Journal of Physics A: Mathematical and Theoretical

PAPER • OPEN ACCESS

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To cite this article: Jiayu He and Matteo G A Paris 2025 *J. Phys. A: Math. Theor.* **58** 325301

View the article online for updates and enhancements.

OPEN ACCESS
IOP Publishing
Journal of Physics A: Mathematical and Theoretical
J. Phys. A: Math. Theor. **58** (2025) 325301 (18pp) https://doi.org/10.1088/1751-8121/adf585

Scrambling for precision: optimizing multiparameter qubit estimation in the face of sloppiness and incompatibility

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Received 13 March 2025; revised 6 July 2025
Accepted for publication 29 July 2025
Published 7 August 2025

CrossMark

Abstract
Multiparameter quantum estimation theory plays a crucial role in advancing quantum metrology. Recent studies focused on fundamental challenges such as enhancing precision in the presence of incompatibility or sloppiness, yet the relationship between these features remains poorly understood. In this work, we explore the connection between sloppiness and incompatibility by introducing an adjustable scrambling operation for parameter encoding. Using a minimal yet versatile two-parameter qubit model, we examine the trade-off between sloppiness and incompatibility and discuss: (1) how information scrambling can improve estimation, and (2) how the correlations between the parameters and the incompatibility between the symmetric logarithmic derivatives impose constraints on the ultimate quantum limits to precision. Through analytical optimization, we identify strategies to mitigate these constraints and enhance estimation efficiency. We also compare the performance of joint parameter estimation to strategies involving successive separate estimation steps, demonstrating that the ultimate precision can be achieved when sloppiness is

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minimized. Our results provide a unified perspective on the trade-offs inherent to multiparameter qubit statistical models, offering practical insights for optimizing experimental designs.

Keywords: multiparameter quantum estimation, quantum metrology, qubit model, incompatibility, sloppiness

1. Introduction

Multiparameter quantum estimation [1–7] is an active area of research for its fundamental interest [8, 9] and due to its wide range of applications in quantum metrology [10–12], quantum imaging [13–16], and other fields [17–25]. Additionally, the recent finding that simultaneous estimation of many parameters can yield a better precision limit than estimating each parameter individually has accelerated the development of this field [26–29]. One of the most important targets in quantum multiparameter metrology is improving parameter estimation precision. The quantum Fisher information matrix (QFIM) and the accompanying quantum Cramér–Rao bound (QCRB) are relevant tools in this endeavor [30–35].

Multiparameter quantum estimation, similarly to the single-parameter case, involves three steps: probe state preparation, parameter encoding via system-probe interaction, and measurement-based information extraction. However, because of the correlation between parameters, designing optimal estimation strategies becomes more challenging. During encoding, a key challenge arises from *sloppiness*, which is a phenomenon where redundant or poorly encoded parameters reduce the efficiency of information extraction. Sloppiness [36–44] occurs when parameters are not independently encoded into the quantum probe state, leading to correlations that obscure individual parameter estimation, which acts as an intrinsic noise source. In contrast, *stiffness* refers to the desirable scenario where parameters are encoded in a way that minimizes redundancy, allowing for efficient and independent estimation of each parameter, thereby enhancing the overall precision of the estimation process.

Meanwhile, the measurement stage introduces a fundamental trade-off: optimizing precision for one parameter often compromises others due to the incompatibility of non-commuting observables. For instance, when the symmetric logarithmic derivatives (SLDs) corresponding to different parameters fail to commute, no single measurement can simultaneously saturate the QCRB for all parameters. Notably, sloppiness (from encoding dynamics) and incompatibility (from measurement) represent distinct sources of estimation uncertainty. Understanding this trade-off is critical for advancing multiparameter quantum metrology and achieving practical precision enhancements.

Incompatibility in multiparameter quantum estimation has been investigated in different systems [45–48], and sloppiness have been discussed in several metrological scenarios [36–44, 49, 50], but little emphasis has been devoted to the link between sloppiness and incompatibility. In this paper, we address a two-parameter scrambling qubit statistical model with tunable sloppiness and present an adjustable scrambling operation to investigate how the correlations between the parameters and the incompatibility between the SLDs influence the precision bounds. The scrambling model generally describes how the unitary operation between the two encodings spreads parameter information across the state in a nontrivial way, affecting both sloppiness and incompatibility.

This work is structured as follows. Section 2 introduces the fundamentals of quantum multiparameter estimation. In section 3, to explore the interplay between sloppiness and incompatibility, we present a two-parameter qubit estimation model that incorporates an information scrambling operation. This operation allows us to tune the correlations between parameters and, consequently, the sloppiness of the model. We also analyze the trade-off between sloppiness and incompatibility. In section 4, we optimize the relevant QCRBs and examine the role of sloppiness in achieving these bounds. Finally, section 5 concludes the paper with a summary of our findings. Details of calculations are provided in the appendices.

2. Multiparameter quantum estimation: precision, sloppiness and incompatibility

In this section, we provide the theoretical framework, definitions and metrics used throughout the paper. We consider finite dimensional systems and a family of quantum states $\rho_{\boldsymbol{\lambda}}$ encoding the values of d real parameters, denoted as a vector $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_d)^{\mathrm{T}}$. If we perform a positive operator-valued measurement (POVM) $\boldsymbol{\Pi}$ with elements $\{\Pi_k\}$ satisfying $\Pi_k \geqslant 0$ and $\sum_k \Pi_k = \mathbb{I}$, the measurement outcome k is obtained with probability $p_{\boldsymbol{\lambda}}(k) = \mathrm{Tr}\left[\rho_{\boldsymbol{\lambda}}\Pi_k\right]$. The estimator function based on the result is denoted as $\hat{\boldsymbol{\lambda}}(k)$. The performance of the estimator is assessed by the covariance matrix $\mathbf{V}(\hat{\boldsymbol{\lambda}})$ with elements

$$V_{\mu\nu} = \sum_k p_{\boldsymbol{\lambda}}(k)\left[\hat{\lambda}_\mu(k) - E_k\left(\hat{\lambda}_\mu\right)\right]\left[\hat{\lambda}_\nu(k) - E_k\left(\hat{\lambda}_\nu\right)\right]$$

where $E_k(\hat{\lambda}_\mu)$ is the expectation value of $\hat{\lambda}_\mu$ over the probability distribution $p_{\boldsymbol{\lambda}}(k)$.

In classical multiparameter estimation, when the estimators satisfying the locally unbiased conditions:

$$\boldsymbol{E}_\nu\left(\hat{\boldsymbol{\lambda}}\right) = \boldsymbol{\lambda} \qquad \partial_\mu E_k\left(\hat{\lambda}_\nu\right) = \delta_\mu^\nu,$$

where $\partial_\mu = \frac{\partial}{\partial\lambda_\mu}$, then the CRB [51] holds

$$\mathbf{V}\left(\hat{\boldsymbol{\lambda}}\right) \geqslant \frac{1}{M\mathbf{F}},$$

where M is the number of repeated measurements and $\mathbf{F}$ is the FI matrix with elements defined by

$$F_{\mu\nu} = \sum_k p_{\boldsymbol{\lambda}}(k)\,\partial_\mu \log p_{\boldsymbol{\lambda}}(k)\,\partial_\nu \log p_{\boldsymbol{\lambda}}(k) = \sum_k \frac{\partial_\mu p_{\boldsymbol{\lambda}}(k)\,\partial_\nu p_{\boldsymbol{\lambda}}(k)}{p_{\boldsymbol{\lambda}}(k)}.$$

The CRB can be saturated in the asymptotic limit of an infinite number of repeated experiments using Bayesian or maximum likelihood estimators [52].

Due to the non-commutativity of the operators, the quantum analogue of the FI cannot be uniquely introduced. In fact, there exist several different definitions of quantum Fisher information. The most celebrated and useful approaches are based on the so-called SLD operators L_μ^{S} [53] and right logarithmic derivative (RLD) operators L_μ^{R} [54, 55], defined as follows

$$\partial_\mu \rho_{\boldsymbol{\lambda}} = \frac{L_\mu^{\mathrm{S}}\rho_{\boldsymbol{\lambda}} + \rho_{\boldsymbol{\lambda}} L_\mu^{\mathrm{S}}}{2}, \tag{1}$$

$$\partial_\mu \rho_{\boldsymbol{\lambda}} = \rho_{\boldsymbol{\lambda}} L_\mu^{\mathrm{R}}. \tag{2}$$

We denote the corresponding SLD and RLD QFIM as $\mathbf{Q}$ and $\mathbf{J}$, respectively, with elements

$$Q_{\mu\nu} = \frac{1}{2}\mathrm{Tr}\left[\rho_{\boldsymbol{\lambda}}\left\{L_\mu^{\mathrm{S}}, L_\nu^{\mathrm{S}}\right\}\right],$$

$$J_{\mu\nu} = \mathrm{Tr}\left[\rho_{\boldsymbol{\lambda}} L_\mu^{\mathrm{R}} L_\nu^{\mathrm{R}\dagger}\right].$$

For pure statistical models, $\rho_{\boldsymbol{\lambda}} = |\psi_{\boldsymbol{\lambda}}\rangle\langle\psi_{\boldsymbol{\lambda}}|$ we have

$$Q_{\mu\nu} = 4\,\mathrm{Re}\left(\langle\partial_\mu\psi_\lambda|\,\partial_\nu\psi_\lambda\rangle - \langle\partial_\mu\psi_\lambda|\,\psi_\lambda\rangle\,\langle\psi_\lambda|\,\partial_\nu\psi_\lambda\rangle\right),$$
$$Q_{\nu\mu} = Q_{\mu\nu}\,,$$

where $\partial_k \equiv \partial_{\lambda_k}$.

2.1. Symmetric and right QCRBs

Using the above SLD and RLD QFIMs, $\mathbf{Q}$ and $\mathbf{J}$, matrix inequalities for the covariance matrix of any set of locally unbiased estimators may be established. Then, in order to obtain a scalar bound and to tailor the optimization of precision according to the different applications, a weight matrix $\mathbf{W}$ (a positive, real $d\times d$ matrix) may be introduced. The corresponding symmetric and right scalar bounds read as follows:

$$C_{\mathrm{S}}\left[W,\hat{\boldsymbol{\lambda}}\right] = \frac{1}{M}\,\mathrm{Tr}\left[\mathbf{W}\,\mathbf{Q}^{-1}\right],$$
$$C_{\mathrm{R}}\left[W,\hat{\boldsymbol{\lambda}}\right] = \frac{1}{M}\left(\mathrm{Tr}\left[\mathbf{W}\,\mathrm{Re}\left(\mathbf{J}^{-1}\right)\right] + \mathrm{Tr}\left[\left|\mathbf{W}\,\mathrm{Im}\left(\mathbf{J}^{-1}\right)\right|_1\right]\right),$$

where $|A|_1 = \sqrt{A^\dagger A}$ and $\mathrm{Re}(A)$ and $\mathrm{Im}(A)$ denote the real and imaginary parts of the complex-valued matrix A, respectively.

2.2. Holevo and Nagaoka CRBs

If the SLDs do not commute, it may happen that measurements that are optimal for different parameters are incompatible, making the symmetric and right QCRB, as well as their scalar counterparts, not achievable. An achievable scalar bound has been derived by Holevo [30]:

$$C_{\mathrm{H}}\left[\mathbf{W},\hat{\boldsymbol{\lambda}}\right] = \min_{\mathbf{X}\in\mathcal{X}}\left\{\mathrm{Tr}\left[\mathbf{W}\,\mathrm{Re}\left(\mathbf{Z}\left[\mathbf{X}\right]\right)\right] + \mathrm{Tr}\left[\left|\mathbf{W}\,\mathrm{Im}\left(\mathbf{Z}\left[\mathbf{X}\right]\right)\right|_1\right]\right\},$$

where the Hermitian $d\times d$ matrix $\mathbf{Z}$ is defined via its elements $\mathbf{Z}_{\mu\nu}[\mathbf{X}] = \mathrm{Tr}\left[\rho_{\boldsymbol{\lambda}} X_\mu X_\nu\right]$ with the collection of Hermitian operators $\mathbf{X}$ belonging to the set $\mathcal{X} = \{\mathbf{X} = (X_1,\ldots,X_d)\,|\,\mathrm{Tr}[(\partial_\mu\rho_{\boldsymbol{\lambda}})X_\nu] = \delta^\mu_\nu\}$. It has been proven that Holevo CRB $C_{\mathrm{H}}[\mathbf{W},\hat{\boldsymbol{\lambda}}]$ becomes attainable by performing a collective measurement on an asymptotically large number of copies of the state $\rho_{\boldsymbol{\lambda}}^{\otimes n}$ with $n\to\infty$. As such, it is typically regarded as the most fundamental scalar bound for multiparameter quantum estimation.

Nagaoka [56] introduced a more informative bound, denoted as $C_{\mathrm{N}}[\mathbf{W},\hat{\boldsymbol{\lambda}}]$, which is particularly valuable for practical experimental measurements. While not as theoretically tight as the Holevo bound $C_{\mathrm{H}}[\mathbf{W},\hat{\boldsymbol{\lambda}}]$, it can be practically achieved using separable measurement strategies, making it more feasible to attain in real-world applications. The bound is defined as:

$$C_{\mathrm{N}}\left[\mathbf{W},\hat{\boldsymbol{\lambda}}\right] = \min_{\boldsymbol{\Pi}}\left\{\mathrm{Tr}\left[\mathbf{W}\mathbf{F}^{-1}\right]\right\}, \tag{3}$$

where the minimization is performed over all possible single-system (non collective) POVMs $\boldsymbol{\Pi}$. As for the other bounds, the optimal POVM generally depends on the true value of the parameters $\boldsymbol{\lambda}$.

2.3. Sloppiness and stiffness

In statistical estimation, degeneracy is an extreme case of sloppiness. Mathematically, degeneracy corresponds to a singular FIM, whereas sloppiness is more concerned with the distribution of the eigenvalues of the FIM. A quantum statistical model is termed sloppy if the QFIM is singular, i.e. $\det[\mathbf{Q}] = 0$. This means that the true parameters describing the system are $m < n$ combinations of the original parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$. The eigenvalues of $\mathbf{Q}$ quantify the sensitivity of the probe state to perturbations along orthogonal parameter directions. A small eigenvalue implies that the state of the probe is insensitive to changes in the corresponding parameter direction, i.e. that combination of parameters is poorly encoded. The degree of sloppiness may be thus quantified by the determinant of QFIM:

$$s := \frac{1}{\det[\mathbf{Q}]} = \det\left[\mathbf{Q}^{-1}\right], \tag{4}$$

which measures how strongly the system depends on a combination of the components of λ rather than on its individual components. Sloppiness may change under reparameterization. A large sloppiness indicates that the model's sensitivity to parameter variations is highly anisotropic. A model with low sloppiness is referred to as a stiff model.

2.4. Compatibility and incompatibility

Due to the non-commutativity of the SLDs associated to different parameters, in a multiparameter scenario QCRB cannot always be achieved. Based on the quantum local asymptotic normality [57–59], it has been shown that the multiparameter SLD-QCRB is attainable if and only if the weak compatibility condition is satisfied [47], defined by

$$\mathrm{Tr}\left[\rho_{\boldsymbol{\lambda}}\left[L^{\mathrm{S}}_{\mu}, L^{\mathrm{S}}_{\nu}\right]\right] = 0. \tag{5}$$

The incompatibility matrix $\mathbf{D}$, also known as mean Uhlmann curvature (MUC), is the antisymmetric matrix defined by

$$D_{\mu\nu} := \frac{1}{2i}\mathrm{Tr}\left[\rho_{\boldsymbol{\lambda}}\left[L^{\mathrm{S}}_{\mu}, L^{\mathrm{S}}_{\nu}\right]\right], \tag{6}$$

and is useful to quantify the incompatibility between the pair of parameters λ_μ and λ_ν. For a two-parameter pure state model we have

$$D_{11} = D_{22} = 0,$$
$$D_{12} = -D_{21} = 4\,\mathrm{Im}\left(\langle\partial_1\psi_\lambda|\partial_2\psi_\lambda\rangle - \langle\partial_1\psi_\lambda|\psi_\lambda\rangle\langle\psi_\lambda|\partial_2\psi_\lambda\rangle\right).$$

A measure of the incompatibility of the model is given by

$$c := \frac{1}{2}\mathrm{Tr}\left[\mathbf{D}^{\dagger}\mathbf{D}\right], \tag{7}$$

which is based on the antisymmetric nature of the MUC. The definition of incompatibility refers to the non-commutativity among different SLDs. Each SLD represents the optimal observable for estimating a given parameter, and their failure to commute reflects a fundamental quantum limitation—the presence of unavoidable noise that restricts the precision of joint estimation. If equation (5) is not met—meaning incompatibility is nonzero—then no single measurement can simultaneously achieve optimal precision for all parameters. In other words, when incompatibility exists, trade-offs in estimation precision are inevitable. This measure is non-negative for all pairs of SLD operators and approaches infinity when all SLD operators are commuting with each other. Incompatibility is affected by the parameters' component, and thus c is not invariant under reparameterization. However, incompatibility is covariant in some natural scenarios in which it should not be possible to generate incompatibility, such as unitary evolution. In our 2-parameter qubit estimation model, to highlight the structural similarity with sloppiness, the incompatibility defined in equation (7) simplifies to

$$c := \det[\mathbf{D}] = -D_{12}^2\,. \tag{8}$$

2.5. Relationship between different bounds

The relationship between the different bounds has been extensively explored widely, providing a classification of quantum statistical models. Specifically, the models can be categorized into four types:

- **Classical** When the quantum state $\rho_{\boldsymbol{\lambda}}$ can be expressed in a diagonal form with parameters $\boldsymbol{\lambda}$ and a $\boldsymbol{\lambda}$-independent unitary U, such that $\rho_{\boldsymbol{\lambda}} = U\Lambda_{\boldsymbol{\lambda}}U^\dagger$, it can be regarded as a classical quantum statistical model. In this scenario, the FI matrix $\mathbf{F}$, the SLD QFIM $\mathbf{Q}$, and the RLD QFIM $\mathbf{J}$ are all identical. Consequently, we obtain $C_\mathrm{S}[\mathbf{W},\hat{\boldsymbol{\lambda}}] = C_\mathrm{R}[\mathbf{W},\hat{\boldsymbol{\lambda}}] = C_\mathrm{H}[\mathbf{W},\hat{\boldsymbol{\lambda}}] = C_\mathrm{N}[\mathbf{W},\hat{\boldsymbol{\lambda}}]$.
- **Quasi-classical** A quantum statistical model is called quasi-classical if all SLD operators commute with each other for every parameter. Thus, the equality $C_\mathrm{S}[\mathbf{W},\hat{\boldsymbol{\lambda}}] = C_\mathrm{H}[\mathbf{W},\hat{\boldsymbol{\lambda}}]$ holds.
- **Asymptotically classical** In asymptotically classical quantum statistical models, all SLD operators satisfy the weak compatibility condition defined in equation (5), and $C_\mathrm{S}[\mathbf{W},\hat{\boldsymbol{\lambda}}]$ is equal to $C_\mathrm{H}[\mathbf{W},\hat{\boldsymbol{\lambda}}]$.
- **D-invariant** When a quantum statistical model is D-invariant, we have [55, 60]
$$C_\mathrm{R}\left[\mathbf{W},\hat{\boldsymbol{\lambda}}\right] = C_\mathrm{H}\left[\mathbf{W},\hat{\boldsymbol{\lambda}}\right] = C_\mathrm{S}\left[\mathbf{W},\hat{\boldsymbol{\lambda}}\right] + \mathrm{Tr}\left[\left|\sqrt{\mathbf{W}}\mathbf{Q}^{-1}\mathbf{U}\mathbf{Q}^{-1}\sqrt{\mathbf{W}}\right|_1\right].$$
This equality illustrates that the RLD bound is achievable by performing a collective measurement on an asymptotically large number of copies.

In [5] a measure has been suggest to quantify the amount of incompatibility (somehow referred to as the *quantumness* of the model [4])

$$R := \|\mathrm{i}\mathbf{Q}^{-1}\mathbf{D}\|_\infty, \tag{9}$$

where $\|A\|_\infty$ denotes the largest eigenvalue of the matrix A. It has been also shown that R is useful in upper bounding the Holevo bound in terms of the SLD-bound, as follows:

$$C_\mathrm{S}\left[\mathbf{W},\hat{\boldsymbol{\lambda}}\right] \leqslant C_\mathrm{H}\left[\mathbf{W},\hat{\boldsymbol{\lambda}}\right] \leqslant C_\mathrm{N}\left[\mathbf{W},\hat{\boldsymbol{\lambda}}\right] \leqslant (1+R)\,C_\mathrm{S}\left[\mathbf{W},\hat{\boldsymbol{\lambda}}\right] \leqslant 2C_\mathrm{S}\left[\mathbf{W},\hat{\boldsymbol{\lambda}}\right]. \tag{10}$$

In the pure-state limit, the Holevo bound is equivalent to the RLD CR bound [60]. When the number of parameters to be estimated is $n = 2$, we have the relation [5]

$$R = \sqrt{\frac{\det[\mathbf{D}]}{\det[\mathbf{Q}]}} = \sqrt{sc}, \tag{11}$$

where s represents the sloppiness and c is the incompatibility of the system. For our qubit multiparameter model, equation (10) can be further rewritten as

$$C_\mathrm{S}\left[\mathbf{W},\hat{\boldsymbol{\lambda}}\right] \leqslant C_\mathrm{H}\left[\mathbf{W},\hat{\boldsymbol{\lambda}}\right] = C_\mathrm{N}\left[\mathbf{W},\hat{\boldsymbol{\lambda}}\right] \leqslant C_\mathrm{S}\left[\mathbf{W},\hat{\boldsymbol{\lambda}}\right]\left(1+\sqrt{sc}\right). \tag{12}$$

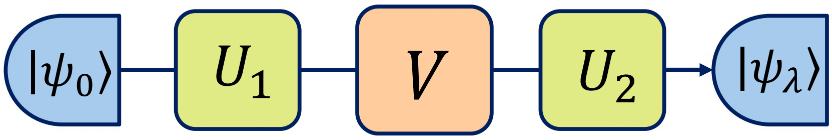

Figure 1. The scrambling model considered in this paper. The model parameters λ_1 and λ_2 are encoded via the unitary operations U_1 and U_2, which represent rotation along the z-axis of the Bloch sphere. To remove sloppiness and adjust correlations between the encoded parameters, we introduce a scrambling operation, represented by the intermediate rotation V.

3. Information scrambling, precision, and the sloppiness-incompatibility trade-off

To systematically investigate the interplay between sloppiness and incompatibility, we consider the two-parameter qubit model illustrated in figure 1. By introducing a tunable scrambling operation during parameter encoding, we control correlations between parameters and quantify their impact on estimation precision. Given the convexity of QFI [61], we consider a pure probe state $|\psi_0\rangle$ defined as

$$|\psi_0\rangle = \cos\frac{\alpha}{2}|0\rangle + \mathrm{e}^{\mathrm{i}\beta}\sin\frac{\alpha}{2}|1\rangle .$$

The model parameters λ_1 and λ_2 are encoded via the unitary operations U_1 and U_2, which represent rotation along the z-axis of the Bloch sphere, and are given by

$$U_k = \mathrm{e}^{-\mathrm{i}\sigma_3\lambda_k},$$

where σ_3 is the Pauli-Z matrix. These rotations imprint λ_1 and λ_2 onto the probe state's phase.

If nothing is done between the two unitaries, the output state depends only on the sum of the two parameters (and not on the difference), the QFIM is thus singular and the model is sloppy. To remove sloppiness in a tunable way, and adjust correlations between the encoded parameters, we introduce an intermediate rotation V between U_1 and U_2:

$$V = \mathrm{e}^{-\mathrm{i}\gamma\vec{\sigma}\cdot\vec{n}}, \qquad \vec{n} = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta).$$

Here, γ controls the rotation strength, θ and ϕ define the rotation axis $\vec{n}$, and $\vec{\sigma}$ is the vector of Pauli matrices. This operation dynamically mixes the parameters, introducing correlations that govern sloppiness. The final state after encoding becomes

$$\begin{aligned}|\psi_\lambda\rangle &= U_2VU_1|\psi_0\rangle \\ &= \begin{pmatrix} \mathrm{e}^{-\mathrm{i}(\lambda_1+\lambda_2)}\cos\frac{\alpha}{2}(\cos\gamma - \mathrm{i}\sin\gamma\cos\theta) - \mathrm{i}\mathrm{e}^{\mathrm{i}(\lambda_1-\lambda_2+\beta-\phi)}\sin\frac{\alpha}{2}\sin\gamma\sin\theta \\ -\mathrm{i}\mathrm{e}^{-\mathrm{i}(\lambda_1-\lambda_2-\phi)}\cos\frac{\alpha}{2}\sin\gamma\sin\theta + \mathrm{e}^{\mathrm{i}(\lambda_1+\lambda_2+\beta)}\sin\frac{\alpha}{2}(\cos\gamma + \mathrm{i}\sin\gamma\cos\theta) \end{pmatrix}.\end{aligned}$$

Explicitly, this state depends on 7 parameters. α and β are probe state initialization's parameters. α balances the superposition weights of $|0\rangle$ and $|1\rangle$. β is the initial phase, influencing interference effects during the parameter encoding. λ_1 and λ_2 are encoding parameters, γ, θ, and ϕ are scrambling parameters.

The rotation angle γ partly governs the strength of parameter mixing. When $\gamma = 0$, $V = \mathbb{I}$, and the parameters perfectly correlated as they combine into a single effective parameter $\lambda_1 + \lambda_2$. This results in maximum sloppiness ($\det\mathbf{Q} \to 0$) because only a single function of the parameters can be estimated. Larger γ may decrease coupling between λ_1 and λ_2, allowing

U_1 and U_2 to imprint independent information the quantum state. The angle θ determines the alignment of rotation axis relative to the z-axis, and ϕ controls the azimuthal orientation of rotation axis, introducing phase-dependent correlations. The scrambling rotation V couples λ_1 and λ_2, enabling control over the model's sloppiness and the non-commutativity of their associated SLDs.

Explicit calculations yield:

$$\begin{aligned} Q_{11} &= 4\sin^2\alpha, \\ Q_{12} &= Q_{21} = 4\left(X\sin^2\alpha - Y\sin 2\alpha\right), \\ Q_{22} &= 4\left[1 - (X\cos\alpha + 2Y\sin\alpha)^2\right], \end{aligned}$$

where

$$\begin{aligned} X &= \cos^2\gamma + \sin^2\gamma\cos 2\theta, \\ Y &= \sin\gamma\sin\theta\left(\sin f\cos\gamma + \sin\gamma\cos\theta\cos f\right), \\ f &= 2\lambda_1 + \beta - \phi. \end{aligned}$$

The measurement incompatibility matrix $\mathbf{D}$ has elements:

$$\begin{aligned} D_{11} &= D_{22} = 0, \\ D_{12} &= -D_{21} = -8Z\sin\alpha, \end{aligned}$$

where

$$Z = \sin\gamma\sin\theta\left(\cos f\cos\gamma - \sin\gamma\cos\theta\sin f\right).$$

From these results, we are to calculate the sloppiness and incompatibility measures as follows

$$1/s = \det[\mathbf{Q}] = 16\sin^2\alpha\left[1 - X^2 - 4Y^2\right] = 64Z^2\sin^2\alpha, \tag{13}$$

$$c = \det[\mathbf{D}] = 64Z^2\sin^2\alpha, \tag{14}$$

uncovering the fundamental trade-off:

$$sc = 1. \tag{15}$$

This equality quantifies the competition between parameter distinguishability and incompatibility in our two-parameter qubit model. In other words, sloppiness and incompatibility cannot be minimized simultaneously. More detailed derivations are provided in appendix A. Notice that equation (15), together with equation (11), is consistent with the observation [48] that when the number of parameters is equal to the dimension of the probe, we have maximum quantumness of the model, i.e. $R = 1$. Here, we chose a uniform weight matrix primarily because both parameters are equally important in our setting. Notice, however, that the main conclusion $sc = 1$ is independent on the choice of the weight matrix, since the quantity R is invariant under reparameterization. In other words, the trade-off is robust against the choice of the weight matrix since it is a general result relying on QFIM and Uhlmann matrix, and it has no relation with optimization.

4. The ultimate bounds to precision

The interplay between sloppiness and incompatibility in multiparameter quantum estimation is not merely a theoretical relation, it also informs strategies for optimizing precision. Here, we explore this optimization for a two-parameter qubit estimation system, revealing how suitable parameter encoding and measurement design achieve these goals.

4.1. Hierachy of QCRBs

In our two-parameter qubit model, we assume that the two parameters are equally relevant, and thus set $\mathbf{W} = \mathbb{I}$. We start by noticing that for a positive definite matrix, which has the property $\frac{1}{n}\,\mathrm{Tr}[A^{-1}] \geqslant \det[A]^{-1/n}$. For $n = 2$, we obtain

$$\frac{2}{\mathrm{Tr}\left[\mathbf{Q}^{-1}\right]} \leqslant \sqrt{\det\left[\mathbf{Q}\right]}.$$

Therefore, in our qubit model, the different QCRBs with $\mathbf{W} = \mathbb{I}$ satisfy

$$2\sqrt{s} \leqslant \mathrm{Tr}\left(\mathbf{Q}^{-1}\right) = C_{\mathrm{S}} \leqslant C_{\mathrm{H}} = C_{\mathrm{N}} \leqslant C_{\mathrm{S}}\left(1 + \sqrt{sc}\right) = 2\,C_{\mathrm{S}}\,, \tag{16}$$

where the dependence of the bounds on the weight matrix and the parameters has been dropped since the weight matrix is set to the identity and the model is unitary (i.e. the bounds do not depends on the parameters). The SLD bound is not achievable, while the larger Holevo and Nagaoka bounds are achievable, and equivalent in this model. The upper bound remains tied to the SLD bound.

4.2. Minimization of SLD bound

$\mathrm{Tr}[\mathbf{Q}^{-1}]$ and $\frac{1}{\det[\mathbf{Q}]}$ are monotonically decreasing functions with respect to the positive definite matrix Q under Loewner order. Therefore, from equation (16), to minimize C_{S}, we need to minimize the sloppiness, i.e. maximize $\det[\mathbf{Q}]$. Given equation (14), the task is linked to maximizing the quantity Z^2. Two parameter regimes achieve this: Case (a): $\gamma = f = \frac{\pi}{2}$, $\theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$, Case (b): $\gamma = \frac{\pi}{4}, f = 0, \theta = \frac{\pi}{2}$. This result underscores the necessity of adjusting γ, θ, f to decouple λ_1 and λ_2. Detailed calculations are provided in appendix B. In both optimal Case (a) and Case (b), the phase parameter f takes the form $f = 2\lambda_1 + \beta - \phi$. Here, β is another parameter related to the initial state besides α. Without loss of generality, we set $\beta = 0$, which effectively fixes the initial state. As a result, we obtain $f = 2\lambda_1 - \phi$. The intermediate unitary rotation involves three parameters: γ, ϕ, and θ. Among these, γ and θ can be fixed, while ϕ must be chosen in a λ_1-dependent manner to optimize the estimation precision. For example, under the first optimal condition, we set $\gamma = \pi/2$ and $\theta = \pi/4$, which gives $\phi = 2\lambda_1 - \pi/2$. Similarly, under the second optimal condition, we take $\gamma = \pi/4$ and $\theta = \pi/2$, leading to $\phi = 2\lambda_1$. These examples also illustrate that the optimal unitary design is inherently parameter-dependent.

Both cases yield $Z^2 = 1/4$, saturating the upper limit of incompatibility. When combined with an optimal probe state ($\alpha = \pi/2$), sloppiness reaches its minimum ($s = 1/16$), and the SLD bound C_{S} attains its lowest value:

$$C_{\mathrm{S}} = \frac{1}{2}.$$

In our two-parameter qubit model, we achieve SLD precision by reducing sloppiness. It means that theoretical maximum precision can only be reached when sloppiness is minimized.

4.3. Minimizing the Holevo and Nagaoka bounds

For a two-parameter pure qubit estimation model, the Holevo bound $C_{\mathrm{H}}[\mathbf{W},\hat{\boldsymbol{\lambda}}]$ is given by

$$C_{\mathrm{H}}\left[\mathbf{W},\hat{\boldsymbol{\lambda}}\right]=C_{\mathrm{R}}\left[\mathbf{W},\hat{\boldsymbol{\lambda}}\right]=C_{\mathrm{S}}\left[\mathbf{W},\hat{\boldsymbol{\lambda}}\right]+\frac{\sqrt{\det\left[\mathbf{W}\right]}}{\det\left[\mathbf{Q}\right]}\mathrm{Tr}\left[\left|\rho_{\boldsymbol{\lambda}}\left[L_1^{\mathrm{S}},L_2^{\mathrm{S}}\right]\right|_1\right].$$

Similarly, the Nagaoka bound $C_{\mathrm{N}}[\mathbf{W},\hat{\boldsymbol{\lambda}}]$ for a two-parameter qubit model is given by

$$C_{\mathrm{N}}\left[\mathbf{W},\hat{\boldsymbol{\lambda}}\right]=C_{\mathrm{S}}\left[\mathbf{W},\hat{\boldsymbol{\lambda}}\right]+\frac{\sqrt{\det\left[\mathbf{W}\right]}}{\det\left[\mathbf{Q}\right]}\mathrm{Tr}\left[\left|\rho_{\boldsymbol{\lambda}}\left[L_1^{\mathrm{S}},L_2^{\mathrm{S}}\right]\right|_1\right].$$

It follows directly that the Holevo and the Nagaoka bounds are identical for a two-parameter pure qubit [60], simplifying to

$$C_{\mathrm{H}}\left[\mathbf{W},\hat{\boldsymbol{\lambda}}\right]=C_{\mathrm{N}}\left[\mathbf{W},\hat{\boldsymbol{\lambda}}\right]=\mathrm{Tr}\left[\mathbf{W}\mathbf{Q}^{-1}\right]+2\sqrt{\det\left[\mathbf{W}\mathbf{Q}^{-1}\right]}.$$

In our model, the Holevo bound, RLD bound and Nagaoka bounds are identical and given by

$$C_{\mathrm{H}}=C_{\mathrm{R}}=C_{\mathrm{N}}=C_{\mathrm{S}}+2\sqrt{s}\geqslant 4\sqrt{s}.$$

We find that all these bounds are expressed as lower bound in terms of sloppiness. When sloppiness achieves minimum, all quantum CR bounds reach their minima simultaneously, reducing to:

$$C_{\mathrm{H}}=C_{\mathrm{R}}=C_{\mathrm{N}}=1.$$

This, together with the minimal condition of the SLD QCRB, indicates that maximum estimation precision can be achieved by minimizing sloppiness.

Holevo bound represents the asymptotically achievable precision limit with collective measurements, performed on all the available copies, say M, of the state encoding the parameters. Upon performing separate measurements (each one performed on one of the M copies) one may reach the Nagaoka bound, which is usually larger than the Holevo bound. However in our model, these two bounds are equivalent, also indicating that our model is D-invariant. Notice that in our model $R=1$, and $C_{\mathrm{H}}=2\,C_{\mathrm{S}}$, i.e. the quantumness R quantifies exactly the additional uncertainty due to the incompatibility between the two SLDs.

4.4. Optimization of bounds for stepwise measurements

Stepwise measurements involve estimating parameters sequentially, rather than estimating them jointly. Having at disposal M repeated preparations of the system, we assume to devote $M/2$ of them to estimate solely λ_1 (assuming λ_2 unknown) and the remaining $M/2$ preparations to estimate λ_2 (assuming λ_1 known from the first step). Of course the role of the two parameters may be exchanged, and we thus have two strategies of this kind.

The (saturable) precision bound on the estimation of λ_1 from the first step is obtained from the SLD-QCRB by choosing a weight matrix of form $\mathbf{W}=\mathrm{Diag}(1,0)$, leading to

$$\mathrm{Var}\lambda_1\geqslant\frac{2\left[\mathbf{Q}^{-1}\right]_{11}}{M},$$

where $[X]_{ij}$ indicates the elements of the matrix X. In the second step, λ_1 is known, and the (achievable) bound to precision in the estimation of λ_2 is given by the single-parameter QCRB

$$\mathrm{Var}\lambda_2\geqslant\frac{2}{MQ_{22}}.$$

The total variance for this estimation strategy is thus bounded by:

$$\mathrm{Var}\lambda_1 + \mathrm{Var}\lambda_2 \geqslant \frac{2}{M}\left(\left[\mathbf{Q}^{-1}\right]_{11} + \frac{1}{Q_{22}}\right) \equiv \frac{1}{M}K_1,$$

where

$$K_1 = 2\left(\left[\mathbf{Q}^{-1}\right]_{11} + \frac{1}{Q_{22}}\right).$$

Similarly, reversing the role of the two parameters, we have $\mathrm{Var}\lambda_1 + \mathrm{Var}\lambda_2 \geqslant K_2/M$, where

$$K_2 = 2\left(\left[\mathbf{Q}^{-1}\right]_{22} + \frac{1}{Q_{11}}\right).$$

For our model, we have

$$\begin{aligned} K_1 &= 2\left(s\,Q_{22} + \frac{1}{Q_{22}}\right) \geqslant 2\times 2\sqrt{s\,Q_{22}\times\frac{1}{Q_{22}}} = 4\sqrt{s}, \\ K_2 &= 2\left(s\,Q_{11} + \frac{1}{Q_{11}}\right) \geqslant 2\times 2\sqrt{s\,Q_{11}\times\frac{1}{Q_{11}}} = 4\sqrt{s}. \end{aligned}$$

The equalities hold if and only if s achieves its minimum, i.e. the minimum values of K_1 and K_2 are equal. Notice that if, instead of dividing the total number of repeated preparations equally between the two estimation procedures, we had chosen an asymmetric allocation, say $M_1 = \gamma M$ measurements for estimating λ_1 and $M_2 = (1-\gamma)M$ for estimating λ_2 (or vice versa), the bounds K_1 and K_2 would have been larger, specifically $K_j \geqslant 2\sqrt{s}/\sqrt{\gamma(1-\gamma)}$, $j = 1,2$. The choice $\gamma = 1/2$ is therefore optimal. When the C_{H} has already reached its lower bound, the relation among the precision bounds is

$$C_{\mathrm{H}} = C_{\mathrm{R}} = C_{\mathrm{N}} = 4\sqrt{s} \leqslant K_1 = K_2,$$

where K_1 and K_2 attain the same optimal value if and only if the sloppiness is minimal. This equality demonstrates that even when the incompatibility is maximal, stepwise measurements can still achieve optimal precision. Sloppiness sets the lower bounds for both the Holevo/Nagaoka bound as well as the bound to precision achievable by successive measurements. The latter bound is identical to the Holevo bound if and only if the sloppiness is minimal, which offers a practical advantage in experiments. In each step, the optimal POVM is the spectral measure of the corresponding SLD. Using equation (1), we have

$$\begin{aligned} L_1 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3, \\ L_2 &= \begin{pmatrix} 0 & -\mathrm{i}e^{-2\mathrm{i}(\lambda_1+\lambda_2)} \\ \mathrm{i}e^{2\mathrm{i}(\lambda_1+\lambda_2)} & 0 \end{pmatrix} \\ &= \cos\left[2\left(\lambda_1+\lambda_2\right)\right]\sigma_2 - \sin\left[2\left(\lambda_1+\lambda_2\right)\right]\sigma_1\,, \end{aligned}$$

meaning that the optimal POVM in the first step is parameter independent, while the second step necessarily involves some adaptive measurement scheme.

5. Conclusions

In this work, we have investigated a two-parameter qubit statistical model with tunable sloppiness to explore the interplay between precision, sloppiness, and incompatibility. By introducing an adjustable scrambling operation during parameter encoding, we have demonstrated how parameter correlations and measurement incompatibility jointly influence the precision bounds.

First, we identified a fundamental trade-off between sloppiness and incompatibility, characterized by the equality $sc = 1$. This result highlights the impossibility of simultaneously minimizing both sloppiness and incompatibility, revealing a key constraint in multiparameter quantum estimation in qubit systems. Notice that by increasing the probe's dimension one can reduce incompatibility while maintaining the same level of sloppiness to achieve SLD bound, as shown in appendix C for qutrit systems. Second, we derived the conditions for optimizing QCRBs. By tuning the parameters of the encoding strategy, we maximized the determinant of the QFIM, thereby minimizing sloppiness.

In our system, the Holevo, Nagaoka, and RLD precision bounds for joint parameter estimation are equivalent and saturable using non-collective measurements. We also compared the performance of joint estimation strategies to those involving successive separate estimation steps, demonstrating that the former can achieve ultimate precision when sloppiness is minimized. Beyond its fundamental significance, this finding offers practical advantages for experimental implementations. Furthermore, our analysis revealed that the minimal achievable precision bounds directly connect sloppiness to the ultimate metrological performance. This underscores the importance of designing probe states and encoding dynamics that emphasize parameter correlations while balancing the effects of non-commutative measurements.

Our results provide new insights into the relationship between sloppiness and incompatibility in two-parameter qubit estimation systems. In future work, we aim to extend this framework to higher-dimensional systems, where careful preparation of the probe state may eliminate incompatibility [48] and render the symmetric CRB achievable. This could lead to fundamentally different trade-offs between sloppiness, precision, and incompatibility compared to the qubit case.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Acknowledgments

This work has been done under the auspices of GNFM-INdAM and has been partially supported by MUR through Project No. PRIN22-2022T25TR3-RISQUE. MGAP thanks Francesco Albarelli, Chiranjib Mukhopadhyay, Abolfazl Bayat, Sara Dornetti, Alessandro Ferraro, Berihu Teklu, Victor Montenegro, Simone Cavazzoni, and Stefano Olivares for useful discussions. J H thanks Shuangshuang Fu for useful discussions.

Appendix A. QFIM and MUC

We first calculate the output state

$$
\begin{aligned}
|\psi_\lambda\rangle &= U_2 V U_1 |\psi_0\rangle \\
&= \begin{pmatrix} e^{-i(\lambda_1+\lambda_2)} \cos\frac{\alpha}{2}(\cos\gamma - i\sin\gamma\cos\theta) - ie^{i(\lambda_1-\lambda_2+\beta-\phi)} \sin\frac{\alpha}{2}\sin\gamma\sin\theta \\ -ie^{-i(\lambda_1-\lambda_2-\phi)}\cos\frac{\alpha}{2}\sin\gamma\sin\theta + e^{i(\lambda_1+\lambda_2+\beta)}\sin\frac{\alpha}{2}(\cos\gamma + i\sin\gamma\cos\theta) \end{pmatrix}.
\end{aligned}
$$

The partial derivatives of the output state ψ_λ with respect to λ_1 and λ_2, respectively, are given by

$$
\begin{aligned}
|\partial_{\lambda_1}\psi_\lambda\rangle &= \begin{pmatrix} -ie^{-i(\lambda_1+\lambda_2)}\cos\frac{\alpha}{2}(\cos\gamma - i\sin\gamma\cos\theta) + e^{i(\lambda_1-\lambda_2+\beta-\phi)}\sin\frac{\alpha}{2}\sin\gamma\sin\theta \\ -e^{-i(\lambda_1-\lambda_2-\phi)}\cos\frac{\alpha}{2}\sin\gamma\sin\theta + ie^{i(\lambda_1+\lambda_2+\beta)}\sin\frac{\alpha}{2}(\cos\gamma + i\sin\gamma\cos\theta) \end{pmatrix}, \\
|\partial_{\lambda_2}\psi_\lambda\rangle &= \begin{pmatrix} -ie^{-i(\lambda_1+\lambda_2)}\cos\frac{\alpha}{2}(\cos\gamma - i\sin\gamma\cos\theta) - e^{i(\lambda_1-\lambda_2+\beta-\phi)}\sin\frac{\alpha}{2}\sin\gamma\sin\theta) \\ e^{-i(\lambda_1-\lambda_2-\phi)}\cos\frac{\alpha}{2}\sin\gamma\sin\theta + ie^{i(\lambda_1+\lambda_2+\beta)}\sin\frac{\alpha}{2}(\cos\gamma + i\sin\gamma\cos\theta) \end{pmatrix}.
\end{aligned}
$$

which lead to

$$
\begin{aligned}
&\langle\partial_{\lambda_1}\psi_\lambda|\,\partial_{\lambda_1}\psi_\lambda\rangle = \langle\partial_{\lambda_2}\psi_\lambda|\,\partial_{\lambda_2}\psi_\lambda\rangle = 1, \\
&\langle\partial_{\lambda_1}\psi_\lambda|\,\partial_{\lambda_2}\psi_\lambda\rangle = \cos^2\gamma + \sin^2\gamma\cos 2\theta + 2i\sin\alpha\sin\gamma\sin\theta\,(\sin\gamma\cos\theta\sin f - \cos f\cos r), \\
&\langle\partial_{\lambda_1}\psi_\lambda|\,\psi_\lambda\rangle = i\cos\alpha, \\
&\langle\partial_{\lambda_2}\psi_\lambda|\,\psi_\lambda\rangle = i\left[\cos\alpha\left(\cos^2\gamma + \sin^2\gamma\cos 2\theta\right) + 2\sin\alpha\sin\gamma\sin\theta\,(\sin f\cos\gamma + \sin\gamma\cos\theta\cos f)\right],
\end{aligned}
$$

where $f = 2\lambda_1 + \beta - \phi$. Then, we calculate the elements of QFIM $\mathbf{Q}$ and the incompatibility matrix $\mathbf{D}$. To make the computation easier, we define

$$
\begin{aligned}
X &= \cos^2\gamma + \sin^2\gamma\cos 2\theta, \\
Y &= \sin\gamma\sin\theta\,(\sin f\cos\gamma + \sin\gamma\cos\theta\cos f), \\
Z &= \sin\gamma\sin\theta\,(\cos f\cos\gamma - \sin\gamma\cos\theta\sin f).
\end{aligned}
$$

We obtain

$$
\begin{aligned}
Q_{11} &= 4\sin^2\alpha, \\
Q_{12} = Q_{21} &= 4\left[\sin^2\alpha\left(\cos^2\gamma + \sin^2\gamma\cos 2\theta\right) - \sin 2\alpha\sin\gamma\sin\theta\,(\sin f\cos\gamma + \sin\gamma\cos\theta\cos f)\right] \\
&= 4\left(X\sin^2\alpha - Y\sin 2\alpha\right), \\
Q_{22} &= 4\left(1 - \left[\cos\alpha\left(\cos^2\gamma + \sin^2\gamma\cos 2\theta\right) + 2\sin\alpha\sin\gamma\sin\theta\,(\sin f\cos\gamma + \sin\gamma\cos\theta\cos f)\right]^2\right) \\
&= 4\left[1 - (X\cos\alpha + 2Y\sin\alpha)^2\right], \\
D_{11} &= D_{22} = 0, \\
D_{12} = -D_{21} &= 4\mathrm{Im}\left(\langle\partial_{\lambda_1}\psi_\lambda|\,\partial_{\lambda_2}\psi_\lambda\rangle - \langle\partial_{\lambda_1}\psi_\lambda|\,\psi_\lambda\rangle\langle\psi_\lambda|\,\partial_{\lambda_2}\psi_\lambda\rangle\right) \\
&= -8\sin\alpha\sin\gamma\sin\theta\,(\cos f\cos\gamma - \sin\gamma\cos\theta\sin f) = -8Z\sin\alpha.
\end{aligned}
$$

The determinant of QFIM can be expressed as:

$$\det[\mathbf{Q}] = Q_{11} \times Q_{22} - Q_{12} \times Q_{21} = 16\sin^2\alpha\left[1 - X^2 - 4Y^2\right],$$

whereas the determinant of the incompatibility matrix is given by

$$\det[\mathbf{D}] = 0 - D_{12}D_{21} = 64Z^2\sin^2\alpha.$$

More simplifications are required to clarify the connection between det[$\mathbf{Q}$] and det[$\mathbf{D}$]. Upon introducing the quantities $A = \sin\gamma\cos\theta$, $B = \sin\gamma\sin\theta$, we find that $A^2 + B^2 = \sin^2\gamma$, so $\cos^2\gamma = 1 - A^2 - B^2$. Besides, $Y^2 + Z^2 = \sin^2\gamma\sin^2\theta(\cos^2\gamma + \sin^2\gamma\cos^2\theta) = B^2(1 - B^2)$ and $X = \cos^2\gamma + \sin^2\gamma(\cos^2\theta - \sin^2\theta) = 1 - 2B^2$, such that

$$\det[\mathbf{Q}] = 16\sin^2\alpha\left[1 - \left(1 - 2B^2\right)^2 - 4F^2\right] = 64\sin^2\alpha\left[B^2\left(1 - B^2\right) - Y^2\right] = 64Z^2\sin^2\alpha.$$

Appendix B. Maximum and minimum of Z^2

To find the stationary points of Z^2, we calculate the partial derivatives of Z^2 with respect to γ, θ, and f, and set them to zero:

$$\begin{aligned}
\frac{\partial Z^2}{\partial\gamma} &= 2\sin^2\theta\sin\gamma\left(\cos f\cos\gamma - \sin\gamma\cos\theta\sin f\right) \\
&\quad\times\left[\cos f\left(\cos^2\gamma - \sin^2\gamma\right) - 2\sin\gamma\cos\gamma\cos\theta\sin f\right] = 0, \\
\frac{\partial Z^2}{\partial\theta} &= 2\sin^2\gamma\sin\theta\left(\cos f\cos\gamma - \sin\gamma\cos\theta\sin f\right) \\
&\quad\times\left[\cos f\cos\gamma\cos\theta - \sin\gamma\sin f\left(\cos^2\theta - \sin^2\theta\right)\right] = 0, \\
\frac{\partial Z^2}{\partial f} &= -2\sin^2\gamma\sin^2\theta\left(\cos f\cos\gamma - \sin\gamma\cos\theta\sin f\right)\left(\cos\gamma\sin f + \sin\gamma\cos\theta\cos f\right) = 0.
\end{aligned}$$

By analyzing these conditions, we find that stationary points occur in the following cases:

Case 1: $\sin\gamma = 0$. In this case, $Z^2 = 0$, which leads to the minimum value of det[$\mathbf{Q}$].
Case 2: $\sin\theta = 0$. This case also leads to $Z^2 = 0$ and to the minimum value of det[$\mathbf{Q}$].
Case 3: $\cos\gamma\cos f = \sin\gamma\cos\theta\sin f$. This case also leads to $Z^2 = 0$ and to the minimum value of det[$\mathbf{Q}$].
Case 4: The system of equations:

$$\begin{cases}
\cos f\left(\cos^2\gamma - \sin^2\gamma\right) = 2\sin\gamma\cos\gamma\cos\theta\sin f, \\
\sin\gamma\sin f\left(\cos^2\theta - \sin^2\theta\right) = \cos f\cos\gamma\cos\theta, \\
\cos\gamma\sin f = -\sin\gamma\cos\theta\cos f,
\end{cases}$$

leads to two solutions that correspond to the maximum value of Z^2.

Those solutions are

(a): $\gamma = f = \frac{\pi}{2}$, $\theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$,
(b): $f = 0, \theta = \frac{\pi}{2}, \gamma = \frac{\pi}{4}$.

In both cases, the maximum value of Z^2 is $1/4$.

Appendix C. Sloppiness and incompatibility with qutrit initial state

We use a model similar to that in figure 1 to study a two-parameter qutrit model. A general qutrit state is defined as:

$$|\psi_0\rangle = \cos\alpha\,|0\rangle + \mathrm{e}^{\mathrm{i}\kappa_1}\sin\alpha\cos\beta\,|1\rangle + \mathrm{e}^{\mathrm{i}\kappa_2}\sin\alpha\sin\beta\,|2\rangle\,,$$

where $0 \geqslant \alpha,\ \beta \geqslant \frac{\pi}{2},\ 0 \geqslant \kappa_1, \kappa_2 \geqslant 2\pi$.

In this model, the initial quantum state simplifies to:

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |2\rangle\right).$$

The unitary operations and rotation are expressed as

$$U_k = \mathrm{e}^{-\mathrm{i}J_z\lambda_k},\ \ V = \mathrm{e}^{-\mathrm{i}\gamma\vec{J}\cdot\vec{n}},$$

where $\vec{J}$ is the vector of generators of SU(3) group in qutrit system, given by

$$J_x = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix},\ J_y = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{pmatrix},\ J_z = \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix},$$

and

$$\vec{n} = \{\cos\phi\,\sin\theta, \sin\phi\,\sin\theta, \cos\theta\}.$$

Explicitly, the rotation operator is expressed as:

$$V = \begin{pmatrix} k^2 & -i\sqrt{2}\mathrm{e}^{-\mathrm{i}\phi}gk & -\mathrm{e}^{-\mathrm{i}2\phi}g^2\\ -i\sqrt{2}\mathrm{e}^{\mathrm{i}\phi}gk & |k|^2 - g^2 & -i\sqrt{2}\mathrm{e}^{-\mathrm{i}\phi}gk^*\\ -\mathrm{e}^{\mathrm{i}2\phi}g^2 & -i\sqrt{2}\mathrm{e}^{\mathrm{i}\phi}gk^* & (k^*)^2 \end{pmatrix},$$

where $k = \cos\frac{\gamma}{2} - \mathrm{i}\sin\frac{\gamma}{2}\cos\theta$ and $g = \sin\frac{\gamma}{2}\sin\theta$.

The output state $|\psi_\lambda\rangle = U_2 V U_1\,|\psi_0\rangle$ is expressed as:

$$|\psi_\lambda\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}}\mathrm{e}^{-\mathrm{i}(\lambda_1+\lambda_2)}\left(k^2 - \mathrm{e}^{-\mathrm{i}(2\phi-2\lambda_1)}g^2\right)\\ -\mathrm{i}g\left(\mathrm{e}^{\mathrm{i}(\phi-\lambda_1)}k + \mathrm{e}^{-\mathrm{i}(\phi-\lambda_1)}k^*\right)\\ \frac{1}{\sqrt{2}}\left(-\mathrm{e}^{\mathrm{i}(2\phi-\lambda_1+\lambda_2)}g^2 + \mathrm{e}^{\mathrm{i}(\lambda_1+\lambda_2)}\left(k^*\right)^2\right) \end{pmatrix}. \tag{C.1}$$

The partial derivatives of the output state $|\psi_\lambda\rangle$ with respect to λ_1 and λ_2:

$$|\partial_{\lambda_1}\psi_\lambda\rangle = \begin{pmatrix} -i\frac{1}{\sqrt{2}}\left(\mathrm{e}^{-\mathrm{i}(\lambda_1+\lambda_2)}k^2 + \mathrm{e}^{\mathrm{i}(-2\phi+\lambda_1-\lambda_2)}g^2\right)\\ g\left(-\mathrm{e}^{\mathrm{i}(\phi-\lambda_1)}k + \mathrm{e}^{-\mathrm{i}(\phi-\lambda_1)}k^*\right)\\ i\frac{1}{\sqrt{2}}\left(\mathrm{e}^{\mathrm{i}(2\phi-\lambda_1+\lambda_2)}g^2 + \mathrm{e}^{\mathrm{i}(\lambda_1+\lambda_2)}\left(k^*\right)^2\right) \end{pmatrix},$$

$$|\partial_{\lambda_2}\psi_\lambda\rangle = \begin{pmatrix} -i\frac{1}{\sqrt{2}}\left(\mathrm{e}^{-\mathrm{i}(\lambda_1+\lambda_2)}k^2 - \mathrm{e}^{\mathrm{i}(-2\phi+\lambda_1-\lambda_2)}g^2\right)\\ 0\\ -i\frac{1}{\sqrt{2}}\left(-\mathrm{e}^{\mathrm{i}(2\phi-\lambda_1+\lambda_2)}g^2 + \mathrm{e}^{\mathrm{i}(\lambda_1+\lambda_2)}\left(k^*\right)^2\right) \end{pmatrix}.$$

Under these conditions, the inner products reduce to

$$\begin{aligned}
&\langle \partial_{\lambda_1}\psi|\,\partial_{\lambda_1}\psi\rangle = 1,\\
&\langle \partial_{\lambda_2}\psi|\,\partial_{\lambda_2}\psi\rangle = |k|^2 - g^2 + 4g^2|k|^2 - 4g^2\cos^2\frac{\gamma}{2},\\
&\langle \partial_{\lambda_1}\psi|\,\psi\rangle = 0,\\
&\langle \partial_{\lambda_2}\psi|\,\psi\rangle = 0,\\
&\langle \partial_{\lambda_1}\psi|\,\partial_{\lambda_2}\psi\rangle = |k|^2 - g^2.
\end{aligned}$$

The elements of the SLD-QFIM $\boldsymbol{Q}$ and Uhlmann curvature $\boldsymbol{D}$ simplify to:

$$\begin{aligned}
&Q_{11} = 4,\\
&Q_{12} = Q_{21} = 4\left(|k|^2 - g^2\right),\\
&Q_{22} = 4\left[\left(|k|^2 - g^2\right)^2 + 4g^2|k|^2 - 4g^2\cos^2\frac{\gamma}{2}\right],\\
&D_{11} = D_{22} = D_{12} = -D_{21} = 0.
\end{aligned} \tag{C.2}$$

The determinants are:

$$\det\boldsymbol{Q} = 16\sin^4\frac{\gamma}{2}\sin^2 2\theta,\quad \det\boldsymbol{D} = 0.$$

The incompatibility, given by $\det\boldsymbol{D} = 0$, indicates that there is no inherent incompatibility between parameters under this initial quantum state. However, to achieve the minimal C_S bound only when sloppiness is minimized ($s = 1/16$), and the corresponding parameters must be $\gamma = \pi$, $\theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$.

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