Small amount of squeezing in high-sensitive realistic interferometry

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Abstract

The input states of an interferometer have been optimized at fixed total photon number N and quantum efficiency η. I show that the phase sensitivity is bounded by $\Delta\phi \sim N^{-3/4}$ for a coherent input signal and by $\Delta\phi \sim N^{-1}$ for squeezed optimized inputs. I also show that in actual non-efficient interferometers only little squeezing is needed to reach the best sensitivity.

1. Introduction

For a long time interferometric detection has been one of the most accurate measurements available in physics. The interferometric measurements concerned the special theory of relativity while interferometry of gravitational waves is expected to change the cosmological view. In an interferometer any fluctuation of some environmental parameter of interest leads to a change in the optical path of a light beam. The fluctuation itself can be measured through the detection of the induced phase shift of the radiation field. Thus it is possible, for example, to monitor small variations in the refraction index of a medium or small displacements induced by weak forces, as in a gravitational antenna [1].

The radiation pressure of the involved light beam unavoidably leads to a back-action effect on the monitored parameter. This, in turn, poses the problem of optimizing the phase sensitivity $\Delta\phi$ as a function of the radiation intensity. The main problem of interferometry is precisely the following [2]: Once the total number of photons N impinging onto the apparatus and the quantum efficiency η of the photodetectors have

been fixed, which are the optimal input states leading to the most accurate phase shift measurements?

In a classical-like Mach–Zehnder interferometer one of the inputs has been placed in a coherent state, whereas the other is not excited. In this case the phase sensitivity is bounded by the shot noise limit $\Delta\phi \sim N^{-1/2}$, which is due to quantum fluctuations at the unused port. Caves [3] first suggested using squeezed states to reduce quantum fluctuations, thus beating the shot noise limit. More recently, Yurke et al. [4] have indicated a scheme involving active elements like four wave mixing, whereas Burnett and Holland [5] have suggested using suitable number states as optimal inputs. In both cases the quantum efficiency of the photodetectors has not been considered, and the resulting phase sensitivity is bounded by $\Delta\phi \sim N^{-1}$. Despite the relevance of the subject, no systematic analysis has been, to my knowledge, presented. In this paper I consider two different realistic working regimes for a Mach–Zehnder interferometer. In the former the signal mode is in a coherent state whereas the previously unused port has been placed in a suitable squeezed vacuum. In the latter, also the signal mode is allowed to be squeezed. Input states

which optimize the sensitivity have been found as a function of N for any value of the quantum efficiency η. Comparisons between the two regimes show that for actual photodetectors ($\eta \simeq 0.9$) only a few percent of the total energy have to be engaged in squeezing to reach the best sensitivity.

The present results for Mach–Zehnder interferometers are also valid for the Michelson interferometer when only two bounces occur. I refer to Ref. [6] and the references therein for the consequences of recycling.

In Section 2 a fully quantum analysis of the Mach–Zehnder interferometer is presented. In Section 3 optimal input states for both considered regimes are found and the corresponding lower bounds for the phase sensitivity are evaluated. The two regimes are also compared to each other concerning their performance in realistic situations. Section 4 concludes the paper with some remarks.

2. Mach–Zehnder interferometer

The Mach–Zehnder interferometer provides a way of monitoring fluctuations of a fixed phase shift ϕ. The schematic diagram of the detector setup is reported in Fig. 1. There are two 50–50 beam splitters and two photocounters of quantum efficiency η. The mode carrying the signal is a whereas mode b is prepared in a squeezed vacuum state. The two modes are synchronous. The monitored quantity is the phase shift ϕ between the two correlated modes c and d exiting from the first beam splitter. The measured quantity is the difference photocurrent

$$\hat{I}_D(\phi) = \hat{I}_1 - \hat{I}_2 = \eta(e^\dagger e - f^\dagger f) , \qquad (1)$$

which can be expressed, in terms of the input modes, in the following form,

$$\hat{I}_D(\phi) = \eta[\cos\phi(\hat{n}_a - \hat{n}_b) + i\sin\phi(ab^\dagger - a^\dagger b)] + \hat{V}_{1\eta} . \qquad (2)$$

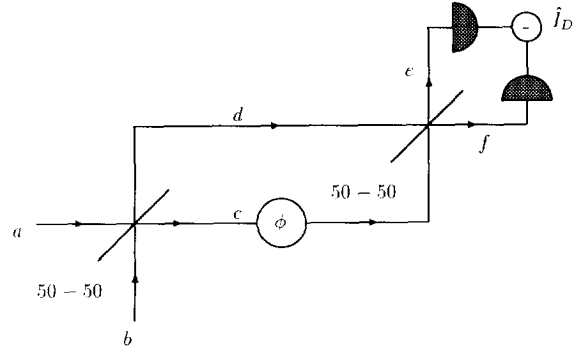The squared photocurrent is given by

Fig. 1. Outline of the Mach–Zehnder interferometer.

$$\begin{aligned}\widehat{I_D^2}(\phi) &= \eta^2 \cos^2\phi[(a^\dagger a)^2 + (b^\dagger b)^2 - 2a^\dagger a b^\dagger b] \\ &\quad - \eta^2 \sin^2\phi[aab^\dagger b^\dagger + a^\dagger a^\dagger bb \\ &\quad - aa^\dagger b^\dagger b - a^\dagger abb^\dagger] \\ &\quad + 2\eta^2 i\sin\phi\cos\phi[a^\dagger aab^\dagger + aa^\dagger ab^\dagger - b^\dagger bb^\dagger a \\ &\quad - b^\dagger b^\dagger ba - a^\dagger aa^\dagger b - a^\dagger a^\dagger ab + b^\dagger bba^\dagger + bb^\dagger ba^\dagger] \\ &\quad + \tfrac{1}{2}\eta(1-\eta)(a^\dagger a + b^\dagger b) + \hat{V}_{2\eta} . \end{aligned} \qquad (3)$$

From Eqs. (2) and (3) one can evaluate the mean value $\overline{I_D}(\phi)$ and the variance $\overline{\Delta I_D^2}(\phi)$ of the probability distribution of the experimental outcomes,

$$\overline{I_D}(\phi) = \mathrm{Tr}\{\hat{\rho}\hat{I}_D(\phi)\} ,$$

$$\overline{\Delta I_D^2}(\phi) = \mathrm{Tr}\{\widehat{I_D^2}(\phi)\} - \overline{I_D}^2(\phi) . \qquad (4)$$

Above $\hat{\rho} = \rho_a \otimes \rho_b$ represents the input density matrix. The operators $\hat{V}_{1\eta}$ and $\hat{V}_{2\eta}$ contain combinations of operators a, $a^\dagger$, b and $b^\dagger$ which, however, lead to vanishing expectation values when the input modes a and b have been placed in a squeezed coherent state $|\alpha, \zeta_1\rangle$ and in a squeezed vacuum $|0, \zeta_2\rangle$, respectively. In formula one has

$$\mathrm{Tr}\{\hat{\rho}\hat{V}_{i\eta}\} = 0, \quad i = 1, 2 ,$$

$$\hat{\rho} = |\alpha, \zeta_1\rangle\langle\zeta_1, \alpha| \otimes |0, \zeta_2\rangle\langle\zeta_2, 0| , \qquad (5)$$

with $\alpha = |\alpha|\, e^{i\psi_\alpha}$, $\zeta_1 = \rho\, e^{2i\psi_1}$ and $\zeta_2 = r\, e^{2i\psi_2}$.

Interferometric detection is based on the ϕ-dependence of the measured photocurrent. Any variation $\Delta\phi$ of ϕ, in fact, leads to a variation in the mean value of $\hat{I}_D(\phi)$. From error propagation calculus one obtains

$$\Delta\phi = [\overline{\Delta I_{\mathrm{D}}^2(\phi)}]^{1/2} \left| \frac{\delta \overline{I_{\mathrm{D}}}(\phi)}{\delta\phi} \right|_{\phi=\phi_0}^{-1} , \qquad (6)$$

where ϕ_0 is the working point of the interferometer, that is, the initially fixed value of the monitored phase shift. The sensitivity $\Delta\phi$ is the smallest fluctuation of the phase ϕ around ϕ_0 which can be found, and depends on the value ϕ_0 itself. Usually, to minimize $\Delta\phi$, the working point is selected which maximizes the derivative in Eq. (6). If the inputs are chosen according to Eq. (5) the mean value of the photocurrent is given by

$$\overline{I_{\mathrm{D}}}(\phi) = \eta \cos\phi(|\alpha|^2 + \sinh^2\rho - \sinh^2 r) . \qquad (7)$$

The best working point is thus given by $\phi_0 = \pi/2$ which, in turn, corresponds to a zero mean value of the photocurrent. In this case the variance of the probability distribution is given by

$$\begin{aligned}
\overline{\Delta I_{\mathrm{D}}^2}(\pi/2) = \eta^2 \{ & |\alpha|^2 [1 + 2\sinh^2 r \\
& - 2\sinh(2r)\cos(\psi_\alpha - 2\psi_2)] + \sinh^2 r \\
& + \sinh^2\rho + 2\sinh^2 r \sinh^2\rho \\
& - \tfrac{1}{2}\sinh(2r)\sinh(2\rho)\cos(2\psi_2 - 2\psi_1) \} \\
& + \tfrac{1}{2}\eta(1-\eta)(|\alpha|^2 + \sinh^2\rho + \sinh^2 r) . \qquad (8)
\end{aligned}$$

This relation shows that the sharpest distribution for the photocurrent I_{D} is obtained for $2\psi_1 = 2\psi_2 = \psi_\alpha$, that is, for squeezing of the two modes parallel to each other, and both in the direction of the signal phase. In the following I consider for simplicity, however without loss of generality, the situation in which $\psi_1 = \psi_2 = \psi_\alpha = 0$. After this assumption the phase sensitivity $\Delta\phi_\eta$ turns out to be

$$\begin{aligned}
\Delta\phi_\eta = & \frac{1}{\alpha^2 + \sinh^2\rho - \sinh^2 r} \\
& \times [\alpha^2 \mathrm{e}^{-2r} + \sinh^2 r + \sinh^2\rho + 2\sinh^2 r \sinh^2\rho \\
& - 2\sinh(2r)\sinh(2\rho) + \sigma_\eta^2(\alpha^2 + \sinh^2\rho \\
& + \sinh^2 r)]^{1/2} , \qquad (9)
\end{aligned}$$

where $\sigma_\eta^2 = (1-\eta)/2\eta$ represents the extra width of the probability distribution of the photocurrent due to the non-unit quantum efficiency of the photodetector.

3. Optimizing the states

Once the working point has been selected, the phase sensitivity $\Delta\phi_\eta$ of the Mach–Zehnder interferometer can be still optimized with respect to the input states. The physical constraint is the total photon number N impinging onto the apparatus, which is expressed by the formula

$$N = \alpha^2 + \sinh^2\rho + \sinh^2 r . \qquad (10)$$

In Eq. (10) α^2 is the number of signal photons whereas $\sinh^2\rho$ and $\sinh^2 r$ denote the number of squeezing photons of modes a and b respectively. Optimizing the inputs thus means finding the best partition of the total photon number N between the signal photons and squeezing photons of the two modes. In this section I consider and compare two distinct working regimes at different values of the quantum efficiency η of the photodetectors. In the first the signal mode a has been placed in a coherent state, i.e. $\rho = 0$, whereas in the other it is allowed to be squeezed. In both cases mode b supplies a suitable squeezed vacuum and the working point is fixed at $\phi_0 = \pi/2$.

3.1. Coherent-signal regime

If the input signal is in a coherent state Eq. (9) reduces to

$$\begin{aligned}
\Delta\phi_\eta = & \frac{1}{\alpha^2 - \sinh^2 r} \\
& \times [\alpha^2 \mathrm{e}^{-2r} + \sinh^2 r + \sigma_\eta^2(\alpha^2 + \sinh^2 r)]^{1/2} , \\
& \qquad (11)
\end{aligned}$$

whereas the energy constraint is expressed by

$$\alpha^2 + \sinh^2 r = N . \qquad (12)$$

Substituting Eq. (11) in Eq. (12) $\Delta\phi_\eta$ as a function of the variable r is obtained. Taking the first derivative with respect to r leads to the optimization equation

$$\begin{aligned}
t^5 + & 2[N(1 + 4\sigma_\eta^2) - 2]t^4 + 6(2N+1)t^3 \\
& - 2N(4N + 4\sigma_\eta^2 + 7)t^2 + t - 4 = 0 , \qquad (13)
\end{aligned}$$

where $t = \mathrm{e}^{2r}$. Eq. (13) can easily be solved numerically for each value of the quantum efficiency η. In
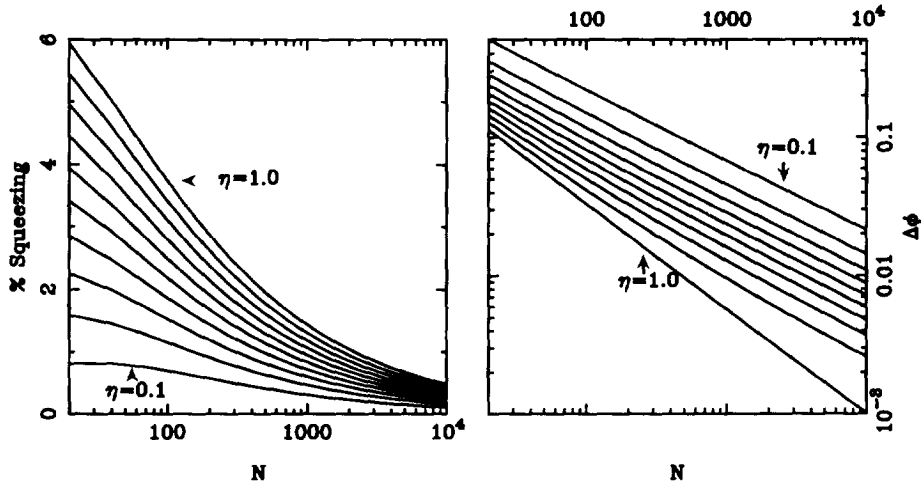
Fig. 2. Optimal squeezing fraction for mode b in the coherent-signal regime and the corresponding sensitivity $\Delta\phi_\eta$ as a function of the total photon number N for quantum efficiency η from 0.1 to 1.0.

Fig. 2 the optimal fraction $\sinh^2 r/N$ of the total photon number N which has to be engaged in the squeezing of the vacuum mode b, is reported for different values of η as a function of N. The corresponding sensitivity $\Delta\phi_\eta$ is also reported. The fraction of squeezing photons decreases with increasing N and decreasing η. For large N it becomes less than 1% for any value of η. The best sensitivity is obviously reached for $\eta = 1$. In this case the lower bound on the phase sensitivity is given by the power law (estimates of numerical errors are given)

$$\Delta\phi_1 = \frac{1.14 \pm 0.01}{N^{0.76 \pm 0.01}} \ . \tag{14}$$

For non-unit quantum efficiency $\eta < 1$ the sensitivity decreases as $\Delta\phi_\eta = C(\eta)N^{-\gamma(\eta)}$ where approximately the proportionality constant $C(\eta)$ is unity and the exponent $\gamma(\eta)$ decreases as $\gamma(\eta) \simeq 0.5 + 0.05\,e^{1.76\eta}$. The precise behaviour of the exponent $\gamma(\eta)$ of the power law as a function of η is reported in Fig. 5.

3.2. Squeezed-signal regime

If the input signal is allowed to be squeezed the phase sensitivity can be optimized also with respect to the variable ρ. The method of Lagrange multipliers reduces the problem of optimizing $\Delta\phi_\eta$ given in Eq.

(9), with the constraint (10), to that of minimizing the function

$$F_\eta(\alpha^2, \rho, r; \lambda) = \Delta\phi_\eta(\alpha^2, \rho, r) + \lambda(\alpha^2 + \sinh^\rho + \sinh^2 r) \ , \tag{15}$$

with respect to α^2, ρ and r, λ being the Lagrange multiplier. The variational problem (15) is equivalent to the equations

$$e^{4\rho} = \tfrac{1}{3}(4\,e^{4r} - 1) \ , \tag{16a}$$

$$(N - 2\sinh^2 r)\{e^{2r-2\rho} - e^{2\rho-2r}$$
$$- e^{-2r}[N - 2\sinh^2 r + \tfrac{1}{2}\sinh(2r)]\}$$
$$+ 2\sinh(2r)[4\sigma_\eta^2 - 2 + e^{2\rho-2r}$$
$$+ e^{2r-2\rho} + e^{-2r}(N - 2\sinh^2 r)] = 0 \ , \tag{16b}$$

$$\alpha^2 + \sinh^2 \rho + \sinh^2 r = 0 \ , \tag{16c}$$

which can be solved numerically at a fixed value of η. For unit quantum efficiency $\eta = 1$ the ideal phase sensitivity [1] is reached,

$$\Delta\phi_1 \sim \frac{1.36 \pm 0.01}{N^{1.00 \pm 0.01}} \ , \tag{17}$$

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[1] An ideal experiment to detect the phase shift of an e.m. field has to be defined in the framework of quantum estimation theory, see Ref. [7]. The phase sensitivity of such a detection has been obtained in Ref. [8].
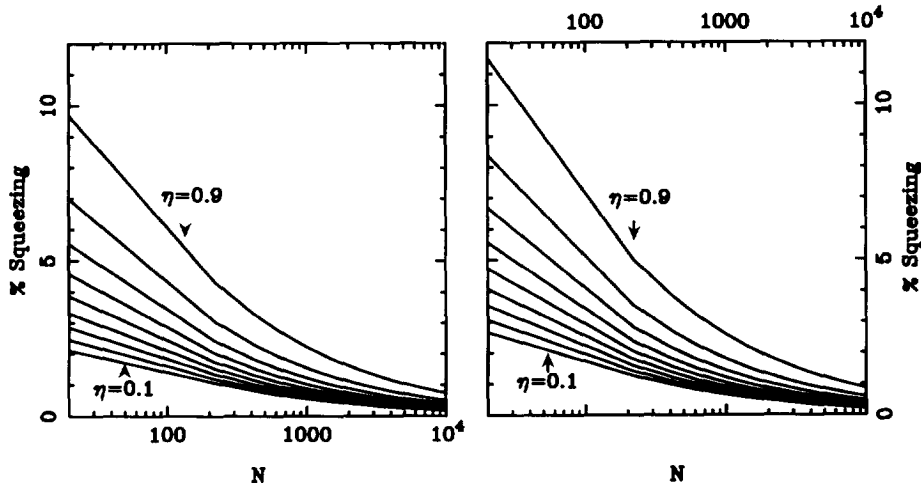
Fig. 3. Optimal squeezing fraction for the two modes in the squeezed-signal regime. Both fractions are reported as a function of the total photon number N for quantum efficiency η from 0.1 to 0.9.

corresponding to an input configuration in which about half of the input energy impinges onto the signal α, while the rest of the energy is distributed, nearly equally, between the squeezing of the two modes,

$$\alpha^2 \lesssim \tfrac{1}{2}N , \quad \sinh^2 \rho \gtrsim \tfrac{1}{4}N ,$$

$$\sinh^2 r \simeq \tfrac{1}{4}N . \tag{18}$$

Unfortunately, the working regime described by Eqs. (18) is not stable with respect to the quantum efficiency η. A small displacement of η from unity, in fact, strongly degrades the power law given in Eq. (17). One has $\Delta\phi_\eta = C(\eta)N^{-\gamma(\eta)}$ where approximately the proportionality constant $C(\eta)$ is unity. The exponent $\gamma(\eta)$ decreases exhibiting a Lorentzian shape versus η. Roughly one has $\gamma(\eta) \simeq 0.5 + 0.175\eta + 0.025\eta^2/(1.84 - 1.76\eta^2)$. The precise behaviour of $\gamma(\eta)$ as a function of η is reported in Fig. 5. In Fig. 3 the optimal fraction of squeezing photons for the two modes is reported, as a function of the total photon number N, for different values of η less than unity. As expected from Eq. (16a) the fraction of squeezing photons suitable for mode a is, independent of the value of η, slightly greater than the corresponding one for mode b. In Fig. 4 the optimal phase sensitivity $\Delta\phi_\eta$ is reported as a function of the total photon number N for different values of η.
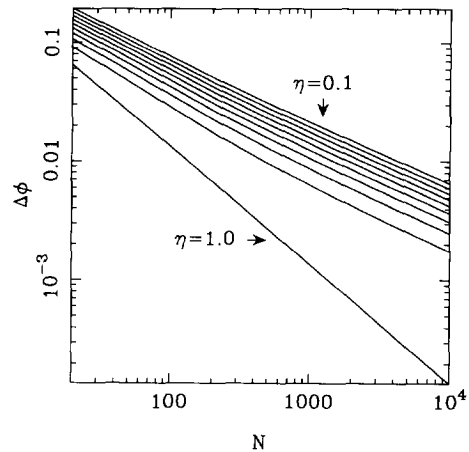

Fig. 4. Optimal phase sensitivity $\Delta\phi_\eta$ for the squeezed-signal regime as a function of the total photon number N for quantum efficiency η from 0.1 to 0.9.

3.3. Comparison between the two regimes

In the coherent-signal regime the phase sensitivity is bounded by the power law $\Delta\phi \sim N^{-3/4}$, whereas in the squeezed-signal regime the bound is given by $\Delta\phi \sim N^{-1}$. These bounds have been obtained for unit quantum efficiency whereas for decreasing η the phase sensitivity decreases. The optimal fraction of squeezing photons also decreases for decreasing η.
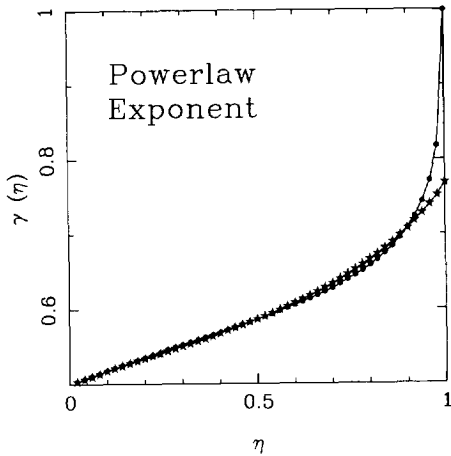
Fig. 5. Comparison between the phase sensitivity power law exponent $\gamma(\eta)$ for the two working regimes. The curves are reported as a function of the quantum efficiency η of the photodetectors, circles are for squeezed regime and squares for the coherent regime.

This is due to the extra width σ_η of the photocurrent probability distribution, which unavoidably limits the benefit induced by squeezing. For any value of the quantum efficiency η the number of squeezing photons needed for the coherent-signal regime is altogether much lower than the corresponding one for the squeezed-signal regime. Also the decrease of the phase sensitivity is different for the two considered regimes. In Fig. 5 I report the behaviour of the exponent $\gamma(\eta)$ of the power law $\Delta\phi \sim N^{-\gamma(\eta)}$, for both regimes. For vanishing η both working regimes show a phase sensitivity which converges to the shot noise limit $\Delta\phi \sim N^{-1/2}$. For η up to $\eta \simeq 0.9$ the two curves are almost indistinguishable, whereas for $\eta \gtrsim 0.9$ the phase sensitivity of the squeezed-signal regime becomes definitely better than that of the coherent-signal regime. The value $\eta \simeq 0.9$ can thus be considered as a threshold, beyond which the squeezing of the input signal can improve the sensitivity of the interferometer.

Up-to-date photodetectors show a quantum efficiency around the threshold value $\eta \simeq 0.9$. Therefore in actual interferometers the coherent-signal working regime has to be preferred because it reaches the best sensitivity using only a few percent of the squeezing photons. Efforts in producing large squeezing will become useful for interferometry only when high efficiency photodetectors ($\eta > 0.9$) will be available.

4. Conclusions

In this paper I have found the optimal input states for best monitoring a fixed phase shift in a Mach–Zehnder interferometer. The input states have been optimized with respect to both the quantum efficiency η of the photodetectors, and the total photon number N impinging onto the apparatus. Two distinct working regimes have been analyzed. In the first the input signal is in a coherent state, whereas in the other it is allowed to be squeezed. In both cases the second port of the interferometer has been placed in a suitable squeezed vacuum.

I have shown that, in the limit of unit quantum efficiency, the phase sensitivity is bounded by the power law $\Delta\phi \sim N^{-3/4}$ in the coherent-signal regime and by $\Delta\phi \sim N^{-1}$ in the squeezed-signal regime. For non-unit quantum efficiency these sensitivities decrease. It has been shown that for a quantum efficiency up to $\eta \lesssim 0.9$ the two regimes lead to the same power law for the sensitivity, however with the coherent-signal regime requiring much less squeezing photons. Beyond this threshold value the squeezing signal regime shows a phase sensitivity definitely better than the coherent-signal regime, thus justifying the need for a larger number of squeezing photons.

In actual interferometric devices the quantum efficiency η is around the threshold value $\eta \simeq 0.9$. Therefore the coherent-signal working regime has to be preferred, assuring the best sensitivity with a few percent of squeezing photons. A large amount of squeezing will improve the phase sensitivity only when high efficiency photodetectors ($\eta > 0.9$) will be available.

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