# Canonical Naimark extension for generalized measurements involving sets of Pauli quantum observables chosen at random 

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#### Abstract

We address measurement schemes where certain observables $X_{k}$ are chosen at random within a set of nondegenerate isospectral observables and then measured on repeated preparations of a physical system. Each observable has a probability $z_{k}$ to be measured, with $\sum_{k} z_{k}=1$, and the statistics of this generalized measurement is described by a positive operator-valued measure. This kind of scheme is referred to as quantum roulettes, since each observable $X_{k}$ is chosen at random, e.g., according to the fluctuating value of an external parameter. Here we focus on quantum roulettes for qubits involving the measurements of Pauli matrices, and we explicitly evaluate their canonical Naimark extensions, i.e., their implementation as indirect measurements involving an interaction scheme with a probe system. We thus provide a concrete model to realize the roulette without destroying the signal state, which can be measured again after the measurement or can be transmitted. Finally, we apply our results to the description of Stern-Gerlach-like experiments on a two-level system.


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## I. INTRODUCTION

In this paper we deal with a specific class of generalized quantum measurements usually referred to as quantum roulettes. These quantum measurements are achieved through the following procedure. Consider $K$ projective measurements, described by the set $\left\{X_{k}\right\}_{k=1, \ldots, K}$ of nondegenerate isospectral observables in a Hilbert space $H$. The system is sent to a detector which, at random, performs the measurement of the observable $X_{k}$. Each observable has a probability $z_{k}$ to be measured, with $\sum_{k} z_{k}=1$. This scheme is referred to as quantum roulette since the measured observable $X_{k}$ is chosen at random, e.g., according to the fluctuating value of a physical parameter, as it happens for the outcome of a roulette wheel. The generalized observable actually measured by the detector is described by a positive operator-valued measure (POVM), which provides the probability distribution of the outcomes and the postmeasurement states [1-3].

As a matter of fact, any POVM on a given Hilbert space may be implemented as a projective measurement in a larger one (e.g., see [4] for single-photon qudits). This measurement scheme is usually referred to as a Naimark extension of the POVM. Indeed, it is quite straightforward to find a Naimark extension for the POVM of any quantum roulette in terms of a joint measurement performed on the system under investigation and an ancillary one.

On the other hand, for any POVM the Naimark theorem [5] ensures the existence of a canonical Naimark extension, i.e., the implementation of the POVM as an indirect measurement involving an independent preparation of an ancillary (probe) system [6], an interaction of the probe with the system under investigation, and a final step in which only the probe is subjected to a (projective) measurement [7,8]. A question thus arises on the canonical implementation of the quantum roulette's POVM and on the resources needed to
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realize the corresponding interaction scheme. This is the main point of this paper. In particular, we focus on quantum roulettes involving the measurements of Pauli matrices on a qubit system and explicitly evaluate their canonical Naimark extensions.

We recall that having the Naimark extension of a generalized measurement is, in general, highly desirable, since it provides a concrete model to realize an apparatus which performs the measurement without destroying the state of the system under investigation. Thus, the state after the measurement can be measured again, or can be transmitted, and the tradeoff between information gain and measurement disturbance may be evaluated [9-12]. Alternatively, the scheme may serve to perform indirect quantum control [13].

It should be emphasized that the concept of quantum roulette provides a natural framework to describe the measurement scheme in which the measured observable depends on an external parameter, which cannot be fully controlled and fluctuates according to a given probability distribution. A prominent example is given by the Stern-Gerlach apparatus, which allows one to measure a spin component of a particle in the direction individuated by an inhomogeneous magnetic field [14,15]. Indeed, whenever the field is fluctuating, or the uncertainty in the splitting force is taken into account [16], the measurement scheme is described by a quantum roulette. Also, in this case, if we have the canonical Naimark extension for the roulette, then we have a concrete way to realize a measurement without destroying the state [17]. We recall that in the continuous variable regime quantum roulettes involving homodyne detection with randomized phase of the local oscillator have been already studied theoretically [18] and realized experimentally [19].

The paper is structured as follows. In the next section we introduce notation, briefly review the Naimark theorem, and gather all the necessary tools, e.g., the Cartan decomposition of $\mathrm{SU}(4)$ transformations, which allows us to greatly reduce the number of parameters involved in the problem of finding
the canonical Naimark extension. In Sec. III we introduce the concept of quantum roulette, derive the corresponding POVM, and illustrate an example of noncanonical Naimark extension. In Sec. IV we derive the canonical extension for one-parameter Pauli quantum roulettes and discuss details of their implementation, whereas Sec. V is devoted to analyze Stern-Gerlach-like experiments as quantum roulettes, i.e., taking into account the possibility that the magnetic field is randomly fluctuating. Finally, Sec. VI closes the paper with some concluding remarks.

## II. NOTATION AND TOOLS

## A. POVMs and the Naimark theorem

When we measure an observable on a quantum system, we cannot predict which outcome we will obtain in each run. What we know is the spectrum of possible outcomes and their probability distribution. Given a system described by a state $\rho$ in the Hilbert space $H$, to obtain the probability distribution of the outcomes $x$ we use the Born Rule:

$$
p_{x}=\operatorname{Tr}\left[\rho \Pi_{x}\right]
$$

In order to satisfy the properties of the distribution $p_{x}$, the operators $\Pi_{x}$ do not need to be projectors. The operators $\Pi_{x}$ have to be positive, $\Pi_{x} \geqslant 0$, since the probability distribution $p_{x}$ has to be positive for every $|\varphi\rangle \in H$, and normalized, $\sum_{x} \Pi_{x}=\mathbb{I}$, since $p_{x}$ is normalized. A decomposition of identity by positive operators $\Pi_{x}$ will be referred to as a positive operator-valued measure ( POVM ), and the operators $\Pi_{x}$ are the elements of the POVM.

We use $\Pi_{x}$ to get information about the probability distribution $p_{x}$, but if we are interested in postmeasurement states we have to introduce the set of operators $M_{x}$, the detection operators. These operators should give the same probability distribution given by $\Pi_{x}$; thus, they are obtained from $p_{x}=$ $\operatorname{Tr}\left[M_{x} \rho M_{x}^{\dagger}\right]=\operatorname{Tr}\left[\rho \Pi_{x}\right]$. Therefore, detection operators that satisfy $\Pi_{x}=M_{x} M_{x}^{\dagger}$ are $M_{x}=U_{x} \sqrt{\Pi_{x}}$, with $U_{x}$ being a unitary operator such that $U_{x} U_{x}^{\dagger}=\mathbb{I}$, and this leaves a residual freedom on the postmeasurement states. The postmeasurement states are then given by

$$
\rho_{x}=\frac{1}{p_{x}} M_{x} \rho M_{x}^{\dagger} .
$$

A measurement described by the operators $\Pi_{x}$ is referred to as generalized measurement.

In order to link general measurements with physical schemes of measurement, we have the Naimark theorem, which states that a generalized measurement in a Hilbert space $H_{A}$ may be always seen as an indirect measurement in a larger Hilbert space given by the tensor product $H_{A} \otimes H_{B}$. This indirect measure is known as canonical Naimark extension for the generalized measurement (see Fig. 1). Conversely, when we perform a projective measure on the subsystem $H_{B}$ of a composite system $H_{A} \otimes H_{B}$, the degrees of freedom of $H_{B}$ may be traced out and we obtain the same probability distribution of the outcomes of the projective measurement and the same postmeasurement states performing a generalized measurement on the subsystem $H_{A}$.


FIG. 1. (Color online) The two measurement schemes linked by the Naimark theorem. (a) A generalized measurement described by the POVM $\Pi_{x}=M_{x}^{\dagger} M_{x}$ and (b) its canonical Naimark extension, defined by the triple $\left\{\rho_{B}, U,\left\{P_{x}\right\}\right\}$, describing the probe state $\rho_{B}$, the evolution operator $U$, and the projective measurement $\left\{P_{x}\right\}$ on the probe system, respectively.

The Naimark theorem gives a practical recipe to evaluate the canonical extension for a generalized measurement in a Hilbert space $H_{A}$ :

$$
\begin{equation*}
\Pi_{x}=\operatorname{Tr}_{B}\left[\mathbb{I} \otimes \rho_{B} U^{\dagger} \mathbb{I} \otimes P_{x} U\right] \tag{1}
\end{equation*}
$$

where $\rho_{B} \in L\left(H_{B}\right)$ describes the state of the probe system (or ancilla), the operators $\left\{P_{x}\right\} \in L\left(H_{B}\right)$ are a set of projectors which describe the measurement on the ancilla, and the unitary operator $U \in L\left(H_{A} \otimes H_{B}\right)$ works on both the system and the ancilla. A canonical Naimark extension for the generalized measurement given by the operators $\left\{\Pi_{x}\right\} \in L\left(H_{A}\right)$ is thus individuated by the triple $\left\{\rho_{B}, U,\left\{P_{x}\right\}\right\}$.

Evaluating the canonical Naimark extension for a generalized measurement is desirable since it gives a concrete model to realize an apparatus which performs the measurement without destroying the state. Then, the postmeasurement state can be transmitted, or measured again.

## B. The Cartan decomposition of $\mathbf{S U}(4)$ transformations

In the following we are going to deal with two-qubit interactions, i.e., unitary operators (with unit determinant) of the group $\mathrm{SU}(4)$, which are individuated by 15 parameters. In order to reduce the number of these parameters we will make use of the Cartan decomposition, which allows us to factor a general operator in $\operatorname{SU}(4)$ into local operators working on single qubits and a single two-qubit operator $V$ individuated by three parameters [20-22] (see Fig. 2).

According to the Cartan decomposition any $X \in \mathrm{SU}(4)$ can be rewritten as $X=\left(R_{1} \otimes R_{0}\right) V\left(S_{1} \otimes S_{0}\right)$, where $R_{1}, R_{0}, S_{1}, S_{0} \in \mathrm{SU}(2)$, and $V=\exp \left\{i \sum_{j=1}^{3} k_{j} \sigma_{j} \otimes \sigma_{j}\right\}$, with $\mathbf{k} \equiv\left(k_{1}, k_{2}, k_{3}\right) \in \mathbb{R}^{3}$. The operators $\sigma_{i}$ are the Pauli matrices.


FIG. 2. (Color online) The Cartan decomposition of the operator $X \in \mathrm{SU}(4)$, given by $\left(R_{1} \otimes R_{0}\right) V\left(S_{1} \otimes S_{0}\right)$.

If we introduce the following equivalence relation in $\mathrm{SU}(4)$,

$$
\begin{equation*}
A \sim B \quad \text { if } \quad A=\left(R_{1} \otimes R_{0}\right) B\left(S_{1} \otimes S_{0}\right) \tag{2}
\end{equation*}
$$

we can split the whole space of unitary operators with unit determinant into equivalence classes. Then, since an operator $X \in \mathrm{SU}(4)$ is represented (by the equivalent relation given here) by the matrix $V$, it is possible to establish a link between operators in $\mathrm{SU}(4)$ and real vectors: $X \sim \mathbf{k}$, where $\mathbf{k}$ is the class vector of $X$. The following operations are class-preserving: (1) shift- $\mathbf{k}$ can be shifted by $\pm \frac{\pi}{2}$ along one of its components; (2) reverse - the sign of two components of $\mathbf{k}$ can be reversed; and (3) swap-two components of $\mathbf{k}$ can be swapped.

By the use of these operations it is always possible to reduce any $\mathbf{k}$ into a bounded region $K$ given by the following:
(1) $\frac{\pi}{2}>k_{1} \geqslant k_{2} \geqslant k_{3} \geqslant 0$.
(2) $k_{1}+k_{2} \leqslant \frac{\pi}{2}$.
(3) If $k_{3}=0$, then $k_{1} \leqslant \frac{\pi}{4}$.
$\mathbf{k} \in K$ are referred to as canonical class vectors.
The expression of the operator $V$ may be simplified using a different set of parameters, e.g.,

$$
\begin{aligned}
& k_{1}=-\frac{\alpha_{1}-\alpha_{2}}{4} \\
& k_{2}=-\frac{\alpha_{1}+\alpha_{2}}{4} \\
& k_{3}=-\frac{\alpha_{3}}{2}
\end{aligned}
$$

Additionally, we introduce the operators $\Sigma_{i}=\frac{1}{2} \sigma_{i} \otimes \sigma_{i}$, which are normalized in the space of $4 \times 4$ operators, with the inner product $\langle A, B\rangle=\operatorname{Tr}\left[B^{\dagger} A\right]$. Eventually, we obtain the following matrix $V$ :

$$
\begin{equation*}
V=\exp \left\{-i\left[\frac{1}{2}\left(\alpha_{1}-\alpha_{2}\right) \Sigma_{1}+\frac{1}{2}\left(\alpha_{1}+\alpha_{2}\right) \Sigma_{2}+\alpha_{3} \Sigma_{3}\right]\right\} \tag{3}
\end{equation*}
$$

In terms of the $\alpha$ parameters the bounded region corresponding to the canonical class vectors is given by

$$
\begin{aligned}
-\pi & \leqslant \alpha_{1} \leqslant 0, \\
0 & \leqslant \alpha_{2} \leqslant-\alpha_{1}, \\
\alpha_{1}+\alpha_{2} & \leqslant 2 \alpha_{3} \leqslant 0, \\
\text { If } \quad \alpha_{3} & =0 \quad \text { then } \quad \alpha_{1}-\alpha_{2} \geqslant-\pi .
\end{aligned}
$$

As we will see in the next section, the Cartan decomposition simplifies the problem of finding the canonical Naimark extension for the quantum roulette, since we will be able to neglect single-qubit operations, and thus reducing the 15-parameters operator $U$ to the 3-parameters operator $V$. Furthermore, we will restrict the interval of the parameters of $V$ to the bounded region defined above.

## III. THE PAULI QUANTUM ROULETTE WHEEL

Let us consider $K$ observables $\left\{X_{k}\right\}$ in a Hilbert space $H_{A}$ with dimension $d=\operatorname{dim}\left(H_{A}\right)$. All the observables are nondegenerate and isospectral. Since the observables are nondegenerate and the Hilbert space is finite dimensional, each of them has $d$ eigenvalues. We use a detector which chooses at random one of these observables and performs a measurement of that observable. Each observable has a probability $z_{k}$ of being selected by the detector and $\sum_{k} z_{k}=1$. This scheme,
denoted by the $K$-tuple $\left\{\left\{X_{1}, z_{1}\right\},\left\{X_{2}, z_{2}\right\}, \ldots,\left\{X_{K}, z_{K}\right\}\right\}$, is referred to as quantum roulette.

If we have a system represented by the state $\rho \in L\left(H_{A}\right)$ and we send it to the detector, the probability distribution of the outcomes is given by

$$
\begin{align*}
p_{x} & =\sum_{k} z_{k} p_{x}^{(k)}=\sum_{k} z_{k} \operatorname{Tr}\left[\rho P_{x}^{(k)}\right] \\
& =\operatorname{Tr}\left[\rho \sum_{k} z_{k} P_{x}^{(k)}\right]=\operatorname{Tr}\left[\rho \Pi_{x}\right], \tag{4}
\end{align*}
$$

where $p_{x}^{(k)}$ is the probability distribution of the outcome $x$ for the observable $X_{k}$ and $P_{x}^{(k)}=|x\rangle^{(k)(k)}\langle x|$ is the onedimensional projector on the eigenspace of the eigenvalue $x$ for the observable $X_{k}$. In the last equality of Eq. (4) we have introduced the POVM of the roulette, whose elements are given by

$$
\begin{equation*}
\Pi_{x}=\sum_{k} z_{k} P_{x}^{(k)} \tag{5}
\end{equation*}
$$

$\Pi_{x}$ are positive operators; indeed, given any $|\varphi\rangle \in H_{A}$, we have $\langle\varphi| \Pi_{x}|\varphi\rangle=\sum_{k} z_{k}\left|\langle\varphi \mid x\rangle^{(k)}\right|^{2} \geqslant 0$; and they represent a decomposition of identity, since

$$
\sum_{x} \Pi_{x}=\sum_{x} \sum_{k} z_{k} P_{x}^{(k)}=\sum_{k} z_{k} \sum_{x} P_{x}^{(k)}=\sum_{k} z_{k} \mathbb{I}=\mathbb{I} .
$$

On the other hand, $\Pi_{x}$ are not orthogonal projectors since $\Pi_{x} \Pi_{x^{\prime}} \neq \delta_{x x^{\prime}} \Pi_{x}$. Indeed,

$$
\Pi_{x} \Pi_{x^{\prime}}=\sum_{k, k^{\prime}} z_{k} z_{k^{\prime}} P_{x}^{(k)} P_{x^{\prime}}^{\left(k^{\prime}\right)} \neq 0
$$

In fact, while ${ }^{(k)}\left\langle x \mid x^{\prime}\right\rangle^{(k)}$ has to be equal to $\delta_{x x^{\prime}}$ for a fixed value of $k$, the quantity ${ }^{(k)}\left\langle x \mid x^{\prime}\right\rangle^{\left(k^{\prime}\right)}$ (with $k \neq k^{\prime}$ ) could be different from zero also when $x \neq x^{\prime}$.

## A. The noncanonical Naimark extension

The quantum roulette has a Naimark extension that is not the canonical one (i.e., the indirect measurement scheme described by the Naimark theorem), that can be obtained as follows: consider an additional Hilbert space $H_{B}$, describing the ancilla, with dimension equal to the number $K$ of observables $X_{k} \in L\left(H_{A}\right)$ and a basis $\left\{\left|\theta_{k}\right\rangle\right\}$ in $H_{B}$. Then we introduce projectors $Q_{x}=\sum_{k} P_{x}^{(k)} \otimes\left|\theta_{k}\right\rangle\left\langle\theta_{k}\right|$ in the larger Hilbert space $H_{A} \otimes H_{B}$ and we prepare the initial state of the ancilla in $\left|\omega_{B}\right\rangle=\sum_{k} \sqrt{z_{k}}\left|\theta_{k}\right\rangle$, obtaining the probability distribution

$$
p_{x}=\operatorname{Tr}_{A B}\left[\rho \otimes\left|\omega_{B}\right\rangle\left\langle\omega_{B}\right| Q_{x}\right]
$$

that gives us the POVM's elements:

$$
\Pi_{x}=\operatorname{Tr}_{B}\left[\mathbb{I} \otimes\left|\omega_{B}\right\rangle\left\langle\omega_{B}\right| Q_{x}\right]=\sum_{k} z_{k} P_{x}^{(k)}
$$

Moreover, the postmeasurement states will be given by

$$
\begin{aligned}
\rho_{x} & =\frac{1}{p_{x}} \operatorname{Tr}_{B}\left[Q_{x} \rho \otimes\left|\omega_{B}\right\rangle\left\langle\omega_{B}\right| Q_{x}\right] \\
& =\frac{1}{p_{x}} \sum_{k} z_{k} P_{x}^{(k)} \rho P_{x}^{(k)} .
\end{aligned}
$$

This measurement scheme does not involve an evolution operator $U \in L\left(H_{A} \otimes H_{B}\right)$ and the projective measure is performed on both system and ancilla, unlike the canonical extension that involves a projective measurement on the sole ancilla.

## IV. THE CANONICAL NAIMARK EXTENSION OF THE PAULI QUANTUM ROULETTE WHEEL

We focus on quantum roulettes which work on qubit systems with Hilbert space $H_{A} \equiv \mathbb{C}^{2}$ and address quantum roulettes involving the measurement of Pauli operators. In order to obtain the canonical Naimark extension for these roulettes we have to add a probe system (the ancilla). We assume a two-dimensional ancilla and show that this is enough to realize the canonical extension using Eq. (1), where the elements of the POVM are given by Eq. (5).

Since $H_{B}$ is a Hilbert space with dimension 2, we choose the following representations: the state $\rho_{B}=\left|\omega_{B}\right\rangle\left\langle\omega_{B}\right|$, and the projector $P_{x}=|x\rangle\langle x|$, where $\left|\omega_{B}\right\rangle=\cos \frac{\theta}{2}|0\rangle+e^{i \varphi} \sin \frac{\theta}{2}|1\rangle$ and $|x\rangle=\cos \frac{\alpha}{2}|0\rangle+e^{i \beta} \sin \frac{\alpha}{2}|1\rangle$, with the parameters $\alpha, \theta \in$ $[0 ; \pi]$ and $\beta, \varphi \in[0 ; 2 \pi$ ). Notice that $|0\rangle,|1\rangle$ comprise a basis in $H_{B}$; we assume that $|0\rangle$ is the eigenvector of $\sigma_{3}$ related to the eigenvalue 1 , while $|1\rangle$ is the eigenvector related to -1 .

The last tool to individuate the canonical extension is the evolution operator $U \in L\left(H_{A} \otimes H_{B}\right)$ which works on the overall state of the composite system. The operator $U \in \mathrm{SU}(4)$, then it is defined by 15 parameters. Therefore, the total number of parameters that defines the Naimark extension is 19 (4 parameters from $\left|\omega_{B}\right\rangle$ and $|x\rangle$ plus 15 from $U$ ). As we will see, this number can be greatly reduced by employing the Cartan decomposition.

## A. Application of the Cartan decomposition to the Naimark extension

The Naimark theorem provides a practical connection between the generalized measurement given by the quantum roulette and the indirect measurement described by the extension. Indeed, both the probability distribution of the outcomes $p_{x}$ and the postmeasurement states $\rho_{A x}$ have to be equal for these two schemes. That is,

$$
\begin{align*}
p_{x}=\operatorname{Tr}_{A B}\left[U \rho_{A} \otimes \rho_{B} U^{\dagger} \mathbb{I} \otimes P_{x}\right]=\operatorname{Tr}_{A}\left[\rho_{A} \Pi_{x}\right],  \tag{6}\\
\rho_{A x}=\frac{1}{p_{x}} \operatorname{Tr}_{B}\left[U \rho_{A} \otimes \rho_{B} U^{\dagger} \mathbb{I} \otimes P_{x}\right]=\frac{1}{p_{x}} M_{x} \rho_{A} M_{x}^{\dagger}, \tag{7}
\end{align*}
$$

where the distribution $p_{x}$ and the state $\rho_{A x}$ in the first equality belong to the projective measurement, while those in the last equality belong to the generalized measurement.

We focus now on the Born rule Eq. (6) in order to evaluate the operators $\Pi_{x}$; after straightforward calculation, we obtain the elements of the POVM:

$$
\Pi_{x}=S_{1}^{\dagger} \operatorname{Tr}_{B}\left[\left(\mathbb{I} \otimes S_{0} \rho_{B} S_{0}^{\dagger}\right) V^{\dagger}\left(\mathbb{I} \otimes R_{0}^{\dagger} P_{x} R_{0}\right) V\right] S_{1}
$$

Consider now $S_{0} \rho_{B} S_{0}^{\dagger}$ and $R_{0}^{\dagger} P_{x} R_{0}$; the operators $R_{0}, S_{0} \in$ $L\left(H_{B}\right)$ represent a rotation in the qubit Hilbert space $H_{B}$. Since both $\rho_{B}$ and $P_{x}$ are not yet defined and depend on some parameters, we can combine the rotation to them, and we are
left with a transformation from $L\left(H_{B}\right)$ to $L\left(H_{B}\right)$ :

$$
\begin{aligned}
& \rho_{B} \rightarrow \rho_{B}^{\prime}=S_{0} \rho_{B} S_{0}^{\dagger} \\
& P_{x} \rightarrow P_{x}^{\prime}=R_{0}^{\dagger} P_{x} R_{0}
\end{aligned}
$$

i.e., we can neglect this transformation by a suitable reparametrization of $\rho_{B}^{\prime}$ and $P_{x}^{\prime}$. Furthermore, we assume the operator $S_{1}$ to be the identity ( $S_{1}=\mathbb{I}$ ). We make this assumption in order to simplify the research of the canonical extension. This ansatz will be justified a posteriori: once we find the Naimark extension, if the probability distribution $p_{x}$ obtained from the extension is equal to the one obtained from the POVM, then the extension is correct and $S_{1}=\mathbb{I}$.

Now we have the POVM's elements obtained by the canonical extension:

$$
\begin{equation*}
\Pi_{x}=\operatorname{Tr}_{B}\left[\left(\mathbb{I} \otimes \rho_{B}\right) V^{\dagger}\left(\mathbb{I} \otimes P_{x}\right) V\right] \tag{8}
\end{equation*}
$$

and these elements have to be equal to those evaluated for the quantum roulette in exam. Using the Cartan decomposition on the canonical Naimark extension reduces the number of parameters to 7 (four from $\rho_{B}$ and $P_{x}$ and three from $V$ ).

## B. Detection operators for the quantum roulette

The operator $R_{1}$ is not involved in the definition of the elements of the POVM, but it is necessary for the evaluation of the postmeasurement state $\rho_{A x}$. Since the operators $R_{0}$ and $S_{0}$ were absorbed into, respectively, $P_{x}$ and $\rho_{B}$ and $S_{1}=\mathbb{I}$, then the decomposition of $U$ is $U=\left(R_{1} \otimes \mathbb{I}\right) V$ and the left part of Eq. (7) becomes

$$
\begin{aligned}
\rho_{A x} & =\frac{1}{p_{x}} \operatorname{Tr}_{B}\left[\left(R_{1} \otimes \mathbb{I}\right) V \rho_{A} \otimes \rho_{B} V^{\dagger}\left(R_{1}^{\dagger} \otimes P_{x}\right)\right] \\
& =\frac{1}{p_{x}} R_{1} \operatorname{Tr}_{B}\left[V \rho_{A} \otimes \rho_{B} V^{\dagger}\left(\mathbb{I} \otimes P_{x}\right)\right] R_{1}^{\dagger}
\end{aligned}
$$

Therefore, the operator $R_{1}$ describes a residual degree of freedom in the design of possible postmeasurement states. This freedom was expected; since when we define a POVM $\Pi_{x}$, the postmeasurement states can be evaluated using the detection operators $M_{x}$. These operators are defined as $M_{x}=U_{x} \sqrt{\Pi_{x}}$, where $U_{x}$ is a unitary operator. The operator $U_{x}$ provides the same freedom given by $R_{1}$ to the postmeasurement states.

## C. The general solution

The problem of finding the canonical Naimark extension for a given quantum roulette is now basically reduced to the solution of four equations dependent on seven parameters. Indeed, the considered roulettes are always in qubit spaces; hence, the elements of the POVM $\Pi_{x}$ are self-adjoint $2 \times 2$ operators on the field $\mathbb{C}$ and the relation given by Eq. (8) provides four equations: one from the element $\Pi_{x 11}$ that is real, two from the element $\Pi_{x 12}$ (the real part and the imaginary part), and one from the element $\Pi_{x 22}$.

## D. Exchange of the parameters

One may wonder if it is possible to look for the canonical extension when the parameters $\alpha_{i}$ get values from all $\mathbb{R}$. The Cartan decomposition does not impose restriction on the range of the components of the class vectors (that is, the parameters
$\alpha_{i}$ ), but we know that each operator in $\mathrm{SU}(4)$ is related [via the equivalence relation Eq. (2)] to a canonical class vector, whose components lie on the bounded region $K \in \mathbb{R}^{3}$. On the other hand, if we find an extension with $\alpha_{i}$ outside of $K$, it is possible to use the three class-preserving operations (shift, reverse, and swap) to bring back the parameters to $K$.

Can the parameters be brought back to $K$ after we have found the canonical extension? This is not possible, unless we also modify the other objects of the extension. Indeed, if we have found the extension, then we have defined both $\alpha_{i}$ and $\theta, \varphi, \alpha, \beta$. But if the $\alpha_{i}$ are modified by one of the three class-preserving operations, then also the other parameters are modified and the state $\left|\omega_{B}\right\rangle$ and the operator $P_{x}$ change. In fact, the operations are class preserving, so they transform the operator $V$ into

$$
V \rightarrow\left(R_{1} \otimes R_{0}\right) V^{\prime}\left(S_{1} \otimes S_{0}\right)
$$

and the operators $R_{0}, S_{0} \in \mathrm{SU}(2)$ modify both the initial state of $H_{B}$ and the orthogonal projector:

$$
\begin{gathered}
\rho_{B}\left(\theta^{\prime}, \varphi^{\prime}\right)=S_{0} \rho_{B}(\theta, \varphi) S_{0}^{\dagger} \\
P_{x}\left(\alpha^{\prime}, \beta^{\prime}\right)=R_{0}^{\dagger} P_{x}(\alpha, \beta) R_{0} .
\end{gathered}
$$

Hence, to transform the $\alpha_{i}$ and keep the correct extension is necessary to modify $\rho_{B}$ and $P_{x}$.

## E. The canonical extension

We now focus on quantum roulettes given by Pauli operators $\sigma_{i}$ and on their canonical Naimark extension. The most general quantum roulette of this kind is $\left\{\sigma_{i}, z_{i}\right\}_{i=1,2,3}$, and its canonical extension depends on two undefined parameters (e.g., $z_{1}$ and $z_{2}$ ). Finding the extension for the general roulette is analytically challenging, and thus we focus on roulettes involving two Pauli operators.

Let us consider the roulette $\left\{\left\{\sigma_{1}, z\right\},\left\{\sigma_{3}, 1-z\right\}\right.$, where $z$ gets values from the interval $(0 ; 1)$. The POVM's elements are given by

$$
\Pi_{1}=\frac{1}{2}\left(\begin{array}{cc}
2-z & z  \tag{9}\\
z & z
\end{array}\right), \quad \Pi_{-1}=\frac{1}{2}\left(\begin{array}{cc}
z & -z \\
-z & 2-z
\end{array}\right)
$$

and the detection operators are given by $M_{x}=U_{x} \sqrt{\Pi_{x}}$, $x= \pm 1$. Upon expanding them on the Pauli basis, i.e., $M_{x}=$ $a_{0} \mathbb{I}+a_{1} \sigma_{1}+a_{2} \sigma_{2}+a_{3} \sigma_{3}$, the coefficients $a_{i}$ are evaluated using the inner product $\langle X, Y\rangle=\operatorname{Tr}\left[X Y^{\dagger}\right]$. For the roulette $\left\{\left\{\sigma_{1}, z\right\},\left\{\sigma_{3}, 1-z\right\}\right.$, we obtain $a_{i}=a_{i}(z)$ (for $\left.i=0,1,3\right)$ and $a_{2}=0$. In other words, the detection operators of a roulette involving the Pauli operators $\sigma_{1}$ and $\sigma_{3}$ have no component on the missing one, i.e., $\sigma_{2}$. This result also holds for the other roulettes depending on two $\sigma$ s; e.g., for $\left\{\left\{\sigma_{2}, z\right\},\left\{\sigma_{3}, 1-z\right\}\right\}$, the detection operators $M_{x}$ have no component by $\sigma_{1}$.

The solution for the canonical extension corresponds to the parameters

$$
\begin{aligned}
\alpha_{1} & =-\pi, \quad \alpha_{2}=0, \quad \alpha_{3}=\arcsin \left(-\sqrt{\frac{1}{1-\frac{z}{2}}-1}\right) \\
\alpha & =\arccos (z-1), \quad \beta=\pi, \quad \theta=\frac{\pi}{2}, \quad \forall \varphi
\end{aligned}
$$

where $\alpha_{1}, \alpha_{2}, \theta, \varphi$, and $\beta$ are in the correct range and we are left to check whether also $\alpha_{3}$ and $\alpha$ lie in the correct range. First,
$\cos \alpha=z-1$, i.e., $\cos \alpha \in(-1 ; 0)$; then, $\alpha \in\left(\frac{\pi}{2} ; \pi\right)$. Finally, $\sin \alpha_{3}=-\sqrt{\frac{1}{1-\frac{\pi}{2}}-1}$, that is, $\sin \alpha_{3} \in(-1 ; 0)$; then $\alpha_{3} \in\left(-\frac{\pi}{2} ; 0\right)$. The parameter $\alpha_{3}$ has to be in $\left[\frac{\alpha_{1}+\alpha_{2}}{2} ; 0\right]$, and, since $\alpha_{1}=-\pi$ and $\alpha_{2}=0$, its greatest range is $\left[-\frac{\pi}{2} ; 0\right]$.

The ingredients of the canonical extension are thus the state $\left|\omega_{B}\right\rangle=\frac{1}{\sqrt{2}}|0\rangle+\frac{e^{i \varphi}}{\sqrt{2}}|1\rangle$, the projectors

$$
\begin{align*}
P_{1} & =\frac{1}{2}\left(\begin{array}{cc}
2-z & \sqrt{z(2-z)} \\
\sqrt{z(2-z)} & z
\end{array}\right) \\
P_{-1} & =\mathbb{I}-P_{1} \tag{10}
\end{align*}
$$

and the unitary

$$
V=\left(\begin{array}{cccc}
f(z) & 0 & 0 & 0  \tag{11}\\
0 & 0 & i f^{*}(z) & 0 \\
0 & i f^{*}(z) & 0 & 0 \\
0 & 0 & 0 & f(z)
\end{array}\right)
$$

with $f(z)=\sqrt{\sqrt{\frac{2-2 z}{2-z}}+\frac{i}{\sqrt{\frac{2}{2}-1}}}$.
In order to obtain the canonical Naimark extension for the roulettes $\left\{\left\{\sigma_{1}, z\right\},\left\{\sigma_{2}, 1-z\right\}\right\}$ and $\left\{\left\{\sigma_{2}, z\right\},\left\{\sigma_{3}, 1-z\right\}\right\}$, we have to remove our previous assumption $S_{1}=\mathbb{I}$. In fact, to rotate a Pauli operator $\sigma_{i}$ by an angle $\theta$ we have to use a rotation operator $W=e^{-i(\mathbf{n} \cdot \sigma) \theta}$, where $\mathbf{n}$ is the versor of the direction around which the rotation is made. Then, to move from a two-Pauli-operators roulette to another, we need to apply the correct rotation in order to modify the $\sigma_{i}$. For example, to move from $\left\{\left\{\sigma_{1}, z\right\},\left\{\sigma_{3}, 1-z\right\}\right\}$ to $\left\{\left\{\sigma_{2}, z\right\},\left\{\sigma_{3}, 1-z\right\}\right\}$ we have to apply the operator $W=e^{-i \frac{\pi}{4} \sigma_{3}}$, which changes $\sigma_{1}$ into $\sigma_{2}$ and leaves $\sigma_{3}$ unchanged.

Therefore, the extensions for the other roulettes depending on two Pauli operators are defined by the same parameters of the extension for $\left\{\left\{\sigma_{1}, z\right\},\left\{\sigma_{3}, 1-z\right\}\right\}$, but the elements $\Pi_{x}$ are rotated by the operator $W$, that is,

$$
\Pi_{x} \rightarrow W \Pi_{x} W^{\dagger}
$$

This means that, while for the first found extension the operator $S_{1}$ can be assumed equal to $\mathbb{I}$, for the extensions of the other roulettes the operator $S_{1}$ has to be equal to the conjugate transpose of the rotation operator $W$. We find that, for the roulette given by $\sigma_{2}$ and $\sigma_{3}, S_{1}=e^{i \frac{\pi}{4} \sigma_{3}}$, while, for the one given by $\sigma_{1}$ and $\sigma_{2}, S_{1}=e^{-i \frac{\pi}{4} \sigma_{1}}$.

## V. THE STERN-GERLACH APPARATUS AS A QUANTUM ROULETTE

The so called Stern-Gerlach apparatus allows one to measure a component (e.g., the component along the $z$-axis $S_{z}$ ) of the quantum observable spin, i.e., the intrinsic angular momentum of a particle. The measurement is usually performed on a collimated beam of particles (e.g., neutral atoms) sent with thermal speed into a region of inhomogeneous magnetic field. Here the particles are deflected by the field in some beams which, after propagating into the vacuum, are collected by a screen. The magnetic field is usually assumed to be of the form $\mathbf{B}=(B-b z) \mathbf{e}_{3}$, where $z$ is the coordinate along the $z$ axis, $B$ is the field in the origin, and $b$ is a constant. Actually, this is an artificial model, since a field like this one does not respect the Maxwell equations, as $\boldsymbol{\nabla} \cdot \mathbf{B}=-b \neq 0$. On the other hand,
we may assume that $b \ll B$ so that we can neglect the other components of $\mathbf{B}$. The interaction Hamiltonian is given by $H=(B-b z) \sigma_{3}$ (neglecting the vacuum permittivity), and the corresponding evolution operator is given by $U=e^{-i \tau(B-b z) \sigma_{3}}$, where $\tau$ is an effective interaction time. Starting from an initial state which is factorized into a spin and a spatial part, i.e., $|\Psi\rangle\rangle=\left(c_{0}|0\rangle+c_{1}|1\rangle\right) \otimes|\psi(\mathbf{q})\rangle$, the evolved state is given by

$$
U|\Psi\rangle\rangle=c_{0}|0\rangle \otimes\left|\psi_{-}(\mathbf{q})\right\rangle+c_{1}|1\rangle \otimes\left|\psi_{+}(\mathbf{q})\right\rangle
$$

where $\left|\psi_{ \pm}(\mathbf{q})\right\rangle=e^{ \pm i \tau(B-b z)}|\psi(\mathbf{q})\rangle$. The evolution is thus coupling the spin and the spatial part. Moving to the momentum representation

$$
\begin{aligned}
\left|\tilde{\psi}_{ \pm}(\mathbf{p})\right\rangle= & \int d^{3} \mathbf{q} e^{-i \mathbf{q} \cdot \mathbf{p}}\left|\psi_{ \pm}(\mathbf{q})\right\rangle \\
& =\left|\tilde{\psi}\left(\mathbf{p} \pm \tau b \mathbf{e}_{3}\right)\right\rangle
\end{aligned}
$$

and tracing out the spin part after the interaction, we have that the motional degree of freedom after the interaction is described by the density operator:

$$
\varrho_{\mathbf{p}}=\left|c_{1}\right|^{2}\left|\tilde{\psi}_{+}(\mathbf{p})\right\rangle\left\langle\tilde{\psi}_{+}(\mathbf{p})\right|+\left|c_{0}\right|^{2}\left|\tilde{\psi}_{-}(\mathbf{p})\right\rangle\left\langle\tilde{\psi}_{-}(\mathbf{p})\right|
$$

As a consequence the beam is divided in two parts, and it is possible to perform measurements on a screen placed at a given distance from the magnetic field, where we can see the beams as two different spots.

If for some reason the direction of the magnetic field is tilted we have $\mathbf{B}=(B-b t) \mathbf{e}_{\alpha}$, where $t$ is a coordinate along the new direction and $\mathbf{e}_{\alpha}=\cos \alpha \mathbf{e}_{3}+\sin \alpha \mathbf{e}_{\perp}, \mathbf{e}_{\perp}$ denoting any direction perpendicular to the $z$ axis, say $\mathbf{e}_{1}$. The above analysis is still valid if we perform the substitution $H \rightarrow(B-b t) \sigma_{\theta}$, where $\sigma_{\theta}$ is a Pauli matrix describing a spin component along a tilted axis. Assuming a rotation along the $x$ axis we have that $\sigma_{\theta}$ corresponds to the rotated operator:

$$
\sigma_{\theta}=U_{\theta} \sigma_{3} U_{\theta}^{\dagger}
$$

where $U_{\theta}=e^{-i \sigma_{1} \theta}$ and $\theta=\alpha / 2$.

## A. The continuous quantum roulette

Usually, the magnetic field of the apparatus is assumed to be a stable classical quantity. However, in any practical situation the magnetic field unavoidably fluctuates. In particular, we focus on Stern-Gerlach apparatuses in which the magnetic field fluctuates in one dimension around a pre-established direction and provide a more detailed analysis of nonideal setups [23,24]. As mentioned above, a measurement of spin made with a tilted magnetic field corresponds to measure the operator $\sigma_{\theta}$. If the magnetic field is fluctuating, then we may describe this situation using a continuous quantum roulette where $\theta$ is randomly fluctuating around the $z$ axis according to a given probability distribution.

In principle, the magnetic field may fluctuate in any direction on the $z-y$ plane; i.e., the angle $\theta$ takes values between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. On the other hand, in a realistic situation, the magnetic field moves away from the pre-established direction (the $z$ axis, in this case) by a small angle. We thus introduce a Gaussian probability distribution $z(\theta)$ for the fluctuating values of $\theta$ :

$$
z(\theta)=\frac{1}{A} \exp \left\{-\frac{\theta^{2}}{2 \Delta^{2}}\right\}
$$

where the normalization $A$ is

$$
A=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \exp \left\{-\frac{\theta^{2}}{2 \Delta^{2}}\right\} d \theta=\sqrt{2 \pi} \Delta \operatorname{erf}\left(\frac{\pi}{2 \sqrt{2} \Delta}\right)
$$

In order to evaluate the elements of the POVM, which describes this continuous quantum roulette, we need the projectors on the eigenspaces of $\sigma_{\theta}$, i.e.,

$$
\begin{align*}
P_{1}(\theta) & =\left(\begin{array}{cc}
\cos ^{2} \theta & i \cos \theta \sin \theta \\
-i \cos \theta \sin \theta & \sin ^{2} \theta
\end{array}\right), \\
P_{-1}(\theta) & =\mathbb{I}-P_{1}(\theta) . \tag{12}
\end{align*}
$$

Therefore, the elements of the POVM are given by

$$
\begin{equation*}
\Pi_{x}=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} z(\theta) P_{x}(\theta) d \theta \tag{13}
\end{equation*}
$$

that is the equation equivalent to Eq. (5) in the continuous case. We can evaluate the elements of the POVM using the distribution $z(\theta)$ and the projectors $P_{x}(\theta)$ given above, and we obtain

$$
\begin{align*}
\Pi_{1} & =\left(\begin{array}{cc}
\frac{1}{2}+f(\Delta) & 0 \\
0 & \frac{1}{2}-f(\Delta)
\end{array}\right) \\
\Pi_{-1} & =\mathbb{I}-\Pi_{1} \tag{14}
\end{align*}
$$

where the function $f(\Delta):[0 ;+\infty) \rightarrow\left[\frac{1}{2} ; 0\right)$ is given by

$$
f(\Delta)=\frac{\operatorname{erf}\left(\frac{\pi-i 4 \Delta^{2}}{2 \sqrt{2} \Delta}\right)+\operatorname{erf}\left(\frac{\pi+i 4 \Delta^{2}}{2 \sqrt{2} \Delta}\right)}{4 e^{2 \Delta^{2}} \operatorname{erf}\left(\frac{\pi}{2 \sqrt{2} \Delta}\right)} .
$$

We have $f(\Delta) \simeq 1 / 2-\Delta^{2}$ for vanishing $\Delta$ and $f(\Delta) \simeq$ $1 / 8 \Delta^{2}$ for $\Delta \rightarrow \infty$.

## B. The canonical extension for the continuous roulette

We look for the canonical extension for this roulette in order to obtain a practicable measurement scheme with the same behavior (same probability distribution and postmeasurement states) as the Stern-Gerlach experiment with fluctuating magnetic field. A canonical extension may be found, corresponding to the parameters

$$
\begin{aligned}
\alpha_{1} & =\arccos [-2 f(\Delta)], \quad \alpha_{2}=\arccos [2 f(\Delta)], \quad \alpha_{3}=0, \\
\alpha & =\pi, \quad \beta=0, \quad \theta=0, \quad \varphi=0 .
\end{aligned}
$$

Let consider the parameters $\alpha_{1}$ and $\alpha_{2}$; the codomain of the function $f(\Delta)$ is $\left(0 ; \frac{1}{2}\right]$. Therefore, if $\cos \alpha_{1}=-2 f(\Delta)$, then $\cos \alpha_{1} \in(0 ;-1]$ and $\alpha_{1} \in\left(-\frac{\pi}{2} ;-\pi\right]$. Instead, $\cos \alpha_{2}=$ $2 f(\Delta)$, i.e., $\cos \alpha_{2} \in(0 ; 1]$ and $\alpha_{2} \in\left(\frac{\pi}{2} ; 0\right]$. Both $\alpha_{1}$ and $\alpha_{2}$ depend on the function $f$, so when we choose a value for $f$ the two parameters are fixed. For example, when $f(\Delta) \rightarrow 0$, then $\alpha_{1} \rightarrow-\frac{\pi}{2}$ and $\alpha_{2} \rightarrow \frac{\pi}{2}$; on the other hand, if $f(\Delta)=\frac{1}{2}$ then $\alpha_{1}=-\pi$ and $\alpha_{2}=0$. The ranges of these two parameters
are correct, and we have $\alpha_{2} \leqslant-\alpha_{1} \forall f(\Delta)$. Finally, since we have fixed $\alpha_{3}=0$, we have to check whether $\alpha_{1}-\alpha_{2} \geqslant-\pi$, and this is the case: as it can be easily checked $\alpha_{1}-\alpha_{2}=-\pi$ for all $\Delta \in[0 ;+\infty)$. The canonical extension is thus given by the state $\left|\omega_{B}\right\rangle=|0\rangle$, the observable $\sigma_{3}$ (measured on the ancilla), and the unitary $V \in L\left(H_{A} \otimes H_{B}\right)$ :

$$
\begin{gather*}
V=\frac{1}{4} \sum_{k=0}^{3} v_{k} \sigma_{k} \otimes \sigma_{k}  \tag{15}\\
v_{0 / 3}=\sqrt{\frac{1}{2}+f(\Delta)} \pm \sqrt{\frac{1}{2}-f(\Delta)},  \tag{16}\\
\stackrel{\Delta \rightarrow 1}{\simeq} 1 \pm \Delta \\
v_{1 / 2}=i v_{0 / 3} \tag{17}
\end{gather*}
$$

For a particle with spin up, represented by the pure state $|0\rangle$, the probability distribution of the outcomes is given by $p_{1}=\frac{1}{2}+f(\Delta), p_{-1}=1-p_{1}$ and, thus, when such a particle is measured, there is always a probability that the apparatus measures the spin down $|1\rangle$.

## VI. CONCLUSIONS

We have addressed Pauli quantum roulettes and found their canonical Naimark extensions. The extensions are minimal, i.e., they involve a single ancilla qubit, and they provide a concrete model to realize the roulettes without destroying the signal state, which can be measured again after the measurement or can be transmitted. Our results provide a natural framework to describe a measurement scheme in which the measured observable depends on an external parameter, which cannot be fully controlled and may fluctuate according to a given probability distribution. As an illustrative example, we have applied our results to the description of Stern-Gerlachlike experiments on a two-level system, taking into account possible uncertainties in the splitting force.

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