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About the quantum Fisher information of nearly pure quantum statistical models

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We address *nearly pure* quantum statistical models, i.e. situations where the information about a parameter is encoded in pure states weakly perturbed by the mixing with a parameter independent state, mimicking a weak source of noise. We show that the symmetric logarithmic derivative is left unchanged, and find an approximate analytic expression for the quantum Fisher information (QFI) which provides bounds on how much a weak source of noise may degrade the QFI.

Keywords: Quantum estimation theory; Quantum Fisher information.

1. Introduction

The quantum Cramér–Rao theorem says that the precision of any regular quantum statistical model is bounded by the inverse of the quantum Fisher information (QFI), which thus represents a central quantity in quantum estimation theory.[1–6] Indeed, evaluating the QFI provides the ultimate quantum limits to precision and, in turn, a general benchmark to assess metrological protocols at the quantum level. Given a quantum statistical model ρ_λ, i.e. a family of quantum states labeled by the value of a parameter of interest, the calculation of the QFI generally involves the diagonalization of ρ_λ, and this fact often prevents the analytic determination of the QFI. For regular models, the QFI may be expressed in terms of the Bures metric or the fidelity, but the evaluation of those quantities also involves the diagonalization of the family ρ_λ.

In this brief communication, we address the evaluation of the QFI for *nearly pure* quantum statistical models, i.e. models where the set ρ_λ is made of weakly mixed states with purity not too far from one. These models may occur in situations, where the information about the parameter λ is encoded in pure states, which are slightly degraded by the mixing with a parameter-independent state, mimicking a weak source of noise. Based on the above assumptions, we find an analytic form of the QFI, and quantify how much it decreases compared to the unperturbed case. Our results are not particularly surprising, but they may be of some interest in some specific situations, since they allow one to evaluate the QFI without the diagonalization of the family of states ρ_λ, and to quantify how much a weak source of noise may degrade the QFI. Potential applications may be found in the characterization of qubit gates in the presence of phase diffusion[7] and in quantum probing techniques for weakly perturbing environments.[8–10]

2. QFI for Nearly Pure Quantum Statistical Models

Let us consider a quantum system with states in the Hilbert space $\mathcal{H}$ and a quantum statistical model $\rho_\lambda \in L(\mathcal{H})$ with $\lambda \in \Lambda$ and

$$\rho_\lambda = \frac{P_\lambda + \epsilon R}{1+\epsilon}, \tag{1}$$

where ϵ is small, $P_\lambda = |\psi_\lambda\rangle\langle\psi_\lambda|$ is a pure state where the information about the parameter λ is encoded, and R is a generic mixed state which describes some form of noise affecting the quantum probe. We assume that in the relevant range of variation for λ, R has support, at least approximately, on the orthogonal complement to that one-dimensional subspaces spanned by P_λ. In other words, we assume $\langle\psi_\lambda|R|\psi_\lambda\rangle \simeq 0$, $\forall\lambda \in \Lambda$. Upon assuming that R does not depend on the parameter to be estimated, i.e. it represents some form of *noise*, then we may find a general form for the QFI of the *nearly pure* quantum statistical model ρ_λ.

The argument proceeds as follows: Since ϵ is small, we may rewrite, the statistical model, up to first order, as $\rho_\lambda \simeq (1-\epsilon)P_\lambda + \epsilon R$. In turn, upon exploiting the relations $P_\lambda^2 = P_\lambda$ and $R \perp P_\lambda$, we have $\rho_\lambda^2 \simeq P_\lambda(1-2\epsilon)$. The purity of model is thus given by $\mu = \mathrm{Tr}[\rho_\lambda^2] = 1-2\epsilon$. Since R does not carry information about the parameter, we have $\partial_\lambda R = 0$ and therefore $\partial_\lambda\rho_\lambda = (1-\epsilon)\partial_\lambda P_\lambda$ and

$$\partial_\lambda\rho_\lambda^2 = \rho_\lambda\partial_\lambda\rho_\lambda + \partial_\lambda\rho_\lambda\rho_\lambda \simeq (1-2\epsilon)\partial_\lambda P_\lambda \simeq (1-\epsilon)\partial_\lambda\rho_\lambda, \tag{2}$$

where the last two equalities are valid up to first order in ϵ. Upon comparing the above equation with the definition of the SLD, i.e. $L_\lambda\rho_\lambda + \rho_\lambda L_\lambda = 2\partial_\lambda\rho_\lambda$, we have

$$L_\lambda(\epsilon) = \frac{2}{1-\epsilon}\partial_\lambda\rho_\lambda \simeq 2\,\partial_\lambda P_\lambda \equiv L_\lambda. \tag{3}$$

Equation (3) says that when the perturbation is small enough, the optimal measurement to estimate the parameter does not change. Note also that this result is consistent with the approximation $R \perp P_\lambda$, being the resulting SLD defined on the sole support of P_λ. We now insert this expression into the definition of the QFI

$H_\epsilon(\lambda) = \mathrm{Tr}[\rho_\lambda L^2_\lambda(\epsilon)]$, thus arriving at

$$H_\epsilon(\lambda) = (1-\epsilon)H_0(\lambda) + 4\epsilon\langle\partial_\lambda\psi_\lambda|R|\partial_\lambda\psi_\lambda\rangle. \tag{4}$$

This is a convex combination of the pure state QFI $H_0(\lambda)$ and the term $h_R(\lambda) = 4\langle\partial_\lambda\psi_\lambda|R|\partial_\lambda\psi_\lambda\rangle$. It may exceed the value $H_0(\lambda)$, thus making noise to increase the QFI), iff $h_R(\lambda) > H_0(\lambda)$. In order to see whether this is really possible, let us rewrite the QFI $H_\epsilon(\lambda)$ in a different form

$$\begin{aligned} H_\epsilon(\lambda) &= H_0(\lambda) + 4\epsilon\left[\langle\partial_\lambda\psi_\lambda|R|\partial_\lambda\psi_\lambda\rangle - \frac{1}{4}H_0(\lambda)\right], \\ &= H_0(\lambda) + 4\epsilon\langle\partial_\lambda\psi_\lambda|R + P_\lambda - \mathbb{I}|\partial_\lambda\psi_\lambda\rangle. \end{aligned} \tag{5}$$

The second term in Eq. (5) is the expectation value, on the unnormalized vector $|\partial_\lambda\psi_\lambda\rangle$, of the operator $N_\lambda = R + P_\lambda - \mathbb{I}$ which, being $R \perp P_\lambda$, is negative semi-definite. In other words, $\langle\partial_\lambda\psi_\lambda|N_\lambda|\partial_\lambda\psi_\lambda\rangle \leq 0$, $\forall\lambda$ with equality achieved iff $R = \mathbb{I} - P_\lambda$. Overall, we have the rather unsurprising conclusion that the QFI may only decrease by adding parameter-independent noise to the quantum state of the probe. On the other hand, Eqs. (4) and (5) may represent a tool to evaluate the scaling of the QFI without resorting to the diagonalization of the quantum statistical model.

The above results may be further validated upon expressing the QFI in terms of fidelity, i.e.

$$H_\epsilon(\lambda) = \lim_{d\lambda\to 0} 8\,\frac{1-\sqrt{F(\rho_\lambda, \rho_{\lambda+d\lambda})}}{d\lambda^2}, \tag{6}$$

where $F(A,B) = \mathrm{Tr}[|\sqrt{A}\sqrt{B}|]^2$. In turn, we have the inequalities[11]

$$E(\rho_1,\rho_2) \leq F(\rho_1,\rho_2) \leq G(\rho_1,\rho_2), \tag{7}$$

where the sub- and super-fidelity are given by

$$E(\rho_1,\rho_2) = \mathrm{Tr}[\rho_1\rho_2] + \sqrt{2(\mathrm{Tr}[\rho_1\rho_2]^2 - \mathrm{Tr}[\rho_1\rho_2\rho_1\rho_2])}, \tag{8}$$

$$G(\rho_1,\rho_2) = \mathrm{Tr}[\rho_1\rho_2] + \sqrt{(1-\mathrm{Tr}[\rho_1^2])(1-\mathrm{Tr}[\rho_2^2])}. \tag{9}$$

Using the above results, up to first order in ϵ, we have

$$E(\rho_\lambda,\rho_{\lambda+d\lambda}) = (1-2\epsilon)|\langle\psi_\lambda|\psi_{\lambda+d\lambda}\rangle|^2, \tag{10}$$

$$G(\rho_\lambda,\rho_{\lambda+d\lambda}) = (1-2\epsilon)|\langle\psi_\lambda|\psi_{\lambda+d\lambda}\rangle|^2 + 2\epsilon. \tag{11}$$

Upon inserting these expression into (6), we obtain

$$(1-\epsilon)H_0(\lambda) - \epsilon K_-(\lambda) \leq H_\epsilon(\lambda) \leq H_0(\lambda) + \epsilon K_+(\lambda), \tag{12}$$

where $K_\pm(\lambda)$ are positive definite functions given by

$$K_+(\lambda) = \lim_{d\lambda\to 0} \frac{8}{d\lambda^2} |\langle\psi_\lambda|\psi_{\lambda+d\lambda}\rangle|^2 \to \infty, \tag{13}$$

$$K_-(\lambda) = \lim_{d\lambda\to 0} \frac{8}{d\lambda^2} \left(\frac{1}{|\langle\psi_\lambda|\psi_{\lambda+d\lambda}\rangle|} - 1\right) = H_0(\lambda). \tag{14}$$

The right side inequality in Eq. (12) is not providing useful information, whereas the left one is instead saying that $H_\epsilon(\lambda) \geq (1-2\epsilon)H_0(\lambda)$. Putting together this result with that coming from Eq. (5), we finally arrive at

$$(1-2\epsilon)H_0(\lambda) \leq H_\epsilon(\lambda) \leq H_0(\lambda), \tag{15}$$

providing a bound on how much a weak source of noise may degrade the QFI.

3. Conclusions

In conclusion, we have derived approximated analytic expressions for the symmetric logarithmic derivative $L_\lambda(\epsilon)$, and the QFI $H_\epsilon(\lambda)$ of quantum statistical models of the form $\rho_\lambda = (1-\epsilon)P_\lambda + \epsilon R$, where ϵ is small and R does not depend on the parameter to be estimated. We found that $L_\lambda(\epsilon) = L_\lambda(0)$, whereas the QFI slightly decreases compared to the unperturbed value $H_0(\lambda)$ and it is bounded by the inequality $(1-2\epsilon)H_0(\lambda) \leq H_\epsilon(\lambda) \leq H_0(\lambda)$. Overall, our results are not unexpected, yet they may be of some interest in some specific situation, since they allow one to evaluate the QFI without the diagonalization of the family of states ρ_λ, and provide a bound on how much a weak source of noise may degrade the QFI.

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