Interferometry as a binary decision problem

Matteo G.A. Paris [1]

Arbeitsgruppe "Nichtklassiche Strahlung" der Max-Planck-Gesellschaft an der Humboldt-Universität zu Berlin,
Rudower Chaussee 5, 12489 Berlin, Germany

Abstract

Binary decision theory has been applied to the general interferometric problem. An optimal detection scheme – according to the Neyman–Pearson criterion – has been considered for different phase-enhanced states of the radiation field, and the corresponding bounds on the minimum detectable phase shift have been evaluated. A general bound on interferometric precision has been also obtained in terms of photon number fluctuations of the signal mode carrying the phase information.

1. Introduction

An interferometer is an optical device devised to detect very small variations in the optical path of a light beam. This is usually accomplished by considering two parts of a quantum state traveling along different routes, accumulating different phases. In such a general scheme the precision in measuring the phase depends not only on the involved quantum state, but also on the specific interferometric setup. In order to derive a general bound on the precision of phase measurement a more abstract scheme has to be considered. Here we will consider interferometry as a binary decision problem, where the two signals are only differentiated by the occurrence of a phase shift. Actually, this is more similar to a communication problem with the phase shift playing the role of encoded information. Nonetheless, it appears intuitively obvious that any form of conventional interferometry cannot lead to a better performance than this communication variety.

An abstract outline of an interferometric detection scheme is shown in Fig. 1. An initially prepared state of radiation, say $\hat{\rho}_0$, travels along the interferometer and it is eventually measured by some detectors, denoted by D. The latter is described by an operator-valued probability measure $d\hat{\mu}(x)$, $x \in X$ being the set of the possible detection outcomes. If some environmental parameter changes then also the optical path is subjected to variation, thus leading to a phase-shift φ on the signal mode.

The aim of the detection scheme $d\hat{\mu}(x)$ is that of discriminating between $\hat{\rho}_0$ and its phase-shifted version $\hat{\rho}_1 = \exp(i\hat{n}\varphi)\hat{\rho}_0\exp(-i\hat{n}\varphi)$, which results if some perturbations have occurred. An optimized interferometer is able to tell the $\hat{\rho}$'s apart for φ as small as possible.

This way of posing the interferometric problem naturally leads to view it as a binary decision problem, to which results and methods from quantum detection theory can be applied [1,2]. Here, the phase-shift φ plays the role of a parameter, labeling one of the two possible outputs from the interferometer, namely the perturbed state $\hat{\rho}_1$. Indeed, this approach can be useful

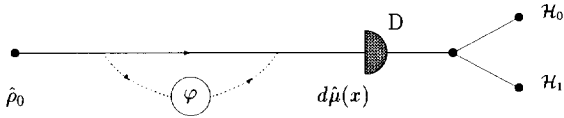[1] E-mail: paris@photon.fta-berlin.de.

Fig. 1. Abstract outline of an interferometric detection scheme.

as it does not refer to any specific detection scheme for the final stage of the interferometer. Thus, an ultimate quantum limit on the interferometric precision can be obtained for specific classes of phase-enhanced states of the radiation field [2].

In this paper we address the interferometric problem as a binary decision one. In the next Section we briefly review the binary decision problem as solved by a Neyman–Pearson optimized strategy. In Section 3 we state the binary interferometric problem. After the illustrative example of coherent states we consider the optimal detection scheme, according to the Neyman–Pearson criterion, for a generic pure state of radiation. A general bound of precision is thus obtained in terms of photon number fluctuations. Two different classes of phase-enhanced states of radiation are then considered: squeezed states and phase-coherent states. The corresponding bounds on the minimum detectable phase-shift are also evaluated. Section 4 closes the paper with some concluding remarks.

2. Neyman–Pearson strategy for binary decision

Our goal is to determine whether or not the initial density matrix has been perturbed. Starting from the outcomes of the detector D we have to infer which is the state of the system, in order to discriminate between the following two hypotheses:

$\mathcal{H}_0$: No perturbation has occurred: true if we infer $\hat{\rho}_0$;

$\mathcal{H}_1$: The system has been perturbed: true if we infer $\hat{\rho}_1$.

We denote by P_{01} the probability of wrong inference for the hypothesis $\mathcal{H}_1$, namely that of inferring $\mathcal{H}_1$ when $\mathcal{H}_0$ is true. In hypothesis testing formulation this is usually referred to as *false alarm probability*. Conversely, we denote by P_{11} the *detection probabil-*

ity, that is the probability of inferring $\mathcal{H}_1$ when it is actually true.

Now, which is the best measurement to discriminate between $\hat{\rho}_0$ and $\hat{\rho}_1$?

If these two states are mutually orthogonal the problem has a trivial solution. It is a matter of measuring the observable for which $\hat{\rho}_0$ and $\hat{\rho}_1$ are eigenstates. However, this is not our case, as it is well known that no orthogonal set of phase-eigenstates is available in quantum optics. In the following we consider nonorthogonal $\hat{\rho}_0$ and $\hat{\rho}_1$ and we focus our attention on pure states $\hat{\rho}_0 = |\psi_0\rangle\langle\psi_0|$ as input for the interferometer.

The optimization problem can be analytically solved, for pure states, by adopting the Neyman–Pearson criterion for binary decision [7]. It reads as follows. First, we have to fix a value for the false alarm probability P_{01}. Then, we have to find the measurement strategy $d\hat{\mu}(x)$ which maximizes the detection probability P_{11}. As a general definition, each measurement strategy which maximizes the detection probability P_{11} for a fixed value of false alarm probability P_{01} is considered as a Neyman–Pearson optimized detection for binary hypothesis testing. It was shown by Helstrom [1] and Holevo [2] that this very general problem could be reduced to solving the eigenvalue problem for the operator

$$d\hat{\mu}(x|\lambda) = \hat{\rho}_1 - \lambda\hat{\rho}_0 , \qquad (1)$$

which represents the optimized measurement scheme. The parameter λ is a Lagrange multiplier. Different values of λ correspond to different values of the false alarm probability, namely to a different Neyman–Pearson strategy.

Once the eigenvalue problem for $d\hat{\mu}(x|\lambda)$ has been solved it results that only positive eigenvectors contribute to the detection probability P_{11} [1,8]. Thus the decision strategy is transparent: after a measurement of the quantity $d\hat{\mu}(x|\lambda)$ if the outcome is positive we infer that the perturbation hypothesis $\mathcal{H}_1$ is true. Conversely, we infer the null hypothesis $\mathcal{H}_0$ when obtaining negative outcome. By expanding the eigenstates of $d\hat{\mu}(x|\lambda)$ in terms of $|\psi_0\rangle$ and $|\psi_1\rangle$ the Lagrange multiplier λ can be eliminated from the expression for the detection probability which results,

$$P_{11} = [\sqrt{P_{01}\kappa} + \sqrt{(1 - P_{01})(1 - \kappa)}]^2,$$

$$0 \leqslant P_{01} \leqslant \kappa,$$

$$P_{11} = 1 \quad \kappa \leqslant P_{01} \leqslant 1 \; . \tag{2}$$

In Eq. (2) κ denotes the square modulus of the overlap between the perturbed and unperturbed states,

$$\kappa = |\langle \psi_0 | \psi_1 \rangle|^2 = |\langle \psi_0 | \exp(i\hat{n}\varphi) | \psi_0 \rangle|^2 \; . \tag{3}$$

The overlap depends both on the initial state and on the phase-shift φ. It is obvious that if the overlap is small, it is easy to discriminate between the two states. Thus, it is possible to obtain strategies with large detection probability without paying the price of an also large false alarm probability. On the contrary, if the overlap becomes appreciable it is difficult to discriminate the states. In the limit of complete overlap the perturbed and the unperturbed states become indistinguishable. The detection probability is now equal to the false alarm probability and the decision strategy is just a matter of guessing after each random measurement outcome.

Choosing a value for the false alarm probability is a matter of convenience, depending on the specific problem to which this approach would be applied. The maximum tolerable value for P_{01} increases with the expected number of measurement outcomes, and conversely a very low rate detection scheme needs a very small false alarm probability. The latter is the case of interferometry, in the following we always will consider a small value for P_{01}.

3. Interferometry as a binary decision problem

Once an input state for the interferometer has been specified, the probability measure in Eq. (1) defines the detection scheme to be performed in order to implement an optimized interferometer. Optimality is in the Neyman–Pearson sense, namely that detection (1) maximizes the detection probability P_{11} at a fixed tolerable value of the false alarm probability P_{01}.

The interferometric strategy is thus the following: the initial state $\hat{\rho}_0$ is prepared and left free to travel along the interferometer. A set of measurements for the quantity (1) is then performed and from the data record we have to infer the state of radiation at the output of the interferometer. From this inference we can discriminate between the two hypotheses, namely we are able to monitor the optical path of the light beam.

The input state is fixed in advance, therefore the detection probability depends only on the accepted value for the false alarm probability and on the actual value of the phase-shift φ. The minimum detectable value of the phase-shift, denoted by φ_M, is defined by the relation

$$P_{11}(\varphi_M; P_{01}) = \tfrac{1}{2} \; . \tag{4}$$

A lower value for P_{11}, in fact, would make the measurements record useless, as no readable information can be extracted in that case.

Let us consider customary coherent states as an illustrative example. Without loss of generality we can set the phase of the initial state to be zero, so that we have

$$|\psi_0\rangle = \exp(-\tfrac{1}{2}\alpha^2) \sum_{k=0}^{\infty} \frac{\alpha^k}{\sqrt{k!}} |k\rangle, \quad \alpha \in \mathbb{R} \; . \tag{5}$$

The photon distribution of a coherent state is Poissonian with mean given by $N \equiv \langle \alpha | \hat{n} | \alpha \rangle = \alpha^2$ and the overlap can be easily evaluated to be

$$\kappa = |\langle \psi_0 | e^{i\hat{n}\varphi} | \psi_0 \rangle|^2 = \exp[-2\alpha^2(1 - \cos\varphi)] \; . \tag{6}$$

Interferometric detection is frequently involved in low rate processes [3]. Therefore, we need, as a general requirement, a small false alarm probability: a convenient setting reads $P_{01} \leqslant \kappa$. Inserting Eq. (2) in Eq. (4) the relation for the minimum detectable phase shift becomes

$$\tfrac{1}{2} = [\sqrt{P_{01}\kappa} + \sqrt{(1 - P_{01})(1 - \kappa)}]^2 \; , \tag{7}$$

that is

$$\kappa = \sqrt{\frac{1 + \sqrt{P_{01}(1 - P_{01})}}{2}} \; . \tag{8}$$

Finally, upon substituting (6) in Eq. (8) and expanding for small φ we have

$$\varphi_M = \sqrt{\log\left(\frac{2}{1 + \sqrt{P_{01}(1 - P_{01})}}\right)} \frac{1}{\sqrt{N}} \; , \tag{9}$$

which represents the lower bound on the minimum detectable phase-shift for *any* interferometer based on

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[3] Among applications of high sensitive interferometry one of the most interesting regards the detection of gravitational waves. The reader may agree that this is a prototype for a *very* low rate process [9].

coherent states. The bound in Eq. (9) is well known and represents the lower bound on the precision also for an interferometer based on classical states of radiation. It is usually termed shot-noise limit.

Let us now consider a generic pure state at the input of the interferometer,

$$|\psi_0\rangle = \sum_{k=0}^{\infty} c_k |k\rangle \, . \tag{10}$$

Still we consider zero initial phase, thus the coefficients $\{c_k\}_{k \in \mathbb{N}}$ are real numbers. For the overlap we have

$$\kappa = \left| \sum_{k=0}^{\infty} c_k^2 e^{ik\varphi} \right|^2$$

$$= \left(\sum_{k=0}^{\infty} c_k^2 \cos k\varphi \right)^2 + \left(\sum_{k=0}^{\infty} c_k^2 \sin k\varphi \right)^2 \, , \tag{11}$$

and, up to second order in the phase-shift,

$$\kappa = 1 - \varphi^2 \Delta N^2 \, , \tag{12}$$

where $\Delta N^2 = \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2$ denotes the photon number fluctuations of the considered state. By substitution in Eq. (8) we obtain the lower bound on the minimum detectable perturbation,

$$\varphi_M = \sqrt{\frac{1 - \sqrt{P_{01}(1 - P_{01})}}{2}} \frac{1}{\Delta N} \, . \tag{13}$$

Eq. (13) represents a quite general result. It indicates that the minimum detectable phase-shift φ_M shows an inverse scaling relative to the photon number fluctuations rather than the photon number intensity. Eq. (13) is not surprising, however, it is worth noticing that we derived it in a direct way from the binary problem approach, i.e. we did not make use of any uncertainty phase–number *pseudo* relation $\Delta N \Delta \varphi \sim 1$. The latter, in fact, can be derived only in a heuristic way [10] and thus possesses only a limited validity.

Eq. (13) suggests to use states with equally probable photon number excitation, namely

$$|\psi\rangle = \sum_{k} c_k |k\rangle, \quad c_k = z \in \mathbb{C}, \quad \forall k \, . \tag{14}$$

Such states, in fact, show infinite photon number fluctuations, thus allowing one, in principle, to monitor an optical path with arbitrary precision. Unfortunately,

the only possibility to construct states as in Eq. (14) is given by the London phase-states [11,12]

$$|\phi\rangle = \sum_{k} e^{ik\phi} |k\rangle \, , \tag{15}$$

which neither possess a finite mean photon number nor are normalizable, namely they are not realistic states of radiation.

A realistic approximation of London phase-states is provided by the so-called phase-coherent states [13]

$$|\chi\rangle = \sqrt{1 - |\chi|^2} \sum_{k} \chi^k |k\rangle \, , \tag{16}$$

where the complex number $\chi = x \exp(i\phi)$ is confined in the unit circle $x < 1$ to assure normalization. A phase-coherent coherent state possesses a mean photon number given by $N = x^2(1 - x^2)^{-1}$ and goes to a London phase-state in the limit $x \to 1$. For a phase-coherent state with zero initial phase the overlap reads

$$\kappa = \left| (1 - x^2) \sum_{k} x^{2k} e^{ik\varphi} \right|^2 = \frac{1 + x^4 - 2x^2}{1 + x^4 - 2x^2 \cos \varphi} \, , \tag{17}$$

leading to a lower bound on the minimum detectable perturbation given by

$$\varphi_M = \sqrt{\frac{1 - \sqrt{P_{01}(1 - P_{01})}}{1 + \sqrt{P_{01}(1 - P_{01})}}} \frac{1}{\sqrt{N(N+1)}} \, . \tag{18}$$

The φ_M scaling in Eq. (18) is much improved relative to the shot-noise limit and shows the benefit of using phase-coherent states.

We now proceed by considering squeezed-coherent states at the input of the interferometer. The use of squeezing in improving precision is quite known for specific setups, as Mach–Zehnder or Michelson interferometers [3,14,15]. Here we obtain a more general bound, which is independent of the measurement strategy.

We consider the interferometer fed by an in-phase squeezed-coherent state, namely a state with parallel signal and squeezing phases. We set this value to zero, so that the initial state is given by

$$|\psi_0\rangle = \hat{D}(x) \hat{S}(r) |0\rangle \, , \tag{19}$$

$\hat{D} = \exp(\alpha a^\dagger - \bar{\alpha} a)$ and $\hat{S} = \exp[\frac{1}{2}(\zeta a^{\dagger 2} - \bar{\zeta} a^2)]$ being the displacement and the squeezing operator re-

spectively. The parameters α and ζ are generally complex, by choosing zero initial phase we can set $\alpha = x \in \mathbb{R}$ and $\zeta = r \in \mathbb{R}$. The quantity x represents the coherent amplitude of the signal whereas r is the squeezing parameter. The mean photon number of such a state is given by $N = x^2 + \sinh^2 r$. We refer to these two terms as the signal and the squeezing photons number, respectively.

A perturbation in the optical path acts differently on the coherent signal relative to the squeezing part, more precisely we have

$$e^{i\hat{n}\varphi}|\alpha, \zeta\rangle = |\alpha e^{i\varphi}, \zeta e^{2i\varphi}\rangle . \tag{20}$$

The overlap is thus expressed by

$$\kappa = |\langle 0|\hat{S}^\dagger(r)\hat{D}^\dagger(x)\hat{D}(xe^{i\varphi})\hat{S}(re^{2i\varphi})|0\rangle|^2 , \tag{21}$$

which is a double Gaussian curve

$$\kappa = \frac{1}{2\sigma_1\sigma_2} \exp\left[-x^2\left(\frac{(1-\cos\varphi)^2}{2\sigma_1^2} + \frac{\sin^2\varphi}{2\sigma_2^2}\right)\right] , \tag{22}$$

with variances given by

$$\sigma_1 = \tfrac{1}{8}[e^{2r}(3 + \cos 2\varphi) + e^{-2r}(1 - \cos 2\varphi)] ,$$
$$\sigma_2 = \tfrac{1}{8}[e^{2r}(1 - \cos 2\varphi) + e^{-2r}(3 + \cos 2\varphi)] . \tag{23}$$

Up to second order in the phase-shift we obtain

$$\kappa \simeq 1 - \varphi^2 x^2 \sigma_1^2 = 1 - 2\varphi^2\beta(1-\beta)N^2 , \tag{24}$$

β being the fraction of the total number of photons which is engaged in squeezing, namely $\sinh^2 r = \beta N$. The lower bound on minimum detectable phase-shift is then obtained by substitution in Eq. (8),

$$\varphi_M = \sqrt{\frac{1 - \sqrt{P_{01}(1 - P_{01})}}{\beta(1 - \beta)}}\frac{1}{2N} . \tag{25}$$

The proportionality constant is of the order of one as a function of P_{01}, whereas the optimum value for the squeezing fraction is given by $\beta = 1/2$.

4. Some remarks

In this paper we have addressed interferometry as a binary decision problem and have derived lower bounds on the minimum detectable phase-shift for some phase-enhanced states of the radiation field. We were not concerned about any specific measurement device and we do not discuss the feasibility of optimized measurement. Actually, the optimal detection, according to the Neyman–Pearson criterion, is generally not available at the present time. Rather, we have attempted to derive the ultimate quantum limit on the detectable phase-shift, which depends only on the initial quantum state at the input of the interferometer. It is worth noticing that for squeezed states the bound in Eq. (25) can actually be approximated by homodyne detection [16,17] or an Mach–Zehnder interferometer [2,3,14,15], however only around a fixed value for the initial phase-shift and for a high efficiency of the involved photodetectors.

An ultimate, state-independent, lower bound on the interferometric precision could be obtained by a further optimization of Eq. (11) over quantum states of radiation, provided that some physical constraints are satisfied. Work along this line is in progress and results will be reported elsewhere.

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