmetric discord. In particular, the main question we want to answer is the following: which is the maximum discord that a gate may produce acting on classical-classical states [30], i.e., states with zero discord?

In order to answer this question, we define below the discording power  $\text{DP}_\mu[U]$  of a given two-qubit gate  $U$ , as the maximum amount of (symmetric) discord produced by  $U$  when acting on the set of classical-classical states  $\rho_{cc}^\mu$ , i.e., the set of states with zero discord and fixed purity  $\mu = \text{Tr}[(\rho_{cc}^\mu)^2]$ . The set of two-qubit unitaries is thus naturally split into equivalence classes, each individuated by the discording power, represented by curves in the purity-discord plane. Notice that the discording power provides a generalization of the so-called entangling power of gate  $U$ , [31,32], which is obtained as  $\text{DP}_{\mu=1}$  since, for pure states, quantum discord coincides with entanglement.

Thanks to the fact that maximally discordant mixed states (MDMS) for a given purity have been identified in

Ref. [33] (see also Ref. [34]), we are able to find analytically the class of gates that are perfect, i.e., the *best discorder*, for a given purity. These are defined as those unitaries that produce a MDMS by acting upon some  $\rho_{cc}^\mu$ . In this way, one also sees that the notion of best discorder is the generalization to any value of the purity of the perfect entangler discussed in Ref. [35]. As a first result, we found that there exist gates which are perfect discorders for any value of purity  $\mu$ . They pertain to a class of operators that includes the  $\sqrt{\text{SWAP}}$ . This is by no means a trivial property, since many gates do not hold it, not even all of the two-qubit gates that are universal for quantum computation. For example, the CNOT gate (as well as the other unitaries to which it is equivalent in the sense specified below) is a perfect discorder for very high or low purity, but not for intermediate values. More specifically, CNOT works as a perfect discorder for rank-4 and rank-3 states, and it is a perfect entangler as well (that is, a perfect discorder) for rank-1 states. However, it fails to achieve the rank-2 MDMS, that are, instead, obtained after the action of  $\sqrt{\text{SWAP}}$  on a suitable rank-2  $\rho_{cc}$  [36].

To proceed with the formal introduction of our figure of merit, we first recall that the discord can be understood as the difference between the mutual information and the classical correlations, [3]. For a generic bipartite state  $\rho_{AB}$ , the mutual information is given by  $I[\rho_{AB}] = S[\rho_A] + S[\rho_B] - S[\rho_{AB}]$ , where  $\rho_A$  and  $\rho_B$  are the local reduced density operators and  $S[\rho]$  the von Neumann entropy of the state  $\rho$ . As for the classical correlation, we suppose that a POVM-measurement  $\mathcal{M}_A$  with elements  $\{M_k\}$  (with  $M_k \geq 0$  and  $\sum_k M_k = \mathbb{1}$ ) is performed on qubit A. It realizes for B the postmeasurement ensemble  $\mathcal{E}_B = \{p_k, \rho_B^k\}$ , where  $p_k = \text{Tr}\{\rho_{AB}(M_k \otimes \mathbb{1})\}$  is the probability for the  $k$ th outcome, while the  $k$ th postmeasurement state of B is  $\rho_B^k = \text{Tr}_A\{\rho_{AB}(M_k \otimes \mathbb{1})\}/p_k$ . The amount of information acquired about qubit B, optimized over all possible POVM, gives the classical correlations  $C^{AB}[\rho_{AB}] = S[\rho_B] - \min_{\mathcal{M}_B} \sum_k p_k S[\rho_B^k]$ . The (one-way) quantum discord is then

$$\delta^{AB}[\rho_{AB}] := I[\rho_{AB}] - C^{AB}[\rho_{AB}]. \quad (1)$$

An analogous procedure leads to the definition of discord with measurement on B,  $\delta^{BA}$ .

Since the conditional entropy is concave over the convex set of POVMs, the minimum is attained on the extreme points of the set, having rank-1 [37,38]. Discord is typically evaluated in a simplified form, where only orthogonal measurements are considered, rather than the more general POVM. For two-qubit states, this is enough to achieve the minimum for rank-2 states, while, for rank-3 and rank-4 states orthogonal measurements give a pretty tight upper bound, as shown in Ref. [39]. Given the numerical evidence provided there, the improvement in doing full minimization is on average at the level of  $10^{-6}$ . Therefore, in the following, we will restrict the evaluation of discord to projective measurements.

We define the *discording power* of a gate  $U$  as the maximum symmetrized quantum discord that can be achieved by such a gate from *any* classical-classical state of a given purity  $\mu$  ( $1/4 \leq \mu \leq 1$ ):

$$\text{DP}_\mu[U] \equiv \max_{\rho_{cc}^\mu} \delta[U\rho_{cc}^\mu U^\dagger], \quad (2)$$

Here,  $\delta$  is the symmetrized discord,

$$\delta[U\rho_{cc}^\mu U^\dagger] = \frac{1}{2}(\delta^{AB}[U\rho_{cc}^\mu U^\dagger] + \delta^{BA}[U\rho_{cc}^\mu U^\dagger]),$$

while a (concordant) classical-classical state  $\rho_{cc}^\mu$  corresponds to a convex combination of orthogonal projectors:

$$\rho_{cc}^\mu = \sum_{r,s} p_{r,s} |\alpha_r\rangle\langle\alpha_r| \otimes |\beta_s\rangle\langle\beta_s|, \quad \text{with} \quad \sum_{r,s} p_{r,s}^2 = \mu, \quad (3)$$

where  $|\alpha_r\rangle, |\beta_s\rangle$  are elements of orthonormal basis for the two qubits, and the  $\{p_{r,s}\}$  are probability distributions.

Any two-qubit unitary may be written in Cartan form [31]  $U = (L_1 \otimes L_2)U_c(\tilde{\theta})(L_3 \otimes L_4)$  with

$$U_c(\tilde{\theta}) = \exp\left(-i \sum_{j=x,y,z} \theta_j \sigma_j \otimes \sigma_j\right), \quad (4)$$

where  $L_i$  are local unitaries, and where (due to symmetry reasons) the independent values of the  $\theta_j$  are constrained by  $0 \leq \theta_z \leq \theta_y \leq \theta_x \leq \pi/4$  [40].

At first, we notice that  $L_1$  and  $L_2$  are not affecting the value of  $\text{DP}_\mu[U]$  since the discord is left unchanged by local unitary operations. The local unitaries  $L_3$  and  $L_4$  are irrelevant as well because  $\text{DP}_\mu[U]$  is the result of a maximization over classical-classical states at fixed purity. This involves a scan over the distributions  $p_{r,s}$  and the orthonormal qubit basis in Eq. (3), and since the latter are just rotations of the logical basis  $\{|0\rangle, |1\rangle\}$ , we can remove  $L_{3,4}$  by combining them with the rotations needed to check all local basis. Overall, we have that all of the gates with a given Cartan kernel  $U_c$  have the same discording power, which, itself, can be intended as a function of the three parameters  $\theta_j$ .

Every two-qubit gate can be associated to a curve in the discord-purity plane, given by  $\text{DP}_\mu[U_c]$ . Examples may be seen in Fig. 1, where we report the symmetrized discord  $\delta$  as function of purity, and in turn the discording power, for several Cartan kernels of the form  $U_c(\alpha, 0, 0)$  and  $U_c(\alpha, \alpha, 0)$  (chosen here because of their relevance for the discussion below). The discording power of these kernels is also plotted in Fig. 2 for a fixed value of the purity,  $\mu = 0.7$  as a function of the angle  $\alpha$ . We can see from these two plots that the gates  $U_c(\pi/4, 0, 0)$  and  $U_c(\pi/8, \pi/8, 0)$  have better performances in generating discord. The first of them is the kernel of the CNOT gate, and, from Fig. 1, it can be appreciated to be a perfect discorder for rank-4 and rank-3 states. The second one

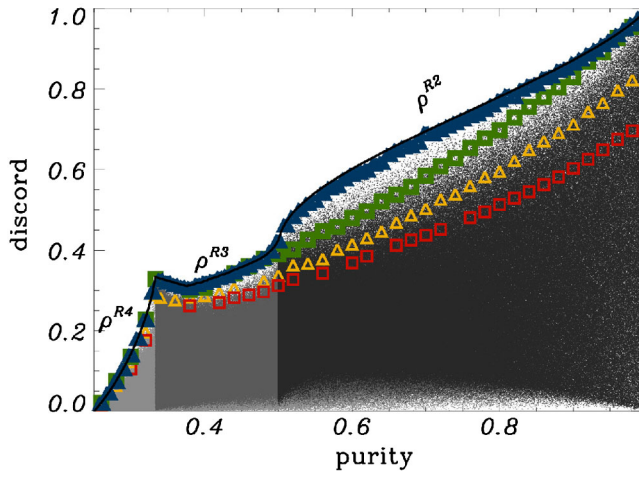


FIG. 1 (color online). Symmetric quantum discord ( $\delta$ ) versus purity ( $\mu$ ) for two-qubit unitary kernels. The maximum discord for a given purity (black continuous line) is obtained for  $\rho^{R2}$ ,  $\rho^{R3}$ ,  $\rho^{R4}$  (see main text). The (numerically evaluated)  $DP_\mu$  for different gates is represented with symbols: from the top,  $U_c(\pi/8, \pi/8, 0)$  (bold triangles),  $U_c(\pi/4, 0, 0)$  (bold squares),  $U_c(\pi/4, \pi/4, 0)$  (triangles), and  $U_c(0.15\pi, 0, 0)$  (squares). The numerical maximization for each gate and for any value of  $\mu$  is performed by considering  $\sim 8 \times 10^6$  classical states, with local rotation angles discretizing  $|\alpha_r\rangle$  and  $|\beta_s\rangle$  by steps of  $0.1\pi$  and  $\sim 5 \times 10^2$  different values for  $p_{r,s}$ . The agreement with the analytic result can be appreciated by considering the two overlapping upper curves, corresponding to the border states and  $DP_\mu[U_c(\pi/8, \pi/8, 0)]$ . This serves the purpose of a consistency check. On the background, discord and purity for layers of  $10^8$  random density matrices of rank-2 (dark points), -3 (intermediate grey) and -4 (lighter grey) are superimposed.

(in fact, the whole class  $U_c(\pi/8, \pi/8, \gamma)$ ,  $\forall \gamma$ ) is a best discorder too; but, this time, for the whole range of purities and all ranks. In particular, the operator corresponding to  $\vec{\theta} = (\pi/8, \pi/8, \pi/8)$  gives the  $\sqrt{\text{SWAP}}$  up to an irrelevant phase.

In order to discuss in more detail the properties that a unitary should have to be a perfect discorder, we need to

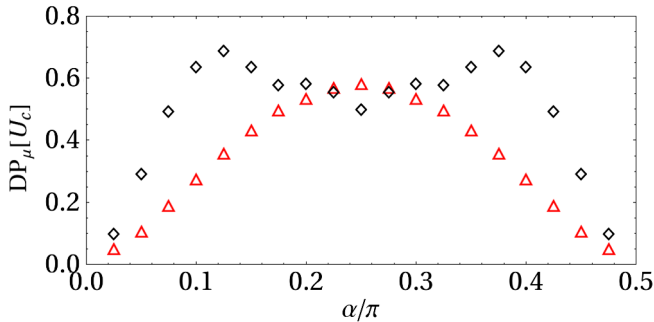


FIG. 2 (color online). Discording power of two families of gates:  $DP_\mu[U_c(\alpha, \beta = \alpha, 0)]$  (diamonds symbols) and  $DP_\mu[U_c(\alpha, 0, 0)]$  (triangle symbols) for a fixed purity  $\mu = 0.7$  as a function of the angle parameterizing the gates.

recall the expressions for the MDMS, separately for each rank. To this end, in Fig. 1, the region of physically admissible states in the  $\mu$ - $\delta$  plane is indicated by the background area. The boundary of this region has been identified in Ref. [33] and is given by states, indicated as  $\rho^{Rn}$ , with  $n = 2, 3, 4$ , that are either symmetric under the exchange of  $A$  and  $B$  or related to their symmetric counterpart by a local rotation. This means that both  $\delta^{AB}$  and  $\delta^{BA}$  (as well as the symmetric discord  $\delta$  that we have chosen to employ) are maximized on this border.

The boundary is quite composed: For low purities, only rank-4 states are present and the maximum discord for a given purity is obtained by Werner states,  $\rho^{R4} \equiv \rho(w) = \frac{1-w}{4} \mathbb{1} + w|\Psi_{\text{me}}\rangle\langle\Psi_{\text{me}}|$ . Here,  $|\Psi_{\text{me}}\rangle$  can be any maximally entangled state, while the Werner parameter, that in principle lays in the range  $[-1/3, 1]$ , is here confined to negative values  $w = -\sqrt{(4\mu - 1)/3}$ , with the purity restricted to  $1/4 \leq \mu \leq 1/3$ . Interestingly, for these values of the purity, the MDMS are separable states, being this lack of entanglement a feature also found when maximizing discord for a given classical correlation [34]. On the other hand, rank-3 states maximizing discord for a given purity are entangled. They are obtained within the family  $\rho^{R3} \equiv \rho(a, b, \varphi)$  [33], where

$$\rho(a, b, \varphi) = \frac{1}{2} \begin{pmatrix} 1 - a + b & 0 & 0 & 0 \\ 0 & a & ae^{-i\varphi} & 0 \\ 0 & ae^{i\varphi} & a & 0 \\ 0 & 0 & 0 & 1 - a - b \end{pmatrix},$$

for a proper choice of  $a$  and  $b$ , with  $a \in [0, 1]$  and  $|b| \leq 1 - a$  (notice that neither the discord nor the purity depend on  $\varphi$ , as it can be canceled by a rotation of qubit  $A$  around the  $z$  axis; we inserted this phase here for future reference). Finally, rank-2 MDMS are obtained from  $\rho^{R3}$  by taking  $b = 1 - a$  and  $1/2 \leq a \leq 1$ , i.e.,  $\rho^{R2} = \rho(a, 1 - a, \varphi)$ . As noted in Ref. [41], these states are quite peculiar as their discord is equal to their concurrence,  $\delta(\rho^{R2}) = \mathcal{C}(\rho^{R2}) = a$ . Both the Werner states and the rank-3 states  $\rho^{R3}$  can be obtained from a classical state under the action of both the CNOT and the  $\sqrt{\text{SWAP}}$ . To show this explicitly, let us start by considering the classical-classical state with the same eigenvalues as the Werner one:  $\rho_{cl}^{R4} = \text{diag}\{(1 - w)/4, (1 - w)/4, (1 - w)/4, (1 + 3w)/4\}$ . It is then easy to see that

$$U_c(\pi/4, 0, 0)\rho_{cl}^{R4}U_c^\dagger(\pi/4, 0, 0) = \frac{1 - w}{4} \mathbb{1} + w|\Phi\rangle\langle\Phi|,$$

which is a Werner state since  $|\Phi\rangle = (|00\rangle + i|11\rangle)/\sqrt{2}$  is maximally entangled. This explains the first part of the plot in Fig. 1 in which the bold squares [representing  $U_c(\pi/4, 0, 0)$ ] fall on the MDMS border. The same is true for any operator with a kernel of the form  $U_c(\pi/8, \pi/8, \gamma)$  as, by considering a rotated classical state  $\rho_{cl}^{R4'} = (\sigma^x \otimes \mathbb{1})\rho_{cl}^{R4}(\sigma^x \otimes \mathbb{1})$ , one has that  $U_c(\pi/8, \pi/8, \gamma)\rho_{cl}^{R4'}U_c^\dagger(\pi/8, \pi/8, \gamma)$  gives a

Werner state with  $|\Psi_{\text{me}}\rangle = (|01\rangle + i|10\rangle)/\sqrt{2}$ . It can be seen by looking at bold triangles in Fig. 1, that this operator is a perfect discorder for any rank. In fact, one has

$$U_c(\pi/8, \pi/8, \gamma) \rho_{cl}^{R3} U_c^\dagger(\pi/8, \pi/8, \gamma) = \rho(a, b, \varphi = \pi/2),$$

where  $\rho_{cl}^{R3} = \text{diag}\{(1-a+b)/2, 0, a, (1-a-b)/2\}$ . This shows that, for any value of  $\gamma$ , the gate  $U_c(\pi/8, \pi/8, \gamma)$  is able to reach the MDMS border of both rank-3 and rank-2, thus showing that this is a perfect discorder for every value of the purity.

On the other hand, the CNOT gate (and all of the operators with the same Cartan kernel), although able to reach the rank-3 MDMS, fails for rank-2 states. This impossibility can be shown analytically by considering its reverse action on  $\rho^{R2}$ : even by allowing local rotations,  $U_c(\pi/4, 0, 0)^\dagger$  applied on  $\rho^{R2}$  gives an entangled (and, thus, nonclassical) state. This clearly implies that the action of  $U_c(\pi/4, 0, 0)$  is not enough to obtain the rank-2 MDMS acting on classical states. We also notice that one of the main differences between discord and entanglement is that the former can be increased by local (nonunitary) operations. On the other hand, MDMS of rank-2 and rank-3 are entangled and thus cannot be created by local operations on classical-classical states. As a consequence, full discording power can only be achieved by nonlocal, two-qubit gates. The case of rank-4 states remains an open question though.

In summary, we have introduced the discording power of a unitary transformation, which measures its capability to produce quantum discord, and have analyzed in detail the generation of symmetric discord by relevant classes of two-qubit gates. Our measure is based on the Cartan decomposition of two-qubit unitaries and on evaluating the maximum discord achievable from classical-classical states at fixed purity. We have identified the gates that are able to generate MDMS, and discussed their performance as a function of the purity. We found that there exist gates which are perfect discorders for any value of purity, and that they belong to a class of operators that includes the  $\sqrt{\text{SWAP}}$ . Even gates that are universal for quantum computation do not share this property; e.g., the CNOT is a perfect discorder for rank-4 and rank-3 states, but it fails to achieve the rank-2 MDMS.

The discording power of a two-qubit unitary provides a generalization of the corresponding measure defined for entanglement on pure states. Our results represent the first attempt to evaluate the correlating power of gates for mixed states, a topic which is largely unexplored even for entanglement [42], and completely absent for other measures not related to the separability paradigm. We also notice that similarly to what is known for entanglement quantifiers, geometric discord and quantum discord yield different states ordering in the mixed setting [43,44],

thus probably inducing inequivalent discording powers, let alone other less related nonclassicality measures, [8]. Our work thus also motivates the search of an entangling power measure starting from mixed states, in order to compare entanglers and discorders at a general value of purity.

The use of quantum discord as a resource for quantum technology is still a debated topic, and a definitive answer may only come from experiments involving carefully tailored states and operations. Our results go in that direction providing a precise characterization of two-qubit unitaries in terms of their ability in generating quantum discord.

This work has been supported by the visiting professorship program of UIB, the CSIC postdoctoral JAE program, the MIUR Project No. FIRB LiCHIS-RBFR10YQ3H, the MICINN Project No. FIS2007-60327, and the MINECO Project No. FIS2011-23526.

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