

Quantifying the non-Gaussian character of a quantum state by quantum relative entropy

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We introduce a measure to quantify the non-Gaussian character of a quantum state: the quantum relative entropy between the state under examination and a reference Gaussian state. We analyze in detail the properties of our measure and illustrate its relationships with relevant quantities in quantum information such as the Holevo bound and the conditional entropy; in particular, a necessary condition for the Gaussian character of a quantum channel is also derived. The evolution of non-Gaussianity is analyzed for quantum states undergoing conditional Gaussification toward twin beams and de-Gaussification driven by Kerr interaction. Our analysis allows us to assess non-Gaussianity as a resource for quantum information and, in turn, to evaluate the performance of Gaussification and de-Gaussification protocols.

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The use of Gaussian states and operations allows the implementation of relevant quantum-information protocols including teleportation, dense coding, and quantum cloning [1]. Indeed, the Gaussian sector of the Hilbert space plays a crucial role in quantum information processing with continuous variables (CVs), especially concerning quantum-optical implementations [2]. On the other hand, quantum-information protocols required for long-distance communication as, for example, entanglement distillation and entanglement swapping, require non-Gaussian operations [3]. In addition, it has been demonstrated that using non-Gaussian states and operations teleportation [4–6] and cloning [7] of quantum states may be improved. Indeed, de-Gaussification protocols for single-mode and two-mode states have been proposed [4–6,8,9] and realized [10]. From a more theoretical point of view, it should be noticed that any strongly superadditive and continuous functional is minimized, at fixed covariance matrix (CM), by Gaussian states. This is crucial to prove extremality of Gaussian states and Gaussian operations [11,12] for various quantities such as channel capacities [13], multipartite entanglement measures [14], and distillable secret keys in quantum key distribution protocols. Overall, non-Gaussianity (nG) appears to be a resource for CV quantum information and a question naturally arises as to whether a convenient measure to quantify the non-Gaussian character of a quantum state may be introduced. Notice that the notion of nG already appeared in classical statistics in the framework of independent component analysis [15].

The first measure of nG of a CV state ϱ was suggested in [16] based on the Hilbert-Schmidt distance between ϱ and a reference Gaussian state. In turn, the HS-based measure has been used to characterize the role of nG as a resource for teleportation [17,18] and in promiscuous quantum correlations in CV systems [19]. Here we introduce a measure $\delta[\varrho]$ based on the quantum relative entropy between ϱ and a reference Gaussian state. This quantity is related to information measures and allows us to assess nG as a resource for quantum information as well as the performances of Gaussification and de-Gaussification protocols. In the following, after introducing its formal definition and showing that it can be

easily computed for any state, either single-mode or multi-mode, we analyze in detail the properties of $\delta[\varrho]$ as well as its dynamics under Gaussification [29] and de-Gaussification protocols.

Let us consider a CV system made of d bosonic modes described by the mode operators a_k , $k=1, \dots, d$, with commutation relations $[a_k, a_j^\dagger] = \delta_{kj}$. A quantum state ϱ of d bosonic modes is fully described by its characteristic function $\chi[\varrho](\lambda) = \text{Tr}[\varrho D(\lambda)]$ where $D(\lambda) = \bigotimes_{k=1}^d D_k(\lambda_k)$ is the d -mode displacement operator, with $\lambda = (\lambda_1, \dots, \lambda_d)^T$, $\lambda_k \in \mathbb{C}$, and where $D_k(\lambda_k) = \exp(\lambda_k a_k^\dagger - \lambda_k^* a_k)$ is the single-mode displacement operator. The canonical operators are given by $q_k = (a_k + a_k^\dagger)/\sqrt{2}$ and $p_k = (a_k - a_k^\dagger)/\sqrt{2}i$ with commutation relations given by $[q_j, p_k] = i\delta_{jk}$. Upon introducing the vector $\mathbf{R} = (q_1, p_1, \dots, q_d, p_d)^T$, the CM $\sigma \equiv \sigma[\varrho]$ and the vector of mean values $\mathbf{X} \equiv \mathbf{X}[\varrho]$ of a quantum state ϱ are defined as $\sigma_{kj} = \frac{1}{2}(\langle R_k R_j + R_j R_k \rangle - \langle R_j \rangle \langle R_k \rangle)$ and $X_j = \langle R_j \rangle$, where $\langle O \rangle = \text{Tr}(\varrho O)$ is the expectation value of the operator O . A quantum state ϱ_G is said to be Gaussian if its characteristic function is Gaussian, that is, $\chi[\varrho_G](\lambda) = \exp(-\frac{1}{2} \lambda^T \sigma \lambda + \mathbf{X}^T \lambda)$, where λ is the real vector $\lambda = (\text{Re } \lambda_1, \text{Im } \lambda_1, \dots, \text{Re } \lambda_d, \text{Im } \lambda_d)^T$. Once the CM and the vectors of mean values are given, a Gaussian state is fully determined. For a system of d bosonic modes the most general Gaussian state is described by $d(2d+3)$ independent parameters.

The von Neumann entropy of a quantum state is defined as $S(\varrho) = -\text{Tr}(\varrho \ln \varrho)$. The von Neumann entropy is non-negative and equals zero if and only if ϱ is a pure state. In order to quantify the non-Gaussian character of a quantum state ϱ we employ the quantum relative entropy (QRE) $S(\varrho \parallel \tau) = \text{Tr}[\varrho(\ln \varrho - \ln \tau)]$ between ϱ and a reference Gaussian state τ . As for its classical counterpart, the Kullback-Leibler divergence, it can be demonstrated that $0 \leq S(\varrho \parallel \tau) < \infty$ when it is definite, i.e., when $\text{supp } \varrho \subseteq \text{supp } \tau$. In particular $S(\varrho \parallel \tau) = 0$ if and only if $\varrho \equiv \tau$. This quantity, though not defining a proper metric in the Hilbert space, has been widely used in different fields of quantum information as a measure of statistical distinguishability for quantum states

[20,21]. Therefore, given a quantum state ϱ with finite first and second moments, we define its nG as $\delta[\varrho] = S(\varrho \parallel \tau)$, where the reference state τ is the Gaussian state with $X[\varrho] = X[\tau]$ and $\sigma[\varrho] = \sigma[\tau]$, i.e., the Gaussian state with the same CM σ and the same vector X of the state ϱ . Finally, since τ is Gaussian, then $\ln \tau$ is a polynomial operator of the second order in the canonical variables which, together with the fact that τ and ρ have the same CM, leads to $\text{Tr}[(\tau - \varrho) \ln \tau] = 0$ [22], i.e., $S(\varrho \parallel \tau) = S(\tau) - S(\varrho)$. Thus we have

$$\delta[\varrho] = S(\tau) - S(\varrho), \quad (1)$$

i.e., nG is the difference between the von Neumann entropies of τ and ϱ . In turn, several properties of the non-Gaussian measure $\delta[\varrho]$ may be derived from the fundamental properties of the QRE [20,21]. In the following we summarize the relevant ones by the following lemmas.

Lemma 1. $\delta[\varrho]$ is a well-defined non-negative quantity, that is, $0 \leq \delta[\varrho] < \infty$ and $\delta[\varrho] = 0$ if and only if ϱ is a Gaussian state.

Proof. Non-negativity is guaranteed by the non-negativity of the quantum relative entropy. Moreover, if $\delta[\varrho] = 0$ then $\varrho = \tau$ and thus it is a Gaussian state. If ϱ is a Gaussian state, then it is uniquely identified by its first and second moments and thus the reference Gaussian state τ is given by $\tau = \varrho$, which, in turn, leads to $\delta[\varrho] = S(\varrho \parallel \tau) = 0$.

Lemma 2. $\delta[\varrho]$ is a continuous functional.

Proof. It follows from the continuity of trace operation and QRE.

Lemma 3. $\delta[\varrho]$ is additive for factorized states: $\delta[\varrho_1 \otimes \varrho_2] = \delta[\varrho_1] + \delta[\varrho_2]$. As a corollary we have that if ϱ_2 is a Gaussian state, then $\delta[\varrho] = \delta[\varrho_1]$.

Proof. The overall reference Gaussian state is the tensor product of the relative reference Gaussian states, $\tau = \tau_1 \otimes \tau_2$. The lemma thus follows from the additivity of QRE and the corollary from Lemma 1.

Lemma 4. For a set of states $\{\varrho_k\}$ having the same first and second moments, then nG is a convex functional, that is, $\delta[\sum_k p_k \varrho_k] \leq \sum_k p_k \delta[\varrho_k]$, with $\sum_k p_k = 1$.

Proof. The states ϱ_k , having the same first and second moments, have the same reference Gaussian state τ which in turn is the reference Gaussian state of the convex combination $\varrho = \sum_k p_k \varrho_k$. Since conditional entropy $S(\varrho \parallel \tau)$ is a jointly convex functional with respect to both states, we have $\delta[\sum_k p_k \varrho_k] = S(\sum_k p_k \varrho_k \parallel \tau) \leq \sum_k p_k S(\varrho_k \parallel \tau) = \sum_k p_k \delta[\varrho_k]$. ■

Notice that, in general, nG is not convex, as may easily be proved upon considering the convex combination of two Gaussian states with different parameters.

Lemma 5. If U_b is a unitary evolution corresponding to a symplectic transformation in the phase space, i.e., if $U_b = \exp(-iH)$ with H at most bilinear in the field operator, then $\delta[U_b \varrho U_b^\dagger] = \delta[\varrho]$.

Proof. Let us consider $\varrho' = U_b \varrho U_b^\dagger$, where U is at most bilinear in the field mode; then its CM transforms as $\sigma[\varrho'] = \Sigma \sigma[\varrho] \Sigma^T$, Σ being the symplectic transformation associated with U . At the same time the vector of mean values simply translates to $X' = X + X_0$. Since any Gaussian state is fully characterized by its first and second moments, then the reference state must necessarily transform as $\tau' = U_b \tau U_b^\dagger$, i.e., with the same unitary transformation U . The lemma follows

from invariance of QRE under unitary transformations.

Lemma 6. nG monotonically decreases under a partial trace, that is, $\delta[\text{Tr}_B[\varrho]] \leq \delta[\varrho]$.

Proof. Let us consider $\varrho' = \text{Tr}_B[\varrho]$. Its CM is the submatrix of $\sigma[\varrho]$ and its first moment vector is the *subvector* of $X[\varrho]$ corresponding to the relevant Hilbert space. As before, also the new reference Gaussian state must necessarily transform as $\tau' = \text{Tr}_B[\tau]$. The QRE monotonically decreases under a partial trace and the lemma is proved.

Lemma 7. nG monotonically decreases under Gaussian quantum channels, that is, $\delta[\mathcal{E}_G(\varrho)] \leq \delta[\varrho]$.

Proof. Any Gaussian quantum channel can be written as $\mathcal{E}_G(\varrho) = \text{Tr}_E[U_b(\varrho \otimes \tau_E)U_b^\dagger]$, where U_b is a unitary operation corresponding to a Hamiltonian at most bilinear in the field modes and where τ_E is a Gaussian state [23]. Then, by using Lemmas 3, 5, and 6 we obtain $\delta[\mathcal{E}_G(\varrho)] \leq \delta[U_b(\varrho \otimes \tau_E)U_b^\dagger] = \delta[\varrho]$. ■

In turn, Lemma 7 provides a necessary condition for a channel to be Gaussian: given a quantum channel \mathcal{E} , and a generic quantum state ϱ , if the inequality $\delta[\mathcal{E}(\varrho)] \leq \delta[\varrho]$ is not satisfied, the channel is non-Gaussian.

Let us now consider a single-mode ($d=1$) system and look for states with the maximum amount of nG at fixed average number of photons $N = \langle a^\dagger a \rangle$. Since $\delta[\varrho] = S(\tau) - S(\varrho)$, we have to maximize $S(\tau)$ and, at the same time, minimize $S(\varrho)$. For a single-mode system the most general Gaussian state can be written as $\varrho_G = D(\alpha)S(\zeta)\nu(n_t)S^\dagger(\zeta)D^\dagger(\alpha)$, $D(\alpha)$ being the displacement operator, $S(\zeta) = \exp(\frac{1}{2}\zeta a^{\dagger 2} - \frac{1}{2}\zeta^* a^2)$ the squeezing operator, $\alpha, \zeta \in \mathbb{C}$, and $\nu(n_t) = (1+n_t)^{-1}[n_t/(1+n_t)]^{a^\dagger a}$ a thermal state with n_t average number of photons. Displacement and squeezing applied to thermal states increase the overall energy, while entropy is an increasing monotonic function of the number of thermal photons n_t and is invariant under unitary operations; thus, at fixed energy, $S(\tau)$ is maximized for $\tau = \nu(N)$. Therefore, the state with the maximum amount of nG must be a pure state [in order to have $S(\varrho) = 0$] with the same CM $\sigma = (N + \frac{1}{2})\mathbb{I}$ of the thermal state $\nu(N)$. These properties individuate the superpositions of Fock states $|\psi_N\rangle = \sum_k \alpha_k |n + l_k\rangle$ where $n \geq 0$, $l_k \geq 1_{k-1} + 3$ or $l_k = 0$, with the constraint $N = \langle a^\dagger a \rangle$, i.e., $n + \sum_k |\alpha_k|^2 l_k = N = \{\det \sigma[\nu(N)]\}^{1/2} - \frac{1}{2}$. These represent maximally non-Gaussian states, and include Fock states $|\psi_N\rangle = |N\rangle$ as a special case. Let us consider now d -mode quantum states with fixed average number of photons $\sum_{k=1}^d \text{Tr}(a_k^\dagger a_k \varrho) = N = \sum_k n_k$. In this case also maximally non-Gaussian states are pure states; the CM being equal to that of a multimode classical state $\tau = R \otimes_k \nu(m_k) R^\dagger$, $\sum_k m_k = N$, where we denote by R a generic set of symplectic passive operations (e.g., beam splitter evolution) which do not increase the energy. In order to maximize $S(\tau) = \sum_k S(\nu(m_k))$ we have to choose $m_k = N/d$ for every k , as, for example, factorized states of the form $|\Psi_N\rangle = |\psi_{N/d}\rangle^{\otimes d}$, whose reference Gaussian states are $\tau = [\nu(N/d)]^{\otimes d}$, are maximally non-Gaussian states at fixed N . Of course for the multimode case there are other more complicated classes of maximally non-Gaussian states that include also entangled pure states. Finally, we observe that the maximum value of nG is a monotonically increasing function of the number of photons N .

Gaussian states are extremal for several functionals in quantum information [11]. In the following we consider two relevant examples, and show how extremality properties may be quantified by the non-Gaussian measure $\delta[\varrho]$. Let us first consider a generic communication channel where the letters from an alphabet are encoded onto a set of quantum states ϱ_k with probabilities p_k . The *Holevo bound* represents the upper bound to the accessible information, and is defined as $\chi(\varrho) = S(\varrho) - \sum_k p_k S(\varrho_k)$ where $\varrho = \sum_k p_k \varrho_k$ is the overall ensemble sent through the channel. Upon fixing the CM (and the first moments) of ϱ , we rewrite the Holevo bound as $\chi(\varrho) = S(\tau) - \delta[\varrho] - \sum_k p_k S(\varrho_k)$, where τ is the Gaussian reference of ϱ . This highlights the role of the nG $\delta[\varrho]$ of the overall state in determining the amount of accessible information: at fixed CM the most convenient encoding corresponds to a set of pure states ϱ_k , $S(\varrho_k) = 0$, forming an overall Gaussian ensemble with the largest entropy. In other words, at fixed CM, we achieve the maximum value of χ upon encoding encoding symbols onto the eigenstates of the corresponding Gaussian state [24]. If the alphabet is encoded onto the eigenstates of a given state ϱ , we have $\chi(\varrho) = S(\tau) - \delta[\varrho]$. This suggests an operational interpretation of nG $\delta[\varrho]$ as the loss of information we get by encoding symbols on the eigenstates of ϱ rather than on those of its reference Gaussian state.

Let us now consider the state ϱ_{AB} describing two quantum systems A and B and define the conditional entropy $S(A|B) = S(\varrho_{AB}) - S(\varrho_B)$. Let us fix the CM of ϱ_{AB} and thus also that of ϱ_B , and consider the reference Gaussian states τ_{AB} and τ_B . We may write $S(A|B) = S_G(A|B) - (\delta[\varrho_{AB}] - \delta[\varrho_B])$ where $S_G(A|B) = S(\tau_{AB}) - S(\tau_B)$, i.e., the conditional entropy evaluated for the reference Gaussian states τ_{AB} and τ_B . Then, upon using Lemma 6 we have $\delta[\varrho_{AB}] - \delta[\varrho_B] \geq 0$ and thus $S(A|B) \leq S_G(A|B)$, i.e., the maximum of conditional entropy at fixed CM is achieved by Gaussian states. In classical information theory the conditional entropy $H(X|Y) = H(X, Y) - H(Y)$, where von Neumann entropies are replaced by Shannon entropies of classical probability distributions, is a positive quantity and may be interpreted [25] as the amount of partial information that Alice must send to Bob so that he gains full knowledge of X given his previous knowledge from Y . When quantum systems are involved the conditional entropy may be negative, negativity being a sufficient condition for the entanglement of the overall state ϱ_{AB} . This *negative* information may be seen as follows [26] for a discrete variable quantum system. Given an unknown quantum state distributed over two systems, we can discriminate between two different cases. If $S(A|B) \geq 0$, as in the classical case, it gives the amount of information that Alice should send to Bob to give him the full knowledge of the overall state ϱ_{AB} . When $S(A|B) < 0$ Alice does not need to send any information to Bob, and moreover they gain $-S(A|B)$ ebits. If we conjecture that this interpretation can be extended to the CV case, the relation $S(A|B) \leq S_G(A|B)$ ensures that, at fixed CM, non-Gaussian states always perform better: Alice needs to send less information, or, for negative values of the conditional entropy, more entanglement is gained. Moreover, since negativity of conditional entropy is a sufficient condition for entanglement [27], we have that for any given bipartite quantum state ϱ_{AB} , if the conditional entropy of the reference Gaussian state τ_{AB} is

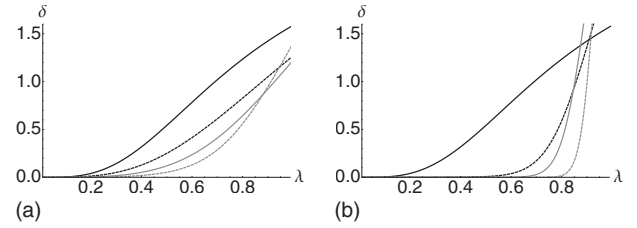


FIG. 1. nG after some steps of the conditional Gaussification protocol of Ref. [29] considering as the initial state the non-Gaussian superposition $|\psi^{(0)}\rangle = (1 + \lambda^2)^{-1/2}(|0, 0\rangle + \lambda|1, 1\rangle)$. (Left) black solid line, initial state; black dashed line, step 1; gray solid line, step 2; gray dashed line, step 3. (Right) Black solid line, initial state; black dashed line step 5; gray solid line, step 10; gray dashed line, step 20.

negative, then ϱ_{AB} is an entangled state. Though being a weaker condition than the negativity of $S(A|B)$, this is a simple and easily computable test for entanglement which is equivalent to evaluating the symplectic eigenvalues [28] of the involved Gaussian states.

Since the amount of nG of a state affects its performance in quantum-information protocols a question naturally arises as to whether this may be engineered or modified at will. As concerns Gaussification, Lemma 7 assures that Gaussian maps do not increase nG. In turn, the simplest example of a Gaussification map is provided by dissipation in a thermal bath [16], which follows from bilinear interactions between the systems under investigation and the environment. On the other hand, a conditional iterative Gaussification protocol has been recently proposed [29] which cannot be reduced to a trace-preserving Gaussian quantum map. It requires only the use of passive elements and on-off photodetectors. Given a bipartite pure state in the Schmidt form, $|\psi^{(k)}\rangle = \sum_{n=0}^{\infty} \alpha_{n,n}^{(k)} |n, n\rangle$, the state at the $(k+1)$ th step of the protocol has the same Schmidt form as $\alpha_{n,n}^{(k+1)} = 2^{-n} \sum_{r=0}^n \binom{n}{r} \alpha_{r,r}^{(k)} \alpha_{n-r,n-r}^{(i)}$. We have considered the initial non-Gaussian superposition $|\psi^{(0)}\rangle = (1 + \lambda^2)^{-1/2}(|0, 0\rangle + \lambda|1, 1\rangle)$ which is asymptotically driven toward the Gaussian twin-beam state $|\psi\rangle = \sqrt{1 - \lambda^2} \sum_{n=0}^{\infty} \lambda^n |n, n\rangle$. We have evaluated the nG at any step of the protocol, for every value of λ .

The results are reported in Fig. 1. For the first steps, the nG decreases monotonically for almost all values of λ (only at the third step, for $\lambda \approx 1$, is the state more non-Gaussian than at the previous steps). Notice that with increase of the number of steps the nG may also increase, e.g., for $\lambda \approx 1$, δ reaches very high values and the maximum value increases. On the other hand, the overall effectiveness of the protocol is confirmed by our analysis, since the range of values of λ for which $\delta \approx 0$ increases at each step of the protocol. In other words, though not being a proper Gaussian map, the conditional protocol of [29] indeed provides an effective Gaussification procedure.

Conditional de-Gaussification procedures have been recently proposed and demonstrated [5,6,8,10]. Here we rather consider the unitary de-Gaussification evolution provided by self-Kerr interaction $U_\gamma = \exp[-i\gamma(a^\dagger a)^2]$ [30,31], which does not correspond to a symplectic transformation and leads to a non-Gaussian state even if applied to a Gaussian state. We have evaluated the nG of the state obtained from a coherent

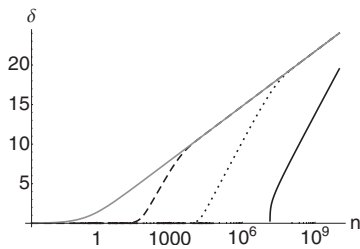


FIG. 2. nG of coherent states undergoing Kerr interaction as a function of the average number of photons and for different values of the coupling constant γ . Black dashed line, $\gamma=10^{-2}$; black dotted line, $\gamma=10^{-4}$; solid black line, $\gamma=10^{-6}$. The gray solid line is the maximum nG at fixed number of photons.

state $|\alpha\rangle$ undergoing Kerr interaction. The results are reported in Fig. 2 as a function of the average number of photons (up to 10^9 photons) and for different values of the coupling constant γ . As is apparent from the plot, nG is an increasing function of the number of photons and the Kerr coupling γ . For $\gamma \approx 10^{-2}$, the maximum nG achievable at fixed energy is quite rapidly achieved. For more realistic values of the nonlinear coupling, i.e., $\gamma \leq 10^{-6}$, non-Gaussian states may be obtained only for a large average number of photons in the output state. In fact, to obtain entanglement, experimental realizations [30,31] involve pulses with an average number of the order of 10^8 photons, which are needed to compensate the almost vanishingly small Kerr nonlinearities of standard glass fiber.

We finally notice that a good measure for the non-Gaussian character of quantum states allows us to define a measure of the non-Gaussian character of a quantum operation. Let us denote by \mathcal{G} the whole set of Gaussian states. A convenient definition for the nG of a map \mathcal{E} reads $\delta[\mathcal{E}] = \max_{\rho \in \mathcal{G}} \delta[\mathcal{E}(\rho)]$, where $\mathcal{E}(\rho)$ denotes the quantum state obtained after the evolution imposed by the map.

In conclusion, we have introduced a measure to quantify the non-Gaussian character of a CV quantum state based on quantum relative entropy. We have analyzed in detail the properties owned by this measure and its relation with some relevant quantities in quantum information. In particular, a necessary condition for the Gaussian character of a quantum channel and a sufficient condition for entanglement of bipartite quantum states can be derived. Our measure is easily computable for any CV state and allows us to assess nG as a resource for quantum technology. In turn, we exploited our measure to evaluate the performances of conditional Gaussification toward twin-beam and de-Gaussification processes driven by Kerr interaction.

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